Eulerian Calculus, II : An Extension of Han's Fundamental Transformation

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In our first Eulerian Calculus paper, we introduced the k-descent and k-excedance statistics on words, considered various bijections associated with them and calculated their distributions. In this paper we consider the k-major and k-den statistics and show that the joint des-maj and exc-den distributions of that k-extension are identical. We also consider the generating functions for these statistics.

Dans notre premier article sur le Calcul Eulérien, nous avons introduit les statistiques k-descentes et k-excédances sur les mots, construit les différentes bijections associées et calculé leurs distributions. Dans le présent article nous considérons l'indice k-majeur et la statistique k-den et montrons que les distributions jointes des-maj et exc-den de cette k-extension sont identiques. Nous considérons aussi les fonctions génératrices de ces statistiques.

1. Introduction

This paper is the second one of a series of three articles (see [6] and [7] for the other two), in which we investigate further extensions of the classical Eulerian Calculus that involves, first, the constructions of transformations on the symmetric group and related structures, then, explicit calculations of distributions of order statistics on those structures.

We use the same notations as in the previous paper, but for convenience sake we recall them here. First, X is a fixed non-empty set, referred to as an alphabet, on which a total ordering D is defined. If $x, y \in X$ and $(x, y) \in D$ we will write $x <_D y$ or simply $x < y$ if no confusion can arise. We will normally take X to be the subset $[\,r\,] = \{1, 2, \ldots, r\}$ $(r \geq 1)$, but D need not be the standard ordering on $[\,r\,]$. We also have a fixed integer, k, such that $0 \leq k \leq r$ and $j = r - k$. The letters $1, \ldots, j$ will be called *small* and the letters $j+1, \ldots, r$ *large*. We say that D is *compatible* with k if, for all x large and y small, we have $x > y$. We also introduce a small letter $\star$ which is greater than any small letter of X.

The free monoid generated by X will be denoted by X^*. The elements

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of X^* are finite *words* $w = x_1x_2\dots x_m$ with letters x_i taken from X. When we wish to stress the fact that a word has both small and large letters, we will speak of *weighted* word. A word w' is said to be a *rearrangement* of the word w if it can be obtained from w by permuting the letters x_1, x_2, ... , x_m in some order.

Let $\mathbf{c} = (c_1, \dots, c_j)$ and $\mathbf{d} = (d_1, \dots, d_k)$ be two vectors with positive integer components. Also let $c = c_1 + \dots + c_j$, $d = d_1 + \dots + d_k$ and $c + d = m$. The class of all $m!/(c_1! \dots c_j!\, d_1! \dots d_k!)$ rearrangements of the word $1^{c_1} \dots j^{c_j} (j+1)^{d_1} \dots r^{d_k}$ will be denoted by $R(\mathbf{c}, \mathbf{d})$ or by $C(v)$, where v is a given word in $R(\mathbf{c}, \mathbf{d})$.

Let $w = x_1x_2\dots x_m$ and let $\overline{w} = y_1y_2\dots y_m$ be its non-decreasing rearrangement (with respect to a given ordering D). As in the previous paper, we say that the word w has a k-*excedance* at i $(1 \le i \le m)$, if either $x_i > y_i$, or $x_i = y_i$ and x_i large. We also say that w has a k-*descent* at i $(1 \le i \le m)$, if either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and x_i large. (By convention, $x_{m+1} = \star$.) The numbers of k-excedances and k-descents of a word w are denoted by $\mathrm{exc}_k\, w$ and $\mathrm{des}_k\, w$. We will sometimes use the notations $\mathrm{exc}_{k,D}$ and $\mathrm{des}_{k,D}$ if we need to draw attention to the ordering D.

In our first paper [6] we showed that for each ordering D compatible with $k \ge 0$ the statistics "des_k" and "exc_k" were equidistributed on each rearrangement class $R(\mathbf{c}, \mathbf{d})$ and proved certain invariance properties on the distributions of those statistics. Actually, we constructed a bijection Φ_k of each rearrangement class $R(\mathbf{c}, \mathbf{d})$ onto itself that satisfied $\mathrm{des}_k\, w = \mathrm{exc}_k\, \Phi_k(w)$, identically. Hence for each rearrangement class $R(\mathbf{c}, \mathbf{d})$ the generating polynomials $\sum_w t^{\mathrm{des}_k\, w}$ and $\sum_w t^{\mathrm{exc}_k\, w}$ $(w \in R(\mathbf{c}, \mathbf{d}))$ are equal. Let $A_{\mathbf{c}}(t)$ be their common value. It was also shown that the generating function for those polynomials could be expressed as

$$(1.1)\qquad \sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s \frac{(1+v_1)^s \dots (1+v_k)^s}{(1-u_1)^{s+1} \dots (1-u_j)^{s+1}},$$

where $\mathbf{u}^{\mathbf{c}} = u_1^{c_1} \dots u_j^{c_j}$ and $\mathbf{v}^{\mathbf{d}} = v_1^{d_1} \dots v_k^{d_k}$. The object of this paper is to extend those results to *joint* statistics involving "des_k" and "exc_k" and thereby propose a natural q-*extension* of all of those results.

As usual, let $(a;q)_n$ denote the q-ascending factorial

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n \ge 1. \end{cases}$$

Then, a natural q-analogue of (1.1) can read

$$(1.2)\qquad \sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s\ge 0} t^s \frac{(-qv_1;q)_s \dots (-qv_k;q)_s}{(u_1;q)_{s+1} \dots (u_j;q)_{s+1}}.$$

One of the purposes of this paper to show that the coefficient $A_{\mathbf{c},\mathbf{d}}(t,q)$ on the left-hand side of (1.2) is actually a *polynomial* with integral coefficients; more essentially, to show that $A_{\mathbf{c},\mathbf{d}}(t,q)$ is the *generating polynomial* for a bivariate statistic $(\mathrm{des}_k, \mathrm{maj}_k)$ over the rearrangement class $R(\mathbf{c},\mathbf{d})$. The statistic "maj_k" will be defined shortly.

The main motivation of the paper is, however, to extend Han's construction [14] to weighted words. This consists, first, of finding an appropriate extension "den_k" of the Denert statistic "den," defined by Han, then, of constructing an explicit bijection ρ of each rearrangement class $R(\mathbf{c},\mathbf{d})$ onto itself such that $(\mathrm{des}_k, \mathrm{maj}_k)(w) = (\mathrm{exc}_k, \mathrm{den}_k)\rho(w)$ holds identically.

In his treatise on Combinatory Analysis, MacMahon [16] defined the *major index*, $\mathrm{maj}\, w$, of a word $w = x_1x_2\ldots x_m$ as the *sum* of the i's such that $1 \le i \le m-1$ and $x_i > x_{i+1}$. As this statistic has been the ideal companion to "des" for deriving q-analogues of the Eulerian numbers (see [2, 3, 4, 9, 13, 17, 19, 20]), it is natural to look for a statistic related to "maj" to obtain a q-extension of the distribution of "des_k." Here we define the *k-major index*, $\mathrm{maj}_k\, w$, of $w = x_1x_2\ldots x_m$ to be the *sum* of all i's $(1 \le i \le m)$ such that i is a k-descent in w.

The crucial problem is to define the appropriate statistic "den_k," the companion of "exc_k." The Mahonian statistic "den" was introduced by Denert [8] on permutations and it was shown by Foata and Zeilberger [10] that the joint distribution of "exc" and "den" was Euler-Mahonian. To be more exact, it was shown that over any rearrangement class $C(v)$ the two polynomials $\sum_w t^{\mathrm{exc}\, w} q^{\mathrm{den}\, w}$ and $\sum_w t^{\mathrm{des}\, w} q^{\mathrm{maj}\, w}$ were equal. The definition of "den" was extended in [14] to the case of words with possible repetition of letters. Han constructed an explicit bijection that transformed the pair $(\mathrm{exc}, \mathrm{den})$ to $(\mathrm{des}, \mathrm{maj})$. Our definition of the new statistic "den_k" (or "$\mathrm{den}_{k,D}$" when we wish to stress the role of the ordering D) rests upon an extension of the notion of "cyclic intervals" already introduced by Han [14] in his definition of "den." That extension, called *k-cyclic interval*, is defined as follows.

Let $S = \{1,\ldots,j\}$ be the set of small elements of X and let $L = \{j+1,\ldots,r\}$ be the set of large elements of X. Let $s_{\max}$ be the largest small letter of X (under the ordering D). Besides the small letter $\star$ that satisfies $s_{\max} <_D \star <_D b$ for any letter b greater than $s_{\max}$, we also adjoin to X a large letter ∞ that is greater than every letter of X. Define X^+ to be $X \cup \{\star, \infty\}$. Similarly, $L^+ = L \cup \{\infty\}$ and $S^+ = S \cup \{\star\}$.

If Y is any totally ordered set and $a, b \in Y$, the *cyclic interval* $\rrbracket a, b \rrbracket$ has been defined in [14] as

$$\rrbracket a, b \rrbracket = \begin{cases} (a,b], & \text{if } a \le b; \\ Y \setminus (b,a], & \text{otherwise.} \end{cases} \tag{1.3}$$

Thus $\rrbracket a, a \rrbracket = \emptyset$.

In our context it is necessary to modify the definition. Let a and b be elements of X^+. Then we define $\rrbracket a, b\rrbracket$ by equation (1.3), where $Y = X^+$. Further, we define $\rrbracket a, b\rrbracket_k$ by

$$\rrbracket a, b\rrbracket_k = \begin{cases} \rrbracket a, b\rrbracket, & \text{if } a, b \in S^+\,; \\ \rrbracket a, b\rrbracket \cup \{a\}, & \text{if } a \in L^+,\ b \in S^+\,; \\ \rrbracket a, b\rrbracket \setminus \{b\}, & \text{if } a \in S^+,\ b \in L^+\,; \\ \rrbracket a, b\rrbracket \cup \{a\} \setminus \{b\}, & \text{if } a, b \in L^+,\ a \neq b\,; \\ X^+, & \text{if } a = b \in L^+. \end{cases} \tag{1.4}$$

The elements of X^+ can be visualized as points on a circle (or a square!) as shown on Fig. 1. The k-cyclic intervals $\rrbracket a, b\rrbracket_k$ must be read *counterclockwise*. The path $-]\!\!\rightarrow$ on the S^+-part shows that whenever a is small, the interval $\rrbracket a, b\rrbracket_k$ is of the form "$(a, \ldots$" (or "$]a, \ldots$" in the French notation) so that $a \notin \rrbracket a, b\rrbracket_k$; also whenever b is small, $\rrbracket a, b\rrbracket_k = \cdots, b]$ and then $b \in \rrbracket a, b\rrbracket_k$. On the contrary, the path $-[\!\!\rightarrow$ on the L^+-part shows that $\rrbracket a, b\rrbracket_k = [a, \ldots$ and so $a \in \rrbracket a, b\rrbracket_k$ whenever a is large; but $\rrbracket a, b\rrbracket_k = \cdots, b)$ (or $\ldots, b[$) whenever b is large and then $b \notin \rrbracket a, b\rrbracket_k$. When D is compatible with k, the small letters lie between ∞ and $\star$, and the large ones between $\star$ and ∞, still reading the square counterclockwise; also $\rrbracket \star, \infty\rrbracket_k = L$.

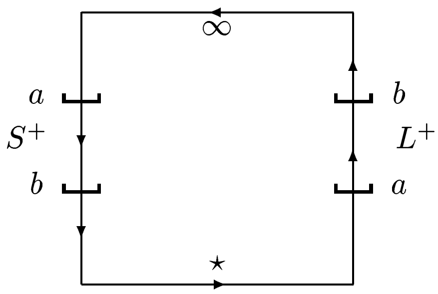


Fig. 1

Let $w = x_1 x_2 \ldots x_m$ be a word on the letters in X. Put $x_{m+1} = \star$, $x_{m+2} = \infty$. For $i = 1, \ldots, m+2$ let $\mathrm{Fact}_i\, w$ be the left factor $x_1 \ldots x_{i-1}$ of w and for each subset B of X let $\mathrm{Fact}_i\, w \cap B$ be the *subword* of $\mathrm{Fact}_i\, w$ consisting only of those letters of $\mathrm{Fact}_i\, w$ that are in B. Furthermore, let $|\mathrm{Fact}_i\, w \cap B|$ denote the *length* of that subword.

Now let $\overline{w} = y_1 y_2 \ldots y_m$ the non-decreasing rearrangement of a word $w = x_1 x_2 \ldots x_m$. The den_k-*coding* of w is defined to be the sequence $(s_i)_{1 \le i \le m+1}$, where

$$s_i = \begin{cases} \left|\mathrm{Fact}_i\, w \cap \rrbracket x_i, y_i\rrbracket_k\right|, & \text{if } 1 \le i \le m\,; \\ |w \cap L|, & \text{if } i = m+1\,; \end{cases} \tag{1.5}$$

and the statistic $\mathrm{den}_k\, w$ to be

$$\mathrm{den}_k\, w = \sum_{i=1}^{m+1} s_i. \tag{1.6}$$

Consider the the set $X = \{1,2,3,4,5\}$ with the ordering $1 < 2 < 4 < 3 < 5$ and assume that 1, 2 and 3 are small, while 4 and 5 are large (i.e., $j = 3$, $k = 2$, $r = 5$). Then $\mathrm{den}_k\, w$ for $k = 2$ and $\begin{pmatrix}\overline{w}\\ w\end{pmatrix} = \begin{pmatrix}1\,1\,2\,4\,4\,3\,3\,5\,5\\ 3\,5\,1\,3\,1\,4\,2\,4\,5\end{pmatrix}$ is easily calculated using Fig. 1. For instance, $\left|\mathrm{Fact}_4\, w \cap \,]\!]3,4]\!]_2\right| = |3,5,1 \cap \{5,1,2\}| = 2$. Hence $(s_i) = (0,0,0,2,0,2,3,3,8,4)$ and $\mathrm{den}_2\, w = 22$.

Our first result is the invariance principle for "maj", a simple extension of the corresponding result for the case $k = 0$ proved in [9, Proposition 2.1]. This result includes of course Theorem 3.1 of [6].

THEOREM 1.1. *Let v be a fixed word in X^* and let D and E be total orderings on $X = [\,r\,]$. Assume that both D and E are compatible with k. Then there is a bijection μ on $C(v) = R(\mathbf{c},\mathbf{d})$ such that for all $w \in C(v)$,*

$$(\mathrm{des}_{k,D}, \mathrm{maj}_{k,D})\ w = (\mathrm{des}_{k,E}, \mathrm{maj}_{k,E})\ \mu(w).$$

The proof of this Theorem occupies section 2. As explained earlier, our main result is the following theorem, proved in section 4.

THEOREM 1.2. *Let v be a fixed word in X^* and let D be an ordering on X compatible with k. Then there is a bijection ρ of $C(v)$ onto itself such that for all $w \in C(v)$,*

$$(\mathrm{des}_k, \mathrm{maj}_k)\ (w) = (\mathrm{exc}_k, \mathrm{den}_k)\ \rho(w). \tag{1.5}$$

To complete this circle of results, we must prove the invariance principle for "den" which may be stated in the following form.

THEOREM 1.3. *Let v be a fixed word in X^* and let D and E be total orderings on $X = [\,r\,]$. Then there is a bijection δ on $C(v) = R(\mathbf{c},\mathbf{d})$ such that for all $w \in C(v)$,*

$$(\mathrm{exc}_{k,D}, \mathrm{den}_{k,D})\ w = (\mathrm{exc}_{k,E}, \mathrm{den}_{k,E})\ \delta(w).$$

This result is proved in section 5 of the current paper. It includes Theorem 3.3 of [6]

There are essentially two parts in the paper. The first sections 2, 3, 4 and 5 are of *algorithmic* nature and rest upon *non-commutative* *algebraic* techniques developed by Schützenberger [18] and discussed in the book by Lothaire [15]. For the constructions of the bijection ρ in section 4 and of ξ_D and ξ_E in section 5 we need two tools : an appropriate *word factorization* and a *commutation rule.* It is noteworthy that the *k-factorisation* introduced in section 4 suggests that further classical word factorizations should be reconsidered in the context of bi-alphabets with two classes of letters, large and small. The second tool is the *Han transposition* discussed in the next section, that is mainly an adaptation of what Han [14] has developed in his seminal paper.

The last sections 6, 7 and 8 deal with classical techniques in the context of q-series. We first calculate the distribution of $(\mathrm{des}_k, \mathrm{maj}_k)$ (and then that of $(\mathrm{exc}_k, \mathrm{den}_k)$ by Theorem 1.2). More precisely, let

$$A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_w t^{\mathrm{des}_k w} q^{\mathrm{maj}_k w} \qquad (w \in R(\mathbf{c},\mathbf{d}))$$

be the generating function for the pair $(\mathrm{des}_k, \mathrm{maj}_k)$ over the class $R(\mathbf{c},\mathbf{d})$. In section 6 we first calculate the *recurrence relations* for $A_{\mathbf{c},\mathbf{d}}(t,q)$. Then, in section 7, we successfully integrate a system of *partial q-difference equations* to prove the following theorem.

