GRAPHICAL MAJOR INDICES, II

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Abstract. Generalizations of the classical statistics "maj" and "inv" (the major index and the number of inversions) on words are introduced that depend on a graph on the underlying alphabet and the behaviour of each letter at the end of a word. The question of characterizing those graphs that lead to equidistributed "maj" and "inv" is posed and answered. This work extends a previous result of Foata and Zeilberger who considered the same problem under the assumption that all letters have the same behaviour at the end of a word.

0. Introduction

Let X be a finite alphabet and U be a relation on X. Without loss of generality we may assume $X = \{1, 2, \dots, r\}$. As U is a subset of $X \times X$, the relation U can also be considered as a directed graph without multiple edges on X. This explains the 'graphical' in our title. Given such a relation U, in [5] extensions maj'_U and inv'_U of the classical major index and the number of inversions, respectively, were introduced for words $w = x_1 x_2 \dots x_m$ with letters from X by

$$\operatorname{maj}'_U w = \sum_{i=1}^{m-1} i \cdot \chi(x_{i+1} U x_i) \tag{0.1a}$$

$$\operatorname{inv}'_U w = \sum_{1 \le i < j \le m} \chi(x_j U x_i). \tag{0.1b}$$

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(Here, as usual, $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true, and $\chi(\mathcal{A}) = 0$ otherwise.) Note that $\mathrm{maj}'_<$ is the classical major index maj (introduced by MacMahon [9, 12] as “greater index”, see also [1, Def. 3.5]),

$$\operatorname{maj} w = \sum_{i=1}^{m-1} i \cdot \chi(x_{i+1} < x_i),$$

and $\mathrm{inv}'_<$ is the usual number of inversions (often attributed to Netto [13, pp. 92f], though he cites earlier occurences; maybe the first occurence under this name is [18]; but probably MacMahon [11, 12] was the first to consider inversions of words instead of just permutations; see also [1, Def. 3.4]),

$$\operatorname{inv} w = \sum_{1 \le i < j \le m} \chi(x_j < x_i).$$

Let $\mathbf{c} = (c(1), c(2), \ldots, c(r))$ be a sequence of r nonnegative integers, and let v be the (non-decreasing) word $v = 1^{c(1)} 2^{c(2)} \ldots r^{c(r)}$. We will denote by $R(v)$ (or by $R(\mathbf{c})$ if there is no ambiguity) the class of all rearrangements of the word v, i.e., the class of all words containing exactly $c(i)$ occurences of the letter i for all $i = 1, 2, \ldots, r$. Then MacMahon [11, 12] showed that maj and inv are equidistributed on each rearrangement class $R(\mathbf{c})$ (see also [1, Cor. 3.8]). This motivates to pose the same question for the generalized major index and generalized number of inversions:

For which relations U on X are maj'_U and inv'_U equidistributed on each rearrangement class $R(\mathbf{c})$?

This question was answered in [5, Theorem 1] by giving a full characterization of all such relations U. More precisely, there it is shown that

Theorem A. *The statistics maj'_U and inv'_U are equidistributed on each rearrangement class $R(\mathbf{c})$ if and only if U is a “bipartitional” relation.*

To make this understandable, we have to explain what a bipartitional relation is.

Definition. A relation U on X is said to be *bipartitional* if there exists a partition $(B_1, B_2, \ldots, B_k)$ of the set X into blocks B_l together with a vector $(\beta_1, \beta_2, \ldots, \beta_k)$ of 0’s and 1’s such that $x\,U y$ if and only if either (1) $x \in B_l$, $y \in B_{l'}$, and $l < l'$, or (2) $x, y \in B_l$, and $\beta_l = 1$.

In words, there are two different types of blocks B_l, those with associated $\beta_l = 0$, let us call them *$\not U$-blocks*, and those with associated $\beta_l = 1$, let us call them *U-blocks*. Then $x\,U y$ if and only if x is in an “earlier” block than y, or if both x and y are in the same U-block. In particular, if x, y are elements of the same $\not U$-block then we have $x \not U y$ and $y \not U x$.

For later use we record that Han [8] has given the following axiomatic characterization of bipartitional relations.

Proposition. *A relation U on X is bipartitional if and only if* (1) $x\,U y$ *and* $y\,U z$ *imply* $x\,U z$, *and* (2) $x\,U y$ *and* $z\,\not U y$ *imply* $x\,U z$.

More general major indices and inversion numbers have been introduced when the underlying alphabet X is partitioned into two subsets. For *signed permutations*, for example, Reiner [14], [15], [16], while developing his theory of B_n P-partitions, was lead to define a major index by making a distinction between *positive* and *negative* integers. This idea was extended to *words* by Clarke and Foata [2], [3], [4] who considered *small* letters and *large* letters and could calculate the corresponding generating functions. Finally, Theorem 2 proved in [5] that involved another definition of the major index closely related to the definitions used by the previous three authors, suggested that a *more general* equidistribution result was to be discovered, if major index and inversion number for words with two kinds of letters could be adequately defined.

The purpose of this paper is to state and prove such a result. It is the content of Theorem B below. This theorem contains both Theorem A and Theorem 2 of [5] as special cases (see the Remark after Theorem B). It can even be considered as a merge of Theorem A and Theorem 2 of [5] into a single theorem.

