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CENTRAL LIMIT THEOREM FOR THE GEODESIC FLOW ASSOCIATED WITH A KLEINIAN GROUP, CASE $\delta > d/2$

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ABSTRACT. – Let Γ be a geometrically finite Kleinian group, relative to the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, and let δ denote the Hausdorff dimension of its limit set, that we suppose here strictly larger than $d/2$. We prove a central limit theorem for the geodesic flow on the manifold $\mathcal{M} := \Gamma \backslash \mathbb{H}$, with respect to the Patterson–Sullivan measure. The argument uses the ground-state diffusion and its canonical lift to the frame bundle, for which the existence of a potential operator is proved. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Consider the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, endowed with some geometrically finite Kleinian group Γ. The Hausdorff dimension $\delta \in [0, d]$ of its limit set (see [13,17] or [18]) plays a fundamental role. When δ is larger than $d/2$, $\delta(\delta - d)$ is the highest eigenvalue of the Laplacian on a fundamental domain. The associated eigenstate Φ plays an important role in the study of the quotient $\mathcal{M} = \Gamma \setminus \mathbb{H}$ and of its geodesic flow. The corresponding ground-state diffusion Z_t^{Φ}, which we call "Φ-diffusion", is then also a natural object and tool in this framework: see [17, 2–4].

As in [3], where the problem of the asymptotic law of windings in cusps of hyperbolic surfaces has been treated, we approximate the Patterson–Sullivan measure m by the images under the geodesic flow θ_t of a quasi-invariant measure ν, which is stationary for the lift of the ground-state diffusion Z_t^{Φ} to $T^1\mathcal{M}$. An important step, which was not needed in [3], but which extends the proof given in [9] for the finite volume case, is the existence of a potential operator V for the lift of the Φ-diffusion.

Denote by $\mathcal{L}_0$ the Lie derivative along the geodesic flow, and by $\mathcal{L}_1, \ldots, \mathcal{L}_d$ the Lie derivatives along the stable horocycle flows (which are no longer defined on the tangent

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bundle $T^1\mathcal{M}$, but only on the frame bundle $O\mathcal{M}$). Let $f_1,\dots,f_d$ denote the conjugate functions of any given function f having bounded derivatives on $O\mathcal{M}$; they are defined by: $f_j := -\int_0^\infty \mathrm{e}^{-s}\mathcal{L}_j f(\cdot\,\theta_s)\,\mathrm{d}s$. Set $f_0 := f$ for convenience. Let us also introduce the canonical projection π_2 from $O\mathcal{M}$ onto $\mathcal{M}$, and the divergence operator K:

$$Kf := \frac{1}{2}\sum_{j=0}^{d}\mathcal{L}_j f_j + \sum_{j=0}^{d}(\mathcal{L}_j \log\Phi\circ\pi_2) f_j - \frac{d}{2}f.$$

Our main result, assuming that $\delta > d/2$, is the following central limit theorem for the geodesic flow on $T^1\mathcal{M}$, relating to the Patterson–Sullivan measure m:

THEOREM. – *Let us fix a real function f on $T^1\mathcal{M}$, such that $\int f\,\mathrm{d}m = 0$, and of class C^2 with bounded and Hölderian derivatives. Then for all $a\in\mathbb{R}$ we have*:

$$\lim_{t\to\infty}\int_{T^1\mathcal{M}} \exp\left(\frac{a\sqrt{-1}}{\sqrt{t}}\int_0^t f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi) = m(T^1\mathcal{M})\times\exp\left(-\frac{a^2}{2}\mathcal{V}(f)\right),$$

where $\mathcal{V}(f) := \sum_{j=0}^d \int (f_j + \mathcal{L}_j VKf)^2\,\mathrm{d}\nu$ vanishes if and only if f is a $\mathcal{L}_0$-derivative.

"F Hölderian on $O\mathcal{M}$" precisely means: there exists some $r>0$ such that $\mathrm{dist}(\xi,\xi')^{-r}|F(\xi)-F(\xi')|$ is bounded on $\{(\xi,\xi')\in O\mathcal{M}^2 \mid 0<\mathrm{dist}(\xi,\xi')<1\}$.

Equivalently, our result reads (with $c(\delta)$ given in Corollary 1): *for all $a\in\mathbb{R}$ we have*:

$$\lim_{t\to\infty} m\left[\xi\in T^1\mathcal{M}\,\middle|\,\int_0^t f(\xi\theta_s)\,\mathrm{d}s \leqslant a\sqrt{t\mathcal{V}(f)}\right] = \|\Phi\|_2^2 c(\delta)^{-1}\times(2\pi)^{-1/2}\int_{-\infty}^{a}\exp\bigl(-s^2/2\bigr)\,\mathrm{d}s.$$

The particular case of $\mathcal{M}$ being convex-cocompact, that is to say without cusp, or associated with a group Γ without parabolic element, can be handled by the coding method of [15]. See also [1,8] and [19].

The particular case of $\mathcal{M}$ having finite volume (corresponding to $\delta = d$) was handled in [9], the Patterson–Sullivan measure being in this case just the Liouville measure. Our result concerns the more general case $d/2<\delta\leqslant d$, in which $\mathcal{M}$ may have both infinite volume and cusps.

For dealing with this new case, we use here globally the same strategy as in [9], which consists roughly in comparing the geodesics with the paths of a diffusion on $T^1\mathcal{M}$, for which the existence of potentials has to be exhibited. But the infinite volume case is much more involved, since m is not invariant under the horocycle flows and is distinct of ν, which is only quasi-invariant under the geodesic and stable horocycle flows.

Finally note that the remaining case $\delta\leqslant d/2$ cannot be handled in the same way, since in that case there does not exist any fundamental diffusion Z_t^Φ associated with δ on the base manifold $\mathcal{M}$.

2. Notations and basic data

Let $\mathbb{H}$ denote the hyperbolic space $\mathbb{H}^{d+1}$, with boundary $\partial\mathbb{H}$, unitary tangent bundle $T^1\mathbb{H}$, and orthonormal frame bundle $O\mathbb{H}$.

Let us identify $\mathbb{H}$ with its Poincaré half-space model $\mathbb{R}^d\times\mathbb{R}_+^*$, and the current point $z\in\mathbb{H}$ with its canonical coordinates $(x,y)=(x_1,\dots,x_d,y)\in\mathbb{R}^d\times\mathbb{R}_+^*$. Set $e_0 := (0,1)$. Recall that the metric of $\mathbb{H}$ is given by: $|\mathrm{d}z|^2 = (|\mathrm{d}x|^2 + \mathrm{d}y^2)/y^2$, that its Riemannian volume measure is given by: $\mathrm{d}\tilde{V} := y^{-d-1}\,\mathrm{d}x\,\mathrm{d}y$, and that its (hyperbolic) Laplacian is given by:

$$\Delta = y^2 \times \left(\frac{\partial^2}{\partial y^2} + \frac{1-d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\right).$$

Let us denote by G the Möbius group of orientation-preserving isometries of $\mathbb{H}$, which is generated by the following elements:

- the translations $\theta_x^+ = \theta_{x_1}^1 \ldots \theta_{x_d}^d$, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$;
- the homotheties θ_t (= the linear dilatation by e^t), for $t \in \mathbb{R}$;
- the Euclidian rotations $R_t^{i,j}$ (= the rotation by t in the plane (x_i, x_j));
- the hyperbolic rotations $R_t^{0,j}$ (defined by: $R_t^{0,j}(x_1, \ldots, x_j, \ldots, x_d, y) = (x_1, \ldots, x_j', \ldots, x_d, y')$,

with $x_j' + y'\sqrt{-1} = [(x_j + y\sqrt{-1})\cos(t/2) + \sin(t/2)]/[\cos(t/2) - (x_j + y\sqrt{-1})\sin(t/2)]$.

Let us set, for any $z = (x, y) \in \mathbb{H}$:

$$T_z := \theta_x^+ \theta_{\operatorname{Log} y}.$$

Observe the following important classical relation:

$$T_{(x,y)} T_{z'} = T_{(x,0)+yz'}. \tag{1}$$

This means in particular that the set $\{T_z \mid z \in \mathbb{R}^d \times \mathbb{R}_+^*\}$ constitutes a group, isomorphic to a subgroup of the affine group of $\mathbb{R}^d$.

Observe that the Euclidian rotations above generate the subgroup SO_d of G, and that SO_d and the hyperbolic rotations together, generate a subgroup of G isomorphic to SO_{d+1}, which we identify with SO_{d+1}. So SO_{d+1} is the subgroup of those $g \in G$ which fix e_0, and SO_d is the subgroup of those $g \in \mathrm{SO}_{d+1}$ whose differential at e_0 fixes $\frac{\partial}{\partial y}$. This identifies $T^1\mathbb{H}$ with $O\mathbb{H}/\mathrm{SO}_d$, and $\mathbb{H}$ with $O\mathbb{H}/\mathrm{SO}_{d+1}$.

