RANDOM PERMUTATIONS AND BERNOULLI SEQUENCES

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Abstract. — The so-called first fundamental transformation provides a natural combinatorial link between statistics involving cycle lengths of random permutations and statistics dealing with runs on Bernoulli sequences.

1. Introduction

Let (X_n) $(n \geq 1)$ be a sequence of independent Bernoulli random variables, the distribution of X_n being $q_n\delta_0 + p_n\delta_1$ with $0 < p_n < 1$ and $q_n = 1 - p_n$. For each $n \geq 2$ let W_n denote the random variable $W_n := X_n + \sum_{1\leq k\leq n-1} X_k X_{k+1}$. In other words, W_n, when applied to a sequence of 0's and 1's, is the number of 1's within the first $(n-1)$ terms, which are themselves immediately followed by 1's, plus one whenever the n-th term is also equal to 1.

Next, let $\mathcal{S}_n$ be the group of the permutations of the interval $[1, n] = \{1, 2, \ldots, n\}$. If σ is such a permutation, let $Z_n(\sigma)$ denote the number of *fixed points* of σ, i.e., the number of integers i such that $1 \leq i \leq n$ and $\sigma(i) = i$. When the equidistribution is taken over $\mathcal{S}_n$, the distribution of the random variable Z_n is common knowledge. For example, the generating function for Z_n is known to be (see [FF] p. 262) $G_{Z_n}(s) = \mathbf{E}[s^{Z_n}] = \sum_{k=0}^{n} (s-1)^k/k!$, so that, when n tends to infinity, Z_n tends to a Poisson random variable with mean equal to 1 in distribution. When $p_k = 1/k$ for $k = 1, 2, \ldots$, the same limiting result holds for the sequence (W_n) $(n \geq 1)$, a result proved by several authors, Diaconis [D], Émery [E], Joffe, Marchand, Perron, Popadiuk [JMPP]. The latter authors also derive the distribution of W_n and notice that it is identical with the distribution of Z_n. They obtain a further extension of this result when the fixed points of the permutations are only recorded until a fixed bound.

The variables W_n and Z_n are defined on two different probability spaces. A priori, there is no reason why their distributions should be equal, so Joffe [J] suggested that some intrinsic link be found to explain that equality. It is the purpose of this note to provide that link by using classical combinatorial techniques.

Assume that n is a fixed integer, also that $p_k = 1/k$ for $k = 1, 2, \ldots, n$ and $p_{n+1} = 1$, thus $X_{n+1} = 1$. With this convention the variable W_n can be renamed as

$$W_{[11]} := \sum_{1 \le k \le n} X_k X_{k+1}.$$

In the same manner, let $Z_{[1]}(\sigma)$ denote the number of fixed points of the permutation σ and let P be the equirepartition on the permutation group $\mathcal{S}_n$. Finally, Q will designate the probability measure of the vector $(X_1, X_2, \ldots, X_n)$, that acts on the set $\{0,1\}^n$.

Finding an intrinsic link boils down to defining a random variable Y such that the following diagramme is commutative.

$$\begin{array}{ccc} (\mathcal{S}_n, \mathrm{P}) & \xrightarrow{\ Y\ } & (\{0,1\}^n, \mathrm{Q}) \\ & {\scriptstyle Z_{[1]}} \searrow & \downarrow {\scriptstyle W_{[11]}} \\ & & [0, n] \end{array}$$

This means that $Z_{[1]} = W_{[11]} \circ Y$ and Q is the image of the measure P under Y, i.e., $\mathrm{Q} = \mathrm{P} \circ Y^{-1}$. The variable Y is defined by means of a classical bijection on the permutation group, referred to as the *first fundamental transformation* (see [L], p. 186–188).

The construction of the first fundamental transformation is recalled in the next section. Then, the variable Y is defined in section 3 and shown to have the properties described in the previous diagramme. In section 4 another variable Y_B is defined that takes the extension proposed by Joffe and his coauthors [JMPP] into account. We conclude the note by calculating various generating functions relevant to the problem under study.