THEOREM 1.4. *The factorial generating function for the polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$ satisfies identity* (1.2).

In section 8 we specialize the calculation of $A_{\mathbf{c},\mathbf{d}}(t,q)$ to the symmetric group and develop an algebra of q-polynomials that *interpolate* the classical *q-Eulerian polynomials* in a very natural manner.

An alternate set-up for sections 2–5 would have been to keep the natural ordering on X, construct the bijection ρ on each rearrangement class $R(\mathbf{c},\mathbf{d})$ and study the invariance of the distributions of the statistics under the action of the *permutations of the multiplicities* $(c_1,\ldots,c_j,d_1,\ldots,d_k)$ of the letters 1, 2,... , r. We have preferred to state and prove our results in the context of total orderings on X, i.e., under the action of the *permutations of the letters* of X.

2. THE INVARIANCE PRINCIPLE FOR THE k-MAJOR INDEX

Let $w = x_1x_2\ldots x_m$ be a word in X^*. Then the k-major index $\mathrm{maj}_{k,D}\, w$ is defined by

$$\mathrm{maj}_{k,D}\, w = \sum\{i : 1 \le i \le m, w \text{ has a } k\text{-descent at } i\}.$$

PROOF OF THEOREM 1.1. We simplify the notation by suppressing the subscript k from “des” and “maj”, and we refer to D-descents and

E-descents of a word w in the obvious way. Without loss of generality, we may assume that D is the standard ordering on $X = [r]$ and that E is obtained from D by interchanging the order of two adjacent letters i and $i+1$. Since E is compatible with k, either $i+1 \le j$, i.e., both i and $i+1$ are small, or $i > j$, i.e., both i and $i+1$ are large.

Case (a) : suppose that $i+1 \le j$. Let w be a word in $C(v)$. Write w in the form $w = u_1 v_1 u_2 v_2 \dots u_n v_n$, where each u_l is a word in the letters i, $i+1$ and each v_i is a word in $X \setminus \{i, i+1\}$. (Note that either or both of u_1, v_n may be empty.)

Consider any one of the subwords u_l. Write in bold-face all the factors of u_l of form $(i+1)i$. Replace each maximal factor of u_l of form $i^p(i+1)^q$ that does not involve any bold-face letters by the factor $(i+1)^q i^p$. Finally, replace each bold-face factor $(i+1)i$ by the factor $i(i+1)$, to obtain a word u'_l. Then the factor $(i+1)i$ occurs in u_l in exactly those places in which the factor $i(i+1)$ occurs in u'_l. Thus u_l has a D-descent in exactly those places at which u'_l has an E-descent.

Now put $\mu(w) = u'_1 v_1 u'_2 v_2 \dots u'_n v_n$. Clearly $\mu(w) \in C(v)$ and

$$(\mathrm{des}_D, \mathrm{maj}_D)\ w = (\mathrm{des}_E, \mathrm{maj}_E)\ \mu(w).$$

Since μ is obviously a bijection, the result follows.

Case (b) : suppose on the other hand that $i > j$. Let $w = u_1 v_1 u_2 v_2 \dots u_n v_n$ as before. For each factor u_l, form u'_l in exactly the same way as in Case (a), *except that* the places of i and $i+1$ in the algorithm should be interchanged. Thus the factor $i(i+1)$ will occur in u_l in exactly those places in which the factor $(i+1)i$ occurs in u'_l. Thus u_l will fail to have a D-descent in exactly those places at which u'_l fails to have an E-descent. Now, putting $\mu(w) = u'_1 v_1 u'_2 v_2 \dots u'_n v_n$, the proof proceeds as before. $\square$

Remark. Define the *D-ligne of route* of w to be the set, $\mathrm{ligne}_D\, w$, of all i such that $1 \le i \le m$ and w has a k-descent at i. The above proof shows that under the same assumptions we also have

$$\mathrm{ligne}_D\, w = \mathrm{ligne}_E\, \mu(w).$$

3. The den statistic

In this section we derive an alternate definition of maj_k, involving the k-cyclic intervals as in the definition of den_k (given in (1.3) and (1.4)). Then we define the statistic "den" on biwords and describe the "Han transposition", a way of manipulating biwords that preserves the statistics "den" and "exc".

3.1. *The k-major index.* Let $w = x_1x_2\ldots x_m$ be a word on the letters in X. As in section 1, put $x_{m+1} = \star$, $x_{m+2} = \infty$ and for $i = 1, \ldots, m+2$ let $\mathrm{Fact}_i\, w$ be the *left factor* $x_1 \ldots x_{i-1}$ of w and let $|\mathrm{Fact}_i\, w \cap B|$ denote the *length* of the *subword* of $\mathrm{Fact}_i\, w$ the letters of which belong to a subset B of X.

Our purpose is to give the following extension of a result of Han [14, Theorem 2.1(i)] that was derived for the case $k = 0$.

THEOREM 3.1. *If D is compatible with k, then for each word $w = x_1x_2\ldots x_m$,*

$$\mathrm{maj}_k\, w = \sum_{i=1}^{m+1} \left|\mathrm{Fact}_i\, w \cap \rrbracket x_i, x_{i+1}\rrbracket_k\right|. \tag{3.1}$$

Theorem 3.1 rests upon the next two lemmas 3.2 and 3.3. In the first one the notation $A \uplus B$ denotes the union of the multisets A and B. For example, $\{1,1,2\} \uplus \{2,3\} = \{1,1,2,2,3\}$.

LEMMA 3.2. *Let $a, b \in X^+$. Then*

(i) $$\rrbracket a, b\rrbracket_k = X^+ \setminus \rrbracket b, a\rrbracket_k, \ \textit{whenever } a \neq b$$

(ii) $$\rrbracket a, b\rrbracket_k \uplus \rrbracket b, \infty\rrbracket_k = \begin{cases} \rrbracket a, \infty\rrbracket_k \uplus X^+, & \textit{if } a > b \textit{ or } a = b \in L^+\,; \\ \rrbracket a, \infty\rrbracket_k, & \textit{otherwise.} \end{cases}$$

Proof. Both parts of this Lemma are easily verified by considering the various possible cases. For part (ii), recall that $\rrbracket a, a\rrbracket_k$ equals X^+ if a is large and is empty otherwise. □

Now let $w = x_1x_2\ldots x_n$ be a word on $X \cup \{\star\}$ and define

$$\mathrm{maj}'\ w = \sum \{i : 1 \le i \le n-1,\ w \text{ has a } k\text{-descent at } i\ \};$$

(i.e., in "maj′" we do not count any descent at the end of w.)

LEMMA 3.3. *Let $w = x_1x_2\ldots x_n$ be a word on $X \cup \{\star\}$. Then*

$$\mathrm{maj}'\ w = \sum_{i=1}^{n-1} \left|\mathrm{Fact}_i\, w \cap \rrbracket x_i, x_{i+1}\rrbracket_k\right| + \left|\,\mathrm{Fact}_n\, w \cap \rrbracket x_n, \infty\rrbracket_k\right|. \tag{3.2}$$

Proof. We use induction on n. The result is clearly true for $n = 1$, so suppose it is true for $n - 1$. Then

$$\mathrm{maj}'\, w - \mathrm{maj}'\, x_1\ldots x_{n-1} = \begin{cases} n-1, & \text{if } w \text{ has a } k\text{-descent at } n-1\,; \\ 0, & \text{otherwise.} \end{cases}$$

Let S_n denote the right-hand side of equation (3.2). Then

$$\begin{aligned}
S_n - S_{n-1} &= \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_{n-1}, x_n\rrbracket_k\right| + \left|\mathrm{Fact}_n\, w \cap \rrbracket x_n, \infty\rrbracket_k\right| \\
&\qquad - \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_{n-1}, \infty\rrbracket_k\right| \\
&= \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_{n-1}, x_n\rrbracket_k\right| + \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_n, \infty\rrbracket_k\right| \\
&\qquad - \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_{n-1}, \infty\rrbracket_k\right| + \left|x_{n-1} \cap \rrbracket x_n, \infty\rrbracket_k\right| \\
&= \left|\mathrm{Fact}_{n-1}\, w \cap (\rrbracket x_{n-1}, x_n\rrbracket_k \uplus \rrbracket x_n, \infty\rrbracket_k)\right| \\
&\qquad - \left|\mathrm{Fact}_{n-1}\, w \cap \rrbracket x_{n-1}, \infty\rrbracket_k\right| + \left|x_{n-1} \cap \rrbracket x_n, \infty\rrbracket_k\right| \\
&= \begin{cases} |\mathrm{Fact}_{n-1} \cap X^+| & \\ \quad + \left|x_{n-1} \cap \rrbracket x_n, \infty\rrbracket_k\right|, & \text{if } w \text{ has a } k\text{-descent at } n-1\,; \\ \left|x_{n-1} \cap \rrbracket x_n, \infty\rrbracket_k\right|, & \text{otherwise}\,; \end{cases} \\
&= \begin{cases} n-1, & \text{if } w \text{ has a } k\text{-descent at } n-1\,; \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

by Lemma 3.2. Hence the result follows. □

The proof of Theorem 3.1 is now completed as follows. Assume without loss of generality that D is the usual ordering on X. Let $w' = x_1x_2\ldots x_m \star$. Then

$$\begin{aligned}
\mathrm{maj}_k\, w &= \mathrm{maj}'\, w' \\
&= \sum_{i=1}^{m} \left|\mathrm{Fact}_i\, w' \cap \rrbracket x_i, x_{i+1}\rrbracket_k\right| + \left|\mathrm{Fact}_{m+1}\, w \cap \rrbracket x_{m+1}, \infty\rrbracket_k\right| \\
&= \sum_{i=1}^{m+1} \left|\mathrm{Fact}_i\, w \cap \rrbracket x_i, x_{i+1}\rrbracket_k\right|,
\end{aligned}$$

since by convention we put $x_{m+1} = \star$, $x_{m+2} = \infty$. □

The above result is *not true* without the assumption of compatibility.

3.2. *Biwords and words.* A *biword* is a two rowed matrix $\alpha = \binom{u}{w}$, where u and w are words in X^* of the same length. The biword α is called a *circuit* if u is a rearrangement of w. A circuit $\alpha = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ x_1 & x_2 & \ldots & x_m \end{pmatrix}$ is a *cycle*, if $y_m = x_1$ and $y_i = x_{i+1}$ for $i = 1, \ldots, m-1$. We will write this cycle as $\alpha = [x_1x_2\ldots x_m] = [w]$, where $w = x_1x_2\ldots x_m$.

Consider the biword $\binom{u}{w}$, where $u = y_1y_2\ldots y_m$ and $w = x_1x_2\ldots x_m$. An ordering D of X being given, we define

$$\mathrm{exc}_k \binom{u}{w} = |\{i : 1 \le i \le m \text{ and } x_i > y_i \text{ or } x_i = y_i \in L^+\}|; \tag{3.3}$$

$$\mathrm{den}_k \binom{u}{w} = \sum_{i=1}^{m} \left|\mathrm{Fact}_i\, w \cap \rrbracket x_i, y_i\rrbracket_k\right|. \tag{3.4}$$

If $\overline{w}$ is the non-decreasing rearrangement of w, then clearly

$$\operatorname{exc}_k \begin{pmatrix} \overline{w} \\ w \end{pmatrix} = \operatorname{exc}_k w; \tag{3.5}$$

and by (1.5) and (1.6)

$$\operatorname{den}_k w = \operatorname{den}_k \begin{pmatrix} \overline{w} \\ w \end{pmatrix} + |w \cap L|. \tag{3.6}$$

Note that if D is compatible with k,

$$\operatorname{den}_k w = \operatorname{den}_k \begin{pmatrix} \overline{w} & \infty \\ w & \star \end{pmatrix}. \tag{3.7}$$

3.3. *The Han Transposition.* Let $x, y, a, b \in X \cup \{\star\}$. Then a and b are *neighbours* with respect to (x, y) if *both* a and b are in $\rrbracket y, x \rrbracket_k$ or *neither* in $\rrbracket y, x \rrbracket_k$. Otherwise, a and b are *strangers* with respect to (x, y).

LEMMA 3.4. *The following conditions are equivalent.*

(i) *a and b are neighbours with respect to (x, y) ;*

(ii) $\left|a \cap \rrbracket y, x \rrbracket_k\right| = \left|b \cap \rrbracket y, x \rrbracket_k\right|$;

(iii) *a and b are neighbours with respect to (y, x) ;*

(iv) $\left|a \cap \rrbracket b, y \rrbracket_k\right| = \left|a \cap \rrbracket b, x \rrbracket_k\right|$;

(v) $\rrbracket a, x \rrbracket_k \uplus \rrbracket b, y \rrbracket_k = \rrbracket a, y \rrbracket_k \uplus \rrbracket b, x \rrbracket_k$.

LEMMA 3.5. *The following conditions are equivalent.*

(i) *a and b are strangers with respect to (x, y) ;*

(ii) $\left|a \cap \rrbracket y, x \rrbracket_k\right| = 1 - \left|b \cap \rrbracket y, x \rrbracket_k\right|$;

(iii) *a and b are strangers with respect to (y, x) ;*

(iv) $\left|a \cap \rrbracket b, y \rrbracket_k\right| = \left|b \cap \rrbracket a, x \rrbracket_k\right|$.

Proof. These two lemmas follow from the similar results [14, Lemma 3.2 and Lemma 3.3]. The only special cases that arise are when two of a, b, x, y are equal and large.

1. If $a = b \in L$ then a and b are neighbours and the results are trivial.

2. If $x = y \in L$ then $\rrbracket y, x \rrbracket_k = X^+$ and a and b are neighbours. The results now follow.

3. Suppose that $a = x \in L$ and $b, y \neq a$. Then $a \notin \rrbracket y, a \rrbracket_k$. So a and b are neighbours if and only if $b \notin \rrbracket y, a \rrbracket_k$. But $b \notin \rrbracket y, a \rrbracket_k$ if and only if $a \notin \rrbracket b, y \rrbracket_k$, as $b \neq a$. Hence a and b are neighbours if and only if $\left|a \cap \rrbracket b, y \rrbracket_k\right| = 0 = \left|a \cap \rrbracket b, x \rrbracket_k\right|$, as $a \notin \rrbracket b, x \rrbracket_k = \rrbracket b, a \rrbracket_k$. Now let a and b be neighbours and suppose that $b < y < a$. Then

$$\rrbracket a, x \rrbracket_k \uplus \rrbracket b, y \rrbracket_k = (X \cup \{\star\}) \uplus \rrbracket b, y \rrbracket_k,$$

and

$$\rrbracket a, y \rrbracket_k \uplus \rrbracket b, x \rrbracket_k = \rrbracket a, y \rrbracket_k \uplus \rrbracket b, a \rrbracket_k = (X \cup \{\star\}) \uplus \rrbracket b, y \rrbracket_k.$$

The other cases may be treated similarly. $\square$

Now let $\begin{pmatrix} x\ y \\ a\ b \end{pmatrix}$ be a biword of length two. Following [14] we define the Han transposition T by

$$T\begin{pmatrix} x\ y \\ a\ b \end{pmatrix} = \begin{cases} \begin{pmatrix} y\ x \\ a\ b \end{pmatrix}, & \text{if } a \text{ and } b \text{ are neighbours;} \\ \begin{pmatrix} y\ x \\ b\ a \end{pmatrix}, & \text{if } a \text{ and } b \text{ are strangers} \end{cases} \tag{3.8}$$

with respect to (x, y). If $\alpha = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} y_1\ y_2\ \dots\ y_m \\ x_1\ x_2\ \dots\ x_m \end{pmatrix}$ is a biword of length m and $1 \le i < m$, we define $T_i\alpha$ to be the biword obtained when the biword $\beta = \begin{pmatrix} y_i\ y_{i+1} \\ x_i\ x_{i+1} \end{pmatrix}$ consisting of the i-th and $i+1$-st columns of α is replaced by $T\beta$.

For instance, using the ordering $1 < 2 < 3 < \star < 4 < 5 < \infty$ with $1, 2, 3$ small and $4, 5$ large we have $T\begin{pmatrix} 3\ 4 \\ 4\ 1 \end{pmatrix} = \begin{pmatrix} 4\ 3 \\ 4\ 1 \end{pmatrix}$, since $1, 4 \notin \rrbracket 3, 4 \llbracket_2 = \{\star\}$, while $T\begin{pmatrix} 2\ 5 \\ 1\ 3 \end{pmatrix} = \begin{pmatrix} 5\ 2 \\ 3\ 1 \end{pmatrix}$, for $\rrbracket 2, 5 \llbracket_2 = \{3, \star, 4\}$.

LEMMA 3.6. *Let $\alpha = \begin{pmatrix} u \\ w \end{pmatrix}$ be a biword of length m and let $1 \le i < m$. Then* $(\mathrm{exc}_k, \mathrm{den}_k)\ T_i\alpha = (\mathrm{exc}_k, \mathrm{den}_k)\ \alpha$.