Of course, the new major index maj_U and inversion number inv_U introduced in this paper will have to generalize the major index and inversion number of [5]. For their definitions we need to partition the underlying alphabet X into two disjoint subsets $X = X_n \dot\cup X_d$. (The subscripts "n" and "d" stand for *no descent* and *descent* at the end of the word, as will be explained in section 1.) Now let U be a relation on X and given a word $w = x_1 x_2 \ldots x_m$ with letters from X define

$$\mathrm{maj}_U\, w = \sum_{i=1}^{m-1} i \cdot \chi(x_{i+1} U x_i) + m \cdot \chi(x_m \in X_d); \tag{0.3a}$$

$$\mathrm{inv}_U\, w = \sum_{1 \le i < j \le m} \chi(x_j U x_i) + |\{i : x_i \in X_d\}|. \tag{0.3b}$$

Note that the difference between $\mathrm{maj}_U\, w$ and $\mathrm{maj}'_U\, w$ equals 0 if the last letter belongs to X_n and equals the length of the word w if the last letter of w belongs to X_d. The difference between $\mathrm{inv}_U\, w$ and $\mathrm{inv}'_U\, w$ is simply the number of letters of the word w that belong to X_d.

As done for the previous pairs of statistics we can ask the following question :

For which relations U on X are maj_U *and* inv_U *equidistributed on each rearrangement class $R(\mathbf{c})$?*

The main purpose of this paper is to answer this question by fully characterizing all such relations U.

Theorem B. *Let $X = X_n \dot{\cup} X_d$ be a given partition. Then the statistics* inv_U *and* maj_U *are equidistributed on each rearrangement class* $R(\mathbf{c})$*, if and only if U is bipartitional and "compatible with the partition $X = X_n \dot{\cup} X_d$."*

To make this statement understandable, we have to explain what "compatible" means.

Definition. A relation U is said to be *compatible with the partition* $X = X_n \dot{\cup} X_d$, if for all $x \in X_n$ and $y \in X_d$ the relations $x\,U y$ and $y \not{U} x$ hold.

Thus, a bipartitional relation U is compatible with the partition $X = X_n \dot{\cup} X_d$ if and only if U is bipartitional on *each* of the subalphabets X_n, X_d, and is such that for all $x \in X_n$ and $y \in X_d$ the relations $x\,U y$ and $y \not{U} x$ hold. This can be made even more explicit. Let $(B_1, B_2, \ldots, B_k)$ be the partition of X and $(\beta_1, \beta_2, \ldots, \beta_k)$ the 0-1 vector associated with the bipartitional relation U. Then there is an integer h, $1 \le h \le k$, such that $X_n = B_1 \dot{\cup} B_2 \dot{\cup} \cdots \dot{\cup} B_h$ and $X_d = B_{h+1} \dot{\cup} B_{h+2} \dot{\cup} \cdots \dot{\cup} B_k$.

Recall [5] that a bipartitional relation U can also be visualized as follows: Again let $U = (B_1, B_2, \ldots, B_k)$, be the partition associated with U and let $(\beta_1, \beta_2, \ldots, \beta_k)$ be the associated 0-1 vector. Rearrange the elements of X in a row in such a way that the elements of B_1 come first, in any order, then the elements of B_2, etc. Then U will consist of all the block products $B_l \times B_{l'}$ with $l < l'$, as well as the block product $B_l \times B_l$ whenever $\beta_l = 1$.

In Figure 1, for instance, the underlying relation consists of four blocks (B_1, B_2, B_3, B_4); the 0-1 vector is $(1, 0, 0, 1)$.

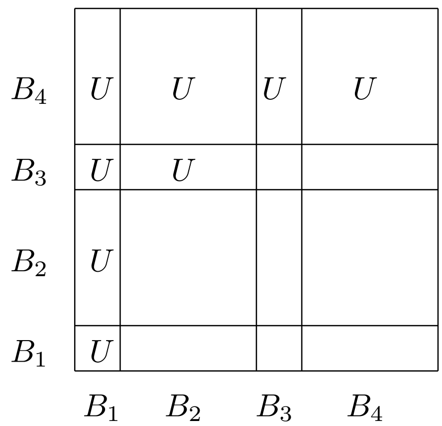

Fig. 1

By the above considerations, this relation is compatible with each partition $X = X_n \dot{\cup} X_d$ provided $X_n = B_1 \cup \cdots \cup B_h$ with $0 \le h \le 4$.

Remark. Notice that Theorem A is the particular case of Theorem B when $X_d = \emptyset$. Theorem 2 of [5] refers to bipartitional relations $(B_1, \ldots, B_k)$, $(\beta_1, \ldots, \beta_k)$ such that $X_n = \bigcup_{l \text{ with } \beta_l = 0} B_l$ and $X_d = \bigcup_{l \text{ with } \beta_l = 1} B_l$.

We shall give two different proofs of the 'if' part of Theorem B. The first proof is by means of generating function techniques and the so-called "MacMahon Verfahren" that essentially gives a general tool for building bijections between words and sets of non-increasing sequences that record the horizontal word statistics such as maj_U (see sections 2, 3). In an earlier version of the paper we have also described two P-partition proofs (not reproduced in the final version) inspired by Stanley's work [19] and Reiner's [14], [15], [16], [17]. However, we decided to give a direct proof instead, avoiding all the definitions that would be necessary to explain the P-partition proofs. The second proof is by means of an explicit bijection (section 4).

In section 5 we prove the 'only if' part of Theorem B. The proof relies on the 'only if' part of Theorem A. In addition, we start section 5 with a new proof for the 'only if' part of Theorem B that is much shorter than the original proof, thus also providing a new proof of the full Theorem A. (We remark that another proof of the 'only if' part of Theorem A, which is computer-assisted, was found by Han [7].) Our proof takes advantage of Han's axiomatic characterization of bipartitional relations given in the Proposition above. All the notation that we are going to use throughout the paper is introduced in section 1.