As usual, we identify any $g \in G$ with $(g, \frac{\mathrm{d}g}{|\mathrm{d}g|})$ operating on $T^1\mathbb{H}$ or $O\mathbb{H}$. As there exists a unique $g \in G$ mapping a given $\xi \in O\mathbb{H}$ to another given frame of $O\mathbb{H}$, we may and do identify G with $O\mathbb{H}$, by identifying any $g \in G$ with the image it gives of the canonical frame at e_0.

We have a unique decomposition of any $g \in G$: $g = T_z R R'$, with $z \in \mathbb{H}$, R an hyperbolic rotation in the plane $(\frac{\partial}{\partial y}, \mathrm{d}g_{e_0}(\frac{\partial}{\partial y}))$, and $R' \in \mathrm{SO}_d$. In particular, we have an identification between $O\mathbb{H}$ and $T^1\mathbb{H} \times \mathrm{SO}_d$.

Note that these identifications have two important consequences:

- Firstly, the base-point of the frame ξ is $\pi_2(\xi) = \xi(e_0)$, π_2 being the canonical projection from $O\mathbb{H}$ onto $\mathbb{H}$. And $\pi_2(\xi T_z) = \xi(z)$, for any $z \in \mathbb{H}$ and any $\xi \in O\mathbb{H}$.
- Secondly, the right multiplication by θ_t moves any frame ξ by the geodesic flow of algebraic length t, in the direction of the vector $\mathrm{d}\xi_{e_0}(\frac{\partial}{\partial y})$. And similarly, the right multiplication by θ_t^j moves the frame ξ by the horocycle flow of algebraic length t, in the direction of the vector $\mathrm{d}\xi_{e_0}(\frac{\partial}{\partial x_j})$.

Indeed, this is clear when ξ is the unit element of G, that is to say the canonical frame at e_0, and this remains clear for the other frames by invariance of π_2 and of the flows under isometries.

Note that since θ_t commutes with the Euclidian rotations, the geodesic flow still makes sense on $T^1\mathbb{H}$. On the contrary, the horocycle flow makes sense only on $O\mathbb{H}$.

Given (z, z', u) in $\mathbb{H} \times \mathbb{H} \times \partial\mathbb{H}$, denote by $\log[B_u(z, z')]$ the Busemann function, that is to say the algebraic hyperbolic distance, on any geodesic ending at u, from the stable horocycle $H(z, u)$ determined by z to the stable horocycle $H(z', u)$.

In the Poincaré half-space model, we have $B_u(z, z') = p(z', u)/p(z, u)$, $p(z, u)$ denoting the Poisson kernel: $p(z, u) = y \times |z - u|^{-2}$ if $u \neq \infty$ and $p(z, \infty) = y$.

We have the cocycle property: $B_u(z, z'') = B_u(z, z') \times B_u(z', z'')$.

We shall use on the unitary tangent bundle $T^1\mathbb{H}$ the two following systems of coordinates:

– firstly, $(z, u) \in \mathbb{H} \times \partial\mathbb{H}$, the geodesic running from z to u determining the unitary tangent vector at the base point z; this identifies $T^1\mathbb{H}$ with $\mathbb{H} \times \partial\mathbb{H}$;
– secondly, given a reference point $z_0 \in \mathbb{H}$, the point (z, u) of $T^1\mathbb{H}$ can be represented by the triple $(u, v, s) \in \partial\mathbb{H} \times \partial\mathbb{H} \times \mathbb{R}$, where
  – v is the starting point of the geodesic ending at u and running through z;
  – s is the algebraic hyperbolic distance from z to the orthogonal projection z_1 of z_0 onto the geodesic $\overrightarrow{vu}$.

Note that the first coordinates above extend to the following global coordinates on $O\mathbb{H}$: $(z, u, r) \in \mathbb{H} \times \partial\mathbb{H} \times \mathrm{SO}_d$, by means of the identification between $O\mathbb{H}$ and $T^1\mathbb{H} \times \mathrm{SO}_d$.

Denote by $\mathrm{dist}(\zeta, uv)$ the hyperbolic distance from $\zeta \in \mathbb{H}$ to the geodesic $\overrightarrow{vu}$.

The following well-known identity is valid for any ζ in $\mathbb{H}$, any distinct u, v in $\partial\mathbb{H}$, and any z on the geodesic $\overrightarrow{vu}$ running from v to u.

$$(*) \qquad ch^2\big(\mathrm{dist}(\zeta, uv)\big) = B_u(\zeta, z) B_v(\zeta, z).$$

(Indeed, since this is an intrinsic formula, we may consider the half-space model with $u = \infty$ and $v = 0$. Denoting then by (X, Y) the Euclidian coordinates of ζ in this model, and by $(0, y)$ those of z, it is elementary that $B_u(\zeta, z) = y/Y$, $B_v(\zeta, z) = (|X|^2 + Y^2)/(yY)$, and, using the classical formula for the distance (see [13]), that $ch^2(\mathrm{dist}(\zeta, uv)) = ch^2 \mathrm{dist}(\zeta, (0, |\zeta|)) = (|X|^2 + Y^2)^2/(Y|\zeta|)^2 = 1 + |X|^2/Y^2 = B_u(\zeta, z)B_v(\zeta, z)$.)

Let Γ be a discrete torsion-free (non-elementary) subgroup of G, that we suppose geometrically finite. Let $\Lambda = \Lambda(\Gamma)$ denote its limit set, with Hausdorff dimension say δ. Recall that $\delta$is also the critical convergence exponent of the Poincaré series relative to Γ (see for example ([18], Theorem 1)). Obviously $\delta \leqslant d$.

We make here the assumption that $\delta > d/2$.

Let $\{\mu_z \mid z \in \mathbb{H}\}$ denote the family of Patterson (finite) measures on Λ associated with Γ. It can be defined, up to a multiplicative constant (that we definitively fix), as the only family of measures on Λ satisfying the following geometric "conformal density" property:

$$\mathrm{d}\mu_{z'}(u) = B_u^\delta(z, z')\,\mathrm{d}\mu_z(u) \quad \text{for any } z, z' \text{ in } \mathbb{H}$$

together with the invariance property by the group Γ, in the sense that:

$$\gamma^*\mu_z = \mu_{\gamma z} \quad \text{for any } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \mathbb{H},$$

with the convention $\gamma^*\mu := \mu \circ \gamma^{-1}$.

See for example ([13], Lecture 2), [18], or ([11], Sections 3.4 and 4.7).

Set

$$\Phi(z) := \int \mathrm{d}\mu_z = \mu_z(\partial\mathbb{H}) = \mu_z(\Lambda), \quad \text{and} \quad \lambda_0 := \delta(\delta - d)/2.$$

It is a Γ-invariant function on $\mathbb{H}$ which verifies $\Delta\Phi = 2\lambda_0\Phi$. See ([13], Theorem 1, p. 301).

Define the Patterson–Sullivan measure $\tilde{m}$ on $T^1\mathbb{H}$ by

$$\mathrm{d}\tilde{m}(u,v,s) := ch^{2\delta}\big(\mathrm{dist}(z_0,uv)\big)\,\mathrm{d}\mu_{z_0}(u)\,\mathrm{d}\mu_{z_0}(v)\,\mathrm{d}s.$$

Note that by the geometric property for (μ_z) and by the identity $(*)$ above, $\tilde{m}$ does not depend on the choice of the reference point z_0. Hence it is intrinsic (it depends only on the subgroup Γ), and then it is Γ-invariant. Moreover it is plainly invariant with respect to the geodesic flow. It is also called Bowen–Margulis measure.

The Liouville measure $\tilde{\lambda}$ on $T^1\mathbb{H}$ can be expressed for any reference point z_0, by:

$$\mathrm{d}\tilde{\lambda}(u,v,s) = ch^{2d}\big(\mathrm{dist}(z_0,uv)\big)\,\mathrm{d}\mu^h_{z_0}(u)\,\mathrm{d}\mu^h_{z_0}(v)\,\mathrm{d}s,$$

where μ^h_z denotes the harmonic measure at z. Recall that we have in the half-space model: $\mathrm{d}\mu^h_z(u) = p^d(z,u)\,\mathrm{d}u$.

Note that the above geometric property holds for harmonic measures, by changing δ into d: $\mathrm{d}\mu^h_{z'}(u) = B^d_u(z,z')\,\mathrm{d}\mu^h_z(u)$ for any z, z' in $\mathbb{H}$.

This and the identity $(*)$ show the irrelevance of the reference point z_0 in the expression of the Liouville measure $\tilde{\lambda}$ above. As can be verified by a direct elementary computation, the expression of $\tilde{\lambda}$ in the (z,u) coordinates is: $\mathrm{d}\tilde{\lambda}(z,u) = \mathrm{d}\mu^h_z(u)\,\mathrm{d}\tilde{V}(z)$.

$\tilde{\lambda}$ is naturally lifted to the Liouville measure λ' on $O\mathbb{H}$, by taking λ' uniform on each fibre SO_d. This Liouville measure λ' is the Haar measure on G, and thus it is invariant by the horocycle and geodesic flows.