Proof. This follows from Lemmas 3.4 and 3.5, exactly as in [14, Lemma 5.2]. □

3.4. *The transformation* $\mathbf{T}_{z_2}$. Let z_1, z_2 be two distinct letters of $X \cup \{\star\}$ and v be a word of length $(m-1)$ $(m \ge 1)$ in the alphabet $X \cup \{\star\} \setminus \{z_2\}$. Denote by $\mathcal{C}(v, z_1, z_2)$ the set of all biwords $\alpha = \begin{pmatrix} u \\ w \end{pmatrix}$, where u is the *non-decreasing rearrangement* of vz_2 and w is any rearrangement of vz_1. Thus u has one occurrence of z_2, while w has none. However the occurrences of the other letters are the same, except for z_1 that occurs one more time in w than in u.

If z_2 is the i-th letter in the word u, the product $T_{m-1} \cdots T_{i+1} T_i$ will transform α into a biword of the form $\alpha' = \begin{pmatrix} u' z_2 \\ w' y_1 \end{pmatrix}$. Then, either u' and has no occurrence of y_1, in which case $y_1 = z_1$ and w' must be a rearrangement of u', or y_1 does occur in u'. In the former case, define $\mathbf{T}_{z_2}(\alpha) = \begin{pmatrix} u' z_2 \\ w' z_1 \end{pmatrix}$. In the latter case, the rightmost occurrence of y_1 in u' is, say, its i'-th letter. Then the product $T_{m-2} \cdots T_{i'+1} T_{i'}$ transforms α' into a biword of the form $\alpha'' = \begin{pmatrix} u'' y_1 z_2 \\ w'' y_2 y_1 \end{pmatrix}$. Again, either u'' and has no occurrence of y_2, in which case $y_2 = z_1$ and w'' must be a rearrangement of u'', or y_2 does occur in u''. In the former case, define $\mathbf{T}_{z_2}(\alpha) = \binom{u''y_1z_2}{w''z_1y_1}$. In the latter case, we continue the same procedure as before by moving the *rightmost* occurrence of y_2 in u'' to the right of u''. After finitely many steps we reach a biword $\alpha^{(l)} = \binom{u^{(l)}y_{l-1}\ldots y_1z_2}{w^{(l)}\ y_l\ \ldots\ y_2y_1}$, where $u^{(l)}$ has no occurrence of y_l. Then necessarily $y_l = z_1$ and $w^{(l)}$ is a rearrangement of $u^{(l)}$. (Note that $u^{(l)}$ may be empty.) Define $u_1 = u^{(l)}$, $w_1 = w^{(l)}$, $v_1 = y_{l-1}\ldots y_2y_1$ and

$$\mathbf{T}_{z_2}(\alpha) = \alpha^{(l)} = \binom{u^{(l)}y_{l-1}\ldots y_1z_2}{w^{(l)}\ y_l\ \ldots\ y_2y_1} = \binom{u_1v_1z_2}{w_1z_1v_1}. \tag{3.9}$$

Thus for each α in $\mathcal{C}(v, z_1, z_2)$ there is a well-defined product of Han transpositions that maps α onto a biword of the form $\binom{u_1v_1z_2}{w_1z_1v_1}$, where u_1 is *the non-decreasing rearrangement of* w_1 *with no occurrence of* z_1.

Denote by $\mathcal{D}(v, z_1, z_2)$ the set of biwords of the previous form $\binom{u_1v_1z_2}{w_1z_1v_1}$.

Lemma 3.7. *The mapping* $\mathbf{T}_{z_2} : \mathcal{C}(v, z_1, z_2) \to \mathcal{D}(v, z_1, z_2)$ *is a bijection.*

Proof. Let $\mathcal{C}(vz_1)$ the set of all rearrangements of vz_1. Then the mapping $w \mapsto \binom{u}{w}$ is evidently a bijection of $\mathcal{C}(vz_1)$ onto $\mathcal{C}(v, z_1, z_2)$. On the other hand, each word w in $\mathcal{C}(vz_1)$ has a unique factorization of the form (w_1, z_1v_1) where w_1 has no occurrence of z_1. Hence the mapping $w \mapsto \binom{u_1v_1z_2}{w_1z_2v_1}$, where u_1 is the non-decreasing rearrangement of w_1, is also a bijection of $\mathcal{C}(vz_1)$ onto $\mathcal{D}(v, z_1, z_2)$. Thus, those two sets, as well as $\mathcal{C}(v, z_1, z_2)$, have the same cardinalities.

Keeping the same notations we see that $\mathbf{T}_{z_2}$ is the product of Han transpositions of the form

$$(T_{m-l}\cdots T_{i^{(l)}+1}T_{i^{(l)}})\cdots(T_{m-2}\cdots T_{i'+1}T_{i'})(T_{m-1}\cdots T_{i+1}T_i).$$

To reverse $\mathbf{T}_{z_2}$ it suffices to determine l and the indices $i^{(l)}, \ldots, i', i$ directly from the biword $\binom{u_1v_1z_2}{w_1z_1v_1}$ in $\mathcal{D}(v, z_1, z_2)$. But l is the length of v_1z_2. Next, when the first letter of v_1 is moved just before the $i^{(l)}$-th letter of u_1, the resulting word $u^{(l-1)}$ is non-decreasing. In the same manner, the other indices, in particular, i', i, indicate where to move the other letters of v_1 and finally z_2 to the left to obtain a non-decreasing word on the top row. ∎

Remark 3.8. The inverse mapping applied to the biword $\binom{u_1v_1z_2}{w_1z_1v_1}$ in $\mathcal{D}(u_1v_1, z_1, z_2)$ is derived by moving to the left, to the first position where the resulting word is not-decreasing, successively the first, the second, ..., the last letter z_2 of the word v_1z_2, the moves being made by means of the Han transpositions as defined in (3.8). The inverse mapping is then independent of z_2 and will be denoted by $\mathbf{T}^{-1}$.

4. The den-maj bijection

In this section we prove Theorem 1.2. We assume that the ordering D is compatible with k, and in fact that D is the standard ordering on $X = [\,r\,]$. It will be convenient to use the terminology of words (see, e.g., [15]) : for instance, if w is the juxtaposition product $w = w_1w_2w_3$, then each w_i $(i = 1,2,3)$ is a *factor* of w ; w_1 (resp. w_3) is a *left factor* (resp. *right factor*) of w. If y is a letter and w a *non-decreasing* word, we will write $y < w$ (resp. $y \le w$), if y is less than (resp. less than or equal to) all the letters in w.

As for the first fundamental transformation described in [5 or 15, chap. 10] and Han's fundamental bijection [14] we need an appropriate *word factorization* besides the Han transposition described in the previous section. That factorization can be built as follows.

Let $z \in X \cup \{\star\}$ and let v be a word in that alphabet; then the word zv is said to be *k-dominant*, if z is large and all letters in v are greater than or equal to z, or if z is small and all small letters in v are less than or equal to z.

Every word in the alphabet $X \cup \{\star\}$ has a unique factorization $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ (the z_i's are letters and the v_i's words), called its *k-factorization*, having the following properties :

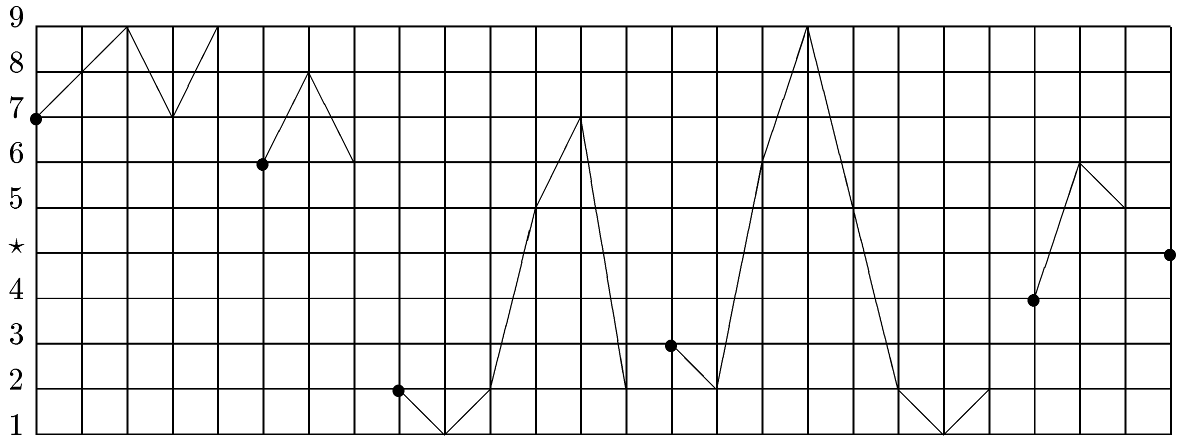
(i) $w = z_1v_1z_2v_2\ldots z_nv_n$;

(ii) each factor z_iv_i is k-dominant $(1 \le i \le n)$;

(iii) there exists an integer l $(1 \le l \le n)$ such that $z_1 > z_2 > \cdots > z_l > \star$ and $z_{l+1} < z_{l+2} < \cdots < z_n \le \star$.

The k-factorization of a word w may be obtained as follows : a letter z of w is called a *k-record*, if either z is large and all letters to the left of z are larger than z, or z is small and all small letters to the left of z are smaller than z. The k-factorization of w is then obtained by cutting w before each k-record.

The notions of k-factorization and k-records are illustrated in Fig. 2. The alphabet consists of five small letters $1, 2, 3, 4, \star$ and $k = 5$ large letters 5, 6, 7, 8, 9 with the ordering $1 < 2 < 3 < 4 < \star < 5 < 6 < 7 < 8 < 9$. The k-factorization of the word w consists of seven factors materialized by continuous polygonal lines, the rightmost one being a single point. Each k-record is the beginning of a continuous polygonal line and is represented by a "bullet" $\bullet$.



$$w = \mathbf{7}\ 8\ 9\ 7\ 9\ \mathbf{6}\ 8\ 6\ \mathbf{2}\ 1\ 2\ 5\ 7\ 2\ \mathbf{3}\ 2\ 6\ 9\ 5\ 2\ 1\ 2\ \mathbf{4}\ 6\ 5\ \star$$

Fig. 2

The k-factorization appears to be the *product* (in a sense that can be precised) of two factorizations, the *decreasing* factorization (the first two factors in the previous example) by the *increasing* factorization (the next five factors), the large letters playing no role in the definitions of those rightmost five factors.

The main property of the k-factorization on which our transformation is based is the following : let $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ be the k-factorization of a word w. Then,

(4.1) *for each $i = 1, \ldots, n-1$ no letter in the left factor $z_1v_1 \ldots z_iv_i$ is equal to z_{i+1} or is strictly between z_i and z_{i+1}. Let w_1 be any rearrangement of that factor; then the k-factorization of $w_1z_{i+1}v_{i+1} \ldots z_nv_n$ has the same rightmost $(n-i)$ factors $z_{i+1}v_{i+1}, \ldots, z_nv_n$ as w and the same rightmost $(n-i+1)$ k-records $z_i, z_{i+1}, \ldots, z_n$ as w.*

Consider a biword $\alpha = \begin{pmatrix} u_1 & u_2\infty \\ w_1 & z\,u_2 \end{pmatrix}$, where : (1) u_1, u_2, w_1 are words in the alphabet $X \cup \{\star\}$; (2) u_1 is the non-decreasing rearrangement of w_1; (3) z is a k-record of the word w_1zu_2. Such a biword is called a *supercycle.* A supercycle is said to be *initial*, if u_1 and w_1 are empty. The notion of *final* supercycle will be defined shortly.

Lemma 4.1. *If α is a supercycle, the factorization*

$$\alpha = \left(\begin{array}{c|c} u_1 & u_2\infty \\ w_1 & z\,u_2 \end{array}\right), \tag{4.2}$$

where u_1 is the non-decreasing rearrangement of some left factor of the bottom word of α and where z is a k-record of the bottom word, is unique. The factorization (4.2) *is called the canonical form of α.*

Proof. If there were two factorizations

$$\begin{pmatrix} u_1 & u_2\infty \\ w_1 & z\,u_2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u'_1 & u'_2\infty \\ w'_1 & z'\,u'_2 \end{pmatrix},$$

we could suppose $|u_1| < |u'_1|$. Then $u_2 = vz'u'_2$ and $zu_2 = zvz'u'_2$ for some k-dominant factor zv and α could be written as $\alpha = \begin{pmatrix} u_1 & vz' & u'_2\infty \\ w_1 & zv & z'\,u'_2 \end{pmatrix}$. As $z \neq z'$ by (4.1), the left factor $u'_1 = u_1vz'$ would not be a rearrangement of $w'_1 = w_1zv$. □

Let $\alpha = \begin{pmatrix} u_1 & u_2\infty \\ w_1 & z\,u_2 \end{pmatrix}$ be a supercycle written in its canonical form and let $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ be the k-factorization of zu_2. Then the following factorization of α, indicated by vertical bars,

$$\alpha = \begin{pmatrix} u_1 & v_1z_2 & v_2z_3 & \ldots & v_n\infty \\ w_1 & z_1v_1 & z_2v_2 & \ldots & z_nv_n \end{pmatrix} \tag{4.3}$$

is well defined. Call it the *k-factorization* of the supercycle α. The (positive) integer n, which is the number of factors in the k-factorization of zu_2, is called the *index* of α and denoted by $\operatorname{index}(\alpha)$. A supercycle α is said to be *final*, if its index is equal to 1.

We first describe a transformation τ on supercycles that decreases their indices whenever they are not final. Let α be a supercycle as shown in (4.3), supposed to be not final, so that $n \geq 2$. With the notations of (3.9) the left factor $\begin{pmatrix} u_1v_1z_2 \\ w_1z_1v_1 \end{pmatrix}$ of the supercycle α is an element of $\mathcal{D}(u_1v_1, z_1, z_2)$. Apply the inverse transformation $\mathbf{T}^{-1}$, as described in Remark 3.8, to that left factor. We get a biword $\begin{pmatrix} u''_1 \\ w'_1 \end{pmatrix} \in \mathcal{C}(u_1v_1, z_1, z_2)$, i.e., a biword such that u''_1 is the non-decreasing rearrangement of $u_1v_1z_2$ and w'_1 is a rearrangement of $u_1v_1z_1$. Then form the supercycle

$$\alpha'' = \begin{pmatrix} u''_1 & v_2z_3 & \ldots & v_n\infty \\ w'_1 & z_2v_2 & \ldots & z_nv_n \end{pmatrix}. \tag{4.4}$$

Replacing the only occurrence of z_2 in u''_1 by z_1 transforms u''_1 into a true *rearrangement* u'_1 of w'_1. Furthermore, u'_1 is *non-decreasing* because of property (4.1). We then obtain a supercycle

$$\alpha' = \begin{pmatrix} u'_1 & v_2z_3 & \ldots & v_n\infty \\ w'_1 & z_2v_2 & \ldots & z_nv_n \end{pmatrix}. \tag{4.5}$$

Moreover, the above expression derived from the k-factorization of α is precisely the k-factorization of α' and the rightmost k-record of w_1' is equal to z_1 (by (4.1)). Finally, $\mathrm{index}(\alpha') = n-1$. Thus the mapping $\tau : \alpha \mapsto \alpha'$ is well defined and satisfies

$$\mathrm{index}\big(\tau(\alpha)\big) < \mathrm{index}(\alpha), \tag{4.6}$$

if α is not final.

To derive the reverse transformation $\alpha' \mapsto \alpha$, the k-record z_1 to the left of z_2 in $w' = w_1' z_2 v_2 \dots z_n v_n$ is to be found first, then the replacement $z_1 \leftarrow z_2$ that maps u_1' onto u_1'' (and therefore α' onto α'') can be defined without ambiguity. But that k-record is necessarily equal to the left-most occurrence of the *smallest letter* y in w_1', if all the letters in w_1' are large, and equal to the left-most occurrence of the *greatest small letter* y in w_1', if some letters of w_1' are small.

In the former case we have $u_1' = y w_3'$ with $z_2 < y \le w_3'$; in the latter case, $u_1' = w_2' y w_3'$, with $y < z_2$ and $w_2' \le y < w_3'$. Then u_1'' can be recovered by

$$u_1'' = \begin{cases} z_2 w_3', & \text{(in the former case)}; \\ w_2' z_2 w_3', & \text{(in the latter case)}. \end{cases} \tag{4.7}$$

Thus the mapping $\alpha'' \mapsto \alpha'$ is perfectly reversible and defined by (4.7).

Now to go from α'' to α we simply have to apply the transformation $\mathbf{T}_{z_2}$ to the leftmost factor $\begin{pmatrix} u_1'' \\ w_1' \end{pmatrix}$ of α''. The index z_2 of the transformation $\mathbf{T}_{z_2}$ is well-defined, as z_2 is the first letter of the second factor of the k-factorization of α''. We then recover α, as written in (4.3). We also have the following property.

PROPERTY 4.2. *Let v be a non-decreasing word in the alphabet $X \cup \{\star\}$ and let $S(v)$ be the set of the supercycles $\alpha = \begin{pmatrix} u_1 & u_2 \infty \\ w_1 & z\, u_2 \end{pmatrix}$, whose bottom word $w_1 z u_2$ is a rearrangement of v. If α is not initial, there is a unique $\beta \in S(v)$ such that $\tau(\beta) = \alpha$ and* $\mathrm{index}(\alpha) < \mathrm{index}(\beta)$.

