Contemporary Mathematics
Volume **00**, XXXX

Word straightening and q-Eulerian Calculus

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This paper is dedicated to Richard Askey for his great scholarship and his successful savoir-faire in promoting the classical theory of Special Functions and making it accessible and useful to new areas of mathematics.

Abstract. This paper contains a new description of the straightening algorithm for pairs of finite words called circuits. The commutation rule underlying the straightening algorithm that keeps invariant a specific word bivariate statistic, is fully characterized. A methodology is developed that makes possible the calculation of the bivariate generating function for that pair of statistics. That generating function appears to be a generalized form of the classical q-Eulerian polynomial.

1. Introduction

The present paper has been motivated by a remark of Richard Askey who asked us to unveil the mysteries hidden in each combinatorial construction, to fully explain how an intricate bijection leading to an explicit calculation dealing with q-series was invented or discovered. We should like to try to match his expectations in this paper by going back to the study of a problem on multipermutation statistics that has kept several of us busy in the past eight years. In doing so we have been lucky to obtain the following new results we will now explain in full details.

In studying the genus zeta function of local minimal hereditary orders Denert [**9**] introduced a new permutation statistic, that was later christened "den." When associated with the classical statistic "exc," the number of excedances (all those terms will be fully redefined further in the paper), the generating function for the pair (exc, den) over the symmetric group $\mathcal{S}_r$ is equal to the q-Eulerian polynomial [**2, 3, 4**], a result proved in [**12**]. The definition of "den" for arbitrary words (with repetitions) whose letters belong to an alphabet X was imagined by Han [**14**]. He proved that the generating function for the pair (exc, den) over each class of

1991 *Mathematics Subject Classification.* Primary 05A15, 33A15; Secondary 05A17, 33A75.
Key words and phrases. Straightening algorithm, rearrangements, permutation statistics, q-Eulerian polynomials.

rearrangements of an arbitrary word was equal to the generalised q-Eulerian polynomial, i.e., the polynomial denoted by $A_{\mathbf{c},\mathbf{d}}(t,q)$ in formula (4.1) when $\mathbf{d}$ is the empty sequence.

When the underlying alphabet X is partitioned into two subalphabets S of *small* letters and L of *large* letters, the statistics "exc" and "den" can be further refined to take large inequalities into account [**6, 7, 8**]. Those statistics are denoted by exc_ℓ, den_ℓ. It was proved in [**7**] that the generating function for the pair $(\mathrm{exc}_\ell, \mathrm{den}_\ell)$ over each class $R(\mathbf{c},\mathbf{d})$ was equal to the further generalized q-Eulerian polynomial $A_{\mathbf{c},\mathbf{d}}(t,q)$, as it appears in (4.1).

It was not easy to prove those results. In each case a transformation on words Φ (resp. Φ_ℓ in the case of a bi-alphabet) was to be constructed. It maps each rearrangement class onto itself and has the further property that

$$(\mathrm{exc}, \mathrm{den})\, w = (\mathrm{des}, \mathrm{maj})\, \Phi(w) \quad \text{and} \quad (\mathrm{exc}_\ell, \mathrm{den}_\ell)\, w = (\mathrm{des}_\ell, \mathrm{maj}_\ell)\, \Phi_\ell(w).$$

Here "des," "maj," "des_ℓ" and "maj_ℓ" are more traditional word statistics, namely, the number of descents, the major index, the number of ℓ-descents, the ℓ-major index, respectively, for which it is more easy to calculate the generating functions for the pairs $(\mathrm{des}, \mathrm{maj})$ and $(\mathrm{des}_\ell, \mathrm{maj}_\ell)$ over each rearrangement class (see, in particular, [**8**]). The derivation of the generating functions for the pairs $(\mathrm{exc}, \mathrm{den})$ and $(\mathrm{exc}_\ell, \mathrm{den}_\ell)$ is then a consequence of what is already known for the pairs $(\mathrm{des}, \mathrm{maj})$ and $(\mathrm{des}_\ell, \mathrm{maj}_\ell)$ using the transformations Φ (resp. Φ_ℓ).

In [**10**] it was shown that the following classical transformations on words, the Cartier-Foata transform [**5**], the transform Φ itself [**14**], as well as its two extensions derived in [**6**] and in [**7**] for Φ_ℓ, could be described by means of a *single* algorithm, the *straightening algorithm.*

In the present paper we will again describe the straightening algorithm (section 2), but we will prove (Theorem 2.1) that it can be further extended to provide a bijective transformation between two classes of special pairs of words, the well-factorized circuits and the well-sorted circuits.

The straightening algorithm can be constructed for each commutation rule. Once the statistics "exc_L" and "den_L" are introduced (see section 3), it is shown that there is one and only one commutation rule that preserves the pair of statistics $(\mathrm{exc}_L, \mathrm{den}_L)$ (Theorem 3.1), making the construction of the desired straightening algorithm unique.

Section 4 is devoted to the algebra of the generalized q-Eulerian polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$. To show in particular that $A_{\mathbf{c},\mathbf{d}}(t,q)$ is the generating polynomial for a pair of statistics (f,g) it suffices to construct an explicit bijection involving finite sequences of integers (see Criterion 4.1).

To apply Criterion 4.1 successfully for the pairs $(\mathrm{exc}_L, \mathrm{den}_L)$ it is essential to find an appropriate partition of the alphabet X into small and large letters and further to derive a total ordering "$\preceq$" such that the second statistic den_L depend only on the L-excedances of each $\preceq$-well-factorized circuit. Such a partition and such a total ordering, denoted by $\preceq_\ell$, are characterized in Section 5.

Criterion 4.1 is fully applied in section 6 in which we show that the generating functions for the classes of $\preceq_\ell$-well-sorted circuits by the pair $(\mathrm{exc}_\ell, \mathrm{den}_\ell)$ are given by the generalized q-Eulerian polynomials presented in section 4.

It will be noticed that all our q-calculations are made directly on the pairs $(\mathrm{exc}, \mathrm{den})$ and $(\mathrm{exc}_\ell, \mathrm{den}_\ell)$ without any reference to the classical pairs $(\mathrm{des}, \mathrm{maj})$

and $(\mathrm{des}_\ell, \mathrm{maj}_\ell)$. In section 7 we show how we can construct a transformation $w \mapsto w'$ having the property that $(\mathrm{des}_\ell, \mathrm{maj}_\ell)\, w = (\mathrm{exc}_\ell, \mathrm{den}_\ell)\, w'$.

2. The straightening algorithm

The straightening algorithm in question acts on *biwords*, defined to be ordered pairs $\alpha = (v, w)$, preferably written as

$$\alpha = \binom{v}{w} = \begin{pmatrix} y_1 \ldots y_{i-1}\, y_i\, y_{i+1}\, y_{i+2} \ldots y_m \\ x_1 \ldots x_{i-1}\, x_i\, x_{i+1}\, x_{i+2} \ldots x_m \end{pmatrix}$$

of words

$$v = y_1 \ldots y_{i-1} y_i y_{i+1} y_{i+2} \ldots y_m, \quad w = x_1 \ldots x_{i-1} x_i x_{i+1} x_{i+2} \ldots x_m,$$

of the same length. The letters x_i, y_i are taken from a finite alphabet X that for convenience we shall take as the subset $\{1, 2, \ldots, r\}$ of the integers, equipped with its standard ordering. The above integer m is said to be the *length* of the biword α. When the words v and w are rearrangements of each other, i.e., when each of them can be derived from the other by permuting the letters in some order, we say that the biword is a *circuit.*

The straightening algorithm is completely described once a Boolean function $Q(x, y; z, t)$ has been defined on quadruples of letters. Starting with Q we define a *commutation rule*, Com_Q, that maps each pair (α, i), where α is a biword and i is an integer less than the length of α, onto a biword

$$\alpha' = \mathrm{Com}_Q(\alpha, i) = \binom{v'}{w'} = \begin{pmatrix} y_1 \ldots y_{i-1}\, y_{i+1}\, y_i\, y_{i+2} \ldots y_m \\ x_1 \ldots x_{i-1} \quad z \quad t\ x_{i+2} \ldots x_m \end{pmatrix},$$

where $z = x_i$ and $t = x_{i+1}$ if $Q(y_i, y_{i+1}; x_i, x_{i+1})$ true; $z = x_{i+1}$ and $t = x_i$ if $Q(y_i, y_{i+1}; x_i, x_{i+1})$ false. Notice that the successive *top* letters y_i, y_{i+1} always commute under the action of Com_Q, but not necessarily the corresponding *bottom* letters x_i, x_{i+1}. Also notice that $\mathrm{Com}_Q(\alpha, i)$ is a circuit if and only if α is a circuit.