We are interested in the quotient manifold $\mathcal{M} := \Gamma \setminus \mathbb{H}$.

It is known that when $\delta > d/2$, Φ is square-integrable with respect to the Riemannian volume measure $\mathrm{d}V$ of $\mathcal{M}$, and is the fundamental eigenstate on $\mathcal{M}$. See ([13], Theorem 1, p. 301).

Note that as a consequence, the volume of $\mathcal{M}$ is finite if and only if $\delta = d$.

Denote by π the canonical projection from the unitary tangent bundle $T^1\mathcal{M} = \Gamma \setminus T^1\mathbb{H}$ onto $\mathcal{M}$, by π_1 the canonical projection from the orthonormal frame bundle $O\mathcal{M} = \Gamma \setminus O\mathbb{H}$ onto $T^1\mathcal{M}$, and by $\pi_2 = \pi \circ \pi_1$ the canonical projection from $O\mathcal{M}$ onto $\mathcal{M}$. By taking the left quotient by Γ, we see that the identifications at the level of $\mathbb{H}$ give at the level of $\mathcal{M}$: $T^1\mathcal{M} \equiv O\mathcal{M}/\mathrm{SO}_d$ and $\mathcal{M} \equiv O\mathcal{M}/\mathrm{SO}_{d+1}$, and

$$\pi_2(\xi T_z) = \xi(z) \quad \text{for any } \xi \in O\mathcal{M} \text{ and } z \in \mathbb{H}. \tag{3}$$

In particular, we identify from now on the functions on $T^1\mathcal{M}$ with the SO_d-invariant functions on $O\mathcal{M}$.

Recall that any Γ-invariant measure $\tilde{n}$ on $T^1\mathbb{H}$ induces a measure n on $T^1\mathcal{M}$.

In particular, denote by λ the Liouville measure induced by $\tilde{\lambda}$, and by m the Patterson–Sullivan measure induced by $\tilde{m}$. Similarly, denote by $\mathrm{d}V = \pi^*\lambda$ the volume measure on $\mathcal{M}$, induced by $\mathrm{d}\tilde{V}$.

Observe that since the flows act on the right-hand side, while Γ acts on the left-hand side, these two actions commute. Thus the geodesic flow makes sense on $T^1\mathcal{M}$ and on $O\mathcal{M}$, and the horocycle flow makes sense on $O\mathcal{M}$.

Let us introduce then the Lie derivatives: for any smooth function F on $O\mathcal{M}$, any ξ in $O\mathcal{M}$, $0 \leqslant i < j \leqslant d$ we set:

$$\mathcal{L}_0 F(\xi) := \frac{\mathrm{d}_0}{\mathrm{d}t} F(\xi\theta_t), \qquad \mathcal{L}_j F(\xi) := \frac{\mathrm{d}_0}{\mathrm{d}t} F\big(\xi\theta^j_t\big), \qquad \mathcal{L}_{i,j} F(\xi) := \frac{\mathrm{d}_0}{\mathrm{d}t} F\big(\xi R^{i,j}_t\big). \tag{4}$$

$\mathrm{d}_0/\mathrm{d}t$ means and will mean the derivative at $t = 0$ with respect to t.

We get from the definitions that for $1 \leqslant i, j \leqslant d$ and $0 \leqslant k \leqslant d$:

$$
\begin{aligned}
&[\mathcal{L}_0, \mathcal{L}_j] = \mathcal{L}_j, \qquad [\mathcal{L}_i, \mathcal{L}_j] = 0, \qquad [\mathcal{L}_k, \mathcal{L}_{i,j}] = 1_{\{i=k\}}\mathcal{L}_j - 1_{\{j=k\}}\mathcal{L}_i, \\
&[\mathcal{L}_0, \mathcal{L}_{0,j}] = \mathcal{L}_j - \mathcal{L}_{0,j}, \qquad [\mathcal{L}_i, \mathcal{L}_{0,j}] = 1_{\{i\neq j\}}\mathcal{L}_{j,i} - 1_{\{i=j\}}\mathcal{L}_0.
\end{aligned} \tag{5}
$$

It also follows immediately from (3) that:

$$
\mathcal{L}_0 F(\xi T_{(x,y)}) = y\frac{\partial}{\partial y}F(\xi T_{(x,y)}), \qquad \mathcal{L}_j F(\xi T_{(x,y)}) = y\frac{\partial}{\partial x_j}F(\xi T_{(x,y)}). \tag{6}
$$

3. An intrinsic measure on $T^1\mathcal{M}$

We introduce an intrinsic measure ν on $T^1\mathcal{M}$, which was already used in [2–4]. Its interest is to be smooth along the stable leaves and quasi-invariant under the geodesic and stable horocycle flows, and to be an invariant measure for two dual diffusions on $O\mathcal{M}$, which are both projected by π_2 onto the Φ-diffusion.

DEFINITION 1. – *Let $\tilde{\nu}$ be the Γ-invariant measure on $T^1\mathbb{H}$ defined by*:

$$
\mathrm{d}\tilde{\nu}(z,u) = \Phi(z)\,\mathrm{d}\mu_z(u)\,\mathrm{d}\tilde{V}(z).
$$

Denote by ν the measure it induces on $T^1\mathcal{M}$.

Denote by ν' the unique measure on $O\mathcal{M}$ which has marginal ν on $T^1\mathcal{M}$ and whose conditional laws on the fibres are the normalized Haar measure on $\mathrm{SO}_d \equiv O\mathcal{M}/T^1\mathcal{M}$.

Set also $\mathrm{d}V^{\Phi}(z) := \Phi^2(z)\,\mathrm{d}V(z)$.

Remark 1. – Observe that by definition of Φ we have $\pi_2^*\nu' = \pi^*\nu = V^{\Phi}$, and then that ν and ν' are finite (since $\delta > d/2$), with mass $\|\Phi\|_2^2$.

Observe also that, due to the geometric property, the Patterson measure μ_z, seen as a measure on $T_z^1\mathbb{H}$ by means of the coordinate system (z,u), makes sense on $T_z^1\mathcal{M}$ as well. So in the preceding definition the second coordinate u can be seen as a unit tangent vector based at z, and the expression for $\tilde{\nu}$ (with $\tilde{V}$ replaced by V) can then be understood directly as the expression for ν.

Remark 2. – In the finite volume case, we have $\delta = d$, Φ constant, $\mathrm{d}\mu(u)$ is proportional to the uniform measure $\mathrm{d}u$, and then our measure ν is proportional to the Liouville measure λ (and to the Patterson–Sullivan measure m).

PROPOSITION 1. – *The measure ν' is quasi-invariant under the geodesic and positive horocycle flows*:

$$
\frac{\mathrm{d}(T_z^*\nu')}{\mathrm{d}\nu'}(\xi) = y^{(d-\delta)} \times \frac{\Phi\circ\pi_2(\xi T_z^{-1})}{\Phi\circ\pi_2(\xi)} \quad \textit{for any } \xi \in O\mathcal{M} \textit{ and } z = (x,y) \in \mathbb{R}^d \times \mathbb{R}_+^*.
$$

Note that this quasi-invariance property is what remains from the invariance of the Liouville measure λ' u nder the flows, in the finite volume case. The proof was already given in [4] (and in [3] in the case of surfaces).

The following theorem appeared already in [3] for $d = 1$. The same proof is valid, with only obvious modifications.

THEOREM 1. – *The measure ν can be expressed by a convolution of m along the stable horocycles of $T^1\mathcal{M}$. Precisely, we have*:

$$\nu = \int_{\mathbb{R}^d} (\theta_x^+)^* m \big(1+|x|^2\big)^{-\delta}\, \mathrm{d}x.$$

Note that the right-hand side above is well defined as a measure on $T^1\mathcal{M}$, although θ_x^+ does not act on $T^1\mathcal{M}$, since one integrates with respect to a SO_d-invariant measure. This right-hand side can also be thought with m replaced by its lift m' to $O\mathcal{M}$, uniform on each fibre SO_d.

Using Remark 1, we immediately deduce from Theorem 1:

COROLLARY 1. – *Let us set $c(\delta) := \int_{\mathbb{R}^d} (1+|x|^2)^{-\delta}\, \mathrm{d}x$.*
Then (for $\delta > d/2$) the mass of the measure m on $T^1\mathcal{M}$ equals $\|\Phi\|_2^2/c(\delta)$.

We also deduce, as in [3], the following approximation result on the measure m:

COROLLARY 2. – *The measure $\theta_S^*\nu$ converges as $S \to +\infty$, in the sense of the evaluation on each bounded function on $T^1\mathcal{M}$ which is continuous along the horocycles, towards the normalized Patterson–Sullivan measure $c(\delta)m$.*

4. Diffusions on $\mathcal{M}$ and on $O\mathcal{M}$

This section is essentially taken from [4].