Finally, in passing, we note that the exponential generating function for the number f_r of all bipartitional relations on $X = \{1, 2, \ldots, r\}$ that are compatible with *some* partition of X into two disjoint subsets equals

$$\sum_{r=0}^{\infty} f_r \frac{u^r}{r!} = \frac{1}{(3-2e^u)^2} = 1 + 4u + 28\frac{u^2}{2!} + 268\frac{u^3}{3!} + 3244\frac{u^4}{4!} + 47404\frac{u^5}{5!} + 810988\frac{u^6}{6!} + \cdots.$$

1. Notation and preliminaries

The q-notations that we use are

$$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j), \quad \text{with } (a;q)_0 = 1,$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$$

for the finite and infinite q-factorials;

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}$$

for the q-binomial coefficient, and

$$\begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1,n_2,\ldots,n_k \end{bmatrix} = \frac{(q;q)_{n_1+n_2+\cdots+n_k}}{(q;q)_{n_1}\,(q;q)_{n_2}\cdots(q;q)_{n_k}},$$

for the q-multinomial coefficient.

We shall use a number of special cases of the q-binomial theorem (cf. [1], Theorem 2.1 or [6], § 1.3),

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} u^n = \frac{(au;q)_\infty}{(u;q)_\infty}. \tag{1.1}$$

First, for $a=0$,

$$\sum_{n=0}^{\infty} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty}, \tag{1.2}$$

and for $u \to -u/a$, $a \to \infty$,

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} u^n}{(q;q)_n} = (-u;q)_\infty. \tag{1.3}$$

Furthermore, with $a = q^{s+1}$,

$$\sum_{n=0}^{\infty} \begin{bmatrix} s+n \\ n \end{bmatrix} u^n = \frac{1}{(u;q)_{s+1}}, \tag{1.4}$$

and with $a = q^{-s}$, $u \to -uq^s$,

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} \begin{bmatrix} s \\ n \end{bmatrix} u^n = (-u;q)_s. \tag{1.5}$$

Finally, extraction of coefficients of u^n in (1.4) gives

$$\begin{bmatrix} s+n \\ n \end{bmatrix} = \sum_{s \ge a_1 \ge \cdots \ge a_n \ge 0} q^{a_1+\cdots+a_n}, \tag{1.6}$$

and in (1.5) gives

$$q^{\binom{n}{2}} \begin{bmatrix} s \\ n \end{bmatrix} = \sum_{s-1 \ge a_1 > \cdots > a_n \ge 0} q^{a_1+\cdots+a_n}. \tag{1.7}$$

Next we review our “U-notations” from the Introduction and add some new ones. Let $w = x_1 x_2 \ldots x_m$ be a word with letters from X. Classically we say that i, $1 \le i \le m-1$, is a *descent* of the word w, if $x_{i+1} < x_i$. The number of descents of w is usually denoted by $\operatorname{des} w$. Furthermore, $\operatorname{maj} w$ is simply the sum of all descents of w.

Now let X be partitioned into two disjoint subsets, $X = X_n \dot\cup X_d$, and let U be a relation on X. In analogy with the above terminology, we call i, $1 \le i \le m$, a *U-descent* of w, if either $x_{i+1} U x_i$, or if $i = m$ and $x_i \in X_d$. (By the way, this makes the explanation for our notation X_n and X_d that was promised in the Introduction. X_n is the set of all letters that create "n"o descent at the end, X_d is the set of all letters that create a "d"escent at the end of a word.) Also, denote by $\operatorname{des}_U w$ the number of all U-descents of w (including a possible "descent at the end of w" when the last letter belongs to X_d). Then $\operatorname{maj}_U w$ is the sum of all U-descents of w.

Finally, we adopt the "bipartitional" notations from [5], section 2. Namely, given a partition $(B_1, B_2, \ldots, B_k)$ of X together with a 0-1 vector $(\beta_1, \beta_2, \ldots, \beta_k)$ we make the following conventions. Let $\mathbf{c}$ be the sequence $\mathbf{c} = (c(1), c(2), \ldots, c(r))$ of nonnegative integers. As before, let $v = 1^{c(1)} 2^{c(2)} \ldots r^{c(r)}$ and denote by $R(v)$ (or by $R(\mathbf{c})$) the class of rearrangements of the word v. If the block B_l consists of the integers $i_1, i_2, \ldots, i_\ell$ and if $u_1, u_2, \ldots, u_r$ are r commuting variables, then we write

$$\begin{aligned}
\mathbf{c} \ge 0 &\text{ for } c(1) \ge 0,\ c(2) \ge 0,\ \ldots,\ c(r) \ge 0;\\
|\mathbf{c}| &\text{ for the } \textit{sum } c(1) + c(2) + \cdots + c(r);\\
\mathbf{u}^{\mathbf{c}} &\text{ for the } \textit{monomial } u_1^{c(1)} u_2^{c(2)} \cdots u_r^{c(r)};\\
c(B_l) &\text{ for the } \textit{sequence } c(i_1), c(i_2), \ldots, c(i_\ell);\\
c(B_l) \ge 0 &\text{ for } c(i_1) \ge 0,\ c(i_2) \ge 0,\ \ldots,\ c(i_\ell) \ge 0;\\
|c(B_l)| &\text{ for the } \textit{sum } c(i_1) + c(i_2) + \cdots + c(i_\ell);\\
u(B_l)^{c(B_l)} &\text{ for the } \textit{monomial } u_{i_1}^{c(i_1)} u_{i_2}^{c(i_2)} \cdots u_{i_\ell}^{c(i_\ell)};\\
\sum u(B_l) &\text{ for the } \textit{sum } u_{i_1} + u_{i_2} + \cdots + u_{i_\ell}.
\end{aligned} \tag{1.8}$$