It is convenient to call *pointed biword* each pair (α, i) where α is a biword of length m and $1 \le i \le m-1$. Now let $\alpha' = \mathrm{Com}_Q(\alpha; i)$. If $1 \le j \le m-1$, we can form the pointed biword (α', j) and further apply Com_Q to (α', j). We obtain the biword $\mathrm{Com}_Q(\alpha', j) = \mathrm{Com}_Q(\mathrm{Com}_Q(\alpha, i), j)$, we shall denote by $\mathrm{Com}_Q(\alpha; i, j)$. By induction it makes sense to define $\mathrm{Com}_Q(\alpha; i_1, \ldots, i_n)$, where $(i_1, \ldots, i_n)$ is a given sequence of integers less than m.

As each commutation always permutes two adjacent letters within the *top* word, we can transform each biword $\alpha = \binom{v}{w}$ into a biword $\alpha' = \binom{v'}{w'}$ whose top word v' is *non-decreasing* by applying a sequence of commutations. We can also say that for each biword $\alpha = \binom{v}{w}$ there exists a sequence $(i_1, \ldots, i_n)$ of integers such that the top word in the resulting biword $\mathrm{Com}_Q(\alpha; i_1, \ldots, i_n)$ is non-decreasing. Such a biword is called a *well-sorted* biword. The sequence $(i_1, \ldots, i_n)$ is called a *commutation sequence.* We define a particular commutation sequence $(i_1, \ldots, i_n)$ called *minimal* by the following two conditions:

(i) it is of minimum length;

(ii) it is minimal with respect to the lexicographic order.

Clearly the minimal sequence is uniquely defined by those two conditions and depends only on the top word v in $\alpha = \binom{v}{w}$. The minimal biword derived from α by using the minimal sequence is called the *straightening* of the biword α and will be denoted by $\mathbf{SORT}_Q(\alpha)$.

EXAMPLE 1. Consider the following Boolean function Q_{CF} defined by

$$Q_{CF}(x,y;z,t) \text{ true, if and only if } x=y.$$

Take $\alpha = \begin{pmatrix} 4\,4\,5\,1\,3\,1\,2\,3\,5 \\ 6\,5\,4\,3\,3\,2\,1\,1\,1 \end{pmatrix}$; then $\mathbf{SORT}_{Q_{CF}}(\alpha) = \begin{pmatrix} 1\,1\,2\,3\,3\,4\,4\,5\,5 \\ 3\,2\,1\,3\,1\,6\,5\,4\,1 \end{pmatrix}$.

The "CF" subscript stands for "Cartier-Foata", as those two authors introduced that commutation in [5]. The present description of Q_{CF} appeared in [10].

Now let $\mathbf{c} = (c_1, c_2, \dots, c_r)$ be a sequence of r nonnegative integers and consider the nondecreasing word $y_1 y_2 \dots y_m = 1^{c_1} 2^{c_2} \dots r^{c_r}$, i.e., $m = c_1 + c_2 + \dots + c_r$ and $y_1 = \dots = y_{c_1} = 1$, $y_{c_1+1} = \dots = y_{c_1+c_2} = 2, \dots, y_{c_1+\dots+c_{r-1}+1} = \dots = y_m = r$. The class of all rearrangements of the word $y_1 y_2 \dots y_m$ will be denoted by $R(\mathbf{c})$.

The set of the *circuits* $\alpha = \binom{v}{w}$, where both words v and w belong to $R(\mathbf{c})$ will be denoted by $C(\mathbf{c})$. When v is the nondecreasing word $1^{c_1} 2^{c_2} \dots r^{c_r}$, we say that the circuit α is *well-sorted.* The subset of all *well-sorted* circuits in $C(\mathbf{c})$ will be denoted by $S(\mathbf{c})$. To each word $w \in R(\mathbf{c})$ there corresponds a unique well-sorted circuit $\Gamma(w) = \binom{\overline{w}}{w} \in S(\mathbf{c})$, where $\overline{w}$ denotes the nondecreasing rearrangement $1^{c_1} 2^{c_2} \dots r^{c_r}$ of w. The natural bijection $\Gamma : R(\mathbf{c}) \to S(\mathbf{c})$ will be one of the ingredients in our derivation.

We consider a second class $F_{\preceq}(\mathbf{c})$ of circuits, called $\preceq$*-well-factorized*, that will also be in bijection with $R(\mathbf{c})$. To this end we suppose given a total order "$\preceq$" on X, not necessarily identical with the standard order and make use of the obvious notations: "$\prec$," "$\succ$," "$\succeq$." A nonempty word $w = x_1 x_2 x_3 \dots x_m$ is said to be $\preceq$-*dominated*, if $x_1 \succ x_2$, $x_1 \succ x_3, \dots, x_1 \succ x_m$. The *right to left cyclic shift* of w is defined to be the word $\delta w = x_2 x_3 \dots x_m x_1$. A circuit of the form $\binom{\delta w}{w}$ with w $\preceq$-dominated is called a $\preceq$*-dominated cycle.*

As it is known (see, e.g. [**15**] (Lemma 10.2.1)) or easily verified, each word w is the juxtaposition product $u^1 u^2 \dots$ of $\preceq$-dominated words whose first letters $\mathrm{pre}(u^1), \mathrm{pre}(u^2), \dots$ are in nondecreasing order with respect to "$\preceq$" :

$$\mathrm{pre}(u^1) \preceq \mathrm{pre}(u^2) \preceq \cdots \tag{2.1}$$

This factorization, called the $\preceq$*-increasing factorization* of w, is unique.

Given the $\preceq$-increasing factorization $u^1 u^2 \dots$ of a word w we can form the juxtaposition product

$$\Delta(w) = \begin{pmatrix} \delta u^1 \; \delta u^2 \dots \\ u^1 \;\; u^2 \;\; \dots \end{pmatrix} \tag{2.2}$$

of the $\preceq$-dominated cycles. Clearly Δ maps each word onto a product of $\preceq$-dominated cycles satisfying inequalities (2.1), in a bijective manner. Such a product, written as a circuit (2.2), will be called a $\preceq$*-well-factorized circuit.* The set of all $\preceq$-well-factorized circuits in $C(\mathbf{c})$ will be denoted by $F_{\preceq}(\mathbf{c})$.

EXAMPLE 2. Consider the total ordering $5 \prec 4 \prec 1 \prec 2 \prec 3$ of the set $X = \{1, \dots, 5\}$ and the word $w = 4\,4\,5\,1\,3\,1\,2\,3\,5$. To w there corresponds the *well-sorted* ciruit $\Gamma(w) = \begin{pmatrix} 1\,1\,2\,3\,3\,4\,4\,5\,5 \\ 4\,4\,5\,1\,3\,1\,2\,3\,5 \end{pmatrix}$. On the other hand, the $\preceq$-increasing factorization of w reads $4 \,|\, 4\,5 \,|\, 1 \,|\, 3\,1\,2 \,|\, 3\,5$, so that the corresponding $\preceq$*-well-factorized* circuit is:

$$\Delta(w) = \left(\begin{array}{c|c|c|c|c} 4 & 5\,4 & 1 & 1\,2\,3 & 5\,3 \\ 4 & 4\,5 & 1 & 3\,1\,2 & 3\,5 \end{array}\right).$$

For every $\mathbf{c}$ the two classes $F_{\preceq}(\mathbf{c})$ and $S(\mathbf{c})$ are equinumerous. For instance, $\Gamma\Delta^{-1}$ is a bijection between the two sets. However we want to build a bijection based on algorithm $\mathbf{SORT}_Q$ that, in general, maps $F_{\preceq}(\mathbf{c})$ *into* $S(\mathbf{c})$. We then impose a condition on the Boolean function Q that will transform the "into" into an "onto". The restriction of $\mathbf{SORT}_Q$ to $F_{\preceq}(\mathbf{c})$ will be denoted by $\mathbf{SORT}_{(\preceq,Q)}$.