4.1. The diffusions Z_t^δ, ξ_t^δ and Z_t^Φ

Let (w_t, W_t) denote a Brownian motion on $\mathbb{R}\times\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$y_t := \exp\big[w_t + (\delta - d/2)t\big], \qquad x_t := \int_0^t y_s\, \mathrm{d}W_s, \qquad Z_t^\delta := (x_t, y_t) \in \mathbb{H}.$$

For all δ, Z_t^δ is the diffusion on $\mathbb{H}$ starting from $e_0 = (0,1)$, with invariant measure $y^{2\delta-2d}\,\mathrm{d}x\,\mathrm{d}y$, and generator:

$$\frac{1}{2}\Delta^\delta := \frac{1}{2}\Delta + \delta y \frac{\partial}{\partial y} = \frac{y^2}{2}\left(\frac{\partial^2}{\partial y^2} + \frac{2\delta+1-d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}\right).$$

Similarly, denote by $Z_t^b = (x_t^b, y_t^b)$ the analogous process with δ replaced by $b \in [0,d]$. In particular, Z_t^0 is the Brownian motion on $\mathbb{H}$.

By formula (2), we see that $\pi_2(\xi T_{Z_t^0}) = \xi(Z_t^0)$ is a Brownian motion on $\mathcal{M}$, started from $\pi_2(\xi)$, for any $\xi \in O\mathcal{M}$. As a consequence, denoting by P_t the Brownian semi-group on $\mathcal{M}$, we have $\mathbb{E}(f\circ\pi_2(\xi T_{Z_t^0})) = P_t f(\pi_2(\xi))$.

Observe then that $T_{Z_t^0}$ is a right Brownian motion on a subgroup of the affine group of $\mathbb{R}^d$. Indeed, for any $b \in [0,d]$,

$$T_{Z_t^b}^{-1} T_{Z_{t+s}^b} = T_{\left(\frac{x_{t+s}^b - x_t^b}{y_t^b}, \frac{y_{t+s}^b}{y_t^b}\right)} = T_{Z_s^b} \circ \Theta_t$$

is independent of the sub-σ-field $\mathcal{F}_t$ generated by the coordinates until time t.

DEFINITION 2. – *For any $\xi \in O\mathcal{M}$, set $\xi_t^\delta := \xi T_{Z_t^\delta}$.*
Set $\Delta^\Phi := \Phi^{-1}\Delta \circ \Phi - 2\lambda_0 = \Delta + 2(\nabla \log \Phi)\cdot\nabla$, and $P_t^\Phi := \exp(\frac{t}{2}\Delta^\Phi)$.
Denote by Z_t^Φ and call "Φ-diffusion" the diffusion on $\mathcal{M}$ with generator $\frac{1}{2}\Delta^\Phi$.

By the preceding observation, ξ_t^δ is a diffusion on $O\mathcal{M}$, starting from ξ.
From (5) we get $\Delta^\delta[F(\xi T_z)] = (D^\delta F)(\xi T_z)$, where

$$D^\delta := \sum_{j=0}^{d} \mathcal{L}_j^2 + (2\delta - d)\mathcal{L}_0 = D^0 + 2\delta\mathcal{L}_0. \tag{7}$$

Then the generator of the diffusion ξ_t^δ is $\frac{1}{2}D^\delta$.

Moreover, note that the Φ-diffusion is symmetrical and has invariant measure V^Φ and semi-group P_t^Φ. In the finite volume case $\delta = d$, this is just the Brownian motion.

Remark 3. – We have for any test-function F on $O\mathcal{M}$:

$$D^0(F\circ\pi_2)(\xi T.) = \Delta\big[F\circ\pi_2(\xi T.)\big] = \Delta(F\circ\xi) = (\Delta F)\circ\xi = (\Delta F)\circ\pi_2(\xi T.),$$

whence $D^0(F\circ\pi_2) = (\Delta F)\circ\pi_2$.

4.2. ν' as an invariant measure

We deduce now from the quasi-invariance property of ν' an adjonction property for ν', and thus its invariance with respect to two dual diffusions on $O\mathcal{M}$.

The following results were already in [4]:

PROPOSITION 2. – *We have for all δ and all test functions F, G on $O\mathcal{M}$:*

$$\int \big(D^\delta F\big) G \,\mathrm{d}\nu' = \int F\big(D^\Phi G\big)\,\mathrm{d}\nu',$$

with $D^\Phi := \sum_{j=0}^d \mathcal{L}_j^2 - d\mathcal{L}_0 + 2\sum_{j=0}^d (\mathcal{L}_j \log\Phi\circ\pi_2)\mathcal{L}_j = (\Phi\circ\pi_2)^{-1} D^0 \circ (\Phi\circ\pi_2) - 2\lambda_0$.

COROLLARY 3. – *For $\xi \in O\mathcal{M}$ and each $\delta > d/2$ we have:*
(i) *under $\mathbb{P}$, the diffusion ξ_t^δ admits the invariant measure ν';*
(ii) *under $\nu'\otimes\mathbb{P}$, ξ_t^δ extends to a stationary diffusion defined for all real t, and ξ_{-t}^δ is the stationary diffusion associated with the infinitesimal generator $\frac{1}{2}D^\Phi$, say ξ_t^Φ.*

Remark 4. – (i) We see from Remark 3 and from the h-process form of D^Φ in Proposition 2 above that we have for any test-function F on $O\mathcal{M}$: $D^\Phi(F\circ\pi_2) = (\Delta^\Phi F)\circ\pi_2$.

(ii) The h-process form of D^Φ shows that ξ_t^Φ can be defined by the following formula, where $0 = t_0 < t_1 < \cdots < t_n$, $F_0, \ldots, F_n$ are test-functions on $O\mathcal{M}$, and $\xi_t^0 = \xi T_{Z_t^0}$:

$$\iint \prod_{j=0}^n F_j\big(\xi_{t_j}^\Phi\big)\,\mathrm{d}\nu'(\xi)\,\mathrm{d}\mathbb{P} = \iint \frac{\mathrm{e}^{-\lambda_0 t_n}}{\Phi\circ\pi_2(\xi)} \times \Phi\circ\pi_2\big(\xi_{t_n}^0\big) \times \prod_{j=0}^n F_j\big(\xi_{t_j}^0\big)\,\mathrm{d}\nu'(\xi)\,\mathrm{d}\mathbb{P}.$$

Letting F_0 go to $1_{\{\xi\}}$, we get the law of ξ_t^Φ started from ξ. Then taking $F_j = f_j\circ\pi_2$ for $j \geqslant 1$, and using that $\pi_2(\xi_t^0) = \xi(Z_t^0)$ is a Brownian motion starting from $\xi(e_0) = \pi_2(\xi)$, and the h-process form of Δ^Φ in Definition 2, we deduce the following:

PROPOSITION 3. – *Under $\mathbb{P}$, the projection $\pi_2(\xi_t^\Phi)$ of the diffusion ξ_t^Φ starting from ξ on $O\mathcal{M}$ is the Φ-diffusion starting from $\pi_2(\xi)$ on $\mathcal{M}$.*

5. Existence of potentials along the stable foliation

Here we establish the existence of a kind of spectral gap for the degenerate foliated diffusion ξ_t^Φ, in order to ensure that enough regular functions possess a potential along its paths. This will be crucial for proving our main result.

Precisely, the aim of this section is to establish the following:

THEOREM 2. – *Let us denote by Q_t^Φ the semi-group of the diffusion ξ_t^Φ.*

For any Borelian bounded function F on $T^1\mathcal{M}$ which is rotationally Hölderian (this means: there exists some $r>0$ such that $|F(\xi g)-F(\xi)|/d(g,\mathrm{SO}_d)^r$ is bounded independently from $g\in\mathrm{SO}_{d+1}-\mathrm{SO}_d$ and $\xi\in T^1\mathcal{M}$), and such that $\int F\,\mathrm{d}\nu=0$, there exists some $\varrho>0$ such that we have: $\|Q_t^\Phi F\|_{L^2(\nu)}\leqslant\varrho^{-1}\mathrm{e}^{-\varrho t}$ for all $t\geqslant 0$.

By means of the coordinates (z,u), we have for each $z\in\mathbb{H}$ an identification between $\partial\mathbb{H}$ and $T_z^1\mathbb{H}$. This allows to consider the Patterson measure μ_z as a measure on $T_z^1\mathcal{M}$, and then to localize the measure ν' as follows:

DEFINITION 3. – *For each $\xi\in O\mathcal{M}$ and each Borelian bounded function F on $O\mathcal{M}$, set*

$$\overline{F}\big(\pi_2(\xi)\big):=\int_{\mathrm{SO}_{d+1}}F(\xi g)\,\mathrm{d}\nu_\xi(g):=\Phi\big(\pi_2(\xi)\big)^{-1}\times\int_{T^1_{\pi_2(\xi)}\mathcal{M}}F\circ\pi_1\,\mathrm{d}\mu_{\pi_2(\xi)}.$$

This defines a probability measure ν_ξ on SO_{d+1} and a function $\overline{F}$ on $\mathcal{M}$.