The statistics "exc_k" and "den_k" on biwords have been defined in (3.3) and (3.4).

PROPERTY 4.3. *For each supercycle α which is not final, we have*

$$(\mathrm{exc}_k, \mathrm{den}_k)\ \tau(\alpha) = (\mathrm{exc}_k, \mathrm{den}_k)\ \alpha.$$

Proof. Use the same notations as in (4.3), (4.4) and (4.5). Lemma 3.6 implies that $(\mathrm{exc}_k, \mathrm{den}_k)\, \alpha'' = (\mathrm{exc}_k, \mathrm{den}_k)\, \alpha$, since α'' is obtained from α by a sequence of Han transpositions.

Let $\binom{y}{x}$ be the (unique) biletter of $\binom{u_1''}{w_1'}$ whose top letter y is equal to z_2 and suppose that $\binom{y}{x}$ is the l-th biletter of the word. That biletter is transformed into $\binom{z_1}{x}$ when going from α'' to α. If $x \neq z_1$, then $\binom{z_1}{x}$ is a k-excedance, if and only if $\binom{z_2}{x}$ is one because of property (4.1). If $x = z_1$, consider the two cases : (i) $z_1 > z_2$ and necessarily z_1 is large; (ii) $z_1 < z_2$ and necessarily z_1 is small. In case (i) (resp. (ii)) $\binom{z_1}{z_1}$ and $\binom{z_2}{z_1}$ are both k-excedances (resp. both non-k-excedances). Hence $\operatorname{exc}_k \alpha'' = \operatorname{exc}_k \alpha'$.

Finally, $\operatorname{Fact}_l w_1' \cap \rrbracket x, z_1 \rrbracket_k = \operatorname{Fact}_l w_1' \cap \rrbracket x, z_2 \rrbracket_k$, as $\operatorname{Fact}_l w_1' \cap (z_1, z_2]$ is empty if z_1 is small and $\operatorname{Fact}_l w \cap [z_2, z_1)$ is empty if z_1 is large. Hence $\operatorname{den}_k \alpha'' = \operatorname{den}_k \alpha'$. $\square$

Now consider an *initial* supercycle of the form $\alpha = \begin{pmatrix} u\star & \infty \\ w & \star \end{pmatrix}$, where w is a word in the alphabet X. Then $w = zu$ for some $z \in X$ and $\star$ is the right-most factor in the k-factorization of $w\star$. For such a supercycle α define

$$(4.8) \qquad (\operatorname{des}_k, \operatorname{maj}_k)\, \alpha = (\operatorname{des}_k, \operatorname{maj}_k)\, w \quad \bigl(= (\operatorname{des}_k, \operatorname{maj}_k)(w\star)\bigr).$$

PROPERTY 4.4. *If w is a word in the alphabet X and $\alpha = \begin{pmatrix} u\star & \infty \\ w & \star \end{pmatrix}$ is an initial supercycle, then*

$$(\operatorname{exc}_k, \operatorname{den}_k)\, \alpha = (\operatorname{des}_k, \operatorname{maj}_k)\, \alpha = (\operatorname{des}_k, \operatorname{maj}_k)\, w.$$

Proof. Let $w = x_1 x_2 \ldots x_m$ and put $x_{m+1} = \star$, $x_{m+2} = \infty$. Then

$$\begin{aligned} &\operatorname{exc}_k \begin{pmatrix} x_2 & x_3 & \ldots & x_m & \star & \infty \\ x_1 & x_2 & \ldots & x_{m-1} & x_m & \star \end{pmatrix} \\ &\quad = \bigl|\{i : 1 \le i \le m, x_i > x_{i+1} \text{ or } x_i = x_{i+1} \in L\}\bigr| = \operatorname{des}_k x_1 x_2 \ldots x_m; \\ &\operatorname{den}_k \begin{pmatrix} x_2 & x_3 & \ldots & x_m & \star & \infty \\ x_1 & x_2 & \ldots & x_{m-1} & x_m & \star \end{pmatrix} \\ &\quad = \sum_{i=1}^{m+1} \bigl|\operatorname{Fact}_i(x_1 x_2 \ldots x_m \star) \cap \rrbracket x_i, x_{i+1} \rrbracket_k\bigr| = \operatorname{maj}_k x_1 x_2 \ldots x_m. \quad \square \end{aligned}$$

The bijection of Theorem 1.2 is constructed as follows :

(a) Let $w = x_1 x_2 \ldots x_m = x_1 u$ be a word in the alphabet $X = [\,r\,]$; form the (initial) supercycle $\alpha = \begin{pmatrix} u\star & \infty \\ w & \star \end{pmatrix}$.

(b) Apply the mapping τ to α iteratively until a final supercycle is reached. This makes sense because of (4.6). Furthermore, when applying

τ iteratively, the letter $\star$ remains the *rightmost* letter in all the supercycles within the iteration. Denote by $\widetilde{\alpha} = \begin{pmatrix} \overline{w} & \infty \\ \widetilde{w} & \star \end{pmatrix}$ the final supercycle obtained. [Then $\overline{w}$ is the non-decreasing rearrangement of w and $\widetilde{w}$.]

(c) Define ρ by $\rho(w) = \widetilde{w}$.

The proof of Theorem 1.2 is now completed as follows. First, to each word $w \in C(v)$ there corresponds one and only one initial supercycle defined by (a). Next the mapping $\alpha \mapsto \widetilde{\alpha}$ is bijective by Property 4.2, as we go from α to $\widetilde{\alpha}$ by applying τ iteratively until a final supercycle is reached. Finally, the mapping $\widetilde{\alpha} \mapsto \widetilde{w}$ is obviously bijective. On the other hand, the property $(\mathrm{exc}_k, \mathrm{den}_k)\, \rho(w) = (\mathrm{des}_k, \mathrm{maj}_k)\, w$ follows from Properties 4.4 and 4.3 and from (3.5) and (3.7). $\square$

Example. Again consider the order $1 < 2 < 3 < \star < 4 < 5 < \infty$ ($k = 2$ and 4, 5 large) and start with the word $w = 4, 4, 5, 1, 3, 1, 2, 3, 5$, so that the initial supercycle is $\alpha = \left(\begin{array}{|ccc|c|ccccc|c} 4&5&1&3&1&2&3&5&\star&\infty \\ 4&4&5&1&3&1&2&3&5&\star \end{array}\right)$ (indicating its k-factorization by vertical bars).

First, $\mathbf{T}^{-1}\begin{pmatrix} 4\,5\,1 \\ 4\,4\,5 \end{pmatrix} = T_1T_2 \begin{pmatrix} 4\,5\,1 \\ 4\,4\,5 \end{pmatrix} = \begin{pmatrix} 1\,4\,5 \\ 4\,5\,4 \end{pmatrix}$, so that $\alpha'' = \left(\begin{array}{|ccc|c|ccccc|c} 1&4&5&3&1&2&3&5&\star&\infty \\ 4&5&4&1&3&1&2&3&5&\star \end{array}\right)$ (in the notations of (4.4)). To obtain $\alpha' = \tau(\alpha)$ we have to replace $z_2 = 1$ by $z_1 = 4$, so that $\alpha_2 = \alpha' = \left(\begin{array}{ccc|c|ccccc|c} 4&4&5&3&1&2&3&5&\star&\infty \\ 4&5&4&1&3&1&2&3&5&\star \end{array}\right)$. Next $\mathbf{T}^{-1}\begin{pmatrix} 4\,4\,5\,3 \\ 4\,5\,4\,1 \end{pmatrix} = T_1T_2T_3 \begin{pmatrix} 4\,4\,5\,3 \\ 4\,5\,4\,1 \end{pmatrix} = \begin{pmatrix} 3\,4\,4\,5 \\ 4\,5\,1\,4 \end{pmatrix}$ and $\alpha'' = \left(\begin{array}{cccc|ccccc|c} 3&4&4&5&1&2&3&5&\star&\infty \\ 4&5&1&4&3&1&2&3&5&\star \end{array}\right)$. To get the next supercycle we have to replace $z_2 = 3$ by $z_1 = 1$, so that $\alpha_3 = \left(\begin{array}{cccc|ccccc|c} 1&4&4&5&1&2&3&5&\star&\infty \\ 4&5&1&4&3&1&2&3&5&\star \end{array}\right)$. Next the transformation $\mathbf{T}^{-1}$ to be applied to $\begin{pmatrix} 1\,4\,4\,5\,1\,2\,3\,5\,\star \\ 4\,5\,1\,4\,3\,1\,2\,3\,5 \end{pmatrix}$ is $(T_5T_6T_7T_8)(T_4T_5T_6)(T_3T_4T_5)(T_2T_3T_4)$, as we have to move the second "1", the "2", the "3" and the "$\star$" to the left. We then get $\mathbf{T}^{-1}\begin{pmatrix} 1\,4\,4\,5\,1\,2\,3\,5\,\star \\ 4\,5\,1\,4\,3\,1\,2\,3\,5 \end{pmatrix} = \begin{pmatrix} 1\,1\,2\,3\,\star\,4\,4\,5\,5 \\ 4\,5\,1\,4\,1\,3\,2\,3\,5 \end{pmatrix}$, so that $\alpha'' = \left(\begin{array}{ccccccccc|c} 1&1&2&3&\star&4&4&5&5&\infty \\ 4&5&1&4&1&3&2&3&5&\star \end{array}\right)$. Finally, the "$\star$" on the top row is to be replaced by the penultimate k-record, i.e., "3." We get $\widetilde{\alpha} = \left(\begin{array}{ccccccccc|c} 1&1&2&3&3&4&4&5&5&\infty \\ 4&5&1&4&1&3&2&3&5&\star \end{array}\right)$. Thus $\rho(w) = 4, 5, 1, 4, 1, 3, 2, 3, 5$. We can verify that $(\mathrm{des}_k, \mathrm{maj}_k)\, w = (\mathrm{des}_k, \mathrm{maj}_k)\, \alpha = (\mathrm{exc}_k, \mathrm{den}_k)\, \alpha = (\mathrm{exc}_k, \mathrm{den}_k)\, \widetilde{\alpha} = (\mathrm{exc}_k, \mathrm{den}_k)\, \rho(w) = (4, 18)$.

5. THE INVARIANCE PRINCIPLE FOR DEN

We now turn to the proof of Theorem 1.3. As before, we use the notations exc_D, exc_E, etc., instead of $\mathrm{exc}_{k,D}$, $\mathrm{exc}_{k,E}$, etc.

Definition. Let D and E be total orderings on X. We say that E is obtained from D by a *vital transposition* if there exist letters a and b in X such that

(i) $b <_D a$ and a covers b (i.e., there is no letter x such that $b <_D x <_D a$);
(ii) $a <_E b$ and b covers a;
(iii) for all $x, y \in X$, if $\{x,y\} \neq \{a,b\}$ then $x <_D y$ if and only if $x <_E y$;
(iv) one of a and b is large and the other is small.

In other words, E is obtained from D by simply interchanging the order of two successive letters, one large and one small.

Now let D be any total ordering on X. It is easy to see that there is a total ordering D' on X, compatible with k, that can be obtained from D by a sequence of vital transpositions. By using the bijection ρ of the previous section, and Theorem 1.1, we can verify that it suffices to prove Theorem 1.3 on the assumption that E can be obtained from D by a vital transposition.

Thus for the remainder of this section we assume that E can be obtained from D by a vital transposition that interchanges the order of b and a. We will assume that $b <_D a$ and $a <_E b$, and that b is small and a is large. In general, x, y, z will designate elements of X, while u, v, w will denote *words* in the alphabet X. We also denote cyclic intervals by $\rrbracket x,y\rrbracket_D$ or $\rrbracket x,y\rrbracket_E$ according as the ordering D or E is used, i.e., we suppress the usual subscript k. If α and β are biwords, we write $\alpha \overset{D}{\sim} \beta$ (resp. $\alpha \overset{E}{\sim} \beta$) to mean that α can be transformed into β by a sequence of D-transpositions (resp. E-transpositions), i.e., Han transpositions defined by means of the ordering D (resp. E).

5.1. *A study of cycles.* The following relations on cyclic intervals based upon orderings D and E are easily checked by looking at Fig. 3, where cyclic intervals are materialized by oriented rectangles.

$$\text{(5.1)}\qquad \rrbracket b,a\rrbracket_D = \emptyset;\quad \rrbracket a,b\rrbracket_D = X;\quad \rrbracket x,b\rrbracket_D = \rrbracket x,a\rrbracket_D \quad (x \in X);$$

$$\text{(5.2)}\qquad \rrbracket b,a\rrbracket_E = X \setminus \{a,b\};\quad \rrbracket a,b\rrbracket_E = \{a,b\};$$
$$\rrbracket x,b\rrbracket_E = \rrbracket x,a\rrbracket_E \uplus \{a,b\} \quad (x \neq a,b);$$

$$\text{(5.3)}\qquad \rrbracket x,a\rrbracket_D = \rrbracket x,a\rrbracket_E \cup \{b\};\quad \rrbracket x,b\rrbracket_E = \rrbracket x,b\rrbracket_D \cup \{a\};$$
$$\rrbracket x,y\rrbracket_D = \rrbracket x,y\rrbracket_E \quad (x,y \neq a,b).$$

We will use the notions of *a-cycle*, *$a \setminus b$-cycle*, *(a,b)-free biword*, defined as follows : an *a-cycle* on X is a biword $\gamma = \begin{pmatrix} u\ a \\ a\ u \end{pmatrix} = [a\,u]$, such that a does *not* occur in the word u. The a-cycle $\gamma = [a\,u]$ is called an *$a \setminus b$-cycle* if furthermore b does not occur in u. The notions of *b-cycle* and *$b \setminus a$-cycle* are defined in a similar way. A biword is said to be *(a, b)-free*, if it does not involve the letters a or b.

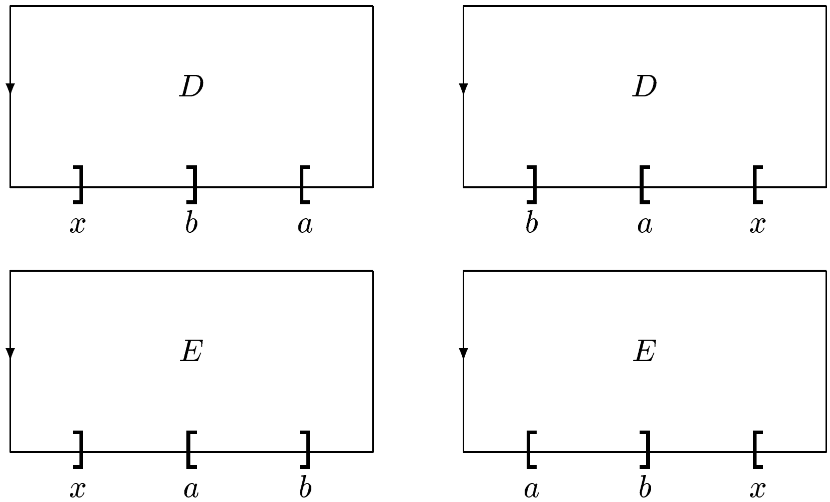


Fig. 3

Lemma 5.1. *If α is an (a, b)-free biword, or an $a \setminus b$-cycle, or a $b \setminus a$-cycle, then*

$$(\mathrm{exc}_D, \mathrm{den}_D)\ \alpha = (\mathrm{exc}_E, \mathrm{den}_E)\ \alpha. \tag{5.4}$$

Proof. Let $\alpha = \binom{u}{v} = \binom{y_1 \ldots y_m}{x_1 \ldots x_m}$ be any one of the forementioned three biwords. If $m = 1$, relation (5.4) is trivial. When $m \geq 2$, the definition of exc_k (given in (3.3)) requires the comparison of the elements in each pair (x_i, y_i) $(1 \leq i \leq m)$. But such pairs involve at most one of the letters a or b. Accordingly, $x_i \geq_D y_i$ if and only if $x_i \geq_E y_i$ and then $\mathrm{exc}_D\, \alpha = \mathrm{exc}_E\, \alpha$.

For the same reason, $z \in \rrbracket x_i, y_i \rrbracket_D$ if and only if $z \in \rrbracket x_i, y_i \rrbracket_E$. From the definition of den_k (given in (3.4)) it follows that $\mathrm{den}_D\, \alpha = \mathrm{den}_E\, \alpha$. $\square$

Lemma 5.2. *Let α be a biword of the form $\binom{w_1}{w_2}\gamma$, where γ is a cycle. Then*

$$\mathrm{exc}_D\, \alpha = \mathrm{exc}_D \binom{w_1}{w_2} + \mathrm{exc}_D\, \gamma; \tag{5.5}$$

$$\mathrm{den}_D\, \alpha = \mathrm{den}_D \binom{w_1}{w_2} + \mathrm{den}_D\, \gamma + l(w_2)\, \mathrm{exc}_D\, \gamma; \tag{5.6}$$

where $l(w_2)$ stands for the length of w_2.