DEFINITION. A Boolean function $Q(x,y;z,t)$ is said to be *bi-symmetric* if it is symmetric in the two sets of parameters $\{x,y\}$ and $\{z,t\}$, i.e., $Q(x,y;z,t) = Q(y,x;z,t)$ and $Q(x,y;z,t) = Q(x,y;t,z)$.

It is straightforward to verify that the commutation rule Com_Q induced by a bi-symmetric Boolean function Q is involutive, i.e., if $\alpha' = \mathrm{Com}_Q(\alpha;i)$, then $\alpha = \mathrm{Com}_Q(\alpha';i)$.

THEOREM 2.1. *Let Q be a bi-symmetric four-variable Boolean function. Then $\mathbf{SORT}_{(\preceq,Q)}$ is a bijection of $F_{\preceq}(\mathbf{c})$ onto $S(\mathbf{c})$.*

PROOF. The proof is essentially the description of the action of $\mathbf{SORT}_{(\preceq,Q)}$ when it is applied to a well-factorized circuit. As will be seen, the bi-symmetry property makes each step reversible. Let $\alpha' = \begin{pmatrix} z_1 \ldots z_{m-1} \\ t_1 \ldots t_{m-1} \end{pmatrix}$ be a biword and the juxtaposition product $\alpha = \alpha' \begin{pmatrix} z_m \\ t_m \end{pmatrix}$ be a $\preceq$-well-factorized circuit in $F_{\preceq}(\mathbf{c})$. Then the letter z_m is necessarily the $\preceq$-greatest letter in the word $z_1 \ldots z_m$ (or in $t_1 \ldots t_m$). To avoid cumbersome notation let $f := \mathbf{SORT}_{(\preceq,Q)}$. By definition of the straightening algorithm we have

$$f(\alpha) = f\Big(f(\alpha')\begin{pmatrix} z_m \\ t_m \end{pmatrix}\Big).$$

Now let $f(\alpha') = \begin{pmatrix} z'_1 \cdots z'_{m-1} \\ x'_1 \ldots x'_{m-1} \end{pmatrix}$, so that $z'_1 \le \cdots \le z'_{m-1}$. There is a unique integer i such that $1 \le i \le m-1$ and $z'_i \le z_m < z'_{i+1}$. To derive $f(\alpha)$ we only have to move z_m to the left (using the commutation rule Com_Q) until z_m lies between z'_i and z'_{i+1}. Using the previous notations we then have

$$f(\alpha) = \mathrm{Com}_Q(f(\alpha')\begin{pmatrix} z_m \\ t_m \end{pmatrix}; m-1, m-2, \ldots, i+1).$$

Let $f(\alpha) = \begin{pmatrix} y_1 \cdots y_{m-1}\, y_m \\ x_1 \ldots x_{m-1}\, x_m \end{pmatrix}$, so that $y_1 \le \cdots \le y_{m-1} \le y_m$. To recover the pair $\Big(f(\alpha'), \begin{pmatrix} z_m \\ t_m \end{pmatrix}\Big)$ from $f(\alpha)$ we only have to locate the letter which is the rightmost one in $y_1 y_2 \ldots y_m$ among those which are the $\preceq$-largest. This defines an integer i without ambiguity. We then define

$$f(\alpha')\begin{pmatrix} z_m \\ t_m \end{pmatrix} = \mathrm{Com}_Q(f(\alpha); i+1, \ldots, m-2, m-1).$$

As the commutation rule Com_Q is involutive, this reverse procedure leads to the initial pair $(f(\alpha'), \begin{pmatrix} z_m \\ t_m \end{pmatrix})$.

In the same manner, $f(\alpha') = f\Bigl(f(\alpha'')\binom{z_{m-1}}{t_{m-1}}\Bigr)$, where α'' is the biword $\begin{pmatrix} z_1 \dots z_{m-2} \\ t_1 \dots t_{m-2} \end{pmatrix}$. If $z_m = t_m$, then z_{m-1} is the $\preceq$-greatest letter in $z_1 \dots z_{m-1}$. If $z_m \neq t_m$, then necessarily $z_{m-1} = t_m$ as α is $\preceq$-well-factorized. As before, there is a unique sequence of commutations that bring z_{m-1} to a position where the resulting top word is nondecreasing.

Conversely, starting with $f(\alpha')$ we know that the letter on the top word that must be brought to the right end is the rightmost letter equal to t_m. We also know that the product of the commutations involved will give back the pair

$$(f(\alpha''), \binom{z_{m-1}}{t_{m-1}}),$$

as the commutation rule is involutive. As each step has a well-defined reverse step uniquely defined, the mapping $\mathbf{SORT}_{(\preceq,Q)}$ is truly bijective. $\square$

3. Unicity of the word transformation

Our next task is to characterize the transformations $\mathbf{SORT}_{(\preceq,Q)}$ that preserves the pair of statistics $(\mathrm{exc}_L, \mathrm{den}_L)$ that will be defined shortly. To this end we will suppose that the alphabet $X = \{1, 2, \dots, r\}$ is made of two disjoint sets S and L of so-called *small* and *large* letters, respectively. Let s (resp. ℓ) be the cardinality of S (resp. of L), so that $s + \ell = r$. With respect to the standard ordering of X the small letters are not necessarily smaller than the large ones. The features "small" and "large" refer to two different ways of considering inequalities, as we will explain below.

Let $\alpha = \binom{v}{w} = \begin{pmatrix} y_1\, y_2 \dots y_m \\ x_1\, x_2 \dots x_m \end{pmatrix}$ be a circuit. The *number of L-excedances* of α, $\mathrm{exc}_L\, \alpha$, is defined be the number of integers i such that $1 \leq i \leq m$ and either $x_i > y_i$, or $x_i = y_i$ and x_i large.

The definition of $\mathrm{den}_L\, \alpha$ is based on the notion of *cyclic interval* defined as follows. Place the r elements $1, 2, \dots, r$ of X on a circle counterclockwise. If x is small (resp. large), place a bracket in the form $-\!\!]\!\!\rightarrow$ (resp. $-\!\![\!\!\rightarrow$) on the vertex x. For $x, y \in X$ $(x \neq y)$ the cyclic interval $\rrbracket x, y \rrbracket_L$ is defined to be the subset of all the elements that lie between x and y when the circle is read counterclockwise. The brackets (in the French notation) indicate if the ends of the interval are to be included or not.

Suppose $x < y$. If x, y are both small, then $\rrbracket x, y \rrbracket_L = \mathbf{]}x, y\mathbf{]} = \{x + 1, \dots, y\}$ (origin excluded and end included); if x small and y large, then $\rrbracket x, y \rrbracket_L = \mathbf{]}x, y\mathbf{[} = \{x + 1, \dots, y - 1\}$ (both ends are excluded); if x large and y small, then $\rrbracket x, y \rrbracket_L = \mathbf{[}x, y\mathbf{]} = \{x, x + 1, \dots, y - 1, y\}$ (both ends are included); if x and y are both large, then $\rrbracket x, y \rrbracket_L = \mathbf{[}x, y\mathbf{[} = \{x, x + 1, \dots, y - 1\}$ (the origin x included, but the end y excluded). Moreover $\rrbracket y, x \rrbracket_L = X \setminus \rrbracket x, y \rrbracket_L$. Finally, define $\rrbracket x, x \rrbracket_L = \emptyset$ or X depending on whether x is small or large.