Remark 5. – (i) The measure ν_ξ depends only on $\pi_1(\xi)$, and then its restriction to any fibre of $\mathrm{SO}_{d+1}/\mathrm{SO}_d$ is uniform.

(ii) We have, for F Borelian and non-negative on $T^1\mathcal{M}$:

$$\int_{\mathcal{M}}\overline{F}\,\mathrm{d}V^\Phi=\int_{\mathcal{M}}\int_{T_z^1\mathcal{M}}F\circ\pi_1\,\mathrm{d}\mu_z\Phi(z)\,\mathrm{d}V(z)=\int_{T^1\mathcal{M}}F\circ\pi_1\,\mathrm{d}\nu=\int_{O\mathcal{M}}F\,\mathrm{d}\nu'.$$

Hence if $F\in L^2(T^1\mathcal{M},\nu)$ and $\int F\,\mathrm{d}\nu=0$, then $\overline{F}\in L^2(\mathcal{M},V^\Phi)$ and $\int\overline{F}\,\mathrm{d}V^\Phi=0$.

We shall use the following commutation relation in G:

LEMMA 1. – *For all $g\in\mathrm{SO}_{d+1}$ and $z\in\mathbb{R}^d\times\mathbb{R}_+^*$, there exists a unique $g_z\in\mathrm{SO}_{d+1}$ such that $gT_z=T_{g(z)}g_z$.*

Denoting by $u=u(g):=g\theta_\infty\in\mathbb{R}^d$ the extremity of the half-geodesic $g\theta_{\mathbb{R}_+}$ started from g, and setting $u'=g_z\theta_\infty$ and $g(z)=(x',y')\in\mathbb{R}^d\times\mathbb{R}_+^*$, we have $u'=(u-x')/y'$.

Proof. – Each element of G can be written $T_{z'}g'$, and in particular gT_z. This implies that $z'=\pi_2(T_{z'}g')=\pi_2(gT_z)=g(z)$, and $u=g\theta_\infty=gT_z\theta_\infty=T_{z'}g'\theta_\infty$.

Let us identify g to $(\phi,\sigma,r)\in[0,\pi]\times\mathbb{S}^{d-1}\times\mathrm{SO}_d$, the point $(\phi,\sigma)\in\mathbb{S}^d$ being the image under g of the north pole of $\mathbb{S}^d$. An elementary computation shows that $u=\sigma\cot g(\phi/2)$, and similarly $u'=\sigma'\cot g(\phi'/2)$ if $g'\equiv(\phi',\sigma',r')$ and $u'=g'\theta_\infty$. It also shows $T_{z'}g'\theta_\infty=x'+y'\sigma'\cot g(\phi'/2)$. Whence $u=x'+y'u'$. □

Remark 6. – If $g\in SO_d$, which is equivalent to $u(g)$ being ∞, we have $g_z\in SO_d$ by Lemma 1, and then if $F=F\circ\pi_1$ we get $Q_t^{\Phi}F(\xi g)=\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\times\mathbb{E}[F\times\Phi\circ\pi_2(\xi T_{g(Z_t^0)})]=Q_t^{\Phi}F(\xi)$, which shows that Q_t^{Φ} acts on $T^1\mathcal{M}$.

By definition of $Q_t^{\Phi}F$ and of $\overline{Q_t^{\Phi}F}$, we have:

$$Q_t^{\Phi}F(\xi)=\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[F\times\Phi\circ\pi_2\big(\xi_t^0\big)\big]\quad\text{and}$$

$$\begin{aligned}\overline{Q_t^{\Phi}F}\big(\pi_2(\xi)\big)&=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi g)}\mathbb{E}\big[F\times\Phi\circ\pi_2(\xi g T_{Z_t^0})\big]\,d\nu_\xi(g)\\&=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[F\times\Phi\circ\pi_2(\xi T_{g(Z_t^0)}g_{Z_t^0})\big]\,d\nu_\xi(g)\\&=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[F\times\Phi\circ\pi_2(\xi T_{Z_t^0}g_{g^{-1}(Z_t^0)})\big]\,d\nu_\xi(g)\\&=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[\Phi\circ\pi_2\big(\xi_t^0\big)\times F\big(\xi_t^0 g_t'\big)\big]\,d\nu_\xi(g),\end{aligned}$$

where $g_t':=g_{g^{-1}(Z_t^0)}$. Therefore we have:

$$Q_t^{\Phi}F(\xi)-\overline{Q_t^{\Phi}F}\big(\pi_2(\xi)\big)=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[\Phi\circ\pi_2\big(\xi_t^0\big)\times\big(F\big(\xi_t^0\big)-F\big(\xi_t^0 g_t'\big)\big)\big]\,d\nu_\xi(g).$$

Let us fix $0<\varepsilon<2\delta-d$, and set for $t\geqslant 0$ and $g\in SO_{d+1}$:

$$a_t^1:=1_{\{y_t^0>\exp[(\delta-d-\varepsilon)t/2]\}},\qquad a_t^2:=\big(1-a_t^1\big)\times 1_{\{|x_t^0-u(g)|\leqslant y_t^0 e^{\varepsilon t/6}]\}},\qquad a_t^3:=1-a_t^1-a_t^2,$$

and for $1\leqslant j\leqslant 3$:

$$A_t^j(\xi):=\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[\Phi\circ\pi_2\big(\xi_t^0\big)\times\big(F\big(\xi_t^0\big)-F\big(\xi_t^0 g_t'\big)\big)\times a_t^j\big]\,d\nu_\xi(g).$$

Of course we have $\|Q_t^{\Phi}F-\overline{Q_t^{\Phi}F}\circ\pi_2\|_{L^2(\nu)}\leqslant\sum_{j=1}^3\|A_t^j\|_{L^2(\nu)}$.

Moreover, using that $Q_t^{\Phi}1=1$, we have:

$$\big\|A_t^j\big\|^2_{L^2(\nu)}\leqslant 4\|F\|^2_\infty\times\int\frac{e^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\big[\Phi\circ\pi_2\big(\xi_t^0\big)\times a_t^j\big]\,d\nu_\xi(g)\,d\nu(\xi).$$

Using Proposition 1, we get first:

$$\begin{aligned}\big\|A_t^1\big\|^2_{L^2(\nu)}&\leqslant 4\|F\|^2_\infty\times\mathbb{E}\bigg[a_t^1\times e^{-\lambda_0 t}\int\frac{\Phi\circ\pi_2(\xi_t^0)}{\Phi\circ\pi_2(\xi)}\,d\nu(\xi)\bigg]\\&=4\|F\|^2_\infty\times e^{-\lambda_0 t}\times\mathbb{E}\big[a_t^1\times\big(y_t^0\big)^{d-\delta}\big]=4\|F\|^2_\infty\times e^{-(d-\delta)^2 t/2}\\&\quad\times\int_{(\delta-\varepsilon)t/2}^{\infty}e^{(d-\delta)y-y^2/2t}\,dy/\sqrt{2\pi t}\end{aligned}$$

$$= 4\|F\|_\infty^2 \times \int_{(2\delta-d-\varepsilon)\sqrt{t}/2}^{\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y/\sqrt{2\pi} = \mathcal{O}\big(\exp\big[-(2\delta-d-\varepsilon)^2 t/8\big]\big).$$

Then for dealing with A_t^2 we need the following:

LEMMA 2. – *We have for any $\xi \in O\mathcal{M}$ and any $z \in \mathbb{R}^d \times \mathbb{R}_+^*$:*

$$\frac{\Phi \circ \pi_2(\xi T_z)}{\Phi \circ \pi_2(\xi)} = \int_{\mathrm{SO}_{d+1}} p^\delta\big(z, u(g)\big) p^{-\delta}\big(e_0, u(g)\big)\,\mathrm{d}\nu_\xi(g),$$

where $u(g) \in \mathbb{R}^d \cup \{\infty\}$ denotes the extremity $g\theta_\infty$ of the half-geodesic $g\theta_{\mathbb{R}_+}$.
(*Let us recall that we have in $\mathbb{R}^d \times \mathbb{R}_+^*$: $\mathrm{SO}_{d+1} \equiv T_{e_0}^1(\mathbb{R}^d \times \mathbb{R}_+^*)$, with $e_0 = (0,1)$.*)

Proof. – Let us choose the half-space model for $\mathbb{H}$, in such a way that ξ be the unit element of G. So that in particular $u(g) = \xi g\theta_\infty$, and then

$$\Phi \circ \pi_2(\xi) \int p^\delta\big(z, u(g)\big) p^{-\delta}\big(e_0, u(g)\big)\,\mathrm{d}\nu_\xi(g) = \int p^\delta(z,u) p^{-\delta}(e_0,u)\,\mathrm{d}\mu_{e_0}(u)$$
$$= \int \mathrm{d}\mu_z = \Phi(z). \qquad \square$$

This lemma implies the following corollary, analogous to the celebrated Sullivan's "shadow lemma".

