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EULERIAN CALCULUS, IV : SPECIALIZATIONS

BY

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ABSTRACT. — Further excedance and descent statistics can be defined on each rearrangement class and their generating functions explicitly calculated. Those generating functions coincide with the classical ones when the rearrangement class is a permutation group, but differ as soon as the class involves repeated elements.

1. Introduction. — In our previous three papers on Eulerian Calculus [ClFo94, ClFo95a, ClFo95b] we have investigated further constructions of transformations on the symmetric group and related structures, and derived explicit formulas for classical order statistics on those structures. Let $(a;q)_n$ denote the q-ascending factorial

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0\,; \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n\ge 1\,; \end{cases}$$

Also let $(u_1,\dots,u_j)$, $(v_1,\dots,v_k)$ be two sequences of commuting variables and $\mathbf{c}=(c_1,\dots,c_j)$ and $\mathbf{d}=(d_1,\dots,d_k)$ be two vectors with non-negative integer components. Write $c=c_1+\dots+c_j$, $d=d_1+\dots+d_k$, then $\mathbf{u}^{\mathbf{c}}$ for $u_1^{c_1}\dots u_j^{c_j}$ and $\mathbf{v}^{\mathbf{d}}$ for $v_1^{d_1}\dots v_k^{d_k}$, finally $(\mathbf{u};q)_{s+1}$ and $(-q\mathbf{v};q)_s$ for the two products

$$(u_1;q)_{s+1}\dots(u_j;q)_{s+1} \qquad \text{and} \qquad (-qv_1;q)_s\dots(-qv_k;q)_s,$$

respectively. In our third paper [ClFo95b] we have considered the identities of the form

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+c+d}} A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s\ge 0} t^s \frac{(-q\mathbf{v};q)_s}{(\mathbf{u};q)_{s+1}}, \tag{1.1}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+c+d}} A^{I}_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s\ge 0} t^s \frac{1}{(\mathbf{u};q)_{s+1}(q\mathbf{v};q)_s}, \tag{1.2}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+c+d}} A^{II}_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s\ge 0} t^s (-\mathbf{u};q)_{s+1}(-q\mathbf{v};q)_s, \tag{1.3}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+c+d}} A^{III}_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s\ge 0} t^s \frac{(-\mathbf{u};q)_{s+1}}{(q\mathbf{v};q)_s}, \tag{1.4}$$

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and showed that the coefficients $A_{\mathbf{c},\mathbf{d}}(t,q)$, $A^{I}_{\mathbf{c},\mathbf{d}}(t,q)$, $A^{II}_{\mathbf{c},\mathbf{d}}(t,q)$ and $A^{III}_{\mathbf{c},\mathbf{d}}(t,q)$ were generating *polynomials* for words that are rearrangements of the word $v = y_1y_2\ldots y_m = 1^{c_1}\ldots j^{c_j}(j+1)^{d_1}\ldots r^{d_k}$ by pairs of the following statistics.

Call $R(\mathbf{c},\mathbf{d})$ the class of those words and let $w = x_1x_2\ldots x_m$ be a word in this class. We say that the word w has a k-*excedance* at i $(1\le i\le m)$, if either $x_i > y_i$, or $x_i = y_i$ and x_i large. We also say that w has a k-*descent* at i $(1\le i\le m)$, if either $x > x_{i+1}$, or $x_i = x_{i+1}$ and x_i large. [By convention, $x_{m+1} = \star$, where $\star$ is an extra letter which is bigger that the *small* letters $1, 2, \ldots, j$, but smaller than the *large* letters $(j+1), \ldots, r$.] The numbers of k-excedances and k-descents of a word w are denoted by $\operatorname{exc}_k w$ and $\operatorname{des}_k w$. The k-*major index* of a word w is also defined to be the *sum*, $\operatorname{maj}_k w$, of all i such that i is a k-descent in w.

The other three pairs of statistics are defined as follows :

$(1.5)_I$ the word $w = x_1x_2\ldots x_m$ has a k-*descent of type I* at i $(1\le i\le m)$, if either $1\le i\le m-1$ and $x_i > x_{i+1}$, or $i = m$ and x_m is large. Thus in case (I) only *strict* descents are counted within the word together with a descent at the end if the last letter is large. The number of k-*descents of type I* in w and the sum of all i such that i is a k-*descent of type I* are respectively denoted by $\operatorname{des}^I_k w$ and $\operatorname{maj}^I_k w$.

