Seidel Triangle Sequences and Bi-Entringer Numbers

Dominique Foata and Guo-Niu Han

En hommage à Pierre Rosenstiehl, Lui, qui dirige avec grand style, Ce journal de combinatoire, Mais sait aussi à l'occasion Nous raconter une belle histoire: Fil d'Ariane et boustrophédon.

Abstract. This Seidel Triangle Sequence Calculus makes it possible to derive several three-variate generating functions, in particular for the Bi-Entringer numbers, which count the alternating permutations according to their lengths, first and last letters. The paper has been motivated by this suprising observation: the number of alternating permutations, whose last letter has a prescribed value and is greater than its first letter, is equal to the Poupard number.

1. Introduction

As they have been reinterpreted in our previous paper [FH13a], the Poupard numbers $g_n(k)$ and $h_n(k)$ for $n \geq 1, 1 \leq k \leq 2n - 1$ can be defined as the coefficients in the following expansions

$$(1.1) \ 1 + \sum_{n \ge 1} \sum_{1 \le k \le 2n+1} g_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!} = \frac{\cos(x-y)}{\cos(x+y)};$$

$$(1.2) \ 1 + \sum_{n>1} \sum_{1 \le k \le 2n+1} h_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!} = \frac{\cos(x-y)}{\cos^2(x+y)}.$$

They are refinements of the tangent and secant numbers

(1.3)
$$\sum_{k} g_n(k) = T_{2n-1} \quad (n \ge 1),$$

(1.4)
$$\sum_{k} h_n(k) = E_{2n} \quad (n \ge 1),$$

which are themselves the coefficients of the Taylor expansions of $\tan u$ and $\sec u$:

$$\tan u = \sum_{n \ge 1} \frac{u^{2n-1}}{(2n-1)!} T_{2n-1} = \frac{u}{1!} 1 + \frac{u^3}{3!} 2 + \frac{u^5}{5!} 16 + \frac{u^7}{7!} 272 + \frac{u^9}{9!} 7936 + \cdots$$

$$\sec u = \sum_{n>0}^{\infty} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!} 1 + \frac{u^4}{4!} 5 + \frac{u^6}{6!} 61 + \frac{u^8}{8!} 1385 + \frac{u^{10}}{10!} 50521 + \cdots$$

$$(See,\,e.g.,\,[Ni23,\,p.\,\,177\text{-}178],\,[Co74,\,p.\,\,258\text{-}259]).$$

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Several combinatorial models have been introduced to interpret the Poupard numbers: see [Po89, FH13a, FH13b, FH13c]. The first numerical values of those numbers are displayed in Fig. 1.1-2.

k =	1	2	3	4	5	6	7	Sum
n = 1	1							1
2	0	2	0					2
3	0	4	8	4	0			16
4	0	32	64	80	64	32	0	272

Fig. 1.1. The Poupard Numbers $g_n(k)$.

k =	1	2	3	4	5	6	7	Sum
n = 1	1							1
2	1	3	1					5
3	5	15	21	15	5			61
4	61	183	285	327	285	183	61	1385

Fig. 1.2. The Poupard Numbers $h_n(k)$.

According to Désiré André [An1879, An1881] each permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$ is said to be *(increasing) alternating* if $x_1 < x_2$, $x_2 > x_3$, $x_3 < x_4$, etc. in an alternating way. Let Alt_n be the set of all alternating permutations of $12\cdots n$. He then proved that $\# \mathrm{Alt}_n = T_n$ (resp. $= E_n$), if n is odd (resp. even). Let $\mathbf{F} w := x_1$ and $\mathbf{L} w := x_n$ be the first and last letters of a permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$.

The numbers $E_n(m) := \#\{w \in Alt_n : \mathbf{F} w = m\}$, now called *Entringer numbers*, were introduced by Entringer himself [En66], who derived their main combinatorial and arithmetical properties. Those numbers are registered as the A008282 sequence in Sloane's On-Line Encyclopedia of Integer Sequences, together with an abundant bibliography [Sl]. They naturally constitute another refinement of the tangent and secant numbers. Their first values are shown in Fig. 1.3.

m =	1	2	3	4	5	6	Sum
n=1	1						1
2	1						1
3	1	1					2
4	2	2	1				5
5	5	5	4	2			16
6	16	16	14	10	5		61
7	61	61	56	46	32	16	272

Fig. 1.3. The Entringer Numbers $E_n(m)$.

We have been led to introduce the *Bi-Entringer numbers*, defined by

(1.5)
$$E_n(m,k) := \#\{w \in Alt_n : \mathbf{F} w = m, \mathbf{L} w = k\},\$$

first, to see whether we could obtain a closed form for their generating function, second, to understand why, and prove that, over the set Alt_{2n-1} and given the event $\{\mathbf{F} < \mathbf{L}\}$, the conditional probability that $\mathbf{F} = k$ is equal to $g_n(k)/T_{2n-1}$, where $g_n(k)$ is the Poupard number defined in (1.1). In Section 5 we shall give two proofs of the latter statement (see Theorem 1.2), a combinatorial one and also an analytic one using the Laplace transform.

Now, to derive the generating function for the Bi-Entringer numbers a study of the so-called *Seidel Triangle Sequences* is to be made and will be developed in Section 2. Roughly speaking, Seidel's memoir [Se1877], as was superbly reactivated by Dumont [Du82], establishes a connection between several sequences of classical numbers and polynomials, by means of a finite difference calculus displayed in matrix form. The method is to be enlarged when dealing with *sequences* of matrices instead of sequences of numbers. It will be seen that with each Seidel Triangle Sequence can be associated an explicit form for its generating function (Theorem 2.2).

The Bi-Entringer numbers, displayed as entries of matrices $M_n := (E_n(m,k))_{1 \le m,k \le n}$ (see Fig. 1.4) give rise to four Seidel Triangle Sequences: the sequences of the upper (resp. lower) triangles of the matrices M_n , for n odd and for n even. It will be shown that each matrix M_n for n odd is symmetric with respect to its diagonal, so that it suffices, when n is odd, to give the expression of the generating function for the upper triangles, as stated in the next theorem.