The *L-Denert statistic*, $\mathrm{den}_L\, \alpha$, of the circuit α is defined to be the number of pairs (i, j) such that $1 \leq i < j \leq m + 1$ and $x_i \in \rrbracket x_j, y_j \rrbracket_L$ (by convention $\rrbracket x_{m+1}, y_{m+1} \rrbracket_L = L$, the subset of all large letters). For each $i = 1, 2, \dots, m$ let p_i be the number of j such that $i + 1 \leq j \leq m + 1$ and $x_i \in \rrbracket x_j, y_j \rrbracket_L$. We also have $\operatorname{den}_L \alpha = p_1 + p_2 + \cdots + p_m$. The sequence $(p_1, p_2, \ldots, p_m)$ is said to be the *L-den-coding* of α and is denoted by L-den-coding(α).

EXAMPLE 3. Take $r = 5$ and suppose that 1, 2, 4 are small and 3, 5 are large, so that the brackets on the circle (here a square!) are displayed as in Fig. 1. There are $s = 3$ small letters and $\ell = 2$ large letters. In particular the small letter 4 is larger than the large letter 3 with respect to the standard ordering.

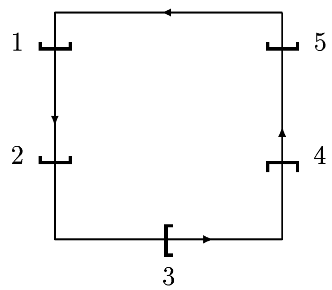


Fig. 1

The following circuit α with letters in X is displayed in the first two rows of the matrix. The third row shows the content of each cyclic interval $\rrbracket x_i, y_i \rrbracket_L$ and the fourth row the L-den-coding of α:

$$\begin{pmatrix} y_i \\ x_i \\ \rrbracket x_i, y_i \rrbracket_L \\ p_i \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ \mathbf{4} & \mathbf{5} & 1 & \mathbf{4} & 1 & 3 & 2 & 3 & \mathbf{5} \\ \{5,1\} & \{5,1\} & \{2\} & \{5,1,2\} & \{2\} & \{3,4\} & \{3,4\} & \{3,4\} & X \\ 4 & 3 & 2 & 4 & 1 & 4 & 1 & 2 & 1 \end{pmatrix}$$

and $\rrbracket x_{m+1}, y_{m+1} \rrbracket_L = L = \{3, 5\}$. There are four excedences indicated in bold-face, so that $\operatorname{exc}_L \alpha = 4$ and $\operatorname{den}_L \alpha = 4 + 3 + 2 + 4 + 1 + 4 + 1 + 2 + 1 = 22$.

Our next task is the following: given a subset L of X of large letters, so that both statistics "exc_L" and "den_L" are well-defined, find a Boolean function Q_L having the property that

> *for every circuit α of length m and every place i $(1 \le i \le m-1)$, the following identity holds*

$$(\operatorname{exc}_L, \operatorname{den}_L)\, \alpha = (\operatorname{exc}_L, \operatorname{den}_L)\ \operatorname{Com}_{Q_L}(\alpha, i). \tag{3.1}$$

Suppose that such a Boolean function $Q = Q_L$ exists and consider a circuit α of length m. Denote the image of the pointed circuit (α, i) under the corresponding commutation rule by $\alpha' = \operatorname{Com}_Q(\alpha, i)$ and take again the notations of section 2 for the circuits α and α'. If (3.1) holds, it follows from the definitions of "exc_L" and "den_L" that

$$\begin{aligned} \operatorname{exc}_L \alpha - \operatorname{exc}_L \alpha' = {} & \chi(x_i > y_i) + \chi(x_{i+1} > y_{i+1}) \\ & + \chi(x_i = y_i \text{ large}) + \chi(x_{i+1} = y_{i+1} \text{ large}) \\ & - \chi(z > y_{i+1}) - \chi(t > y_i) \\ & - \chi(z = y_{i+1} \text{ large}) - \chi(t = y_i \text{ large}) \end{aligned}$$

$$\begin{aligned}\operatorname{den}_L \alpha - \operatorname{den}_L \alpha' = \sum_{1\le h\le i-1} \Bigl(&\chi(x_h \in \rrbracket x_i, y_i \rrbracket_L) + \chi(x_h \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L)\\ &- \chi(x_h \in \rrbracket z, y_{i+1} \rrbracket_L) - \chi(x_h \in \rrbracket t, y_i \rrbracket_L)\Bigr)\\ &+ \chi(x_i \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L) - \chi(z \in \rrbracket t, y_i \rrbracket_L).\end{aligned}$$

If condition $Q(y_i, y_{i+1}, x_i, x_{i+1})$ is true, we must have:

$$\begin{aligned}\operatorname{exc}_L \alpha - \operatorname{exc}_L \alpha' &= \chi(x_i > y_i) + \chi(x_{i+1} > y_{i+1})\\ &\quad + \chi(x_i = y_i \text{ large}) + \chi(x_{i+1} = y_{i+1} \text{ large})\\ &\quad - \chi(x_i > y_{i+1}) - \chi(x_{i+1} > y_i)\\ &\quad - \chi(x_i = y_{i+1} \text{ large}) - \chi(x_{i+1} = y_i \text{ large}) \qquad (3.2)\end{aligned}$$

$$\begin{aligned}\operatorname{den}_L \alpha - \operatorname{den}_L \alpha' = \sum_{1\le h\le i-1} \Bigl(&\chi(x_h \in \rrbracket x_i, y_i \rrbracket_L) + \chi(x_h \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L)\\ &- \chi(x_h \in \rrbracket x_i, y_{i+1} \rrbracket_L) - \chi(x_h \in \rrbracket x_{i+1}, y_i \rrbracket_L)\Bigr)\\ &+ \chi(x_i \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L) - \chi(x_i \in \rrbracket x_{i+1}, y_i \rrbracket_L). \qquad (3.3)\end{aligned}$$

If condition $Q(y_i, y_{i+1}, x_i, x_{i+1})$ is false, we must have:

$$\operatorname{exc}_L \alpha - \operatorname{exc}_L \alpha' = 0, \qquad (3.4)$$

$$\begin{aligned}\operatorname{den}_L \alpha - \operatorname{den}_L \alpha' &= \sum_{1\le h\le i-1} \Bigl(\chi(x_h \in \rrbracket x_i, y_i \rrbracket_L) + \chi(x_h \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L)\\ &\qquad - \chi(x_h \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L) - \chi(x_h \in \rrbracket x_i, y_i \rrbracket_L)\Bigr)\\ &\qquad + \chi(x_i \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L) - \chi(x_{i+1} \in \rrbracket x_i, y_i \rrbracket_L)\\ &= \chi(x_i \in \rrbracket x_{i+1}, y_{i+1} \rrbracket_L) - \chi(x_{i+1} \in \rrbracket x_i, y_i \rrbracket_L). \qquad (3.5)\end{aligned}$$

Now given four elements x, y, z, t of X we say that z, t are *neighbors* with respect to x, y, if both z and t are in $\rrbracket x, y \rrbracket_L$, or neither in $\rrbracket x, y \rrbracket_L$. Otherwise, z and t are said to be *strangers* with respect to x, y.

When expression (3.2) is zero, it is easy to see that x_i and x_{i+1} are neighbors with respect to $\rrbracket y_i, y_{i+1} \rrbracket_L$. In that case expression (3.3) is also equal to zero. On the other hand, when (3.5) is zero, then x_i and x_{i+1} must be strangers with respect to $\rrbracket y_i, y_{i+1} \rrbracket_L$. We have then proved the following theorem.

THEOREM 3.1. *Let Q_L be the Boolean function defined by:*

$$Q_L(x, y, z, t) \textit{ true if and only if } z, t \textit{ are neighbors with respect to } x, y.$$

Then Q_L is the unique Boolean function such that for every circuit α and every place i the following identity holds

$$(\operatorname{exc}_L, \operatorname{den}_L)\, \alpha = (\operatorname{exc}_L, \operatorname{den}_L)\ \operatorname{Com}_{Q_L}(\alpha, i).$$

The Boolean function Q_L here characterized was first introduced in [10] under the name of Q_H.