$(1.5)_{II}$ the word $w = x_1x_2\ldots x_m$ has a k-*descent of type II* at i $(1\le i\le m)$, if either $1\le i\le m-1$ and $x_i \ge x_{i+1}$, or $i = m$ and x_m is large. Thus in case (II) only usual descents and equalities $x_i = x_{i+1}$ are counted within the word and one descent at the end if the last letter is large. In the same manner, we define $\operatorname{des}^{II}_k w$ and $\operatorname{maj}^{II}_k$.

$(1.5)_{III}$ the word $w = x_1x_2\ldots x_m$ has a k-*descent of type III* at i $(1\le i\le m)$, if one of the two conditions (1), (2) holds : (1) $1\le i\le m-1$, $x_i > x_{i+1}$ and x_i large, or $x_i \ge x_{i+1}$ and x_i small; (2) $i = m$ and x_m is large. Thus in case (III) *strict* descents are taken into account together with equalities $x_i = x_{i+1}$ when x_i is small, with a descent at the end if the last letter is large. In the same manner, we define $\operatorname{des}^{III}_k$ and $\operatorname{maj}^{III}_k$.

One of the results of our third paper was to prove the following theorem.

THEOREM 1.1. — *The coefficients $A_{\mathbf{c},\mathbf{d}}(t,q)$ occurring in identity* (1.1) *are the following generating polynomials*

$$A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_w t^{\operatorname{des}_k w} q^{\operatorname{maj}_k w} \qquad (w\in R(\mathbf{c},\mathbf{d})),$$

with analogous results for $A^{I}_{\mathbf{c},\mathbf{d}}(t,q)$, $A^{II}_{\mathbf{c},\mathbf{d}}(t,q)$ and $A^{III}_{\mathbf{c},\mathbf{d}}(t,q)$ concerning identities (1.2), (1.3) *and* (1.4), *respectively.*

We proved Theorem 1.1 in three different ways : first, by finding *recurrence relations* for the generating polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$ that imply a system of *q-difference equations* for their factorial generating functions, a system that can be intregrated to yield (1.1) – (1.4); second, by using a standard rearrangement technique for biwords that goes back to MacMahon, the celebrated *MacMahon Verfahren*; third, by using the symmetric function technique and especially the *Cauchy identity for Schur functions.*

Now if we let $q=1$ in identities (1.1) – (1.4), we obtain

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{1+c+d}} A_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s \frac{(1+\mathbf{v})^s}{(1-\mathbf{u})^{s+1}}, \tag{1.6}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{1+c+d}} A^{I}_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s \frac{1}{(1-\mathbf{u})^{s+1}(1-\mathbf{v})^s}, \tag{1.7}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{1+c+d}} A^{II}_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s (1+\mathbf{u})^{s+1}(1+\mathbf{v})^s, \tag{1.8}$$

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{1+c+d}} A^{III}_{\mathbf{c},\mathbf{d}}(t) = \sum_{s\ge 0} t^s \frac{(1+\mathbf{u})^{s+1}}{(1-\mathbf{v})^s}, \tag{1.9}$$

where $A_{\mathbf{c},\mathbf{d}}(t)$ stands for $A_{\mathbf{c},\mathbf{d}}(t,q)\,\big|_{q=1}$ with analogous expressions for the other polynomials and where $(1+\mathbf{v})^s$ stands for $(1+v_1)^s\ldots(1+v_k)^s$ with an analagous expression for $(1-\mathbf{v})^{s+1}$.

The purpose of this paper is to reprove identities (1.6) – (1.9) by using a fourth technique based on the MacMahon Master Theorem. As there is no q-analogue (so far ?) of this theorem, such a technique is not available for proving their q-versions (1.1) – (1.4).

We conclude the paper by showing the the polynomials $A_{\mathbf{c},\mathbf{d}}(t)$, $A^{I}_{\mathbf{c},\mathbf{d}}(t)$ and $A^{II}_{\mathbf{c},\mathbf{d}}(t)$ also have combinatorial interpretations in terms of *excedances.*

2. Determinantal expressions. — For each j and each k such that $(0\le j,k\le r)$ and $j+k=r$ denote by B_k and B'_j the following two $r\times r$ matrices. The matrix B_k has only 1's under the diagonal and only t's above, but its diagonal, diag B_k, is made of j 1's, followed by k t's, i.e.,

$$\operatorname{diag} B_k = (\underbrace{1,\ldots,1}_{j \text{ times}},\underbrace{t,\ldots,t}_{k \text{ times}}) \tag{2.1}$$

In the matrix B'_j the diagonal entries consists of j t's followed by k 1's, i.e.,

$$\operatorname{diag} B'_j = (\underbrace{t,\ldots,t}_{j \text{ times}},\underbrace{1,\ldots,1}_{k \text{ times}}) \tag{2.2}$$

and the other entries consist of t's above the diagonal and 1's under it. Notice that $B_r = B'_r$ and $B_0 = B'_0$, a matrix that has t's above the diagonal and 1's on and under the diagonal.