Theorem 1.1. The generating functions for the coefficients $E_n(m,k)$ are given by

$$(1.6) \sum_{1 \le m+1 \le k \le 2n-1} E_{2n}(m+1,k+1) \frac{x^{2n-k-1}}{(2n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}$$

$$= \frac{\cos x \cos z}{\cos(x+y+z)};$$

$$(1.7) \sum_{1 \le m+1 \le k \le 2n-1} E_{2n}(k+1,m+1) \frac{x^{2n-k-1}}{(2n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}$$

$$= \frac{\sin x \sin z}{\cos(x+y+z)};$$

$$(1.8) \sum_{1 \le m+1 \le k \le 2n} E_{2n+1}(m+1,k+1) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}$$

$$= \frac{\sin x \cos z}{\cos(x+y+z)}.$$

Theorem 1.1 will be proved in Section 3, once the main arithmetical properties of the Bi-Entringer numbers are given. The first values of the Bi-Entringer numbers are displayed in Fig. 1.4, as entries of the matrices $M_n := (E_n(m, k))_{1 \le m} k n$

Fig. 1.4. The Bi-Entringer Numbers $E_n(m, k)$.

Further arithmetical properties of the Bi-Entringer numbers, in particular involving binomial coefficients, will be given in Section 4. There is also a *linear* connection between Poupard and Bi-Entringer numbers, as stated in the next theorem, which is proved in Section 5.

Theorem 1.2. For $2 \le k \le 2n$ we have:

(1.9)
$$2\sum_{m=1}^{k} E_{2n+1}(m,k) = g_{n+1}(k).$$

The specialization of identities (1.6) and (1.8) for z = 0 provides an expression for the generating function for the *Entringer numbers* $E_n(m)$ themselves, apparently nowhere obtained, to our knowledge. The calculation is banal: just note that $E_{2n}(1, k + 1) = E_{2n-1}(2n - k)$ and $E_{2n+1}(1, k + 1) = E_{2n}(k)$.

Corollary 1.3. We have

(1.10)
$$\sum_{1 \le k \le 2n-1} E_{2n-1}(k) \frac{x^k}{(k-1)!} \frac{y^{2n-k-1}}{(2n-k-1)!} = \frac{\cos x}{\cos(x+y)};$$

(1.11)
$$\sum_{1 \le k \le 2n} E_{2n}(k) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-1}}{(k-1)!} = \frac{\sin x}{\cos(x+y)}.$$

Although the Poupard numbers $h_n(k)$, defined in (1.2), will not be further considered in this paper, it was important to mention that the pairs $(g_n(k), h_n(k))$ and $(E_{2n-1}(k), E_{2n}(k))$ form two refinements of the pairs (T_{2n-1}, E_{2n}) having analogous generating function displayed in (1.1), (1.2), (1.10), (1.11).

2. Seidel Triangle Sequences

Throughout the paper the following exponential generating functions will be attached to each infinite matrix $A = (a(m, k))_{m,k>0}$

$$A(x,y) := \sum_{m,k \ge 0} a(m,k) \frac{x^m}{m!} \frac{y^k}{k!};$$

$$A_{m,\bullet}(y) := \sum_{k \ge 0} a(m,k) \frac{y^k}{k!}; \qquad A_{\bullet,k}(x) := \sum_{m \ge 0} a(m,k) \frac{x^m}{m!};$$

for A itself, its m-th row, its k-th column. As can be found in [Du82], a Seidel matrix A = (a(m, k)) $(m, k \ge 0)$ is defined to be an infinite matrix, whose entries belong to some ring, and obey the following rules:

(SM1) the sequence of the entries from the top row a(0,0), a(0,1), a(0,2),... is given; it is called the *initial sequence*;

(SM2) for $m \ge 1$ and $k \ge 0$ the following relation holds:

$$a(m,k) = a(m-1,k) + a(m-1,k+1).$$

The entries of the Seidel matrix A can be obtained by applying rule (SM2) inductively, starting with the initial sequence. The leftmost column $a(0,0), a(1,0), a(2,0), \ldots$ is called the *final sequence*. As stated in the next proposition, the exponential generating functions for the final sequence $A_{\bullet,0}(x)$ and for the Seidel matrix itself A(x,y) can be derived from the generating function $A_{0,\bullet}(y)$ for the initial sequence. See, e.g., [Du82, DV80].

Proposition 2.1. Let $A = (a_{i,j})$ $(i, j \ge 0)$ be a Seidel matrix. Then,

$$A_{\bullet,0}(x) = e^x A_{0,\bullet}(x)$$
 and $A(x,y) = e^x A_{0,\bullet}(x+y)$.

As noted by Dumont [Du82], the following example of a Seidel matrix, denoted by $H = (h_{i,j})_{i,j>0}$, goes back to Seidel himself [Se1877]. The initial sequence consists of the sequence of the coefficients of the Taylor expansion of $1 - \tanh u = 2/(1 + e^{2u})$, that is, 1, -1, 0, 2, 0, -16, 0, 272, $0, \ldots$ so that

$$\overline{H}_{0,\bullet}(y) = 1 - \tanh y = 1 + \sum_{n \ge 1} \frac{y^{2n-1}}{(2n-1)!} (-1)^n T_{2n-1}$$

$$= 1 - \frac{y}{1!} 1 + \frac{y^3}{3!} 2 - \frac{y^5}{5!} 16 + \frac{y^7}{7!} 272 - \frac{y^9}{9!} 7936 + \cdots$$

It follows from Proposition 2.1 that

(2.1)
$$\overline{H}_{\bullet,0}(x) = \frac{1}{\cosh x} = \frac{2e^x}{1+e^{2x}}; \quad \overline{H}(x,y) = \frac{2e^x}{1+e^{2x+2y}};$$

and the matrix H itself reads:

(2.2)

The Entringer numbers $E_n(m)$ mentioned in the introduction appear as entries of the matrix H, displayed along the counter-diagonals with a given sign. In fact, we have the relation

(2.3)
$$\overline{h}_{i,j} = \begin{cases} (-1)^n E_{i+j+1}(j+1), & \text{if } i+j=2n; \\ (-1)^n E_{i+j+1}(i+1), & \text{if } i+j=2n-1; \end{cases}$$

as can be verified by induction, or sti

(2.4)
$$E_{2n+1}(j+1) = (-1)^n \overline{h}_{2n-j,j} \quad (0 \le j \le 2n);$$

(2.4)
$$E_{2n+1}(j+1) = (-1)^n \overline{h}_{2n-j,j} \quad (0 \le j \le 2n);$$
(2.5)
$$E_{2n}(i+1) = (-1)^n \overline{h}_{i,2n-1-i} \quad (0 \le i \le 2n-1).$$

The matrix \overline{H} will be given a key role in Section 3.