Now it is clear that the above four-variable Boolean function Q_L is bisymmetric. The combination of Theorems 1.1 and 2.1 yields the following result.

THEOREM 3.2. *For every total ordering "$\preceq$" on the alphabet X, for every subset L of large letters of X and every sequence* $\mathbf{c}$, *the mapping* $\mathbf{SORT}_{(\preceq,Q_L)}$ *is a bijection of* $F_{\preceq}(\mathbf{c})$ *onto* $S(\mathbf{c})$ *that satisfies*

$$(\mathrm{exc}_L, \mathrm{den}_L)\, \alpha = (\mathrm{exc}_L, \mathrm{den}_L)\ \mathbf{SORT}_{(\preceq,Q_L)}(\alpha).$$

4. Generalized q-Eulerian polynomials

To introduce those polynomials it is convenient to use the traditional notations of q-calculus, first, the *q-ascending factorial*

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n\geq 1; \end{cases}$$

then the *q-binomial coefficient* (or the Gaussian polynomial)

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0\leq k\leq n; \\ 0, & \text{otherwise.} \end{cases}$$

As before, r will denote a fixed positive integer and s, ℓ two nonnegative integers of sum r. Finally, if $\mathbf{a}=(a_1,\dots,a_m)$ is a sequence of nonnegative integers, we let $\|\mathbf{a}\|$ be the sum $a_1+\cdots+a_m$.

The generalized q-Eulerian polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$, already introduced in [**7**], are polynomials in two variables t, q, indexed by pairs of sequences $\mathbf{c}=(c_1,\dots,c_s)$ and $\mathbf{d}=(d_1,\dots,d_\ell)$ of nonnegative integers. They are defined by

$$\frac{1}{(t;q)_{1+\|\mathbf{c}\|+\|\mathbf{d}\|}} A_{\mathbf{c},\mathbf{d}}(t,q) := \sum_{i\geq 0} t^i \begin{bmatrix} c_1+i \\ i \end{bmatrix} \cdots \begin{bmatrix} c_s+i \\ i \end{bmatrix} \times q^{\binom{d_1+1}{2}} \begin{bmatrix} i \\ d_1 \end{bmatrix} \cdots q^{\binom{d_\ell+1}{2}} \begin{bmatrix} i \\ d_\ell \end{bmatrix}. \tag{4.1}$$

There exist other extensions of q-Eulerian polynomials depending on more parameters, see, e.g. [**11**], but the previous polynomials indexed by pairs of sequences of integers will meet our present needs.

Now let $u_1,\dots,u_s$, $v_1,\dots,v_\ell$ be commuting variables. For convenience we will use the traditional notations $\mathbf{u}^{\mathbf{c}} := u_1^{c_1}\dots u_s^{c_s}$, $\mathbf{v}^{\mathbf{d}} := v_1^{d_1}\dots v_\ell^{d_\ell}$ and

$$(u_1,\dots,u_s;q)_i := (u_1;q)_i\cdots(u_s;q)_i.$$

By means of a simple manipulation on q-series it is readily seen that (4.1) is actually equivalent to the following identity involving the *factorial* generating function for the polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$, namely

$$\sum_{\mathbf{c},\mathbf{d}} A_{\mathbf{c},\mathbf{d}}(t,q)\,\frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{1+\|\mathbf{c}\|+\|\mathbf{d}\|}} = \sum_{i\geq 0} t^i\,\frac{(-qv_1,\dots,-qv_\ell;q)_i}{(u_1,\dots,u_s;q)_{i+1}}. \tag{4.2}$$

Notice that the summation is over all pairs of sequences $\mathbf{c}=(c_1,\dots,c_s)$, $\mathbf{d}=(d_1,\dots,d_\ell)$, including the null sequence. Also notice the occurrences of two different subscripts i and $(i+1)$ in the right-hand side of (4.2) that correspond to the two types of q-binomial coefficients in (4.1).

To be able to make use of identity (4.1) in combinatorial problems and especially in the present paper it is essential to express the q-ascending factorials and the q-binomial coefficients in terms of generating functions for finite sequences of integers. From the classial q-binomial theorem (see, e.g., [**1**], p. 17 or [**13**], § 1.3) we can easily derive the identities:

$$\frac{1}{(t;q)_{1+m}} = \sum_{i\ge 0} t^i \sum_{i\ge a_1\ge\cdots\ge a_m\ge 0} q^{\|\mathbf{a}\|}; \tag{4.3}$$

$$\begin{bmatrix} c+i \\ i \end{bmatrix} = \sum_{i\ge a_1\ge\cdots\ge a_c\ge 0} q^{\|\mathbf{a}\|}; \tag{4.4}$$

$$q^{\binom{d+1}{2}} \begin{bmatrix} i \\ d \end{bmatrix} = \sum_{i\ge a_1>\cdots> a_d\ge 1} q^{\|\mathbf{a}\|}. \tag{4.5}$$

On the other hand, each polynomial $A_{\mathbf{c},\mathbf{d}}(t,q)$ defined by identity (4.1) has positive integral coefficients. Moreover, by multiplying that same identity by $(1-t)$ and making $t=1$ then $q=1$, the expression $A_{\mathbf{c},\mathbf{d}}(1,1)$ is equal to the multinomial coefficient $M(\mathbf{c},\mathbf{d}) = \binom{c_1+\cdots+c_s+d_1+\cdots+d_\ell}{c_1,\ldots,c_s,d_1,\ldots,d_\ell}$. Consequently, if $F(\mathbf{c},\mathbf{d})$ is a finite set of cardinality $M(\mathbf{c},\mathbf{d})$ and if f, g are two integer-valued statistics defined on $F(\mathbf{c},\mathbf{d})$, it makes sense to consider the generating polynomial

$$A^{f,g}_{F(\mathbf{c},\mathbf{d})}(t,q) := \sum_{\alpha\in F(\mathbf{c},\mathbf{d})} t^{f(\alpha)} q^{g(\alpha)}. \tag{4.6}$$

Now the pair (f,g) has a generating function equal to $A_{\mathbf{c},\mathbf{d}}(t,q)$ over $F(\mathbf{c},\mathbf{d})$, i.e., the following identity

$$A^{f,g}_{F(\mathbf{c},\mathbf{d})}(t,q) = A_{\mathbf{c},\mathbf{d}}(t,q)$$

holds, if identity (4.1) is valid when $A_{\mathbf{c},\mathbf{d}}(t,q)$ is replaced by $A^{f,g}_{F(\mathbf{c},\mathbf{d})}(t,q)$. By using the expansions (4.3)-(4.6) we can rewrite that condition as

$$\sum_{(i',\mathbf{a},\alpha)} t^{i'+f(\alpha)} q^{\|\mathbf{a}\|+g(\alpha)} = \sum_{(i,\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(r)})} t^i q^{\|\mathbf{a}^{(1)}\|+\cdots+\|\mathbf{a}^{(r)}\|}. \tag{4.7}$$

In (4.7) the first sum is over all triples $(i',\mathbf{a},\alpha)$ such that

$$\text{(4.8)}\quad \begin{cases} i' \text{ is a nonnegative integer;} \\ \mathbf{a} = (a_1,\ldots,a_{\|\mathbf{c}\|+\|\mathbf{d}\|}) \text{ is a nonincreasing sequence of nonnegative inte-} \\ \quad \text{gers of length } \|\mathbf{c}\|+\|\mathbf{d}\| \text{ satisfying } i'\ge a_1\ge\cdots\ge a_{\|\mathbf{c}\|+\|\mathbf{d}\|}\ge 0; \\ \alpha \text{ is an element of } F(\mathbf{c},\mathbf{d}). \end{cases}$$

The second sum is over all sequences $(i,\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(r)})$, where

$$\text{(4.9)}\quad \begin{cases} i \text{ is a nonnegative integer;} \\ \text{each } \mathbf{a}^{(j)} \text{ is a sequence } \mathbf{a}^{(j)} = (a_{j,1},\ldots,a_{1,c_j}) \text{ of integers satisfying} \\ \quad i\ge a_{j,1}\ge\cdots\ge a_{j,c_j}\ge 0 \ (1\le j\le s); \\ \text{each } \mathbf{a}^{(s+j)} \text{ is a sequence } \mathbf{a}^{(s+j)} = (a_{s+j,1},\ldots,a_{s+j,d_j}) \text{ of integers sat-} \\ \quad \text{isfying } i\ge a_{s+j,1}>\cdots>a_{s+j,d_j}\ge 1 \ (1\le j\le \ell). \end{cases}$$

We then have the following criterion.