COROLLARY 4. – *We have for any $\xi \in O\mathcal{M}$, $a \geqslant 1$, and $z = (x,y) \in \mathbb{R}^d \times \mathbb{R}_+^*$:*

$$\nu_\xi\big(\big\{g \in \mathrm{SO}_{d+1} \mid \big|u(g) - x\big| \leqslant ay\big\}\big) \leqslant \frac{\Phi \circ \pi_2(\xi T_z)}{\Phi \circ \pi_2(\xi)} \times (2y)^\delta \times a^{2\delta}.$$

Proof. – We merely use Lemma 2, observing that:

$$p^\delta(z,u) p^{-\delta}(e_0,u) = y^{-\delta}\big(1 + |x-u|^2 y^{-2}\big)^{-\delta}\big(1+|u|^2\big)^\delta \geqslant 2^{-\delta} y^{-\delta} a^{-2\delta} \quad \text{on} \quad \big\{|u-x| \leqslant ay\big\}. \qquad \square$$

Applying this to $\|A_t^2\|_2^2$, we obtain:

$$\begin{aligned}
\|A_t^2\|_2^2 &\leqslant 4\|F\|_\infty^2 \int \frac{\mathrm{e}^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}\big[\Phi \circ \pi_2\big(\xi_t^0\big) \\
&\qquad \times \big(1 - a_t^1\big) \times \nu_\xi\big(\big\{g \mid \big|u(g) - x_t^0\big| \leqslant y_t^0 \mathrm{e}^{\varepsilon t/6}\big\}\big)\big]\,\mathrm{d}\nu(\xi) \\
&\leqslant 2^{2+\delta}\|F\|_\infty^2 \times \mathrm{e}^{-\lambda_0 t} \int \mathbb{E}\bigg[\frac{\Phi^2 \circ \pi_2(\xi_t^0)}{\Phi^2 \circ \pi_2(\xi)} \times \big(y_t^0\big)^\delta \mathrm{e}^{\varepsilon\delta t/3} \times 1_{\{y_t^0 \leqslant \mathrm{e}^{(\delta-d-\varepsilon)t/2}\}}\bigg]\,\mathrm{d}\nu(\xi) \\
&\leqslant 2^{2+\delta}\|F\|_\infty^2 \times \mathrm{e}^{-\varepsilon\delta t/6} \int \mathbb{E}\bigg[\frac{\Phi^2 \circ \xi(Z_t^0)}{\Phi^2 \circ \xi(Z_0^0)}\bigg]\,\mathrm{d}\nu(\xi) \\
&= 2^{2+\delta}\|F\|_\infty^2 \times \mathrm{e}^{-\varepsilon\delta t/6} \int \mathbb{E}_{\pi_2(\xi)}\bigg[\frac{\Phi^2(Z_t^0)}{\Phi^2(Z_0^0)}\bigg]\,\mathrm{d}\nu(\xi) \\
&= 2^{2+\delta}\|F\|_\infty^2 \times \mathrm{e}^{-\varepsilon\delta t/6} \int P_t\Phi^2\,\mathrm{d}V = 2^{2+\delta}\|F\|_\infty^2 \times \mathrm{e}^{-\varepsilon\delta t/6} \times \|\Phi\|_2^2 = \mathcal{O}\big(\mathrm{e}^{-\varepsilon\delta t/6}\big).
\end{aligned}$$

Finally, using that $F = F \circ \pi_1$, and that $u(g_t') = (u(g) - x_t^0)/y_t^0$, we get:

$$\begin{aligned}\left|A_t^3\right|^2(\xi) &\leqslant \int \frac{\mathrm{e}^{-\lambda_0 t}}{\Phi\circ\pi_2(\xi)}\mathbb{E}\left[\Phi\circ\pi_2\left(\xi_t^0\right)\times\left|F\left(\xi_t^0\right)-F\left(\xi_t^0 g_t'\right)\right|^2 1_{\{|u(g_t')|>\mathrm{e}^{\varepsilon t/6}\}}\right]\mathrm{d}\nu_\xi(g)\\ &\leqslant \mathbb{E}\Big[\sup_{g\in \mathrm{SO}_{d+1}}\left|F\left(\xi_t^\Phi\right)-F\left(\xi_t^\Phi g\right)\right|^2\times 1_{\{d(g,\mathrm{SO}_d)=\mathcal{O}(\mathrm{e}^{-\varepsilon t/6})\}}\Big]=\mathcal{O}\left(\mathrm{e}^{-\varepsilon r t/3}\right),\end{aligned}$$

by the Hölderian hypothesis made on the function F (in Theorem 2).

So far, we just proved the following:

PROPOSITION 4. – *For any bounded measurable function F on $T^1\mathcal{M}$ which is rotationally Hölderian, there exists some $\varrho>0$ such that we have*:

$$\left\| Q_t^\Phi F-\overline{Q_t^\Phi F}\circ\pi_2\right\|_{L^2(\nu)}\leqslant \varrho^{-1}\mathrm{e}^{-\varrho t}\quad \textit{for all } t\geqslant 0.$$

We now complete the proof of Theorem 2.

On one hand we have

$$\begin{aligned}\left\|Q_t^\Phi F\right\|_2 &\leqslant \left\|Q_{t/2}^\Phi\left(Q_{t/2}^\Phi F-\overline{Q_{t/2}^\Phi F}\circ\pi_2\right)\right\|_2+\left\|Q_{t/2}^\Phi\left(\overline{Q_{t/2}^\Phi F}\circ\pi_2\right)\right\|_2\\ &\leqslant \left\|Q_{t/2}^\Phi F-\overline{Q_{t/2}^\Phi F}\circ\pi_2\right\|_2+\left\|P_{t/2}^\Phi\left(\overline{Q_{t/2}^\Phi F}\circ\pi_2\right)\right\|_{L^2(V^\Phi)}\\ &=\mathcal{O}\left(\mathrm{e}^{-\varrho t}\right)+\left\|P_{t/2}^\Phi\left(\overline{Q_{t/2}^\Phi F}\circ\pi_2\right)\right\|_{L^2(V^\Phi)},\end{aligned}$$

by Propositions 3 and 4, and on the other hand we have:

$$\int \overline{Q_{t/2}^\Phi F}\circ\pi_2\,\mathrm{d}V^\Phi=\int \overline{Q_{t/2}^\Phi F}\,\mathrm{d}\nu=\int Q_{t/2}^\Phi F\,\mathrm{d}\nu'=\int F\,\mathrm{d}\nu'=\int F\,\mathrm{d}\nu=0,$$

whence by the spectral gap property of $\mathcal{M}$ (see [12] and [7]), there exists $\eta>0$ such that for any $t\geqslant 0$

$$\begin{aligned}\left\|P_{t/2}^\Phi\left(\overline{Q_{t/2}^\Phi F}\circ\pi_2\right)\right\|_{L^2(V^\Phi)} &\leqslant \mathrm{e}^{-\eta t}\left\|\overline{Q_{t/2}^\Phi F}\circ\pi_2\right\|_{L^2(V^\Phi)}=\mathrm{e}^{-\eta t}\left\|Q_{t/2}^\Phi F\right\|_{L^2(\nu)}\\ &\leqslant \mathrm{e}^{-\eta t}\left\|Q_{t/2}^\Phi F^2\right\|_{L^1(\nu)}^{1/2}=\mathrm{e}^{-\eta t}\|F\|_{L^2(\nu)}.\end{aligned}$$

6. From geodesic flow to stochastic flow

Let us consider a bounded Borelian function f on $T^1\mathcal{M}$, that is to say a SO_d-invariant function on $O\mathcal{M}$, such that $\int f\,\mathrm{d}m=0$ and such that $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded for $1\leqslant j\leqslant d$.

6.1. The conjugate functions

We construct here functions f_j on $O\mathcal{M}$ which are conjugate to the function f, in order to get on $O\mathcal{M}$ a 1-form ω which has f as first coordinate and has a closed restriction to each stable leaf. This will be crucial to replace geodesics by diffusion paths.

DEFINITION 4. –

(i) *Denote by* $\{\mathcal{L}_j'\mid 0\leqslant j\leqslant d\}$ *the dual basis* (*in* $\Lambda^1(O\mathcal{M})$) *of* $\{\mathcal{L}_j\mid 0\leqslant j\leqslant d\}$.

(ii) *Set* $\mathcal{U}^q\phi(\xi):=\int_0^\infty \mathrm{e}^{-qs}\phi(\xi\theta_s)\,\mathrm{d}s$, *for any* $q\in\mathbb{N}^*$, $\xi\in O\mathcal{M}$, *and* ϕ *bounded measurable on* $O\mathcal{M}$.

(iii) *For* $1\leqslant j\leqslant d$, *set* $f_j:=-\mathcal{U}^1\mathcal{L}_j f$, *and set also* (*for convenience*) $f_0:=f$.

(iv) *Set* $\omega:=\sum_{j=0}^d f_j\mathcal{L}_j'$. (*This is a bounded* 1-*form on* $O\mathcal{M}$.)