SEIDEL TRIANGLE MATRICES AND BI-ENTRINGER NUMBERS

We now come to the main definition of this section. A sequence of square matrices (A_n) $(n \ge 1)$ is called a *Seidel triangle sequence* if the following three conditions are fulfilled:

(STS1) each matrix A_n is of dimension n;

(STS2) each matrix A_n has null entries along and below its diagonal; let $(a_n(m,k) \ (0 \le m < k \le n-1)$ denote its entries strictly above its diagonal, so that

$$A_{1} = (\cdot); \quad A_{2} = \left(\begin{array}{c} \cdot & a_{2}(0,1) \\ \cdot & \cdot \end{array}\right); \quad A_{3} = \left(\begin{array}{c} \cdot & a_{3}(0,1) & a_{3}(0,2) \\ \cdot & \cdot & a_{3}(1,2) \\ \cdot & \cdot & \cdot \end{array}\right); \dots;$$

$$A_{n} = \left(\begin{array}{cccc} \cdot & a_{n}(0,1) & a_{n}(0,2) & \cdots & a_{n}(0,n-2) & a_{n}(0,n-1) \\ \cdot & \cdot & a_{n}(1,2) & \cdots & a_{n}(1,n-2) & a_{n}(1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & a_{n}(n-3,n-2) & a_{n}(n-3,n-1) \\ \cdot & \cdot & \cdot & \cdots & a_{n}(n-2,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdots & \vdots \end{array}\right);$$

the dots "." along and below the diagonal referring to null entries.

(STS3) for each $n \geq 2$, the following relation holds:

$$a_n(m,k) - a_n(m,k+1) = a_{n-1}(m,k) \quad (m < k).$$

Record the last columns of the triangles A_2 , A_3 , A_4 , A_5 , ..., read from top to bottom, namely, $a_2(0,1)$; $a_3(0,2)$, $a_3(1,2)$; $a_4(0,3)$, $a_4(1,3)$, $a_4(2,3)$; $a_5(0,4)$, $a_5(1,4)$, $a_5(2,4)$, $a_5(3,4)$; ... as counter-diagonals of an infinite matrix $H = (h_{i,j})_{i,j \geq 0}$, as shown next:

In an equivalent manner, the entries of H are defined by:

$$(2.7) h_{i,j} = a_{i+j+2}(j, i+j+1).$$

The next theorem shows that the three-variable generating function for a Seidel triangle sequence, when suitably normalized, can be expressed in a very closed form.

Theorem 2.2. The three-variable generating function for the Seidel triangle sequence $(A_n = (a_n(m,k)))_{n>1}$ is equal to

$$\sum_{1 \le m+1 \le k \le n-1} a_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = e^x H(x+y,z),$$

where H is the infinite matrix defined in (2.7).

Proof. We set up a sequence of infinite matrices $(\Omega^{(p)} = ((\omega_{i,j}^{(p)})_{i,j\geq 0}))$ $(p\geq 0)$ that record the *rows* of the matrices A_n in the following manner

$$\Omega^{(p)} = \begin{pmatrix} a_{p+2}(p, p+1) & a_{p+3}(p, p+2) & a_{p+4}(p, p+3) & \cdots \\ a_{p+3}(p, p+1) & a_{p+4}(p, p+2) \\ a_{p+4}(p, p+1) & \vdots \end{pmatrix},$$

so that the rows labeled p of the triangles A_n , if they exist, are displayed as counter-diagonals in $\Omega^{(p)}$. Alternatively, the coefficients $\omega_{i,j}^{(p)}$ are defined by

(2.8)
$$\omega_{i,j}^{(p)} = a_{p+i+j+2}(p, p+j+1).$$

By (2.7) and (2.8) $H(x,z) = \sum_{p\geq 0} \frac{z^p}{p!} H_{\bullet,p}(x) = \sum_{p\geq 0} \frac{z^p}{p!} \Omega_{0,\bullet}^{(p)}(x)$. From rule

(STS3) we get $a_{p+k}(p, p+m) - a_{p+k}(p, p+m+1) = a_{p+k-1}(m, p+m)$, so that each matrix $\Omega^{(p)}$ is a Seidel matrix. It follows by Proposition 2.1 that

$$\Omega^{(p)}(x,y) = e^x \Omega_{0,\bullet}^{(p)}(x+y).$$

Define: $\Omega(x, y, z) := \sum_{p>0} \frac{z^p}{p!} \Omega^{(p)}(x, y)$. Then,

$$\Omega(x,y,z) = \sum_{p \ge 0} \frac{z^p}{p!} \Omega^{(p)}(x,y) = \sum_{p \ge 0} \frac{z^p}{p!} e^x \Omega^{(p)}_{0,\bullet}(x+y) = e^x H(x+y,z).$$

On the other hand,

$$\Omega(x, y, z) = \sum_{p \ge 0} \frac{z^p}{p!} \Omega^{(p)}(x, y) = \sum_{i, j, p \ge 0} \frac{z^p}{p!} \frac{x^i}{i!} \frac{y^j}{j!} \omega_{i, j}^{(p)}$$
$$= \sum_{i, j, p \ge 0} \frac{z^p}{p!} \frac{x^i}{i!} \frac{y^j}{j!} a_{p+i+j+2}(p, p+j+1).$$

With the change of variables p + i + j + 2 = n, p = m, p + j + 1 = k, we then get

$$\Omega(x,y,z) = \sum_{1 \le m+1 \le k \le n-1} a_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

This completes the proof of Theorem 2.2. \square

3. The Bi-Entringer numbers

Before proving Theorem 1.1 we give a list of properties involving the Bi-Entringer numbers. The celebrated identities à la boustrophedon (see [MSY96], and [Ro13, p. 95-101] for a more literary approach) satisfied by the Entringer numbers $E_n(m)$ can be extended over to the Bi-Entringer numbers $E_n(m,k)$, as stated in relations (3.1)—(3.4) below.

Proposition 3.1. We have:

- (i) $E_n(m,m) = 0$ for all m and $E_n(n,k) = 0$ for all k and $n \ge 2$;
- (ii) for n odd and $1 \le k, m \le n$

(3.1)
$$E_n(m,k) = \sum_{j=k}^{n-1} E_{n-1}(m,j) \quad \text{if } m < k;$$

(3.2)
$$E_n(m,k) = \sum_{j=k}^{n-1} E_{n-1}(m-1,j) \quad \text{if } m > k.$$

In particular, $E_n(n-2, n-1) = 0$ for $n \ge 5$; $E_n(m, n) = 0$ for all n, m.