CRITERION 4.1. *For the identity $A^{f,g}_{F(\mathbf{c},\mathbf{d})}(t,q) = A_{\mathbf{c},\mathbf{d}}(t,q)$ to hold, it suffices to build a bijection $(i',\mathbf{a},\alpha) \mapsto (i,\mathbf{a}^{(1)},\dots,\mathbf{a}^{(r)})$ of triples having properties* (4.8) *onto sequences having properties* (4.9) *and satisfying*

$$i' + f(\alpha) = i \quad \textit{and} \quad \|\mathbf{a}\| + g(\alpha) = \|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\|. \tag{4.10}$$

5. The well-factorized circuits

Take up again the notations of section 3 where the r elements of the alphabet X have been displayed on a circle. For convenience, insert a new element, called $*$, between the maximum letter r and the minimum 1. With this new convention and when $x > y$, we see that the cyclic interval $\rrbracket x,y\rrbracket_L$ will *always* contain $*$. Actually, $\rrbracket x,y\rrbracket_L$ is reduced to the singleton $\{*\}$ if and only if $x=r$, $y=1$, r small and 1 large. We also see that there is an L-excedance at position i in the circuit $\alpha = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} y_1\, y_2 \cdots y_m \\ x_1\, x_2 \dots x_m \end{pmatrix}$ if and only if the cyclic interval $\rrbracket x_i,y_i\rrbracket_L$ contains $*$.

It is also convenient to introduce the cyclic *sequence* $\rangle\!\rangle x,y\rangle\!\rangle_L$ defined, when $x,y \in X$, $x \neq y$, to be the sequence of the elements of $X \cup \{*\}$ that occur when reading the interval $\rrbracket x,y\rrbracket_L$ counterclockwise. When x is small (resp. large), let $\rangle\!\rangle x,x\rangle\!\rangle_L$ be the empty sequence (resp. the sequence $(x, x+1, \dots, x-1)$.)

For instance, if $x > y$ and if both x, y are small, then

$$\rangle\!\rangle x,y\rangle\!\rangle_L = (x+1, x+2, \dots, r, *, 1, \dots, y-1, y),$$

while if $x < y$ and x is large, but y small, then

$$\rangle\!\rangle x,y\rangle\!\rangle_L = (x, x+1, \dots, y-1, y),$$

with the convention that $x+1 = *$ and $y-1 = *$ when $x=r$ and $y=1$.

Using the notion of cyclic sequences we can also say that there is an L-excedance at position i in the circuit $\alpha = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} y_1\, y_2 \cdots y_m \\ x_1\, x_2 \dots x_m \end{pmatrix}$ if and only if the cyclic sequence $\rangle\!\rangle x_i,y_i\rangle\!\rangle_L$ contains $*$.

Now as the words v and w are rearrangements of each other, there exists a permutation σ of the sequence $(1,2,\dots,m)$ such that $y_1y_2\dots y_m = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(m)}$. Write σ as a product of disjoint cycles:

$$\sigma = \cdots (a, b, \dots, z) \cdots$$

The juxtaposition product $W = \rangle\!\rangle x_a,y_a\rangle\!\rangle_L \rangle\!\rangle x_b,y_b\rangle\!\rangle_L \cdots \rangle\!\rangle x_z,y_z\rangle\!\rangle_L$ is then equal to $\rangle\!\rangle x_a,x_b\rangle\!\rangle_L \cdots \rangle\!\rangle x_z,x_a\rangle\!\rangle_L$. It is empty only if $\begin{pmatrix} x_b \dots x_a \\ x_a \dots x_z \end{pmatrix}$ is equal to $\begin{pmatrix} r \dots r \\ r \dots r \end{pmatrix}$ and r is a small letter. Otherwise, W is equal to the sequence $(x_a+1, \dots, r, *, 1, \dots, x_a)$ (resp. $(x_a, \dots, r, *, 1, \dots, x_a-1)$), if x_a is small (resp. large), repeated as many times as there are L-excedances in the circuit

$$\begin{pmatrix} y_a\, y_b \cdots y_z \\ x_a\, x_b \dots x_z \end{pmatrix} = \begin{pmatrix} x_b & \dots\, x_z\, x_a \\ x_a\, x_b \dots & x_z \end{pmatrix}.$$

Hence, for every $x \in X$ we have:

$$\sum_{j\in\{a,b,\dots,z\}} \chi(x \in \rrbracket x_j,y_j\rrbracket_L) = \sum_{j\in\{a,b,\dots,z\}} \chi(L\text{-excedance at } j). \tag{5.1}$$

By summing the above identity over all the cycles of the permutation σ we deduce that

$$\sum_{1\le j\le m} \chi(x \in]\!]x_j, y_j]\!]_L) = \mathrm{exc}_L\, \alpha.$$

Now if x_1 is small (resp. large), then the cyclic interval $]\!]x_1, y_1]\!]_L$ does not contain (resp. contains) x_1. Hence x_1 occurs in exactly $\mathrm{exc}_L\, \alpha$ cyclic intervals $]\!]x_j, y_j]\!]_L$ for $j = 2, \ldots, m$ and not in $]\!]x_{m+1}, y_{m+1}]\!]_L$ (resp. in exactly $\mathrm{exc}_L\, \alpha - 1$ intervals $]\!]x_j, y_j]\!]_L$ for $j = 2, \ldots, m$ and once in $]\!]x_{m+1}, y_{m+1}]\!]_L$). We have then proved the following proposition.

Proposition 5.1. *Let $(p_1, p_2, \ldots, p_m)$ be the L-den-coding of a circuit α, then $exc_L\, \alpha = p_1$.*

Besides the L-den-coding of a circuit $\alpha = \begin{pmatrix} y_1 \ldots y_m \\ x_1 \ldots x_m \end{pmatrix}$ we also introduce its L-*exc-coding*, denoted by L-exc-coding$(\alpha) = (q_1, q_2, \ldots, q_m)$, where each q_i is defined to be the number of L-excedances in the right factor $\begin{pmatrix} y_i \ldots y_m \\ x_i \ldots x_m \end{pmatrix}$ of α. In particular, $p_1 = q_1 = \mathrm{exc}_L\, \alpha$. We look for a necessary and sufficient condition for the L-den-coding to be identical with the L-exc-coding. As will be seen in the next section, when this property holds, the calculation of the distribution of $(\mathrm{exc}_L, \mathrm{den}_L)$ can easily be made by using Criterion 4.1.

When the set L of large letters is the set of the largest letters in X with respect to the standard ordering, i.e., when $S = \{1, 2, \ldots, s\}$ and $L = \{s+1, s+2, \ldots, r\}$ $(s + \ell = r)$, we write $S < L$ and we speak of "ℓ-den-coding," "exc_ℓ" and "den_ℓ" instead of "L-den-coding," "exc_L" and "den_L." (Remember that ℓ is the number of elements in L.)

Next we introduce the following total ordering $\preceq_\ell$ on X defined by

$$r \prec_\ell (r-1) \prec_\ell \cdots \prec_\ell (s+1) \prec_\ell 1 \prec_\ell 2 \prec_\ell \cdots \prec_\ell s.$$

Proposition 5.2. *If $S < L$, then*

$$L\text{-den-coding}(\alpha) = L\text{-exc-coding}(\alpha) \tag{5.2}$$

holds for every $\preceq_\ell$-well-factorized circuit α.