Let the infinite series occurring in the right-hand sides of (1.6), (1.7), (1.8) and (1.9) be denoted by $A(\mathbf{u},\mathbf{v},t)$, $A^I(\mathbf{u},\mathbf{v},t)$, $A^{II}(\mathbf{u},\mathbf{v},t)$ and $A^{III}(\mathbf{u},\mathbf{v},t)$, respectively. Furthermore, let U be the diagonal matrix $U = \operatorname{diag}(u_1,\ldots,u_j,v_1,\ldots,v_k)$. We first prove the identities :

$$(1-t)\,A(\mathbf{u}(1-t),\mathbf{v}(1-t),t) = \frac{1}{\det(I_r - B_k U)}; \tag{2.3}$$

$$(1-t)\,A^I(\mathbf{u}(1-t),\mathbf{v}(1-t),t) = \frac{(1-\mathbf{v}(1-t))}{\det(I_r - B_0 U)}; \tag{2.4}$$

$$(1-t)\,A^{II}(\mathbf{u}(1-t),\mathbf{v}(1-t),t) = \frac{(1+\mathbf{u}(1-t))}{\det(I_r - B_r U)}; \tag{2.5}$$

$$(1-t)\,A^{III}(\mathbf{u}(1-t),\mathbf{v}(1-t),t) = \frac{(1+\mathbf{u}(1-t))(1-\mathbf{v}(1-t))}{\det(I_r - B'_j U)}. \tag{2.6}$$

Identity (2.3) has been derived in [ClFo94]. The same technique applies for the other three identities. Let us simply prove (2.4). First, it is routine to derive

$$\begin{aligned}\det(I_r &- B_0 U)\\ &= 1 - e_1(\mathbf{u},\mathbf{v}) + (1-t)e_2(\mathbf{u},\mathbf{v}) - \cdots + (-1)^r(1-t)^{r-1}e_r(\mathbf{u},\mathbf{v}),\end{aligned}$$

where the $e_i(\mathbf{u},\mathbf{v})$'s are the elementary symmetric function in the variables $u_1,\ldots,u_j,v_1,\ldots,v_k$. Hence

$$\begin{aligned}(1-t)&\det(I_r - B_0 U)\\ &= (1-t) - e_1(\mathbf{u}(1-t),\mathbf{v}(1-t)) + e_2(\mathbf{u}(1-t),\mathbf{v}(1-t))\\ &\qquad - \cdots + (-1)^r e_r(\mathbf{u}(1-t),\mathbf{v}(1-t)).\\ &= -t + (1-\mathbf{u}(1-t))(1-\mathbf{v}(1-t)),\end{aligned}$$

and then

$$\begin{aligned}\frac{(1-\mathbf{v}(1-t))}{\det(I_r - B_0 U)} &= (1-t)\sum_{s\ge 0} t^s \frac{1}{(1-\mathbf{u}(1-t))^{s+1}(1-\mathbf{v}(1-t))^s}\\ &= (1-t)\,A^I(\mathbf{u}(1-t),\mathbf{v}(1-t),t).\end{aligned}$$

As we will make full use of the MacMahon Master Theorem [Mac15, p. 97] to exploit identities (2.3) - (2.6), it is appropriate to restate this theorem now :

Let $B = (b(i,j))$ $(1 \le i, j \le r)$ *be a square matrix with coefficients in a commutative ring and let* X^* *be the free monoid generated by* $X = [\,r\,]$*. If* $w = x_1 x_2 \ldots x_m$ *is a word in* X^* *whose non-decreasing rearrangement is* $v = y_1 y_2 \ldots y_m = 1^{c_1} \ldots j^{c_j} (j+1)^{d_1} \ldots r^{d_k}$*, define*

$$\beta(w) = b(y_1, x_1) b(y_2, x_2) \ldots b(y_m, x_m) \quad \textit{and} \quad u(w) = \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}}.$$

Then the following identity holds

$$\frac{1}{\det(I_r - BU)} = \sum_w \beta(w)\, u(w), \tag{2.7}$$

where I_r *is the identity matrix and where the sum is over all words* $w \in X^*$*.*