In particular, $\binom{|c(B_l)|}{c(B_l)}$ will denote the multinomial coefficient

$$\binom{c(i_1) + c(i_2) + \cdots + c(i_\ell)}{c(i_1), c(i_2), \ldots, c(i_\ell)}.$$

2. The generating function by inv_U

Let U be a bipartitional relation on X supposed to be compatible with the partition $X = X_n \dot\cup X_d$. Denote by $(B_1, B_2, \ldots, B_k)$ the partition of X and $(\beta_1, \beta_2, \ldots, \beta_k)$ the 0-1 vector associated with U. Furthermore, let h be the integer such that $X_n = B_1 \dot\cup B_2 \dot\cup \cdots \dot\cup B_h$ and $X_d = B_{h+1} \dot\cup B_{h+2} \dot\cup \cdots \dot\cup B_k$. Finally, denote by

$$A^{\operatorname{inv}_U}(q; \mathbf{c}) := \sum_{w \in R(\mathbf{c})} q^{\operatorname{inv}_U w}$$

the generating function for the class $R(\mathbf{c})$ by inv_U. Under those assumptions we have the two identities:

$$A^{\mathrm{inv}_U}(q;\mathbf{c}) = \begin{bmatrix} |\mathbf{c}| \\ |c(B_1)|, |c(B_2)|, \ldots, |c(B_k)| \end{bmatrix} \times \prod_{l=1}^{k} \binom{|c(B_l)|}{c(B_l)}$$
$$\times \prod_{\substack{1\le l\le h\\ \beta_l=1}} q^{\binom{|c(B_l)|}{2}} \times \prod_{\substack{h+1\le l\le k\\ \beta_l=0}} q^{|c(B_l)|} \times \prod_{\substack{h+1\le l\le k\\ \beta_l=1}} q^{|c(B_l)|+\binom{|c(B_l)|}{2}}, \tag{2.1}$$

$$\sum_{\mathbf{c}\ge 0} \frac{A^{\mathrm{inv}_U}(q;\mathbf{c})}{(q;q)_{\mathbf{c}}} \mathbf{u}^{\mathbf{c}}$$
$$= \frac{\prod\limits_{\substack{1\le l\le h\\ \beta_l=1}} (-\sum u(B_l);q)_\infty \prod\limits_{\substack{h+1\le l\le k\\ \beta_l=1}} (-q\sum u(B_l);q)_\infty}{\prod\limits_{\substack{1\le l\le h\\ \beta_l=0}} (\sum u(B_l);q)_\infty \prod\limits_{\substack{h+1\le l\le k\\ \beta_l=0}} (q\sum u(B_l);q)_\infty}. \tag{2.2}$$

Call a pair (i,j) a U*-inversion* in the word $w = x_1x_2\ldots x_m$ if $i<j$ and x_jUx_i. Formula (2.1) follows from the well-known generating function in the ordinary "inv" case. The q-multinomial coefficient is the generating function for the class of words having exactly $|c(B_1)|$ letters equal to 1, … , $|c(B_k)|$ letters equal to k by "inv." Such a word gives rise to exactly $\prod_l \binom{|c(B_l)|}{c(B_l)}$ words in $R(\mathbf{c})$. Now the letters belonging to each block B_l such that $1\le l\le h$ and $\beta_l=0$ provide no further U-inversions and those belonging to each block B_l such that $\beta_l=1$ bring $\binom{|c(B_l)|}{2}$ extra U-inversions when they are compared between themselves. Finally, the term $|\{i : x_i\in X_d\}|$ that is to be added to the number of U-inversions can also be written as $\sum\limits_{h+1\le l\le k} |c(B_l)|$. This proves (2.1).

Finally, we go from (2.1) to (2.2) by a routine q-calculation, using the multinomial theorem. $\square$

3. The generating function by maj_U

We keep the same assumptions on U and the same notations as in the beginning of section 2. Let

$$A^{\mathrm{maj}_U}(q;\mathbf{c}) := \sum_{w\in R(\mathbf{c})} q^{\mathrm{maj}_U w}$$

denote the generating function for the class $R(\mathbf{c})$ by the statistics maj_U. Our purpose is to show that $A^{\mathrm{maj}_U}(q;\mathbf{c})$ is equal to the right-hand side of (2.1), so that inv_U and maj_U are equidistributed on each rearrangement class $R(\mathbf{c})$. We first prove a stronger result, namely, we calculate the generating function for each class $R(\mathbf{c})$ by the *pair* $(\mathrm{des}_U, \mathrm{maj}_U)$ (remember the definition of the statistic des_U in section 1) and show by specialization that the generating polynomial by maj_U is equal to that right-hand side.

Proposition 3.1. *Let*

$$A^{\mathrm{des}_U, \mathrm{maj}_U}(t, q; \mathbf{c}) := \sum_{w \in R(\mathbf{c})} t^{\mathrm{des}_U w} q^{\mathrm{maj}_U w}$$

be the generating polynomial for $R(\mathbf{c})$ by the pair $(\mathrm{des}_U, \mathrm{maj}_U)$. Keeping the assumptions of the beginning of section 2 and the notations (1.8) *we have the identities :*