(iii) For n even and $1 \le m, k \le n$ we further have:

(3.3)
$$E_n(m,k) = \sum_{j=1}^{k-1} E_{n-1}(m,j) \quad \text{if } m < k;$$

(3.4)
$$E_n(m,k) = \sum_{j=1}^{k-1} E_{n-1}(m-1,k) \quad \text{if } m > k.$$

In particular, $E_n(n-1,n)=0$ when $n \geq 4$; $E_n(i,1)=0$ for all i.

(iv) Each matrix M_n is symmetric with respect to its diagonal (resp. its counter-diagonal), whenever n is odd (resp. even), that is,

(3.5)
$$E_n(m,k) = \begin{cases} E_n(k,m), & \text{when } n \text{ is odd,} \\ E_n(n+1-k,n+1-m), & \text{when } n \text{ is even.} \end{cases}$$

Moreover,

(3.6)
$$\sum_{k} E_n(m,k) = E_n(m) \quad (n \ge 1);$$

$$\sum_{m} E_n(m,k) = \begin{cases} E_n(k), & \text{when } n \text{ is odd;} \\ E_n(n+1-k), & \text{when } n \text{ is even.} \end{cases}$$

The proofs of all those properties are easy, by simple manipulations; in particular, (3.5) by using the basic dihedral transformations on alternating permutations. They are omitted.

Proposition 3.2 (The finite difference relations). We have:

(3.7)
$$E_n(m,k) - E_n(m,k+1) = (-1)^{n-1} E_{n-1}(m,k),$$

if $1 \le m < k \le n-1;$

(3.8)
$$E_n(m,k) - E_n(m,k+1) = (-1)^{n-1} E_{n-1}(m-1,k),$$

if $2 \le k+1 \le m \le n.$

Proof. The two identities can be proved by simple iterations of (3.1)-(3.4). Alternatively, we can also proceed as follows. Let $m \operatorname{Alt}_n k$ (resp. $m \operatorname{Alt}_n l k$) designate the number of all σ from Alt_n starting with m and ending with k (resp. ending with the right factor l k). We have:

$$m \operatorname{Alt}_n k - m \operatorname{Alt}_n (k+1) = \begin{cases} -m \operatorname{Alt}_n k (k+1), & \text{if } n \text{ is even;} \\ m \operatorname{Alt}_n (k+1) k, & \text{if } n \text{ is odd.} \end{cases}$$

Next, if n is even,

$$-m \operatorname{Alt}_{n} k (k+1) = \begin{cases} -m \operatorname{Alt}_{n-1} k, & \text{if } 1 \leq m < k \leq n-1; \\ -(m-1) \operatorname{Alt}_{n-1} k, & \text{if } 2 \leq k+1 < m \leq n; \end{cases}$$
 and if n is odd,

$$m \operatorname{Alt}_{n}(k+1) k = \begin{cases} m \operatorname{Alt}_{n-1} k, & \text{if } 1 \leq m < k \leq n-1; \\ (m-1) \operatorname{Alt}_{n-1} k, & \text{if } 2 \leq k+1 < m \leq n. \end{cases}$$

Now, let the sequence of matrices $(W_n) = (e_n(m, k))$ be obtained from the matrices $(M_n) = (E_n(m, k))$ by making the following modifications:

(W1)
$$W_1 := (0)$$
;

(W2)
$$W_n := M_n$$
 for $n \equiv 2, 3 \pmod{4}$ and $n \ge 2$;

(W3)
$$W_n := (-1)M_n \text{ for } n \equiv 0, 1 \pmod{4};$$

(W4) delete the lower triangle from each matrix W_n ;

(W5) make the labels start from $0, 1, 2, \dots$

In other words, for m < k define the normalized Bi-Entringer Numbers $e_n(m, k)$ to be:

(3.9)
$$e_{2n}(m,k) := (-1)^{n+1} E_{2n}(m+1,k+1);$$

$$(3.10) e_{2n+1}(m,k) := (-1)^{n+1} E_{2n+1}(m+1,k+1).$$

Their first values appear in Fig. 3.1.

Fig. 3.1. The normalized Bi-Entringer Numbers $e_n(m, k)$.

By Proposition 3.2, the sequence (W_n) is a Seidel triangle sequence, and the corresponding matrix H, defined by (2.6)–(2.7), is equal to

$$H = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & e_2(0,1) & \cdot & e_4(2,3) & \cdot & e_6(4,5) & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & e_2(0,1) & \cdot & e_4(2,3) & \cdot & e_6(4,5) & \cdot & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 2 & 3 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 & 4 & 4 & 5 \\ \vdots & 2 &$$

the "dots" being written in place of 0's. Note that all the counter-diagonals $e_{2n+1}(0,2n)$, $e_{2n+1}(1,2n)$, ..., $e_{2n+1}(2n-1,2n)$, which are equal to the last columns of the W_{2n+1} 's, have null entries.

It then remains to evaluate $e_{2n}(m, 2n-1)$ for $0 \le m \le 2n-1$. By (3.9) we have: $e_{2n}(m, 2n-1) = (-1)^{n+1} E_{2n}(m+1, 2n)$. But $E_{2n}(m+1, 2n)$ is equal to the Entringer number $E_{2n-1}(m+1)$, as each alternating permutation σ from Alt_{2n} such that $\mathbf{F}\sigma = m+1$ and $\mathbf{L}\sigma = 2n$ can be mapped onto a permutation from Alt_{2n-1} starting with (m+1) by simply deleting the last letter (2n), and this in a bijective manner.

Now, by (2.4), $E_{2n-1}(m+1) = E_{2(n-1)+1}(m+1) = (-1)^{n-1} \overline{h}_{2n-2-m,m}$ for $0 \le m \le 2n-2$. Altogether,

$$(3.11) e_{2n}(m, 2n-1) = \overline{h}_{2n-2-m,m} (0 \le m \le 2n-2),$$

that is, by (2.2),

Thus, H is obtained from the matrix \overline{H} , displayed in (2.2), by replacing all the entries $\overline{h}_{i,j}$ such that i+j is odd by zero. By (2.1) we have $\overline{H}(x,y) = \frac{2e^x}{1+e^{2x+2y}}$, so that

(3.12)
$$H(x,y) = \frac{\overline{H}(x,y) + \overline{H}(-x,-y)}{2} = e^x \frac{1 + e^{2y}}{1 + e^{2x + 2y}}.$$

Let

(3.13)
$$\Omega(x,y,z) = \sum_{1 \le m+1 \le k \le n-1} e_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

be the three-variate generating function for the $e_n(m, k)$'s. By Theorem 2.2

(3.14)
$$\Omega(x,y,z) = e^x H(x+y,z) = e^{2x+y} \frac{1+e^{2z}}{1+e^{2x+2y+2z}}.$$