Proof. This result is essentially [14, lemma 5.5]. Again we just have to verify that if $\gamma = \binom{y_1 \dots y_n}{x_1 \dots x_n}$ is a cycle, then $\rrbracket x_1, y_1 \rrbracket_D \uplus \cdots \uplus \rrbracket x_n, y_n \rrbracket_D$ is the multiset containing each letter of X^+ with multiplicity $\operatorname{exc}_D \gamma$. ∎

Definition. An (a,b)-*sequence* is defined to be a sequence of biwords

$$\sigma = (\widetilde{w_0}, \gamma_1, \ldots, \gamma_h) \qquad (h \geq 0),$$

where $\widetilde{w_0}$ is an (a,b)-free biword (possibly empty) of the form $\binom{\overline{w}_0}{w_0}$ ($\overline{w}_0$ being the non-decreasing rearrangement of the word w_0), and where each γ_i is either an $a \setminus b$-cycle or a $b \setminus a$-cycle.

If $\sigma = (\widetilde{w_0}, \gamma_1, \ldots, \gamma_h)$ is an (a,b)-sequence, the *juxtaposition product* of the biwords $\widetilde{w_0}\gamma_1 \ldots \gamma_h$ will be denoted by $\{\sigma\}$.

LEMMA 5.3. *If σ is an (a,b)-sequence, then*

$$(\operatorname{exc}_D, \operatorname{den}_D)\{\sigma\} = (\operatorname{exc}_E, \operatorname{exc}_E)\{\sigma\}. \tag{5.7}$$

Proof. If $\sigma = (\widetilde{w_0}, \gamma_1, \ldots, \gamma_h)$, then

$$\begin{aligned}
\operatorname{exc}_D\{\sigma\} &= \operatorname{exc}_D \widetilde{w}_0 + \operatorname{exc}_D \gamma_1 + \cdots + \operatorname{exc}_D \gamma_h && \text{[by Lemma 5.2]}\\
&= \operatorname{exc}_E \widetilde{w}_0 + \operatorname{exc}_E \gamma_1 + \cdots + \operatorname{exc}_E \gamma_h && \text{[by Lemma 5.1]}\\
&= \operatorname{exc}_E\{\sigma\}. && \text{[by Lemma 5.2]}
\end{aligned}$$

In the same manner

$$\begin{aligned}
\operatorname{den}_D\{\sigma\} &= \operatorname{den}_D \widetilde{w}_0 + \operatorname{den}_D \gamma_1 + l(w_0) \operatorname{exc}_D \gamma_1\\
&\qquad + \cdots + \operatorname{den}_D \gamma_h + l(w_0\gamma_2 \ldots \gamma_{h-1}) \operatorname{exc}_D \gamma_h && \text{[by (5.6)]}\\
&= \operatorname{den}_E \widetilde{w}_0 + \operatorname{den}_E \gamma_1 + l(w_0) \operatorname{exc}_E \gamma_1\\
&\qquad + \cdots + \operatorname{den}_E \gamma_h + l(w_0\gamma_2 \ldots \gamma_{h-1}) \operatorname{exc}_E \gamma_h && \text{[by (5.4)]}\\
&= \operatorname{den}_E\{\sigma\}. \quad ∎
\end{aligned}$$

Denote by $\Sigma(\mathbf{c},\mathbf{d}; a, b)$ the set of all (a,b)-sequences $\sigma = (\widetilde{w}_0, \gamma_1, \ldots, \gamma_h)$, such that $\widetilde{w}_0 = \binom{\overline{w}_0}{w_0}$, $\gamma_1 = [w_1]$, $\ldots$, $\gamma_h = [w_h]$, and the juxtaposition product of the words $w_0 w_1 \ldots w_h$ belongs to $R(\mathbf{c}, \mathbf{d})$. The (a,b)-sequences in $\Sigma(\mathbf{c},\mathbf{d}; a, b)$ may be regarded as *cycle decompositions* of biwords based on the two letters a, b. To see that $\Sigma(\mathbf{c}, \mathbf{d}; a, b)$ is *equinumerous* with $R(\mathbf{c}, \mathbf{d})$, we can start with a word $w \in R(\mathbf{c}, \mathbf{d})$ and set up a factorization $(u_0, c_1u_1, c_2u_2, \ldots, c_nu_n)$ of w, where the c_i's are letters either equal to a or b, and the u_i's are words with no a's and no b's. We simply get the previous factorization by cutting w just before each occurrence of a or b.

Clearly, $\sigma = (\binom{\overline{u}_0}{u_0}, [c_1u_1], [c_2u_2], \ldots, [c_nu_n])$ belongs to $\Sigma(\mathbf{c},\mathbf{d}; a, b)$ and the mapping $w \mapsto \sigma$ is bijective.

More elaborate bijections will be needed to prove our theorem 1.3. Denote by $\widetilde{R}(\mathbf{c}, \mathbf{d})$ the set of all biwords $\widetilde{w} = \binom{\overline{w}}{w}$, where $w \in R(\mathbf{c}, \mathbf{d})$. The first bijection

$$\xi_D : \widetilde{w} \mapsto \xi_D(\widetilde{w}) = (\widetilde{w}_0, \gamma_1, \ldots, \gamma_h) \tag{5.8}$$

of $\widetilde{R}(\mathbf{c}, \mathbf{d})$ onto $\Sigma(\mathbf{c}, \mathbf{d}; a; b)$, constructed in the next subsection 5.2, will have the property that the juxtaposition product $\{\xi_D(\widetilde{w})\} = \widetilde{w}_0\gamma_1 \ldots \gamma_h$ can be derived from the biword $\widetilde{w} = \binom{\overline{w}}{w}$ by a well-defined sequence of Han D-transpositions (i.e., Han transpositions defined by means of the ordering D). Therefore, by Lemma 3.6

$$(\mathrm{exc}_D, \mathrm{den}_D)\ \widetilde{w} = (\mathrm{exc}_D, \mathrm{den}_D)\ \{\xi_D(\widetilde{w})\}. \tag{5.9}$$

In the last subsection 5.3 another bijection

$$\xi_E : \widetilde{w} \mapsto \xi_E(\widetilde{w}) \tag{5.10}$$

of $\widetilde{R}(\mathbf{c}, \mathbf{d})$ onto $\Sigma(\mathbf{c}, \mathbf{d}; a; b)$ will be constructed. This time the juxtaposition product $\{\xi_E(\widetilde{w})\}$ will be obtainable from $\widetilde{w}$ by a sequence of Han E-transpositions. Therefore

$$(\mathrm{exc}_E, \mathrm{den}_E)\ \widetilde{w} = (\mathrm{exc}_E, \mathrm{den}_E)\ \{\xi_E(\widetilde{w})\}. \tag{5.11}$$

The bijection δ of Theorem 3.1 will then be defined by

$$\widetilde{\delta(w)} = \xi_E^{-1}\xi_D(\widetilde{w}). \tag{5.12}$$

It satisfies our requirements, as

$$\begin{aligned}
(\mathrm{exc}_E, \mathrm{den}_E)\delta(w) &= (\mathrm{exc}_E, \mathrm{den}_E)\ \widetilde{\delta(w)} + |\delta(w) \cap L| && [\text{by } (3.6)]\\
&= (\mathrm{exc}_E, \mathrm{den}_E)\ \{\xi_E\widetilde{\delta(w)}\} + |\delta(w) \cap L| && [\text{by } (5.11)]\\
&= (\mathrm{exc}_E, \mathrm{den}_E)\ \{\xi_D(\widetilde{w})\} + |\delta(w) \cap L| && [\text{by } (5.12)]\\
&= (\mathrm{exc}_D, \mathrm{den}_D)\ \{\xi_D(\widetilde{w})\} + |\delta(w) \cap L| && [\text{by } (5.7)]\\
&= (\mathrm{exc}_D, \mathrm{den}_D)\ \widetilde{w} + |\delta(w) \cap L| && [\text{by } (5.9)]\\
&= (\mathrm{exc}_D, \mathrm{den}_D)\ w. && [\text{by } (3.6)]
\end{aligned}$$

5.2. *Transforming biwords under D.* Throughout this subsection, we use the total ordering D on X and give the construction of ξ_D announced in (5.8).

Let w be a word in $R(\mathbf{c},\mathbf{d})$, and let x be any letter occurring in w. Next, form the biword $\widetilde{w}=\begin{pmatrix}\overline{w}\\ w\end{pmatrix}$. Using the least possible number of Han D-transpositions, move the right-most x in the top row of $\widetilde{w}$ to the right-hand end of that row, forming the biword α. Suppose that the biletter $\begin{pmatrix}x\\ y\end{pmatrix}$ occurs at the right-hand end of α. If $y=x$ then put $\gamma=[x]$. If not, use Han transpositions to move the right-most y in the top row of α to the penultimate position in that row, forming a new biword β with the biword $\begin{pmatrix}y & x\\ z & y\end{pmatrix}$ occupying its right-most two columns. Continue in this way until a cycle γ is obtained. That cycle is necessarily an x-cycle; denote it by γ_1. Thus $\widetilde{w}$ has been transformed into a product $\delta\gamma_1$ for some biword δ. Since the letters in the top row of δ will still be non-decreasing, it follows that $\delta=\widetilde{w'}$ for some word w'. (Actually, the present algorithm is described in detail in [14, Algorithme 4.2] in the context of $(k=0)$-cyclic intervals.) If w' has no occurrence of x, put $\eta_{D,x}(\widetilde{w})=(\widetilde{w'},\gamma_1)$. Otherwise, apply the above algorithm to $\widetilde{w'}$. After finitely many steps $\widetilde{w}$ is transformed into a sequence of the form $(\widetilde{u},\gamma_n,\ldots,\gamma_2,\gamma_1)$, where u has no occurrence of x and the γ_i's are all x-cycles. Again, put

$$\eta_{D,x}(\widetilde{w})=(\widetilde{u},\gamma_n,\ldots,\gamma_2,\gamma_1). \tag{5.13}$$

Finally, let $\eta_{D,x}(\widetilde{w})=\widetilde{w}$ if w has no occurrence of x. Clearly, $\eta_{D,x}$ has a well-defined *inverse mapping* $\eta_{D,x}^{-1}$ that maps each forementioned sequence onto a biword $\widetilde{w}$.

Now if γ is a a-cycle, written as $\gamma=[au_1bu_2b\ldots u_{n-1}bu_n]$, where all the u_i's have no occurrences of b (and of course of a!), define $\zeta_D(\gamma)$ to be the sequence

$$\zeta_D(\gamma)=(\,[au_1],[bu_2],\ldots,[bu_{n-1}],[bu_n]\,). \tag{5.14}$$

If γ has no occurences of b, i.e., γ is an $a\backslash b$-cycle, then we define $\zeta_D(\gamma)=\gamma$.

Again the *inverse mapping* ζ_D^{-1} is obviously defined.

Lemma 5.4. *Let u be a word and y, z be letters. Then*

$$\begin{pmatrix}b & u & a\\ y & z & u\end{pmatrix}\overset{D}{\sim}\begin{pmatrix}a & u & b\\ y & z & u\end{pmatrix}. \tag{5.15}$$

Proof. If u is empty, then $\begin{pmatrix}b & a\\ y & z\end{pmatrix}\overset{D}{\sim}\begin{pmatrix}a & b\\ y & z\end{pmatrix}$ since $\rrbracket b,a\rrbracket_D=\emptyset$. Suppose that (5.15) is true when the length of u is $l\geq 0$. Assume first that y and

z are neighbours with respect to (b,x). Then $\begin{pmatrix} b & x & u & a \\ y & z & x & u \end{pmatrix} \overset{D}{\sim} \begin{pmatrix} x & b & u & a \\ y & z & x & u \end{pmatrix} \overset{D}{\sim} \begin{pmatrix} x & a & u & b \\ y & z & x & u \end{pmatrix}$, by the inductive hypothesis. As y and z are neighbours with respect to (a,x), since $\rrbracket b,x\rrbracket_D = \rrbracket a,x\rrbracket_D$, we have $\begin{pmatrix} x & a \\ y & z \end{pmatrix} \overset{D}{\sim} \begin{pmatrix} a & x \\ y & z \end{pmatrix}$ and so $\begin{pmatrix} b & x & u & a \\ y & z & x & u \end{pmatrix} \overset{D}{\sim} \begin{pmatrix} a & x & u & b \\ y & z & x & u \end{pmatrix}$. The case in which y and z are strangers with respect to (b,x) is treated similarly. $\square$

Let γ be an a-cycle and $\zeta_D(\gamma) = (\delta_0, \delta_1, \ldots, \delta_m)$ as in (5.14). Denote by $\{\zeta_D(\gamma)\}$ the *juxtaposition product* $\delta_0\delta_1\ldots\delta_m$.

PROPOSITION 5.5. *If γ is an a-cycle, then*

$$\gamma \overset{D}{\sim} \{\zeta_D(\gamma)\}. \tag{5.16}$$

Proof. Let $\gamma = [au_1bu_2]$ with no occurrences of b in u_2. Write $au_1 = v_1y$ for $y \in X$. Then $\gamma = \begin{pmatrix} u_1 & b & u_2 & a \\ a & u_1 & b & u_2 \end{pmatrix} = \begin{pmatrix} u_1 & b & u_2 & a \\ v_1 & y & b & u_2 \end{pmatrix} \overset{D}{\sim} \begin{pmatrix} u_1 & a & u_2 & b \\ v_1 & y & b & u_2 \end{pmatrix} = [au_1][bu_2]$, by using (5.15). We then conclude by induction on the number of b's in $[au_1]$. $\square$

The construction of ξ_D.

(1) Start with $w \in R(\mathbf{c}, \mathbf{d})$ and form $\widetilde{w}$ as before.

(2) Form $\eta_{D,a}(\widetilde{w}) = (\widetilde{u}, \gamma_n, \ldots, \gamma_2, \gamma_1)$, as in (5.13) with $x = a$. Note that the sequence reduces to $\widetilde{u} = \widetilde{w}$, if there is no occurrence of a in w. Otherwise, the sequence involves the a-cycles $(\gamma_n, \ldots, \gamma_2, \gamma_1)$.

(3) If necessary, apply ζ_D to each of the a-cycles $\gamma_n, \ldots, \gamma_2, \gamma_1$, to obtain

$$\zeta_D(\gamma_i) = (\,\delta_{i,0}, \delta_{i,1}, \ldots, \delta_{i,m_i-1}, \delta_{i,m_i}\,) \qquad (i = n, \ldots, 2, 1).$$

As each γ_i is an a-sequence, each $\delta_{i,0}$ will be an $a \setminus b$-cycle and the other $\delta_{i,1}, \ldots, \delta_{i,m_i-1}, \delta_{i,m_i}$ will be $b \setminus a$-cycles.

(4) Apply $\eta_{D,b}$ to the biword $\widetilde{u}$ that contains no a, to obtain $\eta_{D,b}(\widetilde{u}) = (\widetilde{u}', \gamma''_{n''}, \ldots, \gamma''_2, \gamma''_1)$, as in (5.13) with $x = b$. Note that the sequence reduces to $\widetilde{u}' = \widetilde{u}$, if there is no occurrence of b in u.

(5) Then $\xi_D(\widetilde{w})$ is the juxtaposition product of the sequences :

$$\begin{aligned}\xi_D(\widetilde{w}) &= (\eta_{D,b}(\widetilde{u}), \zeta_D(\gamma_n), \ldots, \zeta_D(\gamma_1)) \\ &= (\widetilde{u}', \gamma''_{n''}, \ldots, \gamma''_2, \gamma''_1, \delta_{n,0}, \delta_{n,1}, \ldots, \delta_{n,m_n-1}, \delta_{n,m_n}, \\ &\qquad \ldots, \delta_{1,0}, \delta_{1,1}, \ldots, \delta_{1,m_1-1}, \delta_{1,m_1}).\end{aligned}$$

Property (5.9) holds since $\eta_{D,b}(\widetilde{u})$ can be obtained from $\widetilde{w}$ by a sequence of D-transpositions, as well as the $\zeta_D(\gamma_i)$'s by Proposition 5.5.

To recover the biword $\widetilde{w}$ from an (a,b)-sequence $\sigma = (\widetilde{w}_0, \gamma_1, \ldots, \gamma_h)$,

(5′) cut the sequence $(\gamma_1, \ldots, \gamma_h)$ before each $a \setminus b$-cycle to obtain $(\omega_0 \,|\, \omega_1 \,|\, \ldots \,|\, \omega_l)$, where each ω_0 is empty or is a (juxtaposition) product of $b \setminus a$-cycles and the other ω_i $(i \geq 1)$ are juxtaposition products of one $a \setminus b$-cycle followed by $b \setminus a$-cycles.

(4′) Determine $\eta^{-1}_{D,b}(\widetilde{w}_0, \omega_0)$.

(3′) Determine $\zeta^{-1}_D(\omega_i)$ for $i = 1, \ldots, l$.

(2′) The biword $\widetilde{w}$ is the image by $\eta^{-1}_{D,a}$ of the sequence
$\big(\eta^{-1}_{D,b}(\widetilde{w}_0, \omega_0), \zeta^{-1}_D(\omega_1), \ldots, \zeta^{-1}_D(\omega_l)\big)$.