$$\begin{aligned}
&\frac{A^{\mathrm{des}_U, \mathrm{maj}_U}(t, q; \mathbf{c})}{(t; q)_{|\mathbf{c}|+1}} \\
&\quad = \prod_{l=1}^{k} \binom{|c(B_l)|}{c(B_l)} \Biggl(\sum_{s \geq 0} t^s \prod_{\substack{1 \leq l \leq h \\ \beta_l = 0}} \begin{bmatrix} |c(B_l)| + s \\ |c(B_l)| \end{bmatrix} \prod_{\substack{1 \leq l \leq h \\ \beta_l = 1}} q^{\binom{|c(B_l)|}{2}} \begin{bmatrix} s + 1 \\ |c(B_l)| \end{bmatrix} \\
&\quad \times \prod_{\substack{h+1 \leq l \leq k \\ \beta_l = 0}} q^{|c(B_l)|} \begin{bmatrix} |c(B_l)| + s - 1 \\ |c(B_l)| \end{bmatrix} \prod_{\substack{h+1 \leq l \leq k \\ \beta_l = 1}} q^{\binom{|c(B_l)|+1}{2}} \begin{bmatrix} s \\ |c(B_l)| \end{bmatrix} \Biggr)
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\sum_{\mathbf{c} \geq 0} \frac{A^{\mathrm{des}_U, \mathrm{maj}_U}(t, q; \mathbf{c})}{(t; q)_{|\mathbf{c}|+1}} \mathbf{u}^{\mathbf{c}} \\
&\quad = \sum_{s \geq 0} t^s \frac{\displaystyle\prod_{\substack{1 \leq l \leq h \\ \beta_l = 1}} \bigl(-\sum u(B_l); q\bigr)_{s+1} \prod_{\substack{h+1 \leq l \leq k \\ \beta_l = 1}} \bigl(-q \sum u(B_l); q\bigr)_s}{\displaystyle\prod_{\substack{1 \leq l \leq h \\ \beta_l = 0}} \bigl(\sum u(B_l); q\bigr)_{s+1} \prod_{\substack{h+1 \leq l \leq k \\ \beta_l = 0}} \bigl(q \sum u(B_l); q\bigr)_s}.
\end{aligned} \tag{3.3}$$

Suppose that (3.2) has been proved. Multiply both sides of that identity by $(t; q)_{\mathbf{c}+1}$ and let t tend to 1. It is straightforward to see that the right-hand side tends to the right-hand side of (2.1), so that $A^{\mathrm{maj}_U}(q; \mathbf{c}) = A^{\mathrm{inv}_U}(q; \mathbf{c})$, i.e., maj_U and inv_U are equidistributed. Furthermore, it is again a routine q-calculation to derive (3.3) from (3.2). It then suffices to prove (3.2).

Proof of Proposition 3.1. We could accomplish this by appealing to the P-partition theory developed by Stanley [19]; another possibility would be to make use of the B_n P-partition theory developed by Reiner [14]. Both theories can be considered as generalizations of the "MacMahon

Verfahren" (see [9] or [1], chap. 3). Instead of going through the definitions of (B_n) P-partitions, we prefer to give a direct approach, modelled after the MacMahon Verfahren. As in [5] § 4, we proceed as follows:

Let $w = x_1x_2\ldots x_m$ be a word of the class $R(\mathbf{c})$, so that $m = |\mathbf{c}|$. Denote by $w(B_l)$ the *subword* of w consisting of all the letters belonging to B_l $(l = 1,\ldots,k)$. Then replace each letter belonging to B_l by $b_l = \min B_l$ (with respect to the usual order). Call $\overline{w} = \overline{x}_1\overline{x}_2\ldots\overline{x}_m$ the resulting word. Clearly the mapping

$$w \mapsto \bigl(\overline{w}, w(B_1),\ldots,w(B_l)\bigr) \tag{3.4}$$

is bijective. Moreover, $\mathrm{des}_U\, w = \mathrm{des}_U\, \overline{w}$ and $\mathrm{maj}_U\, w = \mathrm{maj}_U\, \overline{w}$. Accordingly, the polynomial $A^{\mathrm{des}_U,\mathrm{maj}_U}(t,q;\mathbf{c})$ is divisible by $\prod\limits_l \binom{|c(B_l)|}{c(B_l)}$.

In the sequel, for each $l = 1,\ldots,k$ let $m_l = |c(B_l)|$. For each $i = 1,2,\ldots,m$ let z_i denote the number of U-descents in the right factor $\overline{x}_i\overline{x}_{i+1}\ldots\overline{x}_m$ of $\overline{w}$ (including one descent in m if $x_m \in X_d$). Clearly, $z_1 = \mathrm{des}_U\,\overline{w}$ and $z_1+\cdots+z_m = \mathrm{maj}_U\,\overline{w}$.

Now let $\mathbf{p} = (p_1,\ldots,p_m)$ be a sequence of m integers satisfying $s' \ge p_1 \ge p_2 \ge \cdots \ge p_m \ge 0$, where s' is a *given* integer. Form the *non-increasing* word $v = y_1y_2\ldots y_m$ defined by $y_i = p_i + z_i$ $(1 \le i \le m)$ and consider the biword

$$\binom{v}{\overline{w}} = \binom{y_1y_2\cdots y_m}{\overline{x}_1\overline{x}_2\ldots\overline{x}_m}.$$

Next rearrange the columns of the previous matrix in such a way that the mutual orders of the columns with the same bottom entries are preserved and the entire bottom row is of the form $b_1^{m_1}b_2^{m_2}\ldots b_k^{m_k}$. We obtain the matrix

$$\begin{pmatrix} a_{1,1}\ldots a_{1,m_1} & \ldots & a_{k,1}\ldots a_{k,m_k} \\ b_1\ \ldots\ b_1 & \ldots & b_k\ \ldots\ b_k \end{pmatrix}.$$

By construction each of the k words $a_{1,1}\ldots a_{1,m_1},\ \ldots\ ,\ a_{k,1}\ldots a_{k,m_k}$ is *non-increasing*.