With $I := \sqrt{-1}$ equation (3.13) reads:

$$\Omega(xI, yI, zI) = \sum_{1 \le m+1 \le k \le n-1} I^{n-2} e_n(m, k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

But, by (3.9) and (3.10)

$$I^{2n-2}e_{2n}(m,k) = E_{2n}(m+1,k+1);$$

$$I^{2n-1}e_{2n+1}(m,k) = IE_{2n+1}(m+1,k+1)$$

Therefore,

$$\Omega(xI, yI, zI) = \sum_{1 \le m+1 \le k \le 2n-1} E_{2n}(m+1, k+1) \frac{x^{2n-k-1}}{(2n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} + I \sum_{1 \le m+1 \le k \le 2n} E_{2n+1}(m+1, k+1) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

Next, (3.14) becomes:

$$\Omega(xI, yI, zI) = e^{2xI + yI} \frac{1 + e^{2zI}}{1 + e^{2xI + 2yI + 2zI}}$$

$$= e^{xI} \frac{e^{-zI} + e^{zI}}{e^{-xI - yI - zI} + e^{xI + yI + zI}}$$

$$= (\cos(x)\cos(z) + I\sin(x)\cos(z))/\cos(x + y + z).$$

By comparing the above two identities for $\Omega(xI, yI, zI)$, we obtain (1.6) and (1.8) in Theorem 1.1.

To prove (1.7) in Theorem 1.1 let $(W'_n) = (e'_n(m, k))$ be the sequence of matrices obtained from $(M_n) = (E_n(m, k))$ by the following modifications:

$$(W'1) W'_1 := (0);$$

(W'2) $W'_n := M^r_n$ for $n \equiv 1, 2 \pmod{4}$ and $n \geq 2$, where M^r_n is obtained from M_n by performing a rotation by 180^o about its center;

(W'3)
$$W'_n := (-1)M^r_n \text{ for } n \equiv 0,3 \pmod{4};$$

(W'4) delete the lower triangle of each matrix W'_n ;

$$(W'5)$$
 start labelling from $0, 1, 2, \dots$

In other words, for m < k define the normalized Bi-Entringer numbers $e'_n(m,k)$ to be:

(3.15)
$$e'_{2n}(m,k) = (-1)^{n+1} E_{2n}(2n-m,2n-k);$$

(3.16)
$$e'_{2n+1}(m,k) = (-1)^n E_{2n+1}(2n+1-m,2n+1-k).$$

The first values of the W'_n are shown in Fig. 3.2.

$$W_{4}' = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & -1 & 0 \\ 2 & \vdots & \vdots & \vdots & 0 \\ 3 & \vdots & \vdots & \vdots & \vdots \end{pmatrix}; \quad W_{5}' = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots$$

$$W_6' = \begin{pmatrix} \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 2 & 2 & 1 & 0 \\ \cdot & \cdot & \cdot & 4 & 2 & 0 \\ \cdot & \cdot & \cdot & 2 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}; W_7' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & -2 & -4 & -5 & -5 \\ \cdot & \cdot & -4 & -8 & -10 & -10 \\ \cdot & \cdot & \cdot & -12 & -14 & -14 \\ \cdot & \cdot & \cdot & \cdot & -16 & -16 \end{pmatrix}.$$

Fig. 3.2. The normalized Bi-Entringer Numbers $e'_n(m, k)$.

By Proposition 3.2, (W'_n) is a Seidel triangle sequence, and the corresponding matrix H defined by (2.6)–(2.7), we shall denote by $H' = (h'_{i,j})$, is equal to:

$$H' = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \vdots & e_3'(1,2) & \vdots & e_5'(3,4) & \vdots & e_7'(5,6) & \cdots \\ e_3'(0,2) & \vdots & e_5'(2,4) & \vdots & e_7'(4,6) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_5'(0,4) & \vdots & e_7'(2,6) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Its counter-diagonals $e'_{2n}(0,2n-1), e'_{2n}(1,2n-1), \ldots, e'_{2n}(2n-2,2n-1)$ have null entries, as it is so for the last columns of the W'_{2n} 's. Furthermore, $e'_{2n+1}(m,2n)=\overline{h}_{2n-m-1,m}$ $(0\leq m\leq 2n-1)$ by (2.5) and (3.16), so that

$$H' = \begin{pmatrix} \cdot & -1 & \cdot & 2 & \cdot & -16 & \cdot & \cdots \\ 0 & \cdot & 2 & \cdot & -16 & \cdot & & \\ \cdot & 1 & \cdot & -14 & \cdot & & & \\ 0 & \cdot & -10 & \cdot & & & & \\ \cdot & -5 & \cdot & & & & & \\ 0 & \cdot & & & & & \\ \vdots & & & & & & \end{pmatrix},$$

which is derived from \overline{H} by replacing the entries $\overline{h}_{i,j}$ such that i+j is even by 0. Therefore,

$$H'(x,y) = \frac{\overline{H}(x,y) - \overline{H}(-x,-y)}{2} = e^x \frac{1 - e^{2y}}{1 + e^{2x + 2y}}.$$

Let

(3.17)
$$\Omega'(x, y, z) = \sum_{1 \le m+1 \le k \le n-1} e'_n(m, k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

be the generating function for the $e'_n(m,k)$'s. By Theorem 2.2

(3.18)
$$\Omega'(x,y,z) = e^x H'(x+y,z) = e^{2x+y} \frac{1-e^{2z}}{1+e^{2x+2y+2z}}.$$

Then,

$$\Omega'(xI, yI, zI) = \sum_{1 \le m+1 \le k \le n-1} I^{n-2} e_n(m, k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

By (3.15) and (3.16)

$$I^{2n-2}e'_{2n}(m,k) = E_{2n}(2n-m,2n-k);$$

$$I^{2n-1}e'_{2n+1}(m,k) = -I E_{2n+1}(2n+1-m,2n+1-k).$$

Therefore,

$$\Omega'(xI, yI, zI) = \sum_{1 \le m+1 \le k \le 2n-1} E_{2n}(2n-m, 2n-k) \frac{x^{2n-k-1}}{(2n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} - I \sum_{1 \le m+1 \le k \le 2n} E_{2n+1}(2n+1-m, 2n+1-k) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

On the other hand, (3.18) becomes:

$$\Omega'(xI, yI, zI) = e^{2xI+yI} \frac{1 - e^{2zI}}{1 + e^{2xI+2yI+2zI}}$$

$$= e^{xI} \frac{e^{-zI} - e^{zI}}{e^{-xI-yI-zI} + e^{xI+yI+zI}}$$

$$= (\sin(x)\sin(z) - I\cos(x)\sin(z))/\cos(x + y + z).$$

Compare the above two identities for $\Omega(xI, yI, zI)$ and use (3.6). This proves (1.8) in Theorem 1.1, and yields another proof of the identity $E_{2n+1}(m,k) = E_{2n+1}(k,m)$.

