A classic proof of a recurrence for a very classical sequence

Dominique Foata[1] and Doron Zeilberger[2]

To Marco Schützenberger, in memoriam.

Richard Stanley [St96] has recently narrated the fascinating story of how the *classical* Schröder [Sch1870] numbers $s(n)$ are even more classical than has been believed before. They (at least $s(10) = 103049$) have been known to Hipparchus (190-127 B.C.). Stanley recalled the three-term linear recurrence ([Co64]; [Co74], p. 57)

$$(1) \qquad 3(2n-1)s(n) = (n+1)s(n+1) + (n-2)s(n-1) \quad (n \geq 2), \\ s(1) = s(2) = 1,$$

and stated that "no direct combinatorial proof of this formula seems to be known." The purpose of this note is to fill this gap.

The present proof reflects the ideas of our great master, Marcel-Paul Schützenberger (1920-1996), who taught us that every algebraic relation is to be given a combinatorial counterpart and vice versa. The methodology has been vigorously and successfully pursued by the École bordelaise (e.g., [Cor75], [Vi85].)

The recurrence (1) is tantalizingly similar to the linear recurrence

$$(2) \qquad 2(2n-1)c(n) = (n+1)c(n+1) \quad (n \geq 1), \qquad c(1) = 1,$$

that is obviously satisfied by the Catalan numbers $c(n) = (1/n)\binom{2n-2}{n-1}$. Our proof is inspired by Rémy's elegant combinatorial proof [Re85] of (2) shown to us by Viennot [Vi82].

Recall ([Co74], pp. 56-57) that a *Schröder tree* T is either the tree consisting of its root alone $T = r$, or an ordered tuple $[r; T_1, \ldots, T_l]$, where $l \geq 2$ and $T_1, \ldots, T_l$ are smaller Schröder trees. The first symbol r is called the *root* of T and the roots of $T_1, \ldots, T_l$ are called the *sons* of T. A sonless node is called a *leaf*. The number of Schröder trees with n leaves is denoted by $s(n)$.

The first values of the $s(n)$'s appear in Table 1.

n	1	2	3	4	5	6	7	8	9	10
$s(n)$	1	1	3	11	45	197	903	4279	20793	103049

Table 1

[1] Département de mathématique, Université Louis Pasteur, 7, rue René-Descartes, F-67084 Strasbourg, France. (`foata@math.u-strasbg.fr`)
[2] Department of mathematics, Temple University, Philadelphia, Pennsylvania 19122. (`zeilberg@math.temple.edu`)

Our proof will be based on another combinatorial model for the Schröder numbers, the *well-weighted binary plane trees.* Recall that a plane tree is said to be *binary*, if each node has either no sons or exactly two sons. As is well-known (see, e.g., [Co74], pp. 52-53), the number of such trees with exactly n leaves is the *Catalan number* $c(n)$. A node is said to be *interior*, if it is not a leaf. The binary trees we will be using are *weighted.* This means that each *interior* node is given a weight equal to 1 or equal to 2. A node is said to be *well-weighted*, if, whenever it has weight 2, its right son is not a leaf.

A binary plane tree is *well-weighted* if it is weighted and if all its interior nodes are well-weighted. In short, we will speak of a *well-weighted tree.* As shown in Table 2, there are $c(4) = 5$ binary plane trees with four leaves, and $s(4) = 11$ well-weighted trees with four leaves. The nodes that can get *either* weight 1 or weight 2 are indicated by the symbol "$\bigcirc$."

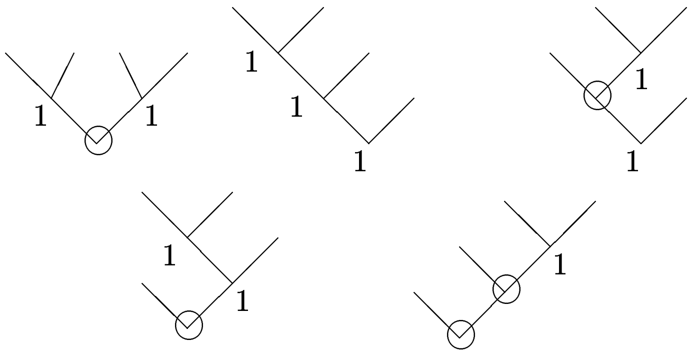


Table 2

A well-weighted tree can be also defined recursively as follows. It is either a single unweighted node (serving both as root and leaf), or a triple $[r; T_1, T_2]$, where r is either 1 or 2, and where T_1 and T_2 are smaller well-weighted trees, with the provision that if T_2 is a mere leaf, then r must be 1.