Next, if $i < i'$, $\overline{x}_i = \overline{x}_{i'} \in B_l$ and $\beta_l = 1$, there is necessarily a U-descent within $\overline{x}_i\overline{x}_{i+1}\ldots\overline{x}_{i'}$. Hence $z_i > z_{i'}$ and so $y_i > y_{i'}$. The word $a_{l,1}\ldots a_{l,m_l}$ corresponding to the block B_l will then be *strictly decreasing*.

On the other hand, $z_m = 1$ iff $\overline{x}_m \in X_d$. Let B_l be a block of X_d, so that $h+1 \le l$ and denote by $\overline{x}_i$ the *rightmost* letter of $\overline{w}$ that belongs to B_l. If $i = m$, then $a_{l,m_l} = p_m + z_m \ge 1$; if $i < m$, then, either there is one letter of X_n in the factor $\overline{x}_{i+1}\ldots\overline{x}_m$ and necessarily one U-descent because U is supposed to be *compatible* with $X_n\dot\cup X_d$, or all the letters in that factor are in X_d and in particular $z_m = 1$. In both cases, $a_{l,m_l} \ge 1$. Also note that

$$a_{l,i} \le y_1 = p_1 + z_1 \le s' + \mathrm{des}_U\,\overline{w}$$

for all l, i. Let then $s = s' + \operatorname{des}_U \overline{w}$. It follows that each of the words $a_{l,1} \ldots a_{l,m_l}$ satisfies

$$
\begin{aligned}
&s \geq a_{l,1} \geq \cdots \geq a_{l,m_l} \geq 0, \text{ if } 1 \leq l \leq h \text{ and } \beta_l = 0;\\
&s \geq a_{l,1} > \cdots > a_{l,m_l} \geq 0, \text{ if } 1 \leq l \leq h \text{ and } \beta_l = 1;\\
&s \geq a_{l,1} \geq \cdots \geq a_{l,m_l} \geq 1, \text{ if } h+1 \leq l \leq k \text{ and } \beta_l = 0;\\
&s \geq a_{l,1} > \cdots > a_{l,m_l} \geq 1, \text{ if } h+1 \leq l \leq k \text{ and } \beta_l = 1.
\end{aligned} \tag{3.5}
$$

The mapping $(s', \mathbf{p}, \overline{w}) \mapsto (s, (a_{l,i}))$ is a bijection satisfying

$$
\begin{gathered}
s = s' + \operatorname{des}_U \overline{w}\,;\\
\sum_{l,i} a_{l,i} = p_1 + \cdots + p_m + z_1 + \cdots + z_m = \sum_i p_i + \operatorname{maj}_U \overline{w}.
\end{gathered} \tag{3.6}
$$

Now it follows from (1.4) and (1.6) that

$$
\frac{1}{(t;q)_{m+1}} = \sum_{s' \geq 0} t^{s'} \sum_{s' \geq p_1 \geq \cdots \geq p_m \geq 0} q^{p_1 + \cdots + p_m},
$$

so that by (3.4)

$$
\begin{aligned}
\frac{1}{\prod\limits_l \binom{m_l}{c(B_l)}} \frac{A^{\operatorname{des}_U, \operatorname{maj}_U}(t, q; \mathbf{c})}{(t;q)_{m+1}} &= \sum_{s' \geq 0} t^{s'} \sum_{s' \geq p_1 \geq \cdots \geq p_m \geq 0} q^{\Sigma\, p_i} \sum_{\overline{w}} t^{\operatorname{des}_U \overline{w}} q^{\operatorname{maj}_U \overline{w}}\\
&= \sum_{(s', \mathbf{p}, \overline{w})} t^{s' + \operatorname{des}_U \overline{w}} q^{\Sigma\, p_i + \operatorname{maj}_U \overline{w}} = \sum_{(s, (a_{l,i}))} t^s q^{\Sigma a_{l,i}} \qquad \text{[by (3.6)]}\\
&= \sum_{s \geq 0} t^s \prod_{\substack{1 \leq l \leq h\\ \beta_l = 0}} \sum_{s \geq a_{l,1} \geq \cdots \geq a_{l,m_l} \geq 0} q^{a_{l,1} + \cdots + a_{l,m_l}} \prod_{\substack{1 \leq l \leq h\\ \beta_l = 1}} \sum_{s \geq a_{l,1} > \cdots > a_{l,m_l} \geq 0} q^{a_{l,1} + \cdots + a_{l,m_l}}\\
&\quad \times \prod_{\substack{h+1 \leq l \leq k\\ \beta_l = 0}} \sum_{s \geq a_{l,1} \geq \cdots \geq a_{l,m_l} \geq 1} q^{a_{l,1} + \cdots + a_{l,m_l}} \prod_{\substack{h+1 \leq l \leq k\\ \beta_l = 1}} \sum_{s \geq a_{l,1} > \cdots > a_{l,m_l} \geq 1} q^{a_{l,1} + \cdots + a_{l,m_l}}\\
&= \sum_{s \geq 0} t^s \prod_{\substack{1 \leq l \leq h\\ \beta_l = 0}} \begin{bmatrix} m_l + s\\ m_l \end{bmatrix} \prod_{\substack{1 \leq l \leq h\\ \beta_l = 1}} q^{\binom{m_l}{2}} \begin{bmatrix} s+1\\ m_l \end{bmatrix}\\
&\qquad \times \prod_{\substack{h+1 \leq l \leq k\\ \beta_l = 0}} q^{m_l} \begin{bmatrix} m_l + s - 1\\ m_l \end{bmatrix} \prod_{\substack{h+1 \leq l \leq k\\ \beta_l = 1}} q^{\binom{m_l+1}{2}} \begin{bmatrix} s\\ m_l \end{bmatrix},
\end{aligned}
$$

by (1.6) and (1.7). Hence (3.2) is established. $\square$

4. The bijective proof of the 'if' part of Theorem B

Again keep for the relation U on X the same assumptions as in the beginning of section 2. In this section we construct a bijection Ψ_U from each class $R(\mathbf{c})$ onto itself, satisfying

$$\operatorname{maj}_U w = \operatorname{inv}_U \Psi_U(w) \tag{4.1}$$

for all words $w \in R(\mathbf{c})$. The construction of the bijection Ψ_U parallels the bijective construction in [5], § 8.