2. The first fundamental transformation

Let σ be a permutation of order n having r orbits $I_1, I_2, \ldots, I_r$ [accordingly, r cycles and we write $\operatorname{cyc} \sigma = r$]. The orbits are numbered in such a way that $\max I_1 < \max I_2 < \cdots < \max I_r$. For each $j = 1, 2, \ldots, r$

let $|I_j|$ be the the cardinality of I_j and let $\check{\sigma}_j$ be the *initially dominated* word

$$\check{\sigma}_j := \sigma^{|I_j|}(\max I_j),\ \sigma^{|I_j|-1}(\max I_j), \ldots, \sigma^2(\max I_j), \sigma(\max I_j).$$

Notice that $\sigma^{|I_j|}(\max I_j) = \max I_j$ and the word $\check{\sigma}_j$ contains all the elements of I_j once and only once. To the permutation σ we now associate the juxtaposition product: $\check{\sigma} := \check{\sigma}_1 \check{\sigma}_2 \ldots \check{\sigma}_r$.

Example. — Consider the permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 7 & 4 & 3 & 9 & 2 & 6 & 1 \end{pmatrix}.$$

The orbits of σ written in increasing order according to their maxima (in boldface) are:

$$I_1 = \{\mathbf{4}\}, \quad I_2 = \{2, 3, 5, \mathbf{7}\}, \quad I_3 = \{1, 6, 8, \mathbf{9}\}.$$

Hence, $\check{\sigma}_1 = \sigma(\mathbf{4}) = \mathbf{4}$; $\check{\sigma}_2 = \sigma^4(\mathbf{7}), \sigma^3(\mathbf{7}), \sigma^2(\mathbf{7}), \sigma(\mathbf{7}) = \mathbf{7}, 3, 5, 2$; $\check{\sigma}_3 = \sigma^4(\mathbf{9}), \sigma^3(\mathbf{9}), \sigma^2(\mathbf{9}), \sigma(\mathbf{9}) = \mathbf{9}, 6, 8, 1$ and

$$\check{\sigma} = \mathbf{4}, \mathbf{7}, 3, 5, 2, \mathbf{9}, 6, 8, 1.$$

Actually, the map $\sigma \mapsto \check{\sigma}$ is a bijection of $\mathcal{S}_n$ onto the set R_n of all the rearrangements of the word $1, 2, \ldots, n$ and referred to as the *first fundamental transformation*. The inverse bijection is defined as follows: start with a rearrangement τ of the word $1, 2, \ldots, n$; cut the word τ before each *upper record*, that is, before each letter greater than all the letters located *on its left*. The factors of τ obtained in that way are *initially dominated*, in the sense that they all begin with their greatest letters. Moreover, the subword of τ consisting of the first letters of those factors is *increasing*. The sequence of those factors is called the *increasing factorization* of τ in *initially dominated words*. The factorization is unique.

Once the factors of the factorisation are given, the cycles of the permutation σ such that $\check{\sigma} = \tau$ can be reconstructed. In the foregoing example the upper records of $\check{\sigma}$ are **4**, **7**, **9**, so that the cycles of σ can be reconstructed as: $\mathbf{4} \leftarrow \mathbf{4}$; $\mathbf{7} \leftarrow 3 \leftarrow 5 \leftarrow 2 \leftarrow \mathbf{7}$; $\mathbf{9} \leftarrow 6 \leftarrow 8 \leftarrow 1 \leftarrow \mathbf{9}$.

Now juxtapose the symbol ∞ to the right of $\check{\sigma} = \check{\sigma}(1)\check{\sigma}(2)\ldots\check{\sigma}(n)$ and let $\check{\sigma}(n+1) := \infty$. The word $\check{\sigma}\infty = \check{\sigma}(1)\check{\sigma}(2)\ldots\check{\sigma}(n)\check{\sigma}(n+1)$ gets one more upper record. A careful study of the previous construction yields the following property.

Property 1. *The element k $(1 \le k \le n)$ is an orbit maximum of σ if and only if k is an upper record in $\check\sigma$. For each $k = 1, 2 \ldots, n$ there is a one-to-one correspondence between the cycles of σ, of length k, and the factors $\check\sigma(i)\check\sigma(i+1)\ldots\check\sigma(i+k)$, of length $(k+1)$, of the word $\check\sigma\infty$, such that $\check\sigma(i)$ and $\check\sigma(i+k)$ are the only upper records of the factor. Finally, the number, $\operatorname{cyc}\sigma$, of cycles of σ is equal to the number of upper records of the word $\check\sigma$.*

The above property is an immediate consequence of the construction of the first fundamental transformation. Using the previous example we see that **4, 7, 9** are the orbit maxima of σ and the upper records of $\check\sigma$. Moreover, the fixed point **4** corresponds to the factor **4,7**, where **4** and **7** are upper records. The cycles of length 4 correspond to the $\mathbf{7}, 3, 5, 2, \mathbf{9}$ and $\mathbf{9}, 6, 8, 1, \infty$, both of length 5.