Make the substitution $B \leftarrow B_k$ in identity (2.7). Then the monomial $\beta(w)$ is simply equal to $t^{\mathrm{exc}_k w}$, so that the MacMahon Master Theorem yields the identity

$$\frac{1}{\det(I_r - B_k U)} = \sum_w t^{\mathrm{exc}_k w} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}}. \tag{2.8}$$

With the substitution $B \leftarrow B'_j$ we get the identity

$$\frac{1}{\det(I_r - B'_j U)} = \sum_w t^{\mathrm{exc}^j w} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}}, \tag{2.9}$$

where

(2.10) $\mathrm{exc}^j w$ is the number of i such that $1 \le i \le m$ and either $x_i > y_i$, or $x_i = y_i$ and $x_i \le j$.

Thus the inverses of the denominators occurring on the right-hand sides of (2.3) - (2.6) are interpreted in terms of excedences. To get an interpretation in terms of descents we need to recall the construction of the transformation that maps excedences onto descents.

3. Descents. — In our first paper [ClFo94] we have constructed a bijection Φ_k of each rearrangement class $R(\mathbf{c}, \mathbf{d})$ onto itself such that

$$\mathrm{des}_k\, w = \mathrm{exc}_k\, \Phi_k(w) \tag{3.1}$$

for each word w in $R(\mathbf{c},\mathbf{d})$. Hence (2.8) and (3.1) imply

$$\frac{1}{\det(I_r - B_k U)} = \sum_w t^{\operatorname{des}_k w} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} = \sum_{\mathbf{c},\mathbf{d}} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} A_{\mathbf{c},\mathbf{d}}(t) \tag{3.2}$$

To do the counterpart for identity (2.9) and write

$$\frac{1}{\det(I_r - B'_j U)} = \sum_w t^{\operatorname{des}^j w} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}}, \tag{3.3}$$

we need an appropriate definition for "des^j" and a new transformation Ψ^j of $R(\mathbf{c},\mathbf{d})$ onto itself having the property

$$\operatorname{des}^j w = \operatorname{exc}^j \Psi^j(w). \tag{3.4}$$

For "des^j we take the definition

(3.5) Let $w = x_1 x_2 \ldots x_m$ be a word; then $\operatorname{des}^j w$ is the number of i such that $0 \le i \le m-1$ and either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and $x_i \le j$. [By convention, $x_0 = x_1$.]

As for Ψ^j we take the *conjugate* of Φ_k in a sense that will be made more precise. First, recall the construction of Φ_k using a running example (see [ClFo94, § 5]).

Suppose $j=2$, $k=4$, $r=6$ and let $\star$ be an extra letter between 2 (the largest small letter) and 3 (the smallest large letter). Next consider the word

$$w = 2,1,1,3,1,3,3,5,5,2,3,3,2,1,4,5,4,6,6,1,3.$$

(a) Add $\star$ at the end of w :

$$w\star = 2,1,1,3,1,3,3,5,5,2,3,3,2,1,4,5,4,6,6,1,3,\star;$$

(b) cut w before each left to right upper record :

$$w\star = \ |2,1,1\,|\,3,1\,|\,3\,|\,3\,|\,5\,|\,5,2,3,3,2,1,4\,|\,5,4\,|\,6\,|\,6,1,3,\star|\,;$$

(c) change the mutual orders of all factors beginning with the *same large* letter (here 3, 5, 6) :

$$w' = \ |2,1,1\,|\,3\,|\,3\,|\,3,1\,|\,5,4\,|\,5,2,3,3,2,1,4\,|\,5\,|\,6,1,3,\star|\,6\,|\,;$$

(d) form the following biword where the top word in each factor is simply equal to bottom factor shifted to the left, the first letter being moved to the end :

$$\begin{pmatrix}\Delta w'\\ w'\end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c|c|c} 1\;1\;2 & 3 & 3 & 1\;3 & 4\;5 & 2\;3\;3\;2\;1\;4\;5 & 5 & 1\;3\;\star\;6 & 6\\ 2\;1\;1 & 3 & 3 & 3\;1 & 5\;4 & 5\;2\;3\;3\;2\;1\;4 & 5 & 6\;1\;3\;\star & 6 \end{array}\right);$$