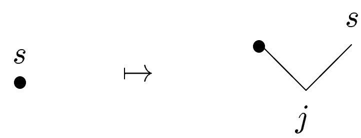
Define a mapping from the set of Schröder trees to the set of well-weighted trees as follows: if $T = r$, then $\Phi(T) := r$; if the root of T has exactly two sons, i.e., $T = [r; T_1, T_2]$, then $\Phi(T) := [1; \Phi(T_1), \Phi(T_2)]$; if the root of T has more than two sons, i.e., $T = [r; T_1, \dots, T_l]$ with $l > 2$, then $\Phi(T) := [2; \Phi(T_1), \Phi([r; T_2, \dots, T_l])\,]$. It is clear that Φ is a bijection that preserves the number of leaves.

Being binary, every well-weighted tree with n leaves, possesses exactly $(n-1)$ interior nodes and hence $(2n-1)$ nodes altogether. A well-weighted tree with n leaves is said to be *pointed*, if exactly one of its $(2n-1)$ nodes is pointed; it is *leaf-pointed* if exactly one of its n leaves is pointed; it is *interior-pointed* if exactly one of its $(n-1)$ interior nodes is pointed. Let $PT(n)$, (resp. $LT(n)$, resp. $IT(n)$) be the set of pointed (resp. leaf-pointed, resp. interior-pointed) well-weighted trees with n leaves.

As $(2n-1)s(n) = \#PT(n)$, $(n+1)s(n+1) = \#LT(n+1)$ and $(n-2)s(n-1) = \#IT(n-1)$, formula (1) will be proved combinatorially (or bijectively, as some people say to-day), if we can construct a bijection σ of $\{1,2,3\} \times PT(n)$ onto the disjoint union $LT(n+1) \cup IT(n-1)$.

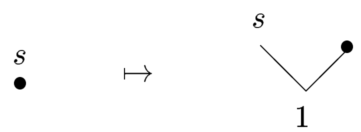
The construction of such a bijection will consist of adding a new leaf to each pointed well-weighted tree of $PT(n)$, in three different ways denoted by L_1, L_2, R_1. We shall get all of $LT(n+1)$ plus a set $B(n+1)$ of leaf-pointed weighted trees, but not well-weighted, which is in one-to-one correspondence with $IT(n-1)$.

To construct the bijection σ we proceed as follows. Start with t in $PT(n)$ and let s be the subtree of t whose root is the pointed node of t. For $j = 1,2$ define $\sigma'(L_j, t)$ to be the leaf-pointed weighted tree with $(n+1)$ leaves obtained from t, by performing the following replacement:



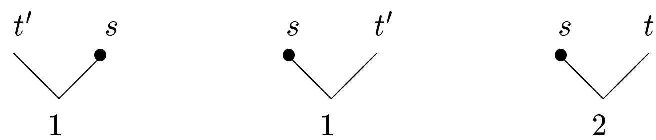
Notice that the new interior node receives weight j and the point gets moved from the root of s to the new leaf.

In the same manner $\sigma'(R_1, t)$ is defined by performing the replacement:



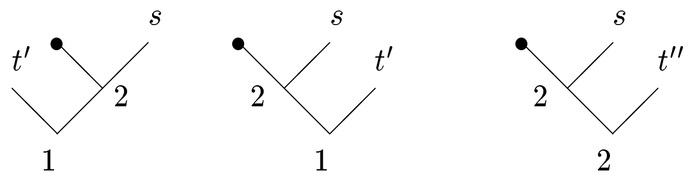
Clearly, σ' is an *injection* of $\{L_1, L_2, R_1\} \times PT(n)$ into the set of leaf-pointed weighted trees with $(n+1)$ leaves. Furthermore, $\sigma'(L_1, t)$ and $\sigma'(R_1, t)$ are always *well-weighted* and $\sigma'(L_2, t)$ is well-weighted when the subtree s is *not* a leaf. Define $\sigma = \sigma'$ in all those cases.

When s is a leaf, the subtree of t whose root is the *father* of the pointed node is one of the following forms



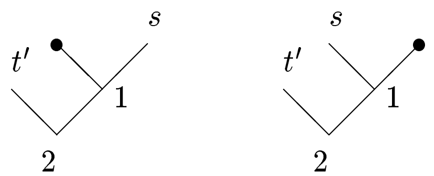
case (a) case (b) case (c)

where the subtree t' may be any tree, while the subtree t'' must *not* be a leaf. When applying σ' to (L_2, t), we will get



case (a) case (b) case (c)

Alas, none of those trees is well-weighted (since s is a leaf). To remedy cases (a) and (b) replace them by:



case (a) case (b)

and define $\sigma(L_2, t)$ to be the tree thereby obtained. It now belongs to $LT(n+1)$. It is straightforward to verify that σ is a *bijection* of the set of all pairs in $\{L_1, L_2, R_1\} \times PT(n)$ that do not belong to case (c) onto $LT(n+1)$.

Finally, to obtain $\sigma(L_2, t)$ in case (c), replace the portion depicted above that belongs to $\sigma'(L_2, t)$ by just the subtree t'' (losing two leaves) and make the root of t'' be the pointed node of the new tree. We get an interior-pointed well-weighted plane tree with $(n-1)$ leaves. Furthermore, the restriction of σ to case (c) is a bijection onto $IT(n-1)$.

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