3. Construction of the random variable Y

Consider a word $\tau\infty$ belonging to $R_n\infty$, i.e., a rearrangement of the word $1, 2, \ldots, n$ to the right of which the letter ∞ has been juxtaposed. Replace each upper record of τ by the letter 1 and the other letters by 0. This yields a word belonging to $1\{0,1\}^{n-1}1$, that is to say, a word, of length $(n+1)$, whose letters belong to the alphabet $\{0,1\}$, beginning and ending with 1. Let $\psi(\tau\infty)$ denote the binary word derived in this manner.

As, by convention, $X_{n+1} \equiv 1$, it makes sense to introduce the random variables

$$W_{[10^i1]} := \sum_{1\le k\le n-i} X_k(1-X_{k+1})(1-X_{k+2})\cdots(1-X_{k+i})X_{k+i+1},$$

for $i = 0, 1, \ldots, n$, all defined on $\{0,1\}^n$. The variable $W_{[10^i1]}$ enumerates the factor of the form 10^i1 in a sequence belonging to $1\{0,1\}^{n-1}1$. For $i = 0$ we get back the variable $W_{[11]}$ that enumerates the factors of the form 11. In a similar way, for $k = 1, 2, \ldots, n$ and for each permutation σ of order n let $Z_{[k]}(\sigma)$ designate the number of cycles of σ of length k.

Proposition 2. *The map $\sigma \mapsto \psi(\check\sigma\infty)$ is a surjection of the permutation group $\mathcal{S}_n$ onto the set $1\{0,1\}^{n-1}1$ having the property that*

$$Z_{[k]}(\sigma) = W_{[10^{k-1}1]}(\psi(\check\sigma\infty)).$$

holds for each $k = 1, 2, \ldots, n$.

From all that has been said above the proposition is evident. What now matters, for every word $v = x_1x_2\ldots x_nx_{n+1} \in 1\{0,1\}^{n-1}1$, is to determine the number of permutations σ such that $\psi(\check\sigma\infty) = v$ or, in an equivalent manner, the number of rearrangements $\tau = y_1 y_2 \ldots y_n$ of the word $12\ldots n$ such that $\psi(\tau\infty) = v$.

As this number depends on n and the sequence $1 = i_r < i_{r-1} < \cdots < i_2 < i_1 < i_0 = n+1$ of the integers i such that $x_i = 1$, denote this number by $N(n; i_r, \ldots, i_1, i_0)$. The enumeration is trivial for $n = 1$. Assume $n \ge 2$.

If $i_1 = 1$ (the word v is simply $10\ldots01$), all the suitable words τ are the rearrangements of $1, 2, \ldots, n$ beginning with n. There are $(n-1)!$ such rearrangements, so that $N(n; 1, n+1) = (n-1)!$

If $2 \le i_1 \le n$, then $x_{i_1} = 1$. In all the suitable rearrangements $\tau = y_1 y_2 \ldots y_n$ we have $y_{i_1} = n$ and the right factor $y_{i_1+1} y_{i_1+2} \ldots y_n$, of length $(n - i_1)$, is made of distinct letters taken from $\{1, 2, \ldots, n-1\}$. There are $(n-1)(n-2)\cdots(n-1-(n-i_1)+1) = (n-1)(n-2)\cdots i_1$ possible factors, so that the following formula holds:

$$N(n; i_r, \ldots, i_2, i_1, n+1) = N(i_1 - 1; i_r, \ldots, i_2, i_1) \times i_1 \cdots (n-2)(n-1);$$

and by induction on r

$$N(n; i_r, \ldots, i_2, i_1, n+1) \\ = N(i_2 - 1; i_r, \ldots, i_3, i_2) \times i_2 \cdots (i_1 - 2) \times i_1 \cdots (n-2)(n-1);$$

and also

$$N(n; i_r, \ldots, i_2, i_1, n+1) = \frac{n!}{(i_{r-1} - 1)\cdots(i_2 - 1)(i_1 - 1)(i_0 - 1)},$$

a formula that holds in all the cases.