(e) remove the vertical bars and rearrange the columns of the latter biword in such a way that the top row is *non-decreasing* and the mutual order of all the biletters having the *same top letter* is preserved :

$$\begin{pmatrix} \ldots \\ \Phi(w') \end{pmatrix} = \begin{pmatrix} 1\,1\,1\,1\,1\,2\,2\,2\,\star\,3\,3\,3\,3\,3\,3\,4\,4\,5\,5\,5\,6\,6 \\ 2\,1\,3\,2\,6\,1\,5\,3\,3\,3\,3\,1\,2\,3\,1\,5\,1\,4\,4\,5\,\star\,6 \end{pmatrix};$$

(e′) cut the biword before each change of top letter :

$$\begin{pmatrix} \ldots \\ \Phi(w') \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c} 1\,1\,1\,1\,1 & 2\,2\,2 & \star & 3\,3\,3\,3\,3\,3 & 4\,4 & 5\,5\,5 & 6\,6 \\ 2\,1\,3\,2\,6 & 1\,5\,3 & 3 & 3\,3\,1\,2\,3\,1 & 5\,1 & 4\,4\,5 & \star\,6 \end{array}\right);$$

(f) within each factor of the biword whose *top word* has only (equal) *large* letters reverse (i.e., take the mirror-image of) the *bottom word* :

$$\begin{pmatrix} \ldots \\ w'' \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c} 1\,1\,1\,1\,1 & 2\,2\,2 & \star & 3\,3\,3\,3\,3\,3 & 4\,4 & 5\,5\,5 & 6\,6 \\ 2\,1\,3\,2\,6 & 1\,5\,3 & 3 & 1\,3\,2\,1\,3\,3 & 1\,5 & 5\,4\,4 & 6\,\star \end{array}\right);$$

(g) remove the vertical bars, delete $\star$ that necessarily occurs at the end of the bottom word; the remaining *bottom* word is, by definition, $\Phi_k(w)$, i.e., $\Phi_k(w)\star = w''$:

$$\begin{pmatrix} \ldots \\ \Phi_k(w) \end{pmatrix} = \begin{pmatrix} 1\,1\,1\,1\,1\,2\,2\,2\,3\,3\,3\,3\,3\,3\,4\,4\,5\,5\,5\,6\,6 \\ 2\,1\,3\,2\,6\,1\,5\,3\,3\,1\,3\,2\,1\,3\,3\,1\,5\,5\,4\,4\,6 \end{pmatrix}.$$

In particular, $\operatorname{des}_k w = \operatorname{exc}_k \Phi_k(w) = 12$.

Furthermore, let $c = c_1 + \cdots + c_j$, $d = d_1 + \cdots + d_k$ and let x be the *last letter* of w, so that $w = w_1 x$. Then, $\Phi_k(w_1 x)$ admits the factorization (w_2, x, w_3), where w_2 and w_3 are words of length $l(w_2) = c$ and $l(w_3) = d - 1$. (If $d = 0$, $l(w_2) = c - 1$.) Thus

(3.5) *the last letter of w is equal to the $(c+1)$-st letter of $\Phi_k(w)$.*

In particular,

$$\Phi_0(w_2 x) = w_3 x \qquad \text{and} \qquad \Phi_r(w_2 x) = x w_4. \tag{3.6}$$

The construction of Ψ^j will also be given by means of an example with $j = 4$, $k = 2$, $r = 6$.

(a) start with a word $w_5 = z_1 z_2 \ldots z_m$ in $R(\mathbf{c}, \mathbf{d})$ (remember that all the letters are taken from the alphabet $[\,r\,]$) and for each $i = 1, 2, \ldots, m$ define

$$x_i = r + 1 - z_{m+1-i} \qquad \text{and form the word} \qquad w_6 = x_1 x_2 \ldots x_m.$$

Clearly $\operatorname{des}^k w_5 = \operatorname{des}_k w_6$. Under this transformation the image of

$$v = 4,6,1,1,3,2,3,6,5,4,4,5,2,2,4,4,6,4,6,6,5$$

is precisely the word w in the preceding example.