(v) *For each* $\xi \in O\mathcal{M}$, *let* $\tilde{\xi}$ *denote the map from* $\mathbb{H}$ *into* $O\mathcal{M}$ *defined by* $\tilde{\xi}(z) := \xi T_z$, *and let* $\omega_\xi := \tilde{\xi}^*\omega$ *denote the pull-back of* ω *by* $\tilde{\xi}$.

LEMMA 3. – *The* 1*-form* ω_ξ *is closed and bounded on* $\mathbb{H}$, *for each* $\xi \in O\mathcal{M}$. *Moreover the* $\mathcal{L}_i f_j$ *exist and are bounded on* $O\mathcal{M}$ *for* $0 \leqslant i \leqslant d$ *and* $1 \leqslant j \leqslant d$, *and we have*:

$$\omega_\xi(z) = y^{-1} f(\xi T_z)\,\mathrm{d}y + y^{-1}\sum_{j=1}^{d} f_j(\xi T_z)\,\mathrm{d}x^j.$$

Proof. – By (5) we have $y\frac{\partial}{\partial y}(f \circ \tilde{\xi}) = (\mathcal{L}_0 f) \circ \tilde{\xi}$, and then $y\frac{\partial}{\partial y} \circ \tilde{\xi}^* = \tilde{\xi}^* \circ \mathcal{L}_0$, whence by duality $y^{-1}\,\mathrm{d}y = \tilde{\xi}^*\mathcal{L}_0'$, and then $(f \circ \tilde{\xi})y^{-1}\,\mathrm{d}y = \tilde{\xi}^*(f\mathcal{L}_0')$. Since the same works with $(f_j, \mathrm{d}x^j, \mathcal{L}_j)$, we get indeed the right expression for ω_ξ.

Then the commutation relation (1) between the flows implies that:

$$\mathcal{L}_j\mathcal{U}^q\phi(\xi) = \frac{\mathrm{d}_0}{\mathrm{d}t}\int_0^\infty \mathrm{e}^{-qs}\phi\big(\xi\theta_t^j\theta_s\big)\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-qs}\frac{\mathrm{d}_0}{\mathrm{d}t}\phi\big(\xi\theta_s\theta^j_{t\mathrm{e}^{-s}}\big)\,\mathrm{d}s = \mathcal{U}^{q+1}\mathcal{L}_j\phi(\xi)$$

for $1 \leqslant j \leqslant d$, and that:

$$\mathcal{L}_0\mathcal{U}^q\phi(\xi) = \int_0^\infty \mathrm{e}^{-qs}\frac{\mathrm{d}}{\mathrm{d}s}\phi(\xi\theta_s)\,\mathrm{d}s = q\mathcal{U}^q\phi(\xi) - \phi(\xi).$$

This implies existence and boundedness of the $\mathcal{L}_i f_k$, by taking $\phi = -\mathcal{L}_k f$, and also that $\mathcal{L}_j f_k - \mathcal{L}_k f_j = \mathcal{U}^2[\mathcal{L}_k, \mathcal{L}_j]f = 0$, and that $\mathcal{L}_0 f_j - \mathcal{L}_j f - f_j = 0$.

Therefore, using (5) again, we have finally:

$$\begin{aligned} y^2\,\mathrm{d}\omega_\xi &= \sum_{j=1}^{d}(\mathcal{L}_0 f_j - \mathcal{L}_j f - f_j)(\xi T_z)\,\mathrm{d}y \wedge \mathrm{d}x^j \\ &\quad + \sum_{1\leqslant j<k\leqslant d}(\mathcal{L}_j f_k - \mathcal{L}_k f_j)(\xi T_z)\,\mathrm{d}x^j \wedge \mathrm{d}x^k = 0. \qquad \square \end{aligned}$$

6.2. A contour deformation

We take here advantage of the closedness of the form ω_ξ to change the integration path in $\int_0^t f(\xi\theta_s)\,\mathrm{d}s$: we substitute the diffusion path $\{\xi_s^\Phi \mid 0 \leqslant s \leqslant (\delta - d/2)^{-1}t\}$ for the geodesic $\xi[0,t] := \{\xi\theta_s \mid 0 \leqslant s \leqslant t\}$. In this contour deformation three residual terms appear, that we prove to be negligible.

PROPOSITION 5. – *For any real* a, *the following difference goes to* 0 *as* $t \to \infty$:

$$c(\delta)\int_{T^1\mathcal{M}} \exp\left(\frac{a\sqrt{-1}}{\sqrt{t}}\int_0^t f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi) - \mathbb{E}\left[\int \exp\left(\frac{a\sqrt{-1}}{\sqrt{(\delta - d/2)t}}\int_{Z^\delta[-t,0]} \omega_\xi\right)\mathrm{d}\nu(\xi)\right].$$

Proof. – Let us set $t' := (\delta - d/2)t$. Using Corollary 2, we have almost surely:

$$c(\delta)\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}}\int_0^{t'} f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi) = \lim_{S\to\infty}\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}}\int_{\log y_S^\delta}^{t'+\log y_S^\delta} f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}\nu(\xi)$$
$$= \lim_{S\to\infty}\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}}\int_{(0,y_S^\delta)}^{(0,y_S^\delta \mathrm{e}^{t'})} \omega_\xi\right)\mathrm{d}\nu(\xi).$$

Then, using Lemma 3, we get:

$$\int_{(0,y_S^\delta)}^{(0,y_S^\delta \mathrm{e}^{t'})} \omega_\xi = \int_{Z_S^\delta}^{Z_{S+t}^\delta} \omega_\xi + \int_{(0,y_S^\delta)}^{Z_S^\delta} \omega_\xi - \int_{(0,y_{S+t}^\delta)}^{Z_{S+t}^\delta} \omega_\xi - \int_{(0,y_S^\delta \mathrm{e}^{t'})}^{(0,y_{S+t}^\delta)} \omega_\xi,$$

and since f_j is bounded:

$$\int_{(0,y_u^\delta)}^{Z_u^\delta} \omega_\xi = \left(y_u^\delta\right)^{-1}\sum_{j=1}^{d}\int_0^1 \left(x_u^\delta\right)^j \times f_j\left(\xi\theta^j_{s(x_u^\delta)^j}\theta_{\log y_u^\delta}\right)\mathrm{d}s = \mathcal{O}\left(\left|\frac{x_u^\delta}{y_u^\delta}\right|\right)$$
$$\overset{\text{law}}{=} |W_1| \times \mathcal{O}\left(\int_0^u \left(y_s^\delta\right)^2\left(y_u^\delta\right)^{-2}\mathrm{d}s\right)^{1/2} \overset{\text{law}}{=} |W_1| \times \mathcal{O}\left(\int_0^u \mathrm{e}^{2w_s-(2\delta-d)s}\,\mathrm{d}s\right)^{1/2},$$

which is almost surely bounded. Hence, uniformly with respect to $S \geqslant 0$ and to $\xi \in T^1\mathcal{M}$, $t^{-1/2}\int_{(0,y_S^\delta)}^{Z_S^\delta}\omega_\xi$ and $t^{-1/2}\int_{(0,y_{S+t}^\delta)}^{Z_{S+t}^\delta}\omega_\xi$ go to zero in probability as $t \to \infty$.

Then we have:

$$\int_{(0,y_S^\delta \mathrm{e}^{t'})}^{(0,y_{S+t}^\delta)} \omega_\xi = \int_{w_S+(\delta-d/2)(S+t)}^{w_{S+t}+(\delta-d/2)(S+t)} f(\xi\theta_s)\,\mathrm{d}s = \int_0^{w_{S+t}-w_S} f(\xi\theta_{s+w_S+(\delta-d/2)(S+t)})\,\mathrm{d}s,$$

and thus for some standard Brownian motion w' independent of w we have:

$$\mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}}\int_{(0,y_S^\delta \mathrm{e}^{t'})}^{(0,y_{S+t}^\delta)} \omega_\xi\right)\mathrm{d}\nu(\xi)\right] = \mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}}\int_0^{w'_t} f(\xi\theta_{s+w_S+(\delta-d/2)(S+t)})\,\mathrm{d}s\right)\mathrm{d}\nu(\xi)\right]$$

which, thanks to Corollary 2, goes as $S \to \infty$ to

$$c(\delta)\mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}}\int_0^{w'_1\sqrt{t}} f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi)\right] \overset{t\to\infty}{\longrightarrow} c(\delta)\mathbb{E}\left[\int \exp\left(\sqrt{-1}w'_1\int f\,\mathrm{d}m\right)\mathrm{d}m\right]$$

by ergodicity, the last quantity being equal to $\|\Phi\|_2^2$, since $\int f\,\mathrm{d}m = 0$. Finally

$$\lim_{t\to\infty} c(\delta)\mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}}\int_0^t f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi)\right]$$
$$=\lim_{t\to\infty}\lim_{S\to\infty}\mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}}\int_{Z^\delta_S}^{Z^\delta_{S+t}}\omega_\xi\right)\mathrm{d}\nu(\xi)\right]$$
$$=\lim_{t\to\infty}\mathbb{E}\left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}}\int_{Z^\delta_{-t}}^{Z^\delta_0}\omega_\xi\right)\mathrm{d}\nu(\xi)\right]$$

by Corollary 3 . □

Now observe that by Definition 4 we have:

$$\int_{Z^\delta[-t,0]}\omega_\xi=\int_{\tilde{\xi}(Z^\delta[-t,0])}\omega=\int_{\xi^\delta[-t,0]}\omega.$$