We can then write

$$\begin{aligned} &\frac{N(n; i_r, \ldots, i_2, i_1, n+1)}{n!} \\ &\quad = \frac{1}{1}\frac{1}{2}\cdots\frac{i_{r-1}-2}{i_{r-1}-1}\frac{1}{i_{r-1}}\frac{i_{r-1}}{i_{r-1}+1}\cdots\frac{i_2-2}{i_2-1}\frac{1}{i_2}\frac{i_2}{i_2+1}\cdots \\ &\qquad\qquad \times\cdots\frac{i_1-2}{i_1-1}\frac{1}{i_1}\frac{i_1}{i_1+1}\cdots\frac{n-1}{n}. \\ &\quad = \mathrm{Q}\{X_1 = 1\}\,\mathrm{Q}\{X_2 = 0\}\cdots \\ &\qquad \times \mathrm{Q}\{X_{i_{r-1}-1} = 0\}\,\mathrm{Q}\{X_{i_{r-1}} = 1\}\,\mathrm{Q}\{X_{i_{r-1}+1} = 0\}\cdots \\ &\qquad \times \mathrm{Q}\{X_{i_1-1} = 0\}\,\mathrm{Q}\{X_{i_1} = 1\}\,\mathrm{Q}\{X_{i_1+1} = 0\}\cdots\mathrm{Q}\{X_n = 0\}. \end{aligned}$$

Now define

$$Y(\sigma) := \psi(\check{\sigma}\infty).$$

The previous calculation shows that

$$\mathrm{P}\{Y = v\} = \frac{N(n; i_r, \ldots, i_2, i_1, n+1)}{n!} = \mathrm{Q}\{v\},$$

so that, when P is the equirepartition on $\mathcal{S}_n$, the distribution of Y is the probability measure Q. Moreover Proposition 2 implies the following result.

Proposition 3. *The two vectors* $(Z_{[1]}, Z_{[2]}, \dots, Z_{[n]})$ *on* $(\mathcal{S}_n, \mathrm{P})$ *and* $(W_{[11]}, W_{[101]}, \dots, W_{[10^{n-1}1]})$ *on* $(\{0,1\}^n, \mathrm{Q})$ *are equally distributed.*

4. Construction of the variable Y_B

Joffe and his coauthors [JMPP] propose the following extension that we will further extend: given an integer $B+n$ such that $B \ge 0$ and $n \ge 1$ consider the permutation group $\mathcal{S}_{B+n}$ of the permutations of order $(B+n)$. Given such a permutation σ, the following operations are successively made: if $\sigma(B+n) = B+n$, put this cycle of length 1 aside. Otherwise, set $\sigma^2(B+n)$, $\sigma^3(B+n)$, $\dots$, until, for the first time, $(B+n)$ occurs. The cycle containing $(B+n)$ is then obtained. Start again with the greatest integer, less than $(B+n)$, not occurring in the cycle already constructed. Continue this procedure until n integers have been extracted. The purpose is to study the distribution of the random vector $(Z_{[1]}, Z_{[2]}, \dots, Z_{[n]})$ on the set of cycles that have been extracted, the last one being possibly incomplete.

To avoid any kind of ambiguity let us re-describe the construction by saying that we make up a sequence of length n in the following way: if $\sigma(B+n) = B+n$, take $\sigma(B+n)$ as the rightmost letter of that sequence. Otherwise, form $\sigma(B+n)$, $\sigma^2(B+n)$, then $\sigma^3(B+n)$, $\dots$, until, for the first time, $(B+n)$ occurs. Then, $(B+n), \dots, \sigma^2(B+n), \sigma(B+n)$ becomes the rightmost factor of that sequence. Start again with the greatest integer, say m, that does not occur in the cycle and form the factor $m, \dots, \sigma^2(m), \sigma(m)$, that is inserted to the left of the previous factor. The procedure is stopped when, for the first time, the length of the sequence is equal to n.

This procedure is nothing but the construction of the word

$$\check\sigma = \check\sigma(1)\check\sigma(2)\dots\check\sigma(B+n)$$

defined in section 2, constructed from right to left, that is stopped as soon as n letters have been obtained, that is, as soon as the right factor $\check\sigma(B+1)\check\sigma(B+2)\dots\check\sigma(B+n)$, of length n, has been obtained. Studying the distribution of the random vector $(Z_{[1]}, Z_{[2]}, \dots, Z_{[n]})$ on the cycles which are actually extracted is equivalent to studying the distribution of the random vector $(\check Z_{[1]}, \check Z_{[2]}, \dots, \check Z_{[n]})$, where for each $k = 1, 2, \dots, n$ the k-th term $\check Z_{[k]}(\sigma)$ designates the number of factors $\check\sigma(i)\check\sigma(i+1)\dots\check\sigma(i+k)$ of the word $\check\sigma\infty$, of length $(k+1)$, such that $i \ge B+1$ and such that the letters $\check\sigma(i)$ and $\check\sigma(i+k)$ are the only upper records of the factor.