Proof. From (5.1) it suffices to prove the property when α is a $\preceq_\ell$-dominated cycle $\begin{pmatrix} y_1 y_2 \ldots y_m \\ x_1 x_2 \ldots x_m \end{pmatrix} = \begin{pmatrix} x_2 x_3 \ldots x_1 \\ x_1 x_2 \ldots x_m \end{pmatrix}$. Identity (5.2) holds if and only if for every $i = 1, \ldots, m$ we have

$$\sum_{i+1\le j\le m+1} \chi(x_i \in]\!]x_j, y_j]\!]_L) = \sum_{i\le j\le m} \chi(L\text{-excedance at } j). \tag{5.3}$$

Let $1 \le i \le m$. As we have noticed earlier, the number of ℓ-excedances in

$$\alpha' = \begin{pmatrix} y_i\, y_{i+1} \ldots y_m \\ x_i\, x_{i+1} \ldots x_m \end{pmatrix} = \begin{pmatrix} x_{i+1}\, x_{i+2} \ldots & x_m & x_1 \\ x_i\ \ x_{i+1} \ldots & x_{m-1} & x_m \end{pmatrix}$$

is equal to the number of occurrences of $*$ in the juxtaposition product

$$W' := \rangle\!\rangle x_i, x_{i+1} \rangle\!\rangle_L \cdots \rangle\!\rangle x_{m-1}, x_m \rangle\!\rangle_L \rangle\!\rangle x_m, x_1 \rangle\!\rangle_L.$$

If x_i is small, then $x_1 = y_m$ is also small and $x_1 \geq x_i$. The product W' is of the form

$$(x_i + 1, \ldots, x_1, \ldots, *, \ldots, x_i, \ldots, x_1).$$

In that sequence the number of occurrences of x_i is equal to the number of occurrences of $*$. As x_i does not occur in $\rrbracket x_i, y_i \rrbracket_L$, identity (5.3) holds.

If x_i is large and x_1 is small, then $x_i > x_1$. The product W' is of the form

$$(x_i, \ldots, *, \ldots, x_1, \ldots, x_i, \ldots, *, \ldots, x_1).$$

If x_i and x_1 both large, then $x_i \geq x_1$, and W' is of the form

$$(x_i, \ldots, *, \ldots, x_1, \ldots, x_i, \ldots, *, \ldots, x_1 - 1).$$

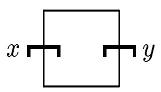
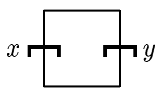
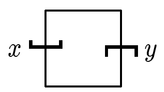
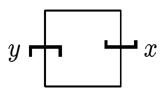
In the last two cases the number of occurrences of x_i is still equal to the number of occurrences of $*$. Moreover, x_i *occurs* in both $\rrbracket x_i, y_i \rrbracket_L$, $\rrbracket x_{m+1}, y_{m+1} \rrbracket_L$, so that

$$\sum_{i+1 \leq j \leq m+1} \chi(x_i \in \rrbracket x_j, y_j \rrbracket_L) = \sum_{i \leq j \leq m} \chi(x_i \in \rrbracket x_j, y_j \rrbracket_L),$$

and (5.3) also holds. $\square$

PROPOSITION 5.3. *If there exists a total ordering $\preceq$ such that for every $\preceq$-well factorized circuit condition* (5.2) *holds, then $S < L$ and the ordering $\preceq$ is identical with $\preceq_\ell$.*

The four columns in the table below refer to the four particular cases considered in the proof. For each circuit appearing in the fifth row the excedances have been indicated in bold-face. The last two rows show the values of the L-den-coding and the L-exc-coding of the circuits.

case 1	case 2	case 3	case 4
$x \in L$, $y \in S$, $x < y$, $x \prec y$	$x \in L$, $y \in S$, $x < y$, $x \succ y$	$x \in S$, $y \in S$, $x < y$, $y \prec x$	$x \in L$, $y \in L$, $x > y$, $y \prec x$
			
$\rrbracket x, y \rrbracket_L = \{x, y\}$ $\rrbracket y, x \rrbracket_L = \emptyset$	$\rrbracket x, y \rrbracket_L = \{x, y\}$ $\rrbracket y, x \rrbracket_L = \emptyset$	$\rrbracket x, y \rrbracket_L = \{y\}$ $\rrbracket y, x \rrbracket_L = \{x\}$	$\rrbracket x, y \rrbracket_L = \{x\}$ $\rrbracket y, x \rrbracket_L = \{y\}$
$\begin{pmatrix} x & y \\ \mathbf{y} & x \end{pmatrix}$ $\emptyset$ $\{x, y\}$ $\{x\}$	$\begin{pmatrix} y & x \\ x & \mathbf{y} \end{pmatrix}$ $\{x, y\}$ $\emptyset$ $\{x\}$	$\begin{pmatrix} y & x \\ x & \mathbf{y} \end{pmatrix}$ $\{y\}$ $\{x\}$ $\emptyset$	$\begin{pmatrix} y & x \\ \mathbf{x} & y \end{pmatrix}$ $\{x\}$ $\{y\}$ $\{x, y\}$
1 1	1 0	1 0	1 1
1 0	1 1	1 1	1 0

PROOF. If $S < L$ does not hold, there exists a pair (x, y) of letters such that $x < y$ with x large and y small. If $x \prec y$ (case 1), then for the $\preceq$-well-factorized circuit $\alpha = \binom{x\,y}{y\,x}$ we have L-den-coding$(\alpha) = (1,1) \neq (1,0) = L$-exc-coding$(\alpha)$, while if $y \prec x$ (case 2) by taking the $\preceq$-well-factorized circuit $\alpha = \binom{y\,x}{x\,y}$ we get L-den-coding$(\alpha) = (1,0) \neq (1,1) = L$-exc-coding$(\alpha)$.

Now suppose $S < L$ and $\preceq \neq \preceq_\ell$. There exists a pair (x, y) such that $y \prec x$ and $x \prec_\ell y$ and x, y are either both small, or both large. Consider the $\preceq$-well-factorized circuit $\binom{y\,x}{x\,y}$. If x, y are both small, then $x < y$ (case 3). Its L-den-coding is $(1,0)$, while its L-exc-coding is $(1,1)$. If x, y are both large, then $x > y$ (case 4). Its L-den-coding is $(1,1)$, while its L-exc-coding is $(1,0)$. See the explicit calculations of the L-den-coding and L-exc-coding in the preceding table. □

6. A q-calculation

When $S < L$ holds, we will adopt a new notation for the classes of the rearrangements of a given word. Let $\mathbf{c} = (c_1, \ldots, c_s)$ and $\mathbf{d} = (d_1, \ldots, d_\ell)$ be two vectors with non negative integer components. The class of all rearrangements of the word $1^{c_1} \ldots s^{c_s} (s+1)^{d_1} \ldots r^{d_l}$ will be denoted by $R(\mathbf{c}, \mathbf{d})$. Also $C(\mathbf{c}, \mathbf{d})$ (resp. $S(\mathbf{c}, \mathbf{d})$, resp. $F_{\preceq_\ell}(\mathbf{c}, \mathbf{d})$) will denote the set of all circuits $\alpha = \binom{v}{w}$ (resp. the subset of well-sorted circuits, resp. the subset of $\preceq_\ell$-well-factorized circuits) with v and w in $R(\mathbf{c}, \mathbf{d})$.

Let $A^{\mathrm{exc}_\ell, \mathrm{den}_\ell}_{F_{\preceq_\ell}(\mathbf{c},\mathbf{d})}(t, q)$ be the generating function for the set $F_{\preceq_\ell}(\mathbf{c}, \mathbf{d})$ by the pair $(\mathrm{exc}_\ell, \mathrm{den}_\ell)$, i.e.,

$$A^{\mathrm{exc}_\ell, \mathrm{den}_\ell}_{F_{\preceq_\ell}(\mathbf{c},\mathbf{d})}(t, q) = \sum_\alpha t^{\mathrm{exc}_\ell \alpha} q^{\mathrm{den}_\ell \alpha} \quad (\alpha \in F_{\preceq_\ell}(\mathbf{c}, \mathbf{d})).$$

THEOREM 6.1. *For each pair* $(\mathbf{c}, \mathbf{d})$ *we have*

$$A^{\mathrm{exc}_\ell, \mathrm{den}_\ell}_{F_{\preceq_\ell}(\mathbf{c},\mathbf{d})}(t, q) = A_{\mathbf{c},\mathbf{d}}(t, q),$$

where $A_{\mathbf{c},\mathbf{d}}(t, q)$ *is the generalized* q*-Eulerian polynomial defined in* (4.1).