As done in previous papers [20], [2], [3], [4], [5] we add a new letter $*$ to X. Then we extend U to the relation U^* on $X \cup \{*\}$ by adding the relations $*Uj$, $j \in X_d$, to U. In other words, U^* is the bipartitional relation on $X \cup \{*\}$ with blocks $(B_1, \ldots, B_h, \{*\}, B_{h+1}, \ldots, B_k)$ and vector $(\beta_1, \ldots, \beta_h, 0, \beta_{h+1}, \ldots, \beta_k)$. Note that for any word w with letters from X we have

$$\operatorname{maj}_U w = \operatorname{maj}_{U^*} w*, \tag{4.2}$$

$$\operatorname{inv}_U w = \operatorname{inv}_{U^*} w*. \tag{4.3}$$

Now we make use of a map that was constructed in [5], § 5. There, given a bipartitional relation, B say, a bijection Φ_B from each rearrangement class $R(\mathbf{c})$ onto itself was constructed that satisfies

$$\operatorname{maj}'_B w = \operatorname{inv}'_B \Phi_B(w) \tag{4.4}$$

for any word $w \in R(\mathbf{c})$. This map Φ_B had the additional property that it fixes the last letter. To be precise, we use the above construction with X replaced by $X \cup \{*\}$ and with the bipartitional relation $B = U^*$.

Let w be a word with letters from X. Form the concatenation $w*$. Then we apply Φ_{U^*} to $w*$. Since Φ_{U^*} fixes the last letter, we have $\Phi_{U^*}(w*) = w'*$ for some word w' with letters from X. This given, we define Ψ_U by $\Psi_U(w) := w'$.

That Ψ_U is a bijection is immediate from the construction. Of course, we also have to check (4.1). Now, we have

$$\begin{aligned}
\operatorname{maj}_U w &= \operatorname{maj}'_{U^*} w* && \text{[by (4.2)]}\\
&= \operatorname{inv}'_{U^*} \Phi_{U^*}(w*) && \text{[by (4.4)]}\\
&= \operatorname{inv}'_{U^*} w'* && \text{[by definition]}\\
&= \operatorname{inv}_U w' && \text{[by (4.3)]}\\
&= \operatorname{inv}_U \Psi_U(w), && \text{[by definition]}
\end{aligned}$$

which is exactly (4.1). $\square$

5. The proof of the 'only if' part of Theorem B

We start this section by giving a new proof of the 'only if' part of Theorem A. This new proof is much shorter than the original one. We remark that this gives also a proof of the complete Theorem A, because the 'if' part of Theorem A is contained in the 'if' part of Theorem B. (To obtain the 'if' part of Theorem A, choose the trivial partition $X = X_n \cup \{\}$ in the 'if' part of Theorem B.) In our new proof we take advantage of Han's axiomatic characterization of bipartitional relations given in the Proposition in the Introduction.

PROOF OF THE 'ONLY IF' PART OF THEOREM A. Let U be a relation on X such that maj'_U and inv'_U are equidistributed on each rearrangement class $R(\mathbf{c})$. We want to show that U is bipartitional. In view of the Proposition in the Introduction, it suffices to show

(U1) $x\,U\,y$ and $y\,U\,z$ imply $x\,U\,z$ for all $x, y, z \in X$,

(U2) $x\,U\,y$ and $z\,\not U\,y$ imply $x\,U\,z$ for all $x, y, z \in X$.

Proof of (U1). Let x, y, z be elements in X such that $x\,U\,y$ and $y\,U\,z$. By way of contradiction, let us assume that $x\,\not U\,z$. We consider the rearrangement class $R(xyz)$. By our assumptions, we have $\mathrm{maj}'_U\, zyx = 3$. Since, also by assumption, maj'_U and inv'_U are equidistributed on $R(xyz)$, there must be a word $v \in R(xyz)$ with $\mathrm{inv}'_U\, v = 3$. We observe that 3 is the maximum value that can be attained. So, since x and z occur in v, and since $x\,\not U\,z$, we must necessarily have $z\,U\,x$, and x must occur before z in v. Thus there are the following possibilities for v:

$$\begin{aligned} v &= xzy, && \text{and in addition } y\,U\,x,\\ v &= xyz, && \text{and in addition } y\,U\,x \text{ and } z\,U\,y,\\ v &= yxz, && \text{and in addition } z\,U\,y. \end{aligned}$$

There remain the following three cases for the relation U on $\{x, y, z\}$,

Case I.	$x\,U\,y\,U\,x$	Case II.	$x\,U\,y\,U\,x$	Case III.	$x\,U\,y\,\not U\,x$
	$x\,\not U\,z\,U\,x$		$x\,\not U\,z\,U\,x$		$x\,\not U\,z\,U\,x$
	$y\,U\,z\,\not U\,y$		$y\,U\,z\,U\,y$		$y\,U\,z\,U\,y$

ad Case I. There holds $\mathrm{maj}'_U\, yzx = 0$, but also $\mathrm{inv}'_U\, w \ge 1$ for any word $w \in R(xyz)$, because of $x\,U\,y\,U\,x$. Thus maj'_U and inv'_U are not equidistributed on $R(xyz)$, which is absurd.

ad Case II. There holds $\mathrm{maj}'_U\, yzx = 1$, but also $\mathrm{inv}'_U\, w \ge 2$ for any word $w \in R(xyz)$, because of $x\,U\,y\,U\,x$ and $y\,U\,z\,U\,y$. Thus maj'_U and inv'_U are not equidistributed on $R(xyz)$, which is absurd.

ad Case III. There holds $\mathrm{maj}'_U\, zxy = 0$, but also $\mathrm{inv}'_U\, w \ge 1$ for any word $w \in R(xyz)$, because of $y\,U\,z\,U\,y$. Thus maj'_U and inv'_U are not equidistributed on $R(xyz)$, which is absurd.