As before, replace the upper records of $\check\sigma\infty$ by 1's and the other letters by 0's. This yields a word belonging to $1\{0,1\}^{B+n-1}1$. Let $\psi_B(\check\sigma\infty)$ denote its right factor of length $(n+1)$, which is then a word belonging to $\{0,1\}^n 1$.

Again, consider the sequence (X_n) $(n \geq 1)$ of independent Bernoulli random variables, such that for each $n = 1, 2, \ldots$ the distribution of X_n is $(1 - 1/n)\delta_0 + (1/n)\delta_1$. Denote the probability measure of the random vector $(X_{B+1}, X_{B+2}, \ldots, X_{B+n})$ by $\mathrm{Q}^{(B)}$ and assume that the variables $W_{[10^i 1]}$ are this time defined with a shift of B units, that is,

$$W_{[10^i 1]} := \sum_{B+1 \leq k \leq B+n-i} X_k(1 - X_{k+1})(1 - X_{k+2}) \cdots (1 - X_{k+i}) X_{k+i+1},$$

where, by convention, $X_{B+n+1} \equiv 1$. Then, the analog of Proposition 2 rewrites as follows.

Proposition 4. *The map $\sigma \mapsto \psi_B(\check{\sigma}\infty)$ is a surjection of the group $\mathcal{S}_{B+n}$ onto the set $\{0,1\}^n 1$, such that for each $k = 1, 2, \ldots, n$ the following identity holds:*

$$\check{Z}_{[k]}(\sigma) = W_{[10^{k-1}1]}(\psi_B(\check{\sigma}\infty)).$$

For each word $v = x_1 x_2 \ldots x_n x_{n+1} \in \{0,1\}^n 1$ let us evaluate the number of permutations σ such that $\psi_B(\check{\sigma}\infty) = v$. This number depends on $B + n$ and on the sequence $1 \leq i_r < i_{r-1} < \cdots < i_2 < i_1 < i_0 = n + 1$ of the integers i such that $x_i = 1$. This number will be denoted by $N(B+n; i_r, \ldots, i_1, i_0)$. Notice that this time $1 \leq i_r$ holds instead of $1 = i_r$ in the case $B = 0$. As

$$N(B + i_r; i_r) = B!\, B(B+1) \cdots (B + i_r - 2),$$

we get the evaluation

$$\begin{aligned} &N(B + n; i_r, \ldots, i_2, i_1, n+1) \\ &\qquad = B!\, B \cdots (B + i_r - 2)(B + i_r) \cdots (B + i_{r-1} - 2)(B + i_{r-1}) \cdots \\ &\qquad\qquad \times \cdots (B + i_1 - 2)(B + i_1) \cdots (B + n - 2)(B + n - 1). \end{aligned}$$

We can then write

$$\begin{aligned} &\frac{N(B + n; i_r, \ldots, i_2, i_1, n+1)}{(B+n)!} \\ &\quad = \frac{B}{B+1}\frac{B+1}{B+2} \cdots \frac{B+i_r-2}{B+i_r-1}\frac{1}{B+i_r}\frac{B+i_r}{B+i_r+1} \cdots \\ &\qquad \cdots \times \frac{B+i_{r-1}-2}{B+i_{r-1}-1}\frac{1}{B+i_{r-1}}\frac{B+i_{r-1}}{B+i_{r-1}+1} \cdots \\ &\qquad \cdots \times \frac{B+i_1-2}{B+i_1-1}\frac{1}{B+i_1}\frac{B+i_1}{B+i_1+1} \cdots \\ &\qquad \cdots \times \frac{B+n-3}{B+n-2}\frac{B+n-2}{B+n-1}\frac{B+n-1}{B+n}. \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Q}\{X_{B+1}=0\}\,\mathrm{Q}\{X_{B+2}=0\}\cdots\\
&\qquad \cdots\times \mathrm{Q}\{X_{B+i_r-1}=0\}\,\mathrm{Q}\{X_{B+i_r}=1\}\,\mathrm{Q}\{X_{B+i_r+1}=0\}\cdots\\
&\qquad \cdots\times \mathrm{Q}\{X_{B+i_{r-1}-1}=0\}\,\mathrm{Q}\{X_{i_{r-1}}=1\}\,\mathrm{Q}\{X_{i_{r-1}+1}=0\}\cdots\\
&\qquad \cdots\times \mathrm{Q}\{X_{B+i_1-1}=0\}\,\mathrm{Q}\{X_{B+i_1}=1\}\,\mathrm{Q}\{X_{B+i_1+1}=0\}\cdots\\
&\qquad \cdots\times \mathrm{Q}\{X_{B+n-2}=0\}\,\mathrm{Q}\{X_{B+n-1}=0\}\,\mathrm{Q}\{X_{B+n}=0\}.
\end{aligned}
$$