PROOF. We will make use of Criterion 4.1 with $F(\mathbf{c}, \mathbf{d})$ replaced by $F_{\preceq_\ell}(\mathbf{c}, \mathbf{d})$, f by exc_ℓ and g by den_ℓ. On the other hand, keep the same notations as in (4.8) and (4.9) and take $\alpha = \begin{pmatrix} y_1 \ldots y_m \\ x_1 \ldots x_m \end{pmatrix}$ in $F_{\preceq_\ell}(\mathbf{c}, \mathbf{d})$. Let $\mathbf{p} = (p_1, \ldots, p_m)$ be its ℓ-den-coding which is also its ℓ-exc-coding. In particular, $p_1 = \mathrm{exc}_\ell\, \alpha$ and the sequence $\mathbf{p}$ is nonincreasing, as well as $\mathbf{b} = (b_1, \ldots, b_m) = \mathbf{a} + \mathbf{p}$.

The construction of the bijection $(i', \mathbf{a}, \alpha) \mapsto (i, \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(r)})$ can be described as follows. Consider the biword $\begin{pmatrix} x_1\, x_2 \ldots x_m \\ b_1\, b_2 \ldots b_m \end{pmatrix}$ to which we apply $\mathbf{SORT}_{Q_{CF}}$ (see Example 1). Let $(z_1, z_2, \ldots, z_m)$ be the bottom word of $\mathbf{SORT}_{Q_{CF}} \begin{pmatrix} x_1\, x_2 \ldots x_m \\ b_1\, b_2 \ldots b_m \end{pmatrix}$. Then define $i := i' + p_1$, $\mathbf{a}^{(1)} := (z_1, \ldots, z_{c_1})$, $\mathbf{a}^{(2)} := (z_{c_1+1}, \ldots, z_{c_1+c_2})$, $\ldots$, $\mathbf{a}^{(r)} := (z_{m-d_\ell+1}, \ldots, z_m)$. Then the two relations in (4.10) rewritten as

$$i = i' + \mathrm{exc}_\ell\, \alpha, \quad \|\mathbf{a}\| + \mathrm{den}_\ell\, \alpha = \|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\|$$

hold, as $p_1 = \mathrm{exc}_\ell\, \alpha$ and $\|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\| = \|\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{p}\| = \|\mathbf{a}\| + \mathrm{den}_\ell\, \alpha$.

For instance, consider $S=\{1,2,3\}$, $L=\{4,5\}$ and take the $\preceq_\ell$-well-factorized circuit $\alpha := \binom{v}{w} = \left(\begin{array}{c|cc|c|ccc|cc} 4 & 5 & 4 & 1 & 1 & 2 & 3 & 5 & 3\\ 4 & 4 & 5 & 1 & 3 & 1 & 2 & 3 & 5\end{array}\right)$. First we get the ℓ-den-coding $\mathbf{p} = (4,3,3,2,2,1,1,1,1)$. Now with $\mathbf{a} := (2,2,1,1,1,1,0,0,0)$ and $i' := 3 > a_1 = 2$ we obtain $\mathbf{b} = \mathbf{a} + \mathbf{p} = (6,5,4,3,3,2,1,1,1)$. Then

$$\mathbf{SORT}_{Q_{CF}}\begin{pmatrix}4\,4\,5\,1\,3\,1\,2\,3\,5\\ 6\,5\,4\,3\,3\,2\,1\,1\,1\end{pmatrix} = \begin{pmatrix}1\,1\,2\,3\,3\,4\,4\,5\,5\\ 3\,2\,1\,3\,1\,6\,5\,4\,1\end{pmatrix}$$

and $(z_1,\dots,z_m) = (3,2,1,3,1,6,5,4,1)$. We get $i = i' + \operatorname{exc}_\ell \alpha = 3+4 = 7$ and $\mathbf{a}^{(1)} = (3,2)$, $\mathbf{a}^{(2)} = (1)$, $\mathbf{a}^{(3)} = (3,1)$, $\mathbf{a}^{(4)} = (6,5)$, $\mathbf{a}^{(5)} = (4,1)$.

By construction all the sequences $\mathbf{a}^{(j)}$ are nonincreasing. Moreover, if x_k, $x_{k'}$ $(k<k')$ are two occurrences of the same large letter $s+j$ in the word w, there is necessarily an ℓ-excedance in the factor $\begin{pmatrix}y_k \dots y_{k'}\\ x_k \dots x_{k'}\end{pmatrix}$ of the circuit α, so that all the sequences $\mathbf{a}^{(s+1)}, \dots, \mathbf{a}^{(r)}$ are *decreasing*. Finally, if x_k is the rightmost occurrence of a given large letter $s+j$ in the word w, then the right factor $\begin{pmatrix}y_k \dots y_m\\ x_k \dots x_m\end{pmatrix}$ of α necessarily contains an ℓ-excedance, so that all the terms in the sequences $\mathbf{a}^{(s+1)}$, $\dots$, $\mathbf{a}^{(r)}$ are greater than or equal to 1. ∎

7. The transformations

Let $w = x_1x_2\dots x_m$ be a word in $R(\mathbf{c},\mathbf{d})$. Supposing $S<L$ form the $\preceq_\ell$-well-factorized circuit $\alpha = \Delta(w)$ (see section 2). Next apply the transformation $\mathbf{SORT}_{(\preceq,Q_L)}$ to α (see section 3), where Q_L is the Boolean function derived when $S<L$. From Theorem 3.2 we know that

$$(\operatorname{exc}_L, \operatorname{den}_L)\,\alpha = (\operatorname{exc}_L, \operatorname{den}_L)\ \mathbf{SORT}_{(\preceq,Q_L)}(\alpha). \tag{7.1}$$

The circuit $\alpha' = \mathbf{SORT}_{(\preceq,Q_L)}(\alpha)$ is well-sorted. Using the transformation Γ (see section 2) we know that to α there corresponds a unique word w' in $R(\mathbf{c},\mathbf{d})$ such that $\Gamma(w') = \alpha'$. We then have the chain

$$w \mapsto \Delta(w) = \alpha \mapsto \mathbf{SORT}_{(\preceq,Q_L)}(\alpha) = \alpha' \mapsto \Gamma^{-1}(\alpha') = w'. \tag{7.2}$$

Let $w = x_1x_2\dots x_m$ be a word in $R(\mathbf{c},\mathbf{d})$. The statistic $\operatorname{des}_\ell w$ (resp. $\operatorname{maj}_\ell w$) is defined (see, e.g. [**7**]) as the number (resp. the sum) of the integers i such that $1\le i\le m$ and either $x_i > x_{i+1}$ or $x_i = x_{i+1}$ and x_i large (by convention $x_{m+1} = s$). It is easy to verify that

$$(\operatorname{des}_\ell, \operatorname{maj}_\ell)\,w = (\operatorname{exc}_\ell, \operatorname{den}_\ell)\,\Delta(w). \tag{7.3}$$

On the other hand, the definitions of the number of ℓ-excedances and the ℓ-Denert statistic for the words w' can be taken as

$$(\operatorname{exc}_\ell, \operatorname{den}_\ell)\,w' := (\operatorname{exc}_\ell, \operatorname{den}_\ell)\,\Gamma(w'). \tag{5.4}$$

It then follows that the chain displayed in (7.2) provides a bijection $w \mapsto w'$ of $R(\mathbf{c},\mathbf{d})$ onto itself having the property that

$$(\operatorname{des}_\ell, \operatorname{maj}_\ell)\,w = (\operatorname{exc}_\ell, \operatorname{den}_\ell)\,w'. \tag{5.5}$$

Acknowledgements. The authors should like to thank one of the referees for his careful reading of the paper that improves the tenor of the final version.

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