(b) apply Φ_k to w_6 to get

$$\begin{pmatrix} \cdots \\ \Phi_k(w) \end{pmatrix} = \begin{pmatrix} \overline{w}_7 \\ w_7 \end{pmatrix} = \begin{pmatrix} 1\,1\,1\,1\,1\,2\,2\,2\,3\,3\,3\,3\,3\,3\,4\,4\,5\,5\,5\,6\,6 \\ 2\,1\,3\,2\,6\,1\,5\,3\,3\,1\,3\,2\,1\,3\,3\,1\,5\,5\,4\,4\,6 \end{pmatrix};$$

(c) replace each entry z in the above biword by $(r+1-z)$ to obtain

$$\alpha = \begin{pmatrix} 6\,6\,6\,6\,6\,5\,5\,5\,4\,4\,4\,4\,4\,4\,3\,3\,2\,2\,2\,1\,1 \\ 5\,6\,4\,5\,1\,6\,2\,4\,4\,6\,4\,5\,6\,4\,4\,6\,2\,2\,3\,3\,1 \end{pmatrix}$$

(d) consider the above biword as a *circuit* in the terminology developed in [CaFo69]. Such a circuit can be expressed as a product of true cycles, and the factorization is unique except for the order of the factors (see [CaFo69, chap. 4, Proposition 4.1]). The true cycles are sorted out to the left one by one. With the running example we get :

$$\begin{bmatrix} 6\,5 \\ 5\,6 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 6\,4 \\ 4\,6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 5\,2\,3\,4 \\ 2\,3\,4\,5 \end{bmatrix}\begin{bmatrix} 6\,5\,4 \\ 5\,4\,6 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 6\,1\,3 \\ 1\,3\,6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

(e) take the inverse of each true cycle, i.e., exchange top and bottom words within each true cycle :

$$\begin{bmatrix} 5\,6 \\ 6\,5 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 4\,6 \\ 6\,4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 2\,3\,4\,5 \\ 5\,2\,3\,4 \end{bmatrix}\begin{bmatrix} 5\,4\,6 \\ 6\,5\,4 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 1\,3\,6 \\ 6\,1\,3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

(f) suppress the brackets in the above product and reorder the columns to get a circuit whose top row is non-decreasing, i.e.,

$$\begin{pmatrix} \overline{w}_8 \\ w_8 \end{pmatrix} = \begin{pmatrix} 1\,1\,2\,2\,2\,3\,3\,4\,4\,4\,4\,4\,4\,5\,5\,5\,6\,6\,6\,6\,6 \\ 6\,1\,2\,2\,5\,2\,1\,4\,6\,4\,3\,5\,4\,6\,4\,6\,5\,6\,4\,4\,3 \end{pmatrix}.$$

The bottom word is by definition $\Psi^k(v)$.

It is readily verified that $\operatorname{exc}_k w_7 = \operatorname{exc}^k w_6$, and so $\operatorname{des}^k v = \operatorname{des}_k w_6 = \operatorname{exc}_k \Psi_k(w_6)$ $(= \operatorname{exc}_k w_7) = \operatorname{exc}^k w_8 = \operatorname{exc}^k \Psi^k(v)$. In the running example all those quantities are equal to 12.

Thus (3.4) is verified and identity (3.3) holds.

4. The proofs of (1.6) - (1.9). — Identity (1.6) is a consequence of (2.3) and (3.2).

For the proof of (1.7) we proceed as follows. Let $\operatorname{des} = \operatorname{des}_0$ and $D = \det(I_r - B_0 U)$. Then $(1/D)v_k = \sum_w t^{\operatorname{des} w} u(w) v_k$ by (3.2) for $k = 0$. As r is the maximum letter of the alphabet, it does not create a descent, if it is placed at the end of a word : $\operatorname{des} wr = \operatorname{des} w$. But $w \mapsto wr$ is a bijection of the set X^* of all words onto the set $X^* r$ of words ending with r. Therefore,

$$\begin{aligned}\frac{1}{D} - \frac{1}{D} v_k &= \sum_w t^{\operatorname{des} w} u(w) - \sum_w t^{\operatorname{des} w} u(w) v_k \\ &= \sum_w t^{\operatorname{des} w} u(w) - \sum_w t^{\operatorname{des} wr} u(wr) = \sum_{w \in X^* \setminus X^* r} t^{\operatorname{des} w} u(w),\end{aligned}$$

where the last summation is over all words *not* ending with r.