Then, define

$$Y_B(\sigma) := \psi_B(\check\sigma\infty).$$

The previous calculation shows that, when the equirepartition P is taken over the group $\mathcal{S}_n$, the distribution of the variable Y_B is, indeed, the probability measure $\mathrm{Q}^{(B)}$. Moreover, Proposition 4 implies the following result.

Proposition 5. *The two vectors* $(\check Z_{[1]}, \check Z_{[2]}, \ldots, \check Z_{[n]})$ *on* $(\mathcal{S}_{B+n}, \mathrm{P})$ *and* $(W_{[11]}, W_{[101]}, \ldots, W_{[10^{n-1}1]})$ *on* $(\{0,1\}^n, \mathrm{Q}^{(B)})$ *are equally distributed.*

5. The generating functions

The distribution of the vector $(Z_{[1]}, Z_{[2]}, \ldots, Z_{[n]})$ (in the case $B=0$) is a classic (see, e.g., [R] p. 68). It can be expressed in three different ways. First, the generating function for the vector is simply

$$G_n = \sum \frac{1}{k_1!\,k_2!\,\cdots k_n!}\Bigl(\frac{t_1}{1}\Bigr)^{k_1}\Bigl(\frac{t_2}{2}\Bigr)^{k_2}\cdots\Bigl(\frac{t_n}{n}\Bigr)^{k_n}, \tag{5.1}$$

where the sum is over all sequences $(k_1, k_2, \ldots, k_n)$ on nonnegative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$. This formula (valid for every $n \ge 0$) is still *equivalent* to the exponential identity

$$\sum_{n\ge 0} G_n\, u^n = \exp\bigl(ut_1 + u^2\frac{t_2}{2} + u^3\frac{t_3}{3} + \cdots\bigr), \tag{5.2}$$

or still equivalent to the induction formula (that can be obtained by taking the derivative of the previous identity)

$$G_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} t_{k+1}\, G_{n-k}. \tag{5.3}$$

By Proposition 3 the above G_n is also the generating function for the vector $(W_{[11]}, W_{[101]}, \ldots, W_{[10^{n-1}1]})$. Notice that the generating function

$\sum_{k=0}^{n}(s-1)^k/k!$ for the variable $Z_{[11]}$, mentioned in the introduction, is a banal consequence of (5.3).

When B is positive, only the induction formula (5.3) can be extended. Again, let G_n denote the generating function for either the vector $(\check Z_{[1]},\check Z_{[2]},\ldots,\check Z_{[n]})$, or the vector $(W_{[11]},W_{[101]},\ldots,W_{[10^{n-1}]})$. We make the calculation for the latter one. By induction, the generating polynomial for the binary sequences of 0's and 1's ending with 1, of length $(n+1)$, ending with 1 is $G_n\dfrac{t_1}{B+n+1}$; for those ending with 10 $G_{n-1}\dfrac{t_2}{B+n}\dfrac{B+n}{B+n+1}$; … ; for those of the form 10^n et 0^{n+1}, respectively

$$G_0\frac{t_{n+1}}{B+1}\frac{B+1}{B+2}\cdots\frac{B+n}{B+n+1}\quad\text{and}\quad\frac{B}{B+1}\frac{B+1}{B+2}\cdots\frac{B+n}{B+n+1}.$$

This yields the analog of (5.3), that is,

$$G_{n+1}=\frac{1}{B+n+1}\Bigl(\sum_{k=0}^{n}t_{k+1}\,G_{n-k}+B\Bigr). \tag{5.4}$$

Now, when we plug $t_1:=s$, $t_k:=1$ for $k\ge 2$ in the previous formula, we immediately obtain the generating polynomial for $\check Z_{[1]}$ or for $W_{[11]}$ in the form

$$\mathbf{E}^{\mathrm{P}}[s^{\check Z_{[1]}}]=\mathbf{E}^{\mathrm{Q}^{(B)}}[s^{W_{[11]}}]=\sum_{k=0}^{n}\frac{(s-1)^k}{(B+1)_k}, \tag{5.5}$$

a formula that was derived by Joffe et al. [JMPP] by means of a probabilistic argument.

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