4. Row sums with binomial coefficients

We next show that the closed forms for the generating functions for the Bi-Entringer numbers derived in (1.6)–(1.8) provide several identities for the numbers themselves, all involving binomial coefficients.

Proposition 4.1. We have:

$$\sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k}{m} E_{2n+2}(m+1,k+2) = (-1)^n \chi(m=0).$$

Proof. Identity (1.6) may be rewritten as:

$$(4.1) \sum_{0 \le m \le k \le 2n} \frac{z^m}{m!} \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m}}{(k-m)!} E_{2n+2}(m+1,k+2) = \frac{\cos(x)\cos(z)}{\cos(x+y+z)}.$$

With y = -x and z = -xz identity (4.1) becomes:

$$(4.2) \sum_{0 \le m \le k \le 2n} \frac{z^m}{m!} \frac{x^{2n}}{(2n-k)!} \frac{(-1)^k}{(k-m)!} E_{2n+2}(m+1, k+2) = \frac{\cos(x)\cos(-xz)}{\cos(-xz)}.$$

Let $\alpha_n(m)$ be the left-hand side of the identity to prove. Then, (4.2) can be expressed as: $\sum_{n,m>0} z^m \frac{x^{2n}}{(2n)!} \alpha_n(m) = \cos(x)$.

Example. n=2.

m = 1 :

$$-\binom{4}{1}\binom{1}{1}2 + \binom{4}{2}\binom{2}{1}4 - \binom{4}{3}\binom{3}{1}5 + \binom{4}{4}\binom{4}{1}5 = -8 + 48 - 60 + 20 = 0;$$

$$m = 0:$$

$$+\binom{4}{0}0 - \binom{4}{1}2 + \binom{4}{2}4 - \binom{4}{3}5 + \binom{4}{4}5 = 0 - 8 + 24 - 20 + 5 = 1.$$

Proposition 4.2. We have

$$\sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k}{m} E_{2n+2}(k+2, m+1) = (-1)^{n-(m-1)/2} \binom{2n}{m} T_m,$$

with the convention that $T_m = 0$, if m is even, and equal to the tangent number T_m otherwise.

Proof. Rewrite identity (1.7) as

$$(4.3) \sum_{0 \le m \le k \le 2n} \frac{z^m}{m!} \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m}}{(k-m)!} E_{2n+2}(k+2, m+1) = \frac{\sin(x)\sin(z)}{\cos(x+y+z)},$$

and let y = -x and z = -xz in (4.3), to get:

$$(4.4) \sum_{0 \le m \le k \le 2n} \frac{z^m}{m!} \frac{x^{2n}}{(2n-k)!} \frac{(-1)^k}{(k-m)!} E_{2n+2}(k+2, m+1) = \frac{\sin(x)\sin(-xz)}{\cos(-xz)}.$$

Let $\alpha_n(m)$ be the left-hand side of the identity to be proved. Then,

$$\sum_{n,m\geq 0} z^m \frac{x^{2n}}{(2n)!} \alpha_n(m) = -\sin(x) \tan(xz) = -\sin(x) \sum_m T_m \frac{(xz)^m}{m!}.$$

Hence,

$$\sum_{n>0} \frac{x^{2n}}{(2n)!} \alpha_n(m) = -\sin(x) T_m \frac{x^m}{m!} = \sum_{k>0} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} T_m \frac{x^m}{m!}. \quad \Box$$

Example. n = 3.

$$m = 1:$$

$$-\binom{6}{1}\binom{1}{1}16 + \binom{6}{2}\binom{2}{1}16 - \binom{6}{3}\binom{3}{1}14 + \binom{6}{4}\binom{4}{1}10 + \binom{6}{5}\binom{5}{1}5 = -6;$$

$$m = 2:$$

$$+\binom{6}{2}\binom{2}{2}32 - \binom{6}{3}\binom{3}{2}28 + \binom{6}{4}\binom{4}{2}20 - \binom{6}{5}\binom{5}{2}10 = 0.$$

Proposition 4.3. We have, for $m \ge 1$

$$\sum_{k=m}^{2n-1} (-1)^k {2n-1 \choose k} {k \choose m} E_{2n+1}(m+1,k+2) = (-1)^{n+1} \chi(m=0).$$

Proof. Rewrite identity (1.8) as

$$(4.5) \sum_{0 \le m \le k \le 2n-1} \frac{z^m}{m!} \frac{x^{2n-1-k}}{(2n-1-k)!} \frac{y^{k-m}}{(k-m)!} E_{2n+1}(m+1,k+2) = \frac{\sin(x)\cos(z)}{\cos(x+y+z)};$$

and let y = -x and z = -xz in (4.5), to get:

$$\sum_{0 \le m \le k \le 2n-1} \frac{z^m}{m!} \frac{x^{2n-1}}{(2n-1-k)!} \frac{(-1)^k}{(k-m)!} E_{2n+1}(m+1,k+2) = \sin(x).$$

Then.

(4.6)
$$\sum_{0 \le m \le k \le 2n-1} \frac{z^m}{m!} \frac{1}{(2n-1-k)!} \frac{(-1)^k}{(k-m)!} E_{2n+1}(m+1,k+2) = (-1)^{n+1} \frac{1}{(2n-1)!}.$$

Let $\alpha_n(m)$ be the left-hand side of the identity to prove. By (4.6), $\alpha_n(0) = (-1)^{n+1}$ and $\alpha_n(m) = 0$ for $m \ge 1$.