In the same manner, $(1/D) t v_k$ is the generating function for words ending with r by "$1 + \operatorname{des}$." Define

$$\operatorname{des}' w = \begin{cases} 1 + \operatorname{des} w, & \text{if } w \text{ ends with } r\,; \\ \operatorname{des} w, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1 - v_k(1-t)}{D} = \frac{1}{D} - \frac{1}{D} v_k + \frac{1}{D} t v_k = \sum_w t^{\operatorname{des}' w} u(w). \tag{4.1}$$

Now continue with other factors :

$$\frac{(1 - v_{k-1}(1-t))(1 - v_k(1-t))}{D} = \Bigl(\sum_w t^{\operatorname{des}' w} u(w)\Bigr)(1 - v_{k-1} + t v_{k-1}).$$

In the same manner, $w \mapsto w(r-1)$ maps X^* onto the set $X^*(r-1)$ of all words ending with $(r-1)$. Furthermore, $\operatorname{des}' w = \operatorname{des}' w(r-1)$, for, either w ends with r and its last letter is counted as a descent, or w ends with a letter $\le (r-1)$ and no other descent is created. Again,

$$\sum_w t^{\operatorname{des}' w} u(w) - \sum_w t^{\operatorname{des}' w} u(w) v_{k-1} = \sum_{w \in X^* \setminus X^*(r-1)} t^{\operatorname{des}' w} u(w)$$

and the factor $\bigl(\sum_w t^{\operatorname{des}' w} u(w)\bigr) t v_{k-1}$ adds back all the words ending with $(r-1)$ and an extra descent is to be counted for all those words. Define

$$\operatorname{des}'' w = \begin{cases} 1 + \operatorname{des}' w, & \text{if } w \text{ ends with } (r-1)\,; \\ \operatorname{des}' w, & \text{otherwise}\,; \end{cases}$$

or, in an equivalent manner,

$$\operatorname{des}'' w = \begin{cases} 1 + \operatorname{des} w, & \text{if } w \text{ ends with } (r-1),\, r\,; \\ \operatorname{des} w, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1 - v_{k-1}(1-t)(1-v_k(1-t))}{D} = \sum_w t^{\operatorname{des}'' w} u(w).$$

More generally, let

$$\text{(4.2)} \qquad \operatorname{des}^I w = \begin{cases} 1 + \operatorname{des} w, & \text{if } w \text{ ends with a } \textit{large} \text{ letter}\,; \\ \operatorname{des} w, & \text{if } w \text{ ends with a } \textit{small} \text{ letter}\,; \end{cases}$$

and let

$$\text{(7.11)} \qquad A^I_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\operatorname{des}^I w} \qquad (w \in R(\mathbf{c},\mathbf{d})).$$

Then, by induction, we can prove the identity

$$\text{(4.3)} \ \sum_{\mathbf{c},\mathbf{d}} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} A^I_{\mathbf{c},\mathbf{d}}(t) = (1-t)\, A^I(\mathbf{u}(1-t), \mathbf{v}(1-t), t) = \frac{(1-\mathbf{v}(1-t))}{\det(I_r - B_0 U)},$$

i.e., we have proved (1.7).

Next, $1/(\det(I_r - B_r U))$ is the generating function for X^* by "des_r" and the statistic "des_r" always includes one descent at the end of each non-empty word w. In the same manner, the product $(1 - tu_1 + u_1)\ldots(1 - tu_j + u_j)$ may be regarded as an operator that kills the ultimate descent in each non-empty word, if the last letter is small.

Hence if we let

$$A^{II}_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\operatorname{des}^{II}} w \qquad (w \in R(\mathbf{c},\mathbf{d})),$$

the following identity holds

$$\sum_{\mathbf{c},\mathbf{d}} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} A^{II}_{\mathbf{c},\mathbf{d}}(t) = (1-t)\, A^{II}(\mathbf{u}(1-t), \mathbf{v}(1-t), t) = \frac{(1+\mathbf{u}(1-t))}{\det(I_r - B_r U)},$$

i.e., identity (1.8) is proved.

Now $1/\det(I_r - B'_j U) = \sum_w t^{\operatorname{des}^j w} u(w)$. The numerator of the fraction on the right-hand side of (2.6) is an operator that kills a descent at the beginning of a non-empty word w, if its first letter is small, but adds a descent at the end if its last letter is large.

Let

$$A^{III}_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\operatorname{des}^{III}} w \qquad (w \in R(\mathbf{c},\mathbf{d}))$$

Then the following identity holds :

$$\sum_{\mathbf{c},\mathbf{d}} \mathbf{u}^{\mathbf{c}} \mathbf{v}^{\mathbf{d}} A^{III}_{\mathbf{c},\mathbf{d}}(t) = (1-t)\, A^{III}(\mathbf{u}(1-t), \mathbf{v}(1-t), t) = \sum_{s \ge 0} t^s \frac{(1+\mathbf{u})^{s+1}}{(1-\mathbf{v})^s},$$

and this proves (1.9).