(1′) The bottom row of the latter biword is the word w.

Example. Let $X = [\,5\,]$ and let D be the natural ordering on X. Let 1, 2 and 3 be small and 4 and 5 be large. Thus $a = 4$, $b = 3$. Let $w = 3, 2, 2, 4, 3, 5, 1, 5, 3, 1$. Then

(1) $\overline{w} = 1, 1, 2, 2, 3, 3, 3, 4, 5, 5$ and $\widetilde{w} = \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,4\,5\,5 \\ 3\,2\,2\,4\,3\,5\,1\,5\,3\,1 \end{pmatrix}$.

(2) Sort out the 4-cycle : $\eta_{D,4}(\widetilde{w}) = \left(\begin{pmatrix} 2\,2\,3\,5 \\ 2\,3\,2\,5 \end{pmatrix}, [4, 1, 5, 3, 3, 1]\right)$.

(3) Then break that 4-cycle : $\zeta_D([4, 1, 5, 3, 3, 1] = [4, 1, 5][3][3, 1]$.

(4) Sort out the 3-cycles : $\eta_{D,3}(\widetilde{u}) = \eta_{D,3}\begin{pmatrix} 2\,2\,3\,5 \\ 2\,3\,2\,5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}[3, 2, 5]$.

(5) Thus $\xi_D(\widetilde{w}) = \big(\tilde{2}, [3, 2, 5], [4, 1, 5], [3], [3, 1]\big)$.

5.3. *Transforming biwords under E.* Throughout this section, we use the total ordering E on X. Our aim is to show that the algorithm described above for the construction of ξ_D also applies to ξ_E, once the *subscript* D is replaced by E. Accordingly, the mappings $\eta_{E,x}$ and ζ_E are to be defined, as well as their inverses.

The *definition* of $\eta_{E,x}$ is the *same* as the definition of $\eta_{D,x}$ shown in (5.13), the Han E-transpositions replacing the Han D-transpositions. The reverse mapping $\eta^{-1}_{E,x}$ is also derived in the same way.

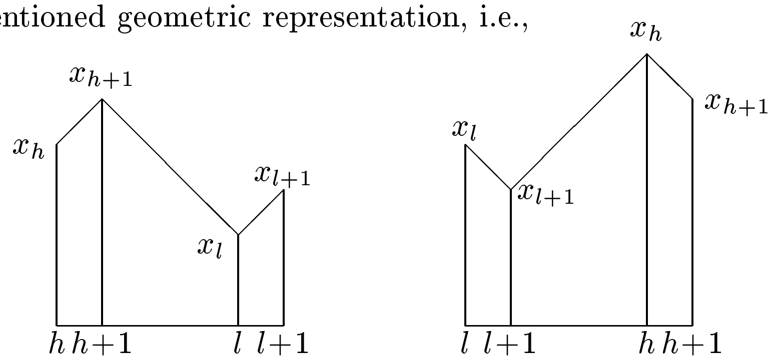
The definition of ζ_E requires a more detailed analysis of the ordering E. For $x, y \notin \{a, b\}$ define $x \prec y$ if and only if a and x are neighbours with respect to (a, y). Then $x \prec x$ if x is small, while $x \not\prec x$ if x is large. Next, we have $x \prec y$ and $x \neq y$ if and only if one of the three conditions holds : (i) $x <_E y <_E a <_E b$; (ii) $a <_E b <_E x <_E y$; (iii) $y <_E a <_E b <_E x$. Note that by (5.2), $x \prec y$ if and only if a and x are strangers with respect to (b, y).

Let $w = x_1 x_2 \ldots x_m$ be a non-empty word in the alphabet $X \setminus \{a, b\}$. We define a bijection $h \mapsto l$ of the interval $\{0, 1, \ldots, m\}$ onto $\{0, 1, \ldots, m\}$ as follows [by convention : $x_0 = x_{m+1} = \infty$] :

(i) if $h = 0$, or if $1 \le h \le m-1$ and $x_h \prec x_{h+1}$, define l to be the smallest integer such that $h+1 \le l$ and $x_l \prec x_{l+1}$.

(ii) if $1 \le h \le m-1$ and $x_h \not\prec x_{h+1}$, or if $h = m$, define l to be the greatest integer such that $l+1 \le h$ and $x_l \not\prec x_{l+1}$.

On Fig. 4 part of the word w is represented as a polygonal line in the Euclidean plane joining the points (h, x_h) to $(h+1, x_{h+1})$, etc. In case (i) the integer l is the abscissa of the nearest *rise* $x_l \prec x_{l+1}$ *to the right* of the rise $x_h \prec x_{h+1}$, while in case (ii) l is the abscissa of the nearest *descent* $x_l \not\prec x_{l+1}$ *to the left* of the descent $x_h \not\prec x_{h+1}$. This suffices to show that $h \mapsto l$ is bijective, and the inverse $l \mapsto h$ is defined in the same way, using the forementioned geometric representation, i.e.,



case (i) case (ii)

Fig. 4

(i) if $l = m$, or if $1 \le l \le m-1$ and $x_l \prec x_{l+1}$, define h to be the greatest integer such that $h+1 \le l$ and $x_h \prec x_{h+1}$;

(ii) if $1 \le h \le m-1$ and $x_l \not\prec x_{l+1}$, or if $l = 0$, define h to be the smallest integer such that $l+1 \le h$ and $x_h \not\prec x_{h+1}$.

Let φ be the bijection that maps the ordered pair of words determined by h onto the pair of words determined by l, i.e.,

$$\varphi(x_1 \dots x_h,\, x_{h+1} \dots x_m) = (x_1 \dots x_l,\, x_{l+1} \dots x_m) \tag{5.17}$$

and denote by φ^{-1} its inverse. The bijection φ is used in the definition of the following transformation Φ. Let $n \ge 0$ and $U = (u_0, u_1, \dots, u_n)$ be a sequence of words in the alphabet $X \setminus \{a, b\}$ (some possibly empty). If $n = 0$, define $\Phi(U) = (u_0)$. Suppose $n \ge 1$; then $\Phi(U)$ is defined recursively by :

$$\begin{gathered}\varphi(u_{n-1}, u_n) = (u'_{n-1}, v_n);\\ \Phi(u_0, u_1, \dots, u_{n-2}, u'_{n-1}) = (v_0, v_1, \dots, v_{n-1});\\ \Phi(U) = \Phi(u_0, u_1, \dots, u_{n-2}, u_{n-1}, u_n) = (v_0, v_1, \dots, v_{n-1}, v_n).\end{gathered} \tag{5.18}$$

As φ is bijective, Φ is also bijective with Φ^{-1} given by :

if $\Phi^{-1}(v_0, v_1, \dots, v_{n-2}, v_{n-1}) = (u_0, u_1, \dots, u_{n-2}, u'_{n-1})$,

and $\varphi^{-1}(u'_{n-1}, v_n) = (u_{n-1}, u_n)$,
then $\Phi^{-1}(v_0, v_1, \ldots, v_{n-1}, v_n) = (u_0, u_1, \ldots, u_{n-2}, u_{n-1}, u_n)$.

Example. Consider the ordering $1 <_E 2 <_E 4 <_E 3 <_E 5$, where $1, 2, 3$ are small, while $4, 5$ are large. Let $a = 4$ and $b = 3$, so that $5 \prec 1 \prec 2$. Next consider the sequence of words $U = (1,2,2,5\,;\ 5,5,2\,;\ 1,2,5\,;\ e\,;\ 2,1,5,1)$ separated by semi-colons, with e denoting the empty word. We have successively :

$\varphi(e\,;\ 2 \not\prec 1 \not\prec 5 \prec 1) = (2,1,5\,;\ 1)$, so that $u'_3 = 2,1,5$ and $v_4 = 1$;
as $5 \prec 2$, $\varphi(1,2,5\,;\ 2 \not\prec 1 \not\prec 5) = (1,2,5,2,1,5\,;\ e)$ and $v_3 = e$;
as $2 \not\prec 1$, $\varphi(5 \not\prec 5 \prec 2\,;\ 1,2,5,2,1,5) = (5\,;\ 5,2,1,2,5,2,1,5)$ and $v_2 = 5,2,1,2,5,2,1,5$;
as $5 \not\prec 5$, $\varphi(1 \prec 2 \prec 2 \not\prec 5\,;\ 5) = (1,2,2\,;\ 5,5)$ and $v_1 = 5,5$, $v_0 = 1,2,2$.
Thus $V = (1,2,2\,;\ 5,5\,;\ 5,2,1,2,5,2,1,5\,;\ e\,;\ 1)$.

We are now ready to define the bijection ζ_E. Let $\gamma = [bu_0au_1a\ldots au_n]$ be a b-cycle, the words u_i's having no occurrences of a and b. Let $\Phi(u_0, u_1, \ldots, u_n) = (v_0, v_1, \ldots, v_n)$. Then put

$$\zeta_E(\gamma) = \bigl([bv_0], [av_1], \ldots, [av_n]\bigr). \tag{5.19}$$

Following the recursive definition of Φ we can also write when $n \geq 1$:
if $\varphi(u_{n-1}, u_n) = (u'_{n-1}, v_n)$ and $\gamma' = [bu_0au_1\ldots au_{n-2}au'_{n-1}]$, then

$$\zeta_E(\gamma) = \bigl(\zeta_E(\gamma'), [av_n]\bigr). \tag{5.20}$$

From what has been said about φ and Φ the mapping ζ_E is clearly bijective and its inverse ζ_E^{-1} well-defined. The crucial point is to prove the analogue of Proposition 5.5, i.e., Proposition 5.7 that we will state after the following lemma.

Lemma 5.6. *Let u be a word with no occurrences of a and b and $y, z \neq a, b$. Then*

$$\begin{pmatrix} a & u & b \\ y & z & u \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} b & u & a \\ y & z & u \end{pmatrix}. \tag{5.21}$$

Proof. The proof is very similar to the proof of Lemma 5.4 with the sole difference that $\rrbracket a, b \llbracket_E = \{a, b\}$, while $\rrbracket a, b \llbracket_D = \emptyset$, but the same steps hold, because u has no occurrences of a and b and $y, z \neq a, b$. $\square$

Proposition 5.7. *If γ is a b-cycle, then*

$$\gamma \overset{E}{\sim} \{\zeta_E(\gamma)\}. \tag{5.22}$$

Proof. Let $\gamma = [bu_0au_1a\dots au_n]$ with no occurrences of a and b in the u_i's. If $n = 0$, there is nothing to prove since $\zeta_E(\gamma) = (\gamma)$. Suppose $n \ge 1$ and take up again the notations of (5.19). Also, let $u_{n-1} = x_1\dots x_h$, $u_n = x_{h+1}\dots x_m$, $v'_{n-1} = x_1\dots x_l$, $v_n = x_{l+1}\dots x_m$. Finally, let $w_{n-2} = u_0au_1a\dots au_{n-2}$, if $n \ge 2$ and $w_{n-2} = e$, if $n = 1$. Consider the two cases (i) and (ii) introduced in the definition of φ :

Case (i) reads either $h = 0$, i.e., u_{n-1} empty, or $1 \le h \le m-1$ and $x_h \prec x_{h+1}$; furthermore l satisfies the relations : $h+1 \le l$, $x_{h+1} \not\prec x_{h+2} \not\prec \cdots \not\prec x_{l-1} \not\prec x_l$, but $x_l \prec x_{l+1}$. If $h = 0$ and $z = a$ or b, we have $\begin{pmatrix} a & x_{h+1} \\ z & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} x_{h+1} & a \\ z & a \end{pmatrix}$; also $\begin{pmatrix} a & x_{h+1} \\ x_h & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} x_{h+1} & a \\ x_h & a \end{pmatrix}$ if $1 \le h$. Moreover, as $\begin{pmatrix} a & x_{h+2} \\ a & x_{h+1} \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} x_{h+2} & a \\ x_{h+1} & a \end{pmatrix}, \dots, \begin{pmatrix} a & x_l \\ a & x_{l-1} \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} x_l & a \\ x_{l-1} & a \end{pmatrix}$, but $\begin{pmatrix} a & x_{l+1} \\ a & x_l \end{pmatrix} \overset{E}{\not\sim} \begin{pmatrix} x_{l+1} & a \\ a & x_l \end{pmatrix}$, moving a (using Han E-transpositions) to the right, just after x_{l+1}, transforms γ into the biword :

$$\gamma'' = \begin{pmatrix} w_{n-2} & x_1 & \dots & x_h & x_{h+1} & x_{h+2} & \dots & x_l & x_{l+1} & a & x_{l+2} & \dots & b \\ b & w_{n-2} & \dots & x_{h-1} & x_h & x_{h+1} & \dots & x_{l-1} & a & x_l & x_{l+1} & \dots & x_m \end{pmatrix}.$$

Finally,

$$\begin{aligned} \gamma'' &\overset{E}{\sim} \begin{pmatrix} w_{n-2} & x_1 & \dots & x_l & x_{l+1} & b & x_{l+2} & \dots & a \\ b & w_{n-2} & \dots & x_{l-1} & a & x_l & x_{l+1} & \dots & x_m \end{pmatrix} && \text{[by Lemma 5.6]} \\ &\overset{E}{\sim} \begin{pmatrix} w_{n-2} & x_1 & \dots & x_l & b & x_{l+1} & x_{l+2} & \dots & a \\ b & w_{n-2} & \dots & x_{l-1} & x_l & a & x_{l+1} & \dots & x_m \end{pmatrix} && [\text{as } x_l \prec x_{l+1}] \\ &= [bw_{n-2}v'_{n-1}][av_n] = \gamma'[av_n] \overset{E}{\sim} \{\zeta_E(\gamma')[av_n]\} = \{\zeta_E(\gamma)\}, \end{aligned}$$

by induction on n (see (5.20) and (5.22)).

Case (ii) reads either $h = m$, i.e., u_n empty, or $1 \le h \le m-1$ and $x_h \not\prec x_{h+1}$; this time l satisfies the relations : $l \le h-1$, $x_{l+1} \prec x_{l+2} \prec \cdots \prec x_{h-1} \prec x_h$, but $x_l \not\prec x_{l+1}$. If $h = m$, then $\begin{pmatrix} a & b \\ x_h & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} b & a \\ a & x_h \end{pmatrix}$, but $\begin{pmatrix} a & x_{h+1} \\ x_h & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} x_{h+1} & a \\ a & x_h \end{pmatrix}$ if $h \le m-1$. First we apply either one of those Han transpositions on γ and make use of Lemma 5.6 to get the biword

$$\gamma'' = \begin{pmatrix} w_{n-2} & x_1 & \dots & x_h & x_{h+1} & b & x_{h+2} & \dots & x_m & a \\ b & w_{n-2} & \dots & x_{h-1} & a & x_h & x_{h+1} & \dots & x_{m-1} & x_m \end{pmatrix}.$$

Now, as $x_h \not\prec x_{h+1}$, we have $\begin{pmatrix} x_{h+1} & b \\ a & x_h \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} b & x_{h+1} \\ a & x_h \end{pmatrix}$. Also the relations $x_{h-1} \prec x_h, \dots, x_{l+1} \prec x_{l+2}$ imply : $\begin{pmatrix} x_h & b \\ x_{h-1} & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} b & x_h \\ a & x_{h-1} \end{pmatrix}, \dots,$ $\begin{pmatrix} x_{l+2} & b \\ x_{l+1} & a \end{pmatrix} \overset{E}{\sim} \begin{pmatrix} b & x_{l+2} \\ a & x_{l+1} \end{pmatrix}$. Thus γ'' is transformed into the biword

$$\gamma''' = \begin{pmatrix} w_{n-2} & x_1 & \dots & x_l & x_{l+1} & b & x_{l+2} & \dots & x_m & a \\ b & w_{n-2} & \dots & x_{l-1} & x_l & a & x_{l+1} & \dots & x_{m-1} & x_m \end{pmatrix}$$

$$\overset{E}{\sim}\begin{pmatrix} w_{n-2} & x_1 & \dots & x_l & b & x_{l+1} & x_{l+2} & \dots & x_m & a \\ b & w_{n-2} & \dots & x_{l-1} & x_l & a & x_{l+1} & \dots & x_{m-1} & x_m \end{pmatrix}$$
$$= \gamma'[av_n] \overset{E}{\sim} \{\zeta_E(\gamma')[av_n]\} = \{\zeta_E(\gamma)\},$$

using the fact that $x_l \not\prec x_{l+1}$ in the second step. ∎

The construction of ξ_E.

(1) Start with $w \in R(\mathbf{c}, \mathbf{d})$ and form $\widetilde{w}$.

(2) Form $\eta_{E,b}(\widetilde{w}) = (\widetilde{u}, \gamma_n, \dots, \gamma_2, \gamma_1)$, as in (5.13) with $x = b$ using Han E-transpositions instead of Han D-transpositions.

(3) If necessary, apply ζ_E to each of the b-cycles γ_n, …, γ_2, γ_1.

(4) Apply $\eta_{E,a}$ to the biword $\widetilde{u}$.