Example. n = 3.

$$m = 0:$$

$$\binom{5}{0}\binom{0}{0}16 - \binom{5}{1}\binom{1}{0}16 + \binom{5}{2}\binom{2}{0}14 - \binom{5}{3}\binom{3}{0}10 + \binom{5}{4}\binom{4}{0}5 = 1;$$

$$m = 1:$$

$$-\binom{5}{1}\binom{1}{1}16 + \binom{5}{2}\binom{2}{1}14 - \binom{5}{3}\binom{3}{1}10 + \binom{5}{4}\binom{4}{1}5 = 0.$$

5. Proofs of Theorem 1.2

In section 4 we have derived several identities for the Bi-Entringer numbers all involving binomial coefficients, in contrast to identity (1.9) that linearly relates Poupard numbers to tangent numbers and Bi-Entringer numbers. The next analytical proof makes use of the Laplace transform

$$\mathcal{L}(f(x), x, s) := \int_0^\infty f(x)e^{-xs} dx,$$

which, in particular, maps $x^k/k!$ onto $1/s^{k+1}$:

$$\mathcal{L}(\frac{x^k}{k!}, x, s) = \frac{1}{s^{k+1}}.$$

To illustrate this Laplace transform method we first give a proof of (1.3). Apply the Laplace transform twice to the left-hand side of (1.1), first, with respect to x, s, then to y, t. We get

$$1 + \sum_{n>1} \sum_{1 \le k \le 2n+1} g_{n+1}(k) \frac{1}{t^{2n+2-k}} \frac{1}{s^k},$$

an expression which becomes

$$1 + \sum_{n \ge 1} \sum_{1 \le k \le 2n+1} g_{n+1}(k) \frac{1}{s^{2n+2}}$$

for s = t. We need prove that

$$1 + \sum_{n \ge 1} \sum_{1 \le k \le 2n+1} g_{n+1}(k) \frac{1}{s^{2n+2}} = 1 + \sum_{n \ge 1} T_{2n+1} \frac{1}{s^{2n+2}},$$

which is equivalent to

$$\int_0^\infty \int_0^\infty \frac{\cos(x-y)}{\cos(x+y)} \, e^{-t(x+y)} dx \, dy = \int_0^\infty \tan(x) \, e^{-tx} dx.$$

But this identity is true, since by letting r = x + y:

$$\int_0^\infty \int_0^\infty \frac{\cos(x-y)}{\cos(x+y)} e^{-t(x+y)} dx \, dy = \int_0^\infty \int_0^r \frac{\cos(r-2y)}{\cos(r)} e^{-tr} dy \, dr$$

$$= \int_0^\infty \frac{e^{-tr}}{\cos(r)} \int_0^r \cos(r-2y) dy \, dr$$

$$= \int_0^\infty \frac{e^{-tr}}{\cos(r)} \sin(r) dr$$

$$= \int_0^\infty \tan(r) e^{-tr} dr,$$

so that identity (1.3) is proved.

Analytical proof of Theorem 1.2. Start with identity (4.5), which is another form of (1.8) and apply the Laplace transform to its left-hand side three times with respect to (x, s), (y, t), (z, u), respectively. We get

$$\sum_{0 \le m \le k \le 2n-1} \frac{1}{u^{m+1}} \frac{1}{s^{2n-k}} \frac{1}{t^{k-m+1}} E_{2n+1}(m+1, k+2),$$

which becomes

(5.1)
$$\sum_{0 \le m \le k \le 2n-1} \frac{1}{s^{2n+2}} \frac{1}{t^{k+2}} E_{2n+1}(m+1, k+2),$$

when $t \leftarrow st$ and $u \leftarrow st$. Apply the Laplace transform to the right-hand side of (4.5) three times with respect to (x, s), (y, t), (z, u), respectively, and let $t \leftarrow st$, $u \leftarrow st$. With r = y + z we get:

$$\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\sin(x)\cos(z)}{\cos(x+y+z)} e^{-xs-yst-zst} dx dy dz$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{r} \frac{\sin(x)\cos(z)}{\cos(x+r)} e^{-xs-rst} dz dr dx$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \frac{\sin(x)\sin(r)}{\cos(x+r)} e^{-xs-rst} dr dx$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{2} \left(\frac{\cos(x-r)}{\cos(x+r)} - 1 \right) e^{-xs-rst} dr dx$$

$$= \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} \left(\sum_{n\geq 1} \sum_{1\leq k\leq 2n+1} g_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{r^{k-1}}{(k-1)!} \right) e^{-xs-rst} dr dx$$

$$= \frac{1}{2} \sum_{n\geq 1} \sum_{1\leq k\leq 2n+1} g_{n+1}(k) \frac{1}{s^{2n+2-k}} \frac{1}{(st)^k} =$$

$$(5.2) \qquad \frac{1}{2} \sum_{n\geq 1} \sum_{1\leq k\leq 2n+1} g_{n+1}(k) \frac{1}{s^{2n+2}} \frac{1}{t^k}.$$

Then, (1.9) is a consequence of the identity (5.1)=(5.2).

Combinatorial proof of Theorem 1.2. We make use of the greater neighbor statistic "grn," which was defined in our previous paper [FH13a] as follows: let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be an alternating permutation from Alt_n, so that $\sigma(i) = n$ for a certain i $(1 \le i \le n)$. By convention, let $\sigma(0) = \sigma(n+1) := 0$. Then, its definition reads:

$$grn(\sigma) := \max\{\sigma(i-1), \sigma(i+1)\}.$$

Let \mathfrak{A}_{2n+1} be the set of all *decreasing* alternating permutations of $\{1, 2, \dots, 2n+1\}$, i.e., permutations $w = x_1 x_2 \dots x_{2n+1}$ such that $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, etc. and let $\mathfrak{A}_{2n+1,k} := \{\sigma \in \mathfrak{A}_{2n+1} : \operatorname{grn}(\sigma) = k\}$. It was proved in [FH13a] (Theorem 1.4) that

$$(5.3) g_{n+1}(k+1) = \# \mathfrak{A}_{2n+1,k}.$$

For $n \geq 1$ each set $\mathfrak{A}_{2n+1,k}$ can be split into two subsets of the same cardinality $\mathfrak{A}_{2n+1,k}^{\leq} + \mathfrak{A}_{2n+1,k}^{\geq}$, depending on whether the greater neighbor k is on the left, or on the right of (2n+1).

On the other hand, let

$$G_{2n+1,k} := \{ \sigma \in \operatorname{Alt}_{2n+1} : \mathbf{L} \, \sigma = k > \mathbf{F} \, \sigma \}$$

be the set of all *increasing* alternating permutations from Alt_{2n+1} having their last letter equal to k and greater than their first letter. Then, each permutation

$$\sigma = x_1 \cdots x_{j-1} (2n+1) x_{j+1} \cdots x_{2n+1}$$

from $\mathfrak{A}_{2n+1,k}^{\leq}$, which is such that $x_{j-1} = k > x_{j+1}$ can be mapped onto the permutation $\tau = (x_{j+1}+1)\cdots(x_{2n+1}+1)\,1\,(x_1+1)\cdots(x_{j-1}+1)$ from $G_{2n+1,k+1}$ in a bijective manner. Thus, $\#G_{2n+1,k+1} = \#\mathfrak{A}_{2n+1,k}^{\leq}$ and $2\#G_{2n+1,k+1} = \#\mathfrak{A}_{2n+1,k}^{\leq} + \#\mathfrak{A}_{2n+1,k}^{\geq} = \#\mathfrak{A}_{2n+1,k}$, so that

$$g_{n+1}(k+1) = 2 \# G_{2n+1,k+1} = 2 \sum_{l \le k} E_{2n+1}(k+1,l),$$

as $E_{2n+1}(k+1, k+1) = 0$. This implies identity (1.9) since $E_{2n+1}(m, l) = E_{2n+1}(l, m)$ for all m, l (see Proposition 3.1(iv)).