5. Excedance statistics. — The final problem is to associate *excedance statistics* to those descent statistics. We will only do it for des^I and des^{II}. Let $\Phi = \Phi_0$.

It follows from (3.1) and $(1.5)_I$ that

$$\operatorname{des}^I w'x = \begin{cases} 1 + \operatorname{des} w'x = 1 + \operatorname{exc} \Phi(w'x) = 1 + \operatorname{exc} w_1 x, & \text{if } x \text{ is large;} \\ \operatorname{des} w'x = \operatorname{exc} \Phi(w'x) = \operatorname{exc} w_1 x, & \text{if } x \text{ is small.} \end{cases}$$

We are then led to define

$$\operatorname{exc}^I w = \begin{cases} 1 + \operatorname{exc} w, & \text{if the last letter of } w \text{ is large;} \\ \operatorname{exc} w, & \text{if the last letter of } w \text{ is small.} \end{cases} \tag{5.1}$$

We have then proved the following proposition

PROPOSITION 5.1. — *The transformation Φ also satisfies $\operatorname{exc}^I \Phi(w) = \operatorname{des}^I w$, and consequently*

$$A^I_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\operatorname{des}^I w} = \sum_w t^{\operatorname{exc}^I w} \qquad (w \in R(\mathbf{c},\mathbf{d})).$$

Now using (3.6), $\operatorname{des}^{II} w'x$ is equal to

$$\begin{cases} \operatorname{des}_r w'x - 1 = \operatorname{exc}_r \Phi_r(w'x) - 1 = \operatorname{exc}_r x w_2 - 1, & \text{if } x \text{ is small;} \\ \operatorname{des}_r w'x = \operatorname{exc}_r \Phi_r(w'x) = \operatorname{exc}_r x w_2, & \text{if } x \text{ is large.} \end{cases}$$

Again, for each word $w = x_1 x_2 \ldots x_m$ whose non-decreasing rearrangement is $\overline{w} = y_1 y_2 \ldots y_m$, we are led to define

(5.2) $\operatorname{exc}^{II} w$ to be the number of i such that $1 \le i \le m$ and $x_i \ge y_i$, *minus one* if x_1 is small.

We have then the following result.

PROPOSITION 5.2. — *The transformation Φ_r also satisfies*

$$\operatorname{exc}^{II} \Phi_r(w) = \operatorname{des}^{II} w,$$

and consequently

$$A^{II}_{\mathbf{c},\mathbf{d}}(t) = \sum_w t^{\operatorname{des}^{II} w} = \sum_w t^{\operatorname{exc}^{II} w} \qquad (w \in R(\mathbf{c},\mathbf{d})).$$

Remark. — Keep the same notations, in particular, consider the letters $1, \ldots, j$ as being small, and the other letters large. Furthermore, keep the natural ordering on $[r]$. When restricted to permutations, the statistics "des^I," "des^{II}" "des^{III}" coincide with the definition of "des_k." However

"exc_k," "exc^I," and "exc^{II}" differ from one another. Consequently, Φ_k, Φ and Φ_r provide three different bijections that map the same "des_k" onto three different excedance statistics.

In the following table we have displayed the actions of those three transformations on the elements of $\mathcal{S}_3$ with one small letter ($j = 1$) and two large letters 2 and 3 ($k = 2$) The letters giving rise to excedances in the various cases have been printed in bold-face.

w	$\mathrm{des}_2\, w$	Φ_2	Φ	Φ_3
$1, 2, 3$	1	$1, \mathbf{3}, 2$	$1, 2, \mathbf{3}$	$\mathbf{3}, 1, 2$
$1, 3, 2$	2	$1, \mathbf{2}, \mathbf{3}$	$1, \mathbf{3}, \mathbf{2}$	$\mathbf{2}, 1, \mathbf{3}$
$2, 1, 3$	2	$\mathbf{2}, \mathbf{3}, 1$	$\mathbf{2}, 1, \mathbf{3}$	$\mathbf{3}, \mathbf{2}, 1$
$2, 3, 1$	1	$\mathbf{3}, 1, 2$	$\mathbf{3}, 2, 1$	$1, \mathbf{3}, 2$
$3, 1, 2$	2	$\mathbf{3}, \mathbf{2}, 1$	$\mathbf{3}, 1, \mathbf{2}$	$\mathbf{2}, \mathbf{3}, 1$
$3, 2, 1$	2	$\mathbf{2}, 1, \mathbf{3}$	$\mathbf{2}, \mathbf{3}, 1$	$1, \mathbf{2}, \mathbf{3}$

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