(5) Then $\xi_E(\widetilde{w})$ is the juxtaposition product of the sequences :
$\xi_E(\widetilde{w}) = (\eta_{E,a}(\widetilde{u}), \zeta_E(\gamma_n), \dots, \zeta_D(\gamma_1))$.

Property (5.11) holds since $\eta_{E,a}(\widetilde{u})$ can be obtained from $\widetilde{w}$ by a sequence of E-transpositions, as well as the $\zeta_E(\gamma_i)$'s by Proposition 5.7.

We recover the biword $\widetilde{w}$ from each (a, b)-sequence σ by reversing the previous steps, as was explained after the definition of ξ_D given in the previous subsection.

Example. Let $X = [\,5\,]$ and let D be the natural ordering on X. Let 1, 2 and 3 be small and 4 and 5 be large. Thus $a = 4$, $b = 3$. It was shown in the previous subsection that the word $w = 3, 2, 2, 4, 3, 5, 1, 5, 3, 1$ was mapped onto the (a, b)-sequence : $\xi_D(\widetilde{w}) = \bigl(\tilde{2}, [3, 2, 5], [4, 1, 5], [3], [3, 1]\bigr)$.

Reverse the orders of 3 and 4 to obtain the E-ordering as in the previous example. The construction of $\xi_E^{-1}\xi_D(\widetilde{w})$ is made as follows :

(5′) Cut the sequence of a- or b-cycles just before each $b \setminus a$-cycle : $\mid [3, 2, 5], [4, 1, 5] \mid [3] \mid [3, 1]$.

(4′) The sequence of cycles does not start with an a-cycle : $\eta_{E,4}^{-1}(\tilde{2}) = \tilde{2}$.

(3′) Apply ζ_E^{-1} to each sequence determined by the factorization in (5′) : as $5 \prec 1$, $\varphi^{-1}(2, 5\,; 1, 5) = (e\,; 2, 5, 1, 5)$, so that $\zeta_E^{-1}([3, 2, 5], [4, 1, 5]) = [3, 4, 2, 5, 1, 5]$; also $\zeta_E^{-1}([3]) = [3]$ and $\zeta_E^{-1}([3, 1]) = [3, 1]$.

(2′) Determine $\eta_{E,3}^{-1}\bigl(\tilde{2}, [3, 4, 2, 5, 1, 5], [3], [3, 1]\bigr)$, i.e., form the juxtaposition product $\begin{pmatrix} 2\,4\,2\,5\,1\,5\,3\,3\,1\,3 \\ 2\,3\,4\,2\,5\,1\,5\,3\,3\,1 \end{pmatrix}$ and transform the top row into a non-decreasing row using as few Han E-transpositions as possible (move the second leftmost 2 to the second position, then the leftmost 1 to the first position, …) to obtain the biword $\begin{pmatrix} 1\,1\,2\,2\,4\,3\,3\,3\,5\,5 \\ 3\,2\,2\,4\,5\,3\,1\,5\,3\,1 \end{pmatrix}$.

(1′) Then, as defined in (5.12), $\delta(w) = 3, 2, 2, 4, 5, 3, 1, 5, 3, 1$.
We can verify that $(\mathrm{exc}_D, \mathrm{den}_D)(w) = (\mathrm{exc}_E, \mathrm{den}_E)\bigl(\delta(w)\bigr) = (5, 24)$.

6. Recurrence relations

In this section we give recurrence relations for the generating function $A_{\mathbf{c},\mathbf{d}}(t,q)$. Write

$$A_{\mathbf{c},\mathbf{d}}(t,q)=\sum_{s\ge 0} A_{\mathbf{c},\mathbf{d},s}(q)t^s, \tag{6.1}$$

so that $A_{\mathbf{c},\mathbf{d},s}(q)$ is the generating polynomial for the words $w\in R(\mathbf{c},\mathbf{d})$ such that $\operatorname{des}_k w=s$ by the k-major index. We will use the notation $[s]_q$ for $1+q+q^2+\cdots+q^{s-1}$.

Proposition 6.1. *With $m=c+d$ the following relations hold*

$$\begin{aligned}(6.2)\quad &(1-q^{c_j+1})A_{\mathbf{c}+1_j,\mathbf{d}}(t,q)\\ &\qquad=(1-tq^{1+m})A_{\mathbf{c},\mathbf{d}}(t,q)-q^{c_j+1}(1-t)A_{\mathbf{c},\mathbf{d}}(tq,q);\\ (6.3)\quad &(1-q^{-d_k-1})A_{\mathbf{c},\mathbf{d}+1_k}(t,q)\\ &\qquad=-(1-tq^{1+m})A_{\mathbf{c},\mathbf{d}}(t,q)+q^{-d_k}(1-t)A_{\mathbf{c},\mathbf{d}}(tq,q);\\ (6.4)\quad &[c_j+1]_qA_{\mathbf{c}+1_j,\mathbf{d},s}(q)\\ &\qquad=[c_j+1+s]_qA_{\mathbf{c},\mathbf{d},s}(q)+q^{s+c_j}[1+m-s-c_j]_qA_{\mathbf{c},\mathbf{d},s-1}(q);\\ (6.5)\quad &[d_k+1]_qA_{\mathbf{c},\mathbf{d}+1_k,s}(q)\\ &\qquad=q^{d_k+1}[s-d_k]_qA_{\mathbf{c},\mathbf{d},s}(q)+q^s[2+m-s+d_k]_qA_{\mathbf{c},\mathbf{d},s-1}(q);\end{aligned}$$

Proof. The latter two identities are equivalent to the former two ones. Consider relation (6.4) first. By the invariance principle for maj, this relation is equivalent to the relation formed when j is replaced by any i such that $1\le i\le j$. It is convenient to prove the relation for $i=1$, and we rewrite the relation in the form

$$\begin{aligned}(6.6)\quad &(1+q+\cdots+q^{c_1})A_{\mathbf{c}+1_1,\mathbf{d},s}(q)\\ &=(1+q+\cdots+q^{c_1+s})A_{\mathbf{c},\mathbf{d},s}(q)+(q^{c_1+s}+\cdots+q^m)A_{\mathbf{c},\mathbf{d},s-1}(q).\end{aligned}$$

Consider the set $R^*(\mathbf{c}+1_1,\mathbf{d},s)$ of *1-marked* words, i.e., rearrangements w^* of $1^{c_1+1}\dots r^{d_k}$ with s k-descents such that exactly one letter equal to 1 has been marked. Each word $w\in R(\mathbf{c}+1_1,\mathbf{d})$ that has s k-descents gives rise to c_1+1 marked words $w^{(0)},\dots,w^{(c_1)}$. Define

$$\operatorname{maj}^* w^{(i)}=\operatorname{maj}_k w+n_1,$$

where n_1 is the number of letters equal to 1 to the *right* of the marked 1. Then clearly

$$\sum_{i=0}^{c_1}\operatorname{maj}^* w^{(i)}=(1+q+\cdots+q^{c_1})\operatorname{maj}_k w.$$

Hence

$$(1+q+\cdots+q^{c_1})A_{\mathbf{c}+1_1,\mathbf{d},s}(q)=\sum_{w\in R^*(\mathbf{c}+1_1,\mathbf{d},s)} q^{\operatorname{maj}^* w}.$$

Let the word $w=x_1x_2\ldots x_m \in R(\mathbf{c},\mathbf{d})$ have s k-descents. We say that w has $m+1$ *slots* x_ix_{i+1}, $i=0,\ldots,m$ (where we put $x_0=\emptyset$ and $x_{m+1}=\star$). Call the slot x_ix_{i+1} *green* if either x_ix_{i+1} is a k-descent, $x_i=1$ or $i=0$. Call the other slots *red*. Then there are $1+s+c_1$ green slots and $m-s-c_1$ red slots. Label the green slots $0,1,\ldots,c_1+s$ from right to left, and label the red slots $c_1+s+1,\ldots,m$ from left to right.

For example, with $X=[3]$, $j=2$ and $k=1$, the word $w=2\ 2\ 1\ 3\ 2\ 1\ 2\ 3\ 3$ has 5 1-descents and 10 slots. As $c_1=2$, there are 8 green slots and 2 red slots, labelled as follows

$$\begin{array}{lccccccccccccccccccc}
\text{slot} & | & 2 & | & 2 & | & 1 & | & 3 & | & 2 & | & 1 & | & 2 & | & 3 & | & 3 & |\ \star\\
\text{label} & 7 & & 8 & & 6 & & 5 & & 4 & & 3 & & 2 & & 9 & & 1 & & 0
\end{array}$$

Denote by $w^{(i)}$ the word obtained from w by inserting a marked 1 into the i-th slot. Then it may be verified that

$$\operatorname{des}_k w^{(i)}=\begin{cases}\operatorname{des}_k w, & \text{if } i\le c_1+s\,;\\ \operatorname{des}_k w+1, & \text{otherwise.}\end{cases}\tag{6.7}$$

$$\operatorname{maj}^* w^{(i)}=\operatorname{maj}_k w+i.\tag{6.8}$$

Example. Consider the above word w. The following table shows the values of des_k and maj^* on $w^{(i)}$. We show k-descents thus : $2^\frown 1$. The marked 1 is shown in italics.

i	$w^{(i)}$	$\operatorname{des}_1 w^{(i)}$	$\operatorname{maj}^* w^{(i)}$
0	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown \mathit{1}$	5	28
1	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown \mathit{1}\ \ 3^\frown$	5	29
2	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ \mathit{1}\ \ 2\ \ 3^\frown 3^\frown$	5	30
3	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown \mathit{1}\ \ 1\ \ 2\ \ 3^\frown 3^\frown$	5	31
4	$2\ \ 2^\frown 1\ \ 3^\frown \mathit{1}\ \ 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	32
5	$2\ \ 2^\frown 1\ \ \mathit{1}\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	33
6	$2\ \ 2^\frown \mathit{1}\ \ 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	34
7	$\mathit{1}\ \ 2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	35
8	$2^\frown \mathit{1}\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	6	36
9	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2^\frown \mathit{1}\ \ 3^\frown 3^\frown$	6	37

So each word $w\in R(\mathbf{c},\mathbf{d})$ with s k-descents and $\operatorname{maj}_k w=n$ gives rise to c_1+s+1 marked words in $R^*(\mathbf{c}+1_1,\mathbf{d},s)$ with maj^* equal to

$n, n+1, \ldots, n+c_1+s$; and to $m-s-c_1$ marked words in $R^*(\mathbf{c}+1_1, \mathbf{d}, s+1)$ with maj^* equal to $n+c_1+s+1, \ldots, n+m$. Hence a word w in $R(\mathbf{c}, \mathbf{d})$ with $s-1$ k-descents gives rise to $m-s+1-c_1$ marked words in $R^*(\mathbf{c}+1_1, \mathbf{d}, s)$ with maj^* equal to $\mathrm{maj}_k w+c_1+s, \ldots, \mathrm{maj}_k w+c$. This now proves relation (6.6).

We now consider relation (6.5), which we will rewrite in the form

$$\begin{aligned}(6.9)\quad &(1+q+\cdots+q^{d_k})A_{\mathbf{c},\mathbf{d}+1_k,s}(q)\\ &=(q^{d_k+1}+\cdots+q^s)A_{\mathbf{c},\mathbf{d},s}(q)+(q^s+\cdots+q^{1+m+d_k})A_{\mathbf{c},\mathbf{d},s-1}(q).\end{aligned}$$

Consider the set $R^*(\mathbf{c}, \mathbf{d}+1_k, s)$ of *r-marked* words, i.e., rearrangements w^* of $1^{c_1}\ldots r^{d_k+1}$ with s k-descents such that exactly one letter equal to r has been marked. Each word $w \in R(\mathbf{c}, \mathbf{d}+1_k)$ that has s k-descents gives rise to d_k+1 marked words $w^{(0)}, \ldots, w^{(d_k)}$. Define

$$\mathrm{maj}^*\, w^{(i)} = \mathrm{maj}_k\, w + n_r,$$

where n_r is the number of letters equal to r to the *left* of the marked r. Then clearly

$$\sum_{i=0}^{d_k} \mathrm{maj}^*\, w^{(i)} = (1+q+\cdots+q^{d_k})\,\mathrm{maj}_k\, w.$$

Hence

$$(1+q+\cdots+q^{d_k})A_{\mathbf{c},\mathbf{d}+1_k,s}(q) = \sum_{w\in R^*(\mathbf{c},\mathbf{d}+1_k,s)} q^{\mathrm{maj}^* w}.$$

Let the word $w = x_1x_2\ldots x_m \in R(\mathbf{c}, \mathbf{d})$ have s k-descents. Then, as before, w has $m+1$ *slots* x_ix_{i+1}, $i=0,\ldots,m$ (where we put $x_0=\emptyset$ and $x_{m+1}=\star$). Call the slot x_ix_{i+1} *green* if x_ix_{i+1} is a k-descent *and* $x_i<r$. Call the other slots *red*. Then there are $s-d_k$ geen slots and $m+1-s+d_k$ red slots. Label the green slots $0,1,\ldots,s-d_k-1$ from right to left, and label the red slots $d_k-s,\ldots,m$ from left to right.

For example, with $X=[3]$, $j=2$ and $k=1$, the word $w=$ 2 2 1 3 2 1 2 3 3 has 5 1-descents and 10 slots. As $d_1=3$, there are 2 green slots and 8 red slots, labelled as follows

$$\begin{array}{ll} \text{slot} & \mid 2 \mid 2 \mid 1 \mid 3 \mid 2 \mid 1 \mid 2 \mid 3 \mid 3 \mid \star \\ \text{label} & \;2\;\;\;3\;\;\;1\;\;\;4\;\;\;5\;\;\;0\;\;\;6\;\;\;7\;\;\;8\;\;\;9 \end{array}$$

Denote by $w^{(i)}$ the word obtained from w by inserting a marked r into the i-th slot. Then it may be verified that

$$(6.10)\qquad \mathrm{des}_k\, w^{(i)} = \begin{cases} \mathrm{des}_k\, w, & \text{if } i \le s-d_k-1\,; \\ \mathrm{des}_k\, w+1, & \text{otherwise.}\end{cases}$$

$$(6.11)\qquad \mathrm{maj}^*\, w^{(i)} = \mathrm{maj}_k\, w + i + d_k + 1.$$

Each word $w \in R(\mathbf{c},\mathbf{d})$ with s k-descents and $\operatorname{maj}_k w = n$ gives rise to $s-d_k$ marked words in $R^*(\mathbf{c},\mathbf{d}+1_k,s)$ with maj^* equal to $n, n+1, \ldots, n+s-d_k-1$; and to $m+1-s+d_k$ marked words in $R^*(\mathbf{c},\mathbf{d}+1_k,s+1)$ with maj^* equal to $n+s+1, \ldots, n+m+d_k+1$. Hence a word w in $R(\mathbf{c},\mathbf{d})$ with $s-1$ k-descents gives rise to $m+2-s-d_k$ marked words in $R^*(\mathbf{c},\mathbf{d}+1_k,s)$ with maj^* equal to $\operatorname{maj}_k w+s, \ldots, \operatorname{maj}_k w+m+d_k+1$. This now proves relation (6.7).

Example. The following table shows as before the values of des_k and maj^* when $r=3$ is inserted in w.

i	$w^{(i)}$	$\operatorname{des}_1 w^{(i)}$	$\operatorname{maj}^* w^{(i)}$
0	$2\ \ 2^\frown 1\ \ 3^\frown 2\ \ \mathit{3}^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	32
1	$2\ \ 2\ \ \mathit{3}^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	5	33
2	$\mathit{3}^\frown 2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	6	34
3	$2\ \ \mathit{3}^\frown 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	6	35
4	$2\ \ 2^\frown 1\ \ \mathit{3}^\frown 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	6	36
5	$2\ \ 2^\frown 1\ \ 3^\frown \mathit{3}^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown$	6	37
6	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ \mathit{3}^\frown 2\ \ 3^\frown 3^\frown$	6	38
7	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ \mathit{3}^\frown 3^\frown 3^\frown$	6	39
8	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown \mathit{3}^\frown 3^\frown$	6	40
9	$2\ \ 2^\frown 1\ \ 3^\frown 2^\frown 1\ \ 2\ \ 3^\frown 3^\frown \mathit{3}^\frown$	6	41

A short table for the polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$:
$A_{(0),(0)}(t,q) = 1$, $A_{(1),(0)}(t,q) = 1$, $A_{(0),(1)}(t,q) = tq$,
$A_{(2),(0)}(t,q) = 1$, $A_{(1,1),(0)}(t,q) = 1+qt$, $A_{(1),(1)}(t,q) = tq+tq^2$,
$A_{(0),(1,1)}(t,q) = tq^2+t^2q^3$, $A_{(0),(2)}(t,q) = t^2q^3$,
$A_{(3),(0)}(t,q) = 1$, $A_{(2,1),(0)}(t,q) = 1+tq+tq^2$,
$A_{(1,1,1),(0)}(t,q) = 1+2tq+2tq^2+t^2q^3$, $A_{(2),(1)}(t,q) = tq+tq^2+tq^3$,
$A_{(1,1),(1)} = tq+2tq^2+tq^3+t^2q^3+t^2q^4$,
$A_{(1),(1,1)}(t,q) = tq^2+tq^3+t^2q^3+2t^2q^4+t^2q^5$,
$A_{(1),(2)}(t,q) = t^2q^3+t^2q^4+t^2q^5$, $A_{(0),(1,1,1)}(t,q) = tq^3+2t^2q^4+2t^2q^5+t^3q^6$,
$A_{(0),(2,1)}(t,q) = t^2q^4+t^2q^5+t^3q^6$, $A_{(0),(3)}(t,q) = t^3q^6$.