Dumont [Du14] drew our attention to the following relation between Poupard and Bi-Entringer numbers, namely,

(5.4)
$$g_n(k) = 2 E_{2n}(k, k+1) = 2 E_{2n}(k+1, k).$$

Before giving a combinatorial proof of that identity we state and prove a property on alternating permutations, both increasing and decreasing, that involves the statistics \mathbf{F} , \mathbf{L} , and also two other statistics attached to the *left* \mathbf{f} and *right* \mathbf{l} neighbors of the maximum. Each permutation σ of $12 \cdots n$ may be written

(5.5)
$$\sigma = x_1 x_2 \cdots x_n = \mathbf{F}(\sigma) \cdots \mathbf{f}(\sigma) \max(\sigma) \mathbf{l}(\sigma) \cdots \mathbf{L}(\sigma),$$

where $\mathbf{F}(\sigma) = x_1$, $\mathbf{L}(\sigma) = x_n$ and, if $\max(\sigma) := n = x_k$, then $\mathbf{f}(\sigma) = x_{k-1}$, $\mathbf{l}(\sigma) = x_{k+1}$, where the convention $x_0 = x_{n+1} = 0$ still holds. For each finite word $w = y_1 y_2 \cdots y_m$, whose letters are integers, it is convenient to use the notation: $(w+1) := (y_1+1)(y_2+1)\cdots(y_m+1)$, when w is not the empty word e, and e when it is.

Property 5.1. Let $\sigma = w_1 \max(\sigma) w_2$ be a permutation of $12 \cdots (2n-1)$, so that $\max(\sigma) = 2n-1$. Then, the mapping

$$(5.6) \phi_1: w_1 \max(\sigma) w_2 \mapsto w_2 \max(\sigma) w_1$$

is a bijection of Alt_{2n-1} onto itself having the property

(5.7)
$$(\mathbf{f}, \mathbf{l})\sigma = (\mathbf{L}, \mathbf{F})\phi_1(\sigma),$$

while the mapping

(5.8)
$$\phi_2: w_1 \max(\sigma) w_2 \mapsto (w_2 + 1) 1 (w_1 + 1)$$

is a bijection of \mathfrak{A}_{2n-1} onto Alt_{2n-1} having the property

(5.9)
$$(\mathbf{f}, \mathbf{l})\sigma = (\mathbf{L} - 1, \mathbf{F} - 1)\phi_2(\sigma).$$

The proof of Property 5.1 is straightforward. Just mention three examples: (i) $\phi_1(3427561) = 5617342$ and $(\mathbf{f}, \mathbf{l})(34\mathbf{2}7\mathbf{5}61) = (2, 5) = (\mathbf{L}, \mathbf{F})(\mathbf{5}617342)$; (ii) $\phi_2(5471326) = 2437165$ and $(\mathbf{f}, \mathbf{l})(5471326) = (4, 1) = (\mathbf{L} - 1, \mathbf{F} - 1)(\mathbf{2} 43716\mathbf{5})$; (iii) $\phi_2(7461325) = 5724361$ and $(\mathbf{f}, \mathbf{l})(7\mathbf{4}61325) = (0, 4) = (\mathbf{L} - 1, \mathbf{F} - 1)(\mathbf{5} 72436\mathbf{1})$.

It follows from (5.6)–(5.9) that the product $\phi_1 \circ \phi_2 : \mathfrak{A}_{2n-1} \to \operatorname{Alt}_{2n-1}$ has the property that for every σ from \mathfrak{A}_{2n-1} we have: $(\mathbf{f}-1,\mathbf{l}-1)(\phi_1 \circ \phi_2)(\sigma) = (\mathbf{L}-1,\mathbf{F}-1)\phi_2(\sigma) = (\mathbf{f},\mathbf{l})\sigma$. In view of (5.3) and since $\operatorname{grn} = \max(\mathbf{l},\mathbf{f})$, this implies the identity:

(5.10)
$$q_n(k) = \# \mathfrak{A}_{2n-1,k-1} = \# \operatorname{Alt}_{2n-1,k}$$

For $n \geq 1$ each set $Alt_{2n-1,k}$ can be split into two subsets of the same cardinality $Alt_{2n-1,k}^{\leq} + Alt_{2n-1,k}^{\geq}$, depending on whether the greater neighbor k is on the left, or on the right of (2n+1). Now, with each permutation

$$\sigma = x_1 x_2 \cdots x_{i-2} k (2n-1) x_{i+1} x_{i+2} \cdots x_{2n-1}$$

from $Alt_{2n-1,k}^{\leq}$ associate the permutation

$$\phi_3(\sigma) := k x'_{j-2} \cdots x'_2 x'_1(2n) x'_{2n-1} \cdots x'_{j+2} x'_{j+1}(k+1),$$

where

$$x_j' = \begin{cases} x_j, & \text{if } x_j \le k; \\ x_j + 1, & \text{if } x_j > k. \end{cases}$$

It is obvious that ϕ_3 is a bijection of $\mathrm{Alt}_{2n-1,k}$ onto the set $\mathfrak{E}_{2n,k}$ of all permutations τ from Alt_{2n} such that $\mathbf{F}\tau=k$, $\mathbf{L}\tau=k+1$. As $\#\mathfrak{E}_{2n,k}=E_{2n}(k,k+1)$, it follows that

(5.11)
$$\# \operatorname{Alt}_{2n-1,k} = 2\# \operatorname{Alt}_{2n-1,k}^{\leq} = 2E_{2n}(k, k+1).$$

This proves identity (5.2) in view of (5.10) and (5.11).

Final Remarks. The Seidel Matrix method developed in this paper can also be used to derive the three-variable generating functions obtained in our previous two papers [FH13b] and [FH13c]. It is also the main tool in our next paper [FH13d].

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Dominique Foata Institut Lothaire 1, rue Murner F-67000 Strasbourg, France

foata@unistra.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu.han@unistra.fr