Altogether this shows that we obtain a contradiction in any case. Hence we must have $x\,U z$.

Proof of (U2). Let x, y, z be elements in X such that $x\,U y$ and $z \not{U} y$. By way of contradiction, let us assume that $x \not{U} z$. Now observe that (U1), which was already established, implies $y \not{U} z$. This is because in case $y\,U z$ we would have $x\,U y$ and $y\,U z$, and thus $x\,U z$, in contradiction with our assumption.

Now we consider again the rearrangement class $R(xyz)$. By our assumptions, we have $\mathrm{maj}'_U\, yzx = 0$. Since, also by assumption, maj'_U and inv'_U are equidistributed on $R(xyz)$, there must be a word $v \in R(xyz)$ with $\mathrm{inv}'_U\, v = 0$. Since x and y occur in v, and since $x\,U y$, we must necessarily have $y \not{U} x$ (and x must occur before y in v).

There remain the following two cases for the relation U on $\{x, y, z\}$,

$$\begin{array}{llcll} \text{Case I.} & x\,U y \not{U} x & \qquad & \text{Case II.} & x\,U y \not{U} x \\ & x \not{U} z\,U x & & & x \not{U} z \not{U} x \\ & y \not{U} z \not{U} y & & & y \not{U} z \not{U} y \end{array}$$

ad Case I. There holds $\mathrm{maj}'_U\, yxz = 3$, but also $\mathrm{inv}'_U\, w \le 2$ for any word $w \in R(xyz)$, because of $y \not{U} z \not{U} y$. Thus maj'_U and inv'_U are not equidistributed on $R(xyz)$, which is absurd.

ad Case II. There holds $\mathrm{maj}'_U\, zyx = 2$, but also $\mathrm{inv}'_U\, w \le 1$ for any word $w \in R(xyz)$, because of $x \not{U} z \not{U} x$ and $y \not{U} z \not{U} y$. Thus maj'_U and inv'_U are not equidistributed on $R(xyz)$, which is absurd.

Altogether this shows that we obtain a contradiction in any case. Hence we must have $x\,U z$. $\square$

Remark. The argument above shows that it is even sufficient to require equidistribution of maj'_U and inv'_U only on rearrangement classes containing only three letters. Han [7] has given another proof of the 'only if' part of Theorem A based on his axiomatic characterization of bipartitional relations (the Proposition in the Introduction). In fact, our proof is directly inspired by his. Instead of letting the computer verify *all* possible relations U on the letters $\{x, y, z\}$ (there are $2^9 = 512$ such relations!) and sorting out the "right" ones, as he did, we have just provided an argument to reduce the number of cases to consider.

Proof of the 'only if' part of Theorem B. Let U be a relation on $X = X_n \dot{\cup} X_d$ such that maj_U and inv_U are equidistributed on each rearrangement class $R(\mathbf{c})$. First choose $\mathbf{c}$ such that only letters from X_n are chosen. For these rearrangement classes, maj_U and inv_U reduce to maj'_U and inv'_U, respectively. Hence, by the 'only if' part of Theorem A, we infer that U is bipartitional on X_n. Secondly, choose $\mathbf{c}$ such that only letters from X_d are chosen. For these rearrangement classes, maj_U and

inv_U reduce to $\mathrm{maj}'_U + |\mathbf{c}|$ and $\mathrm{inv}'_U + |\mathbf{c}|$, respectively. Again, by the 'only if' part of Theorem A, we infer that U is bipartitional on X_d.

Thus it remains to see how letters from X_n are related to letters from X_d. Take a letter $x \in X_n$ and a letter $y \in X_d$. We want to show that we must have $x\,U y$ and $y \not{U} x$. We establish this by excluding the other three cases.

Case I: $x \not{U} y \not{U} x$. Then $\mathrm{maj}_U\, xy = 2$, $\mathrm{maj}_U\, yx = 0$, but $\mathrm{inv}_U\, xy = \mathrm{inv}_U\, yx = 1$. Thus maj_U and inv_U are not equidistributed on $R(xy)$, which is absurd.

Case II: $x \not{U} y\, U x$. Then $\mathrm{maj}_U\, xy = 3$, $\mathrm{maj}_U\, yx = 0$, but $\mathrm{inv}_U\, xy = 2$, $\mathrm{inv}_U\, yx = 1$. Thus maj_U and inv_U are not equidistributed on $R(xy)$, which is absurd.

Case III: $x\, U y\, U x$. Then $\mathrm{maj}_U\, xy = 3$, $\mathrm{maj}_U\, yx = 1$, but $\mathrm{inv}_U\, xy = \mathrm{inv}_U\, yx = 2$. Thus maj_U and inv_U are not equidistributed on $R(xy)$, which is absurd.

This shows that the only legal possibility is $x\, U y \not{U} x$. Hence, U is a bipartitional relation on X that is compatible with the partition $X = X_n \dot{\cup} X_d$. $\square$

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