7. Generating functions

In this section we use the recurrence relations (6.2) and (6.3) to prove Theorem 1.4. Thus let

$$A(t,q;\mathbf{u},\mathbf{v}) := \sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+m}} A_{\mathbf{c},\mathbf{d}}(t,q). \tag{7.1}$$

We will consider the partial q-difference

$$D_{u_j} = A(t,q;u_1,\ldots,u_j;\mathbf{v}) - A(t,q;u_1,\ldots,u_{j-1},u_jq;\mathbf{v}).$$

Directly from equation (7.1) we obtain

$$\begin{aligned} D_{u_j} &= \sum_{\mathbf{c},\mathbf{d}} (1-q^{c_j+1}) \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{2+m}} A_{\mathbf{c}+1_j,\mathbf{d}}(t,q) \\ &= \sum_{\mathbf{c},\mathbf{d}} (1-tq^{m+1}) \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{2+m}} A_{\mathbf{c},\mathbf{d}}(t,q) \\ &\qquad\qquad - \sum_{\mathbf{c},\mathbf{d}} q^{c_j+1}(1-t) \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{2+m}} A_{\mathbf{c},\mathbf{d}}(tq,q) \end{aligned}$$

using relation (6.2). Now

$$\begin{aligned} \sum_{\mathbf{c},\mathbf{d}} (1-tq^{m+1}) \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{2+m}} A_{\mathbf{c},\mathbf{d}}(t,q) &= \sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+m}} A_{\mathbf{c},\mathbf{d}}(t,q) \\ &= u_j A(t,q;\mathbf{u},\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} \sum_{\mathbf{c},\mathbf{d}} q^{c_j+1}(1-t) \frac{\mathbf{u}^{\mathbf{c}+1_j}\mathbf{v}^{\mathbf{d}}}{(t;q)_{2+m}} A_{\mathbf{c},\mathbf{d}}(tq,q) &= \sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}+1_j} q^{c_j+1}\mathbf{v}^{\mathbf{d}}}{(tq;q)_{1+m}} A_{\mathbf{c},\mathbf{d}}(tq,q) \\ &= u_j q A(tq,q;u_1,\ldots,u_{j-1},u_jq,\mathbf{v}). \end{aligned}$$

Hence

$$\begin{aligned} (7.2)\quad & A(t,q;\mathbf{u},\mathbf{v}) - A(t,q;u_1,\ldots,u_{j-1},u_jq,\mathbf{v}) \\ &\qquad = u_j A(t,q;\mathbf{u},\mathbf{v}) - u_j q A(tq,q;u_1,\ldots,u_{j-1},u_jq,\mathbf{v}). \end{aligned}$$

The (partial) q-difference equation with respect to each u_i $(i=1,\ldots,j)$ has the form

$$\begin{aligned} (7.3)\quad & A(t,q;\mathbf{u},\mathbf{v}) - A(t,q;u_1,\ldots,u_iq,\ldots,u_j,\mathbf{v}) \\ &\qquad = u_i A(t,q;\mathbf{u},\mathbf{v}) - u_i q A(tq,q;u_1,\ldots,u_iq,\ldots,u_j,\mathbf{v}). \end{aligned}$$

In the same way, (6.3) leads to the q-difference equation

$$\begin{aligned} (7.4)\quad & A(t,q;\mathbf{u},v_1,\ldots,v_lq,\ldots,v_k) - A(t,q;\mathbf{u},\mathbf{v}) \\ &\qquad = -v_l q A(t,q;\mathbf{u},v_1,\ldots,v_lq,\ldots,v_k) + v_l q A(tq,q;\mathbf{u},\mathbf{v}), \end{aligned}$$

for $l = 1, \ldots, k$. Now let

$$A(t, q; \mathbf{u}, \mathbf{v}) = \sum_{s \geq 0} t^s G_s(\mathbf{u}, \mathbf{v}, q).$$

From (7.3) we get

$$\sum_{s \geq 0} t^s (1 - u_i) G_s(\mathbf{u}, \mathbf{v}, q) = \sum_{s \geq 0} t^s (1 - u_i q^{s+1}) G_s(u_1, \ldots, u_i q, \ldots, u_j, \mathbf{v}, q).$$

Taking the coefficient of t^s in both members yields the relation

$$G_s(\mathbf{u}, \mathbf{v}, q) = \frac{1 - u_i q^{s+1}}{1 - u_i} G_s(u_1, \ldots, u_i q, \ldots, u_j, \mathbf{v}, q), \tag{7.5}$$

for $i = 1, \ldots, j$. Similarly, from equation (7.4) we get the relation

$$G_s(\mathbf{u}, \mathbf{v}, q) = \frac{1 + v_l q}{1 + v_l q^{s+1}} G_s(\mathbf{u}, v_1, \ldots, v_l q, \ldots, v_k, q) \tag{7.6}$$

for $l = 1, \ldots, k$.

Now put

$$F_s(\mathbf{u}, \mathbf{v}, q) = G_s(\mathbf{u}, \mathbf{v}, q) \frac{(\mathbf{u}; q)_{s+1}}{(-\mathbf{v}q; q)_s}, \tag{7.7}$$

where we have used the notation $(\mathbf{u}; q)_{s+1} = (u_1; q)_{s+1} \ldots (u_j; q)_{s+1}$ and a similar notation for $(-\mathbf{v}q; q)_s$. Then from equations (7.5) and (7.6) we deduce that for $i = 1, \ldots, j$ and $l = 1, \ldots, k$,

$$\begin{aligned} F_s(\mathbf{u}, \mathbf{v}, q) &= F_s(u_1, \ldots, u_i q, \ldots, u_j, \mathbf{v}, q) \\ F_s(\mathbf{u}, \mathbf{v}, q) &= F_s(\mathbf{u}, v_1, \ldots, v_l q, \ldots, v_k, q) \end{aligned} \tag{7.8}$$

But $F_s(\mathbf{u}, \mathbf{v}, q)$ can be expressed as $F_s(\mathbf{u}, \mathbf{v}, q) = \sum_{\mathbf{c}, \mathbf{d}} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} F_{s, \mathbf{c}, \mathbf{d}}(q)$, where $F_{s, \mathbf{c}, \mathbf{d}}(q)$ is a power series in non-negative powers of q. Fix a non-zero pair $(\mathbf{c}, \mathbf{d})$ and let a be a non-zero component of $(\mathbf{c}, \mathbf{d})$. Then relation (7.8) implies that $F_{s, \mathbf{c}, \mathbf{d}}(q) = q^a F_{s, \mathbf{c}, \mathbf{d}}(q)$. Therefore, $F_{s, \mathbf{c}, \mathbf{d}}(q) = 0$. Hence $F_s(\mathbf{u}, \mathbf{v}, q) = F_{s,0,0}(q)$. It remains to evaluate $F_{s,0,0}(q)$.

From (7.7)

$$\begin{aligned} F_{s,0,0}(q) &= F_s(\mathbf{u}, \mathbf{v}, q) \Big|_{\mathbf{u} = 0, \mathbf{v} = 0} \\ &= G_s(\mathbf{u}, \mathbf{v}, q) \frac{(\mathbf{u}; q)_{s+1}}{(-\mathbf{v}q; q)_s} \Big|_{\mathbf{u} = 0, \mathbf{v} = 0} = G_s(0, 0, q) = 1, \end{aligned}$$

as $\sum_{s \geq 0} t^s G_s(0, 0, q) = A(t, q; 0, 0) = 1/(t; q)_1 = \sum_{s \geq 0} t^s$. Thus $G_s(\mathbf{u}, \mathbf{v}, q) = \dfrac{(-\mathbf{v}q; q)_s}{(\mathbf{u}; q)_{s+1}}$ by (7.7). This proves theorem 1.4.

8. Interpolating q-Eulerian polynomials

Let $A_r^k(t,q) = A_{1^j,1^k}(t,q)$ $(j+k=r)$, which is the generating polynomial for the symmetric group $\mathcal{S}_r$ by the two-variable statistic $(\mathrm{des}_k, \mathrm{maj}_k)$. We wish to derive the following equation.

$$\frac{1}{(t;q)_{1+r}} A_r^k(t,q) = \sum_{s\ge 0} t^s [s+1]_q^j q^k [s]_q^k. \tag{8.1}$$

First, let

$$A_r^k(t,q) = \sum_{s=0}^{r} A_{r,s}^k(q) t^s \qquad (0\le k\le r). \tag{8.2}$$

Then relations (6.4) and (6.5) are rewritten, the first one with $\mathbf{c} = 1^{j-1}$, $c_j = 0$, $m = r-1$ and $\mathbf{d} = 1^k$,

$$A_{r,s}^k(q) = [1+s]_q A_{r-1,s}^k + q^s [r-s]_q A_{r-1,s-1}^k(q); \tag{8.3}$$

the second one with $\mathbf{c} = 1^j$, $d_k = 0$, $m = r-1$ and $\mathbf{d} = 1^{k-1}$,

$$A_{r,s}^k(q) = q[s]_q A_{r-1,s}^{k-1}(q) + q^s [1+r-s]_q A_{r-1,s-1}^{k-1}(q). \tag{8.4}$$

As the polynomial $A_r^k(t,q)$ is also the generating polynomial for $\mathcal{S}_r$ by any pair $(\mathrm{des}_{k,D}, \mathrm{maj}_{k,D})$, where D is an arbitrary total order on $[\,r\,]$ (see [6]), it makes sense to consider the polynomial

$$\begin{aligned} A_r(x,y;t,q) = \sum_s A_{r,s}(x,y;q) t^s &= \sum_k \binom{r}{k} x^k y^{r-k} A_r^k(t,q) \\ &= \sum_x \binom{r}{k} x^k y^{r-k} \sum_s A_{r,s}^k(q) t^s. \end{aligned} \tag{8.5}$$

Using (8.3) and (8.4) it is easy to derive the following recurrence relation for the polynomials $A_{r,s}(x,y;q)$. We find

$$\begin{aligned} A_{r,s}(x,y;q) &= (q[s]_q x + [s+1]_q y) A_{r-1,s}(x,y;q) \\ &\quad + q^s([r-s+1]_q x + [r-s]_q y) A_{r-1,s-1}(x,y;q). \end{aligned} \tag{8.6}$$

Following Garsia's method [11] introduce the Eulerian differential operator δ_t (see also [1]) defined by

$$\delta_t P(t) = \frac{P(qt) - P(t)}{t(q-1)}.$$

In particular, $\delta_t t^s = [s]_q t^{s-1}$ and $\delta_t(1/(t;q)_r) = [s]/(t;q)_{r+1}$. Also note the rule for δ_t-differentiating a product :

$$\delta_t\big(P(t)Q(t)\big) = \delta_t P(t).Q(qt) + P(t).\delta_t Q(t).$$

Using this operator (8.6) may be rewritten as

$$\begin{aligned}(8.7)\quad A_r(x,y;t,q) = A_{r-1}(x,y;tq,q)(y + ytq[r-1]_q + xtq[r]_q)\\ + \delta_t A_{r-1}(x,y;t,q)t(1-tq^r)(qx+y).\end{aligned}$$

Next introduce
(8.8)
$$G = G(x,y;t,q;u) = \sum_{r\ge 0}\sum_{s\ge 0}\frac{u^r}{r!}t^s G_{r,s}(x,y;q) = \sum_{r\ge 0}\frac{u^r}{r!}\frac{A_r(x,y;t,q)}{(t;q)_{r+1}}.$$

The forementioned relation (8.7) may be expressed as

$$\begin{aligned}&t(qx+y)(1-tq^{r+1})\delta_t A_r(x,y;t,q)\\ &\qquad = A_{r+1}(x,y;t,q) - A_r(x,y;tq,q)(y + ytq[r]_q + xtq[r+1]_q).\end{aligned}$$

Nw δ_t-differentiate (8.8) and make use of the previous relation. An easy calculation leads to

$$(8.9)\quad t(qx+y)\delta_t G(x,y;t,q;u) + yG(x,y;tq,q;u) = D_u G(x,y;t,q;u),$$

where "D_u" is the usual derivative with respect to u. Consider the coefficients of $u^r t^s/r!$ in both members of (8.9). We get

$$(qx+y)[s]_q G_{r,s}(x,y;q) + yq^s G_{r,s}(x,y;q) = G_{r+1}(x,y;q).$$

On the other hand, $G(x,y;t,q;u=0) = A_0(x,y;t,q)/(t;q)_1 = 1/(1-t)$. Therefore, $G_{0,s}(x,y;q) = 1$ for all $s \ge 0$. Hence $G_{r+1,s}(x,y;q) = (qx[s]_q + y[s]_q + yq^s)G_{r,s}(x,y;q) = (xq[s]_q + y[s+1]_q)G_{r,s}(x,y;q)$ and $G_{r,s}(x,y;q) = \big(xq[s]_q + y[s+1]_q\big)^s$ $(r \ge 0)$. Thus

$$(8.10)\quad \sum_{r\ge 0}\frac{u^r}{r!}\frac{A_r(x,y;t,q)}{(t;q)_{r+1}} = \sum_{r\ge 0}\sum_{s\ge 0}\frac{u^r}{r!}t^s(xq[s]_q + y[s+1]_q)^r$$

$$(8.11)\quad = \sum_{s\ge 0} t^s \exp\big(u(xq[s]_q + y[s+1]_q)\big).$$

Note that (8.1) is equivalent to (8.10).

Remark. For $k=0$ the polynomial $A_r^k(t,q)$ is the usual q-Eulerian polynomial (see, e.g. [2], [3], [4], [11], [12], [13])

$$A_r^0(t,q)=\sum_w t^{\operatorname{des} w} q^{\operatorname{maj} w} \qquad (w\in\mathcal{S}_r),$$

which is the generating polynomial for $\mathcal{S}_r$ by the usual number of descents "des" and the major index "maj." The generating function for those polynomials is given by (see op. cit.)

$$\sum_{r\ge 0}\frac{u^r}{r!}\frac{A_r^0(t,q)}{(t;q)_{r+1}}=\sum_{s\ge 0}t^s\exp(u[s+1]_q). \tag{8.12}$$

Also the polynomials $tA_r^0(t,q)$ have the generating function

$$\sum_{r\ge 0}\frac{u^r}{r!}\frac{tA_r^0(t,q)}{(t;q)_{r+1}}=\sum_{s\ge 0}t^s\exp(u[s]_q). \tag{8.13}$$

Comparing (8.12) and (8.13) with (8.11) we see that the polynomials $A_r(x,y;t,q)$ *interpolate* the usual q-Eulerian polynomials $A_r^0(t,q)=A_r(x=0,y=1;t,q)$ and $q^r tA_r^0(t,q)=A_r(x=1,y=0;t,q)$.

There is also a method of deriving the generating function for the polynomials $A_r(x,y;t,q)$, i.e., (8.10)–(8.11), directly from (8.12). From the combinatorial interpretation of $A_r^k(t,q)$ as a generating polynomial for $\mathcal{S}_r$ by $(\operatorname{des}_k,\operatorname{maj}_k)$, it is easy to obtain the identity

$$A_r^k(t,q)=A_r^{k-1}(t,q)+(tq^r-1)A_{r-1}^{k-1}(t,q) \qquad (1\le k\le r)$$

By iteration we get

$$A_r^k(t,q)=\sum_{s=0}^{k}\binom{k}{s}(tq^r-1)(tq^{r-1}-1)\dots(tq^{r-s+1}-1)A_{r-s}^0(t,q).$$

Hence

$$\begin{aligned}A_r(x,y;t,q)=\sum_{0\le s\le k\le r}&\frac{r!}{(r-k)!\,s!\,(k-s)!}x^k y^{r-k}\\ &\times(tq^r-1)(tq^{r-1}-1)\dots(tq^{r-s+1}-1)A_{r-s}^0(t,q)\end{aligned}$$

and

$$\begin{aligned}\sum_{r\ge 0}\frac{u^r}{r!}\frac{A_r(x,y;t,q)}{(t;q)_{r+1}}&=\sum_{s,n,m\ge 0}\frac{u^{s+n+m}x^{s+n}y^m}{s!\,n!\,m!}\frac{(-1)^s}{(t;q)_{n+m+1}}A_{n+m}^0(t,q)\\ &=\exp(-ux)\sum_{s\ge 0}\frac{(u(x+y))^s}{s!}\frac{1}{(t;q)_{s+1}}A_s^0(t,q)\\ &=\sum_{r\ge 0}t^s\exp\bigl(u(xq[s]_q+y[s+1]_q)\bigr),\end{aligned}$$

using (8.12) in the last step.

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