Hence by reversing the time we deduce from Corollary 3 and Proposition 5 the following:

COROLLARY 5. – *We have for any real a,*

$$\lim_{t\to\infty}\left\{c(\delta)\int_{T^1\mathcal{M}}\exp\left(\frac{a\sqrt{-1}}{\sqrt{t}}\int_0^t f(\xi\theta_s)\,\mathrm{d}s\right)\mathrm{d}m(\xi)-\nu\otimes\mathbb{E}\left[\exp\left(\frac{-a\sqrt{-1}}{\sqrt{(\delta-d/2)t}}\int_{\xi^\Phi[0,t]}\omega\right)\right]\right\}=0.$$

7. The central limit theorem on $T^1\mathcal{M}$

Let us fix a Borelian function f of class C^2 on $T^1\mathcal{M}$, with bounded and Hölderian derivatives, and such that $\int f\,\mathrm{d}m=0$.

Remark 7. – A careful reading of our arguments below shows that indeed the following slightly weaker regularity hypothesis is sufficient to guarantee our result:

(H) f and $\mathcal{L}_0 f$ are bounded, rotationally Hölderian on $T^1\mathcal{M}$, and continuous along the stable leaves, and, for $1\leqslant j\leqslant d$, $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded and Hölderian on $O\mathcal{M}$.

Recall that "F rotationally Hölderian on $T^1\mathcal{M}$" means: there exists some $r>0$ such that $d(g,\mathrm{SO}_d)^{-r}|F(\xi g)-F(\xi)|$ is bounded independently from $g\in\mathrm{SO}_{d+1}-\mathrm{SO}_d$ and $\xi\in T^1\mathcal{M}$, d denoting here any distance on SO_{d+1}, and that "F Hölderian on $O\mathcal{M}$" precisely means: there exists some $r>0$ such that $\mathrm{dist}(\xi,\xi')^{-r}|F(\xi)-F(\xi')|$ is bounded on $\{(\xi,\xi')\in O\mathcal{M}^2 \mid 0<\mathrm{dist}(\xi,\xi')<1\}$.

7.1. The divergence of ω with respect to ξ_t^Φ

LEMMA 4. – *If ϕ is a bounded Hölderian function on $O\mathcal{M}$, then $\mathcal{U}^1\phi$ and $\mathcal{U}^2\phi$ are bounded and Hölderian on $O\mathcal{M}$.*

Proof. – Let us consider $\varphi_s := \exp[s\sum_{j=1}^d a_j\mathcal{L}_{0,j}]$, for $s>0$ and for fixed real a_j's. Fix an Hölder exponent for ϕ, say $r\in]0,1[$. We have for $q=1$ or 2:

$$s^{-r}\left|\mathcal{U}^q\phi(\xi\varphi_s)-\mathcal{U}^q\phi(\xi)\right| \leqslant \int_0^\infty e^{-qt}s^{-r}\left|\phi(\xi\varphi_s\theta_t)-\phi(\xi\theta_t)\right| dt,$$

which is bounded with respect to (s,ξ) if $\int_0^\infty e^{(r-q)t}s^{-r}|\phi(\xi\varphi_{se^{-t}}\theta_t)-\phi(\xi\theta_t)|\,dt$ is, and thus if $s^{-r}|\phi(\xi\varphi_{se^{-t}}\theta_t)-\phi(\xi\theta_t)|$ is bounded with respect to (s,t,ξ).

Now this will be so if we prove that $D(\theta_{-t}\varphi_{se^{-t}}\theta_t,\mathrm{Id})=\mathcal{O}(s)$, u niformly with respect to t, D denoting some metric on G, which we can choose left invariant.

Then observe that it is sufficient to consider the case of $\varphi_s=\exp[s\mathcal{L}_{0,j}]$.

Now we have the Campbell–Hausdorff formula: $\frac{d_0}{ds}(\theta_{-t}\exp(s\mathcal{L}_{0,j})\theta_t)=\exp[\mathrm{ad}(-t\mathcal{L}_0)]\mathcal{L}_{0,j}$, and (4) gives the matrix of $\mathrm{ad}(\mathcal{L}_0)$ in the base $(\mathcal{L}_{0,j},\mathcal{L}_j)$, which has its square equal to the unit matrix. So we see that $\exp[\mathrm{ad}(-t\mathcal{L}_0)]\mathcal{L}_{0,j}=e^t\mathcal{L}_{0,j}-(\mathrm{sh}\,t)\mathcal{L}_j$, and thus that

$$\frac{d}{ds}(\theta_{-t}\varphi_{se^{-t}}\theta_t)=\theta_{-t}\varphi_{se^{-t}}\theta_t\times\frac{d_0}{ds}(\theta_{-t}\varphi_{se^{-t}}\theta_t)=\theta_{-t}\varphi_{se^{-t}}\theta_t\times\left(\mathcal{L}_{0,j}-\left(\frac{1-e^{-2t}}{2}\right)\mathcal{L}_j\right),$$

which by left invariance of D shows the boundedness of $\frac{d}{ds}D(\theta_{-t}\varphi_{se^{-t}}\theta_t,\mathrm{Id})$. This shows that $\mathcal{U}^q\phi$ is rotationally Hölderian on $O\mathcal{M}$. Finally, observe that we get from (4):

$$\frac{d_0}{ds}\left(\theta_{-t}\exp(s\mathcal{L}_j)\theta_t\right)=e^{-t}\mathcal{L}_j$$

if $1\leqslant j\leqslant d$ (and $\mathcal{L}_0$ if $j=0$). Thus we get also the boundedness of $\frac{d}{ds}D(\theta_{-t}\exp(se^{-t}\mathcal{L}_j)\theta_t,\mathrm{Id})$, and of $\frac{d}{ds}D(\theta_{-t}\exp(se^{-t}\mathcal{L}_{i,k})\theta_t,\mathrm{Id})$, for $1\leqslant i<k\leqslant d$, since $[\mathcal{L}_0,\mathcal{L}_{i,k}]=0$. The result follows. □

Recall that the diffusion ξ_t^Φ admits the generator $\frac{1}{2}D^\Phi=\frac{1}{2}\sum_{j=0}^d\mathcal{L}_j^2+\sum_{j=0}^d h_j\mathcal{L}_j$, where $h_j:=\mathcal{L}_j(\log\Phi\circ\pi_2)-\frac{d}{2}1_{\{j=0\}}$, for $0\leqslant j\leqslant d$.

Now we have the following general lemma, more or less known:

LEMMA 5. – *Consider a* 1*-form Ω of class C^1 on $O\mathcal{M}$. We have for any $t\geqslant 0$*:

$$\int_{\xi^\Phi[0,t]}\Omega=M_t^\Omega+\int_0^t \mathrm{div}\,\Omega\left(\xi_s^\Phi\right)ds,\quad \textit{with}\quad \mathrm{div}\,\Omega=\frac{1}{2}\sum_{j=0}^d\mathcal{L}_j\left(\Omega(\mathcal{L}_j)\right)+\sum_{j=0}^d h_j\Omega(\mathcal{L}_j);$$

M_t^Ω is a continuous martingale having increasing process:

$$\left\langle M_t^\Omega\right\rangle=\int_0^t\sum_{j=0}^d\Omega(\mathcal{L}_j)^2\left(\xi_s^\Phi\right)ds.$$

Proof. – By linearity, it is sufficient to consider $\Omega = G\,\mathrm{d}F$, for F of class C^1 and G of class C^2 on $O\mathcal{M}$. Now we have by Itô formula:

$$\mathrm{d}F\big(\xi_s^{\Phi}\big) = \mathrm{d}M_s^F + \frac{1}{2}D^{\Phi}F\big(\xi_s^{\Phi}\big)\,\mathrm{d}s, \quad \text{with} \quad \mathrm{d}\big\langle M_s^F\big\rangle = \sum_{j=0}^{d}(\mathcal{L}_jF)^2\big(\xi_s^{\Phi}\big)\,\mathrm{d}s$$

and then

$$\int\limits_{\xi^{\Phi}[0,t]} \Omega = \int_0^t G\big(\xi_s^{\Phi}\big)\circ \mathrm{d}F\big(\xi_s^{\Phi}\big) = \int_0^t G\big(\xi_s^{\Phi}\big)\,\mathrm{d}M_s^F + \int_0^t \frac{G\times D^{\Phi}F}{2}\big(\xi_s^{\Phi}\big)\,\mathrm{d}s + \frac{1}{2}\big\langle M_t^G, M_t^F\big\rangle.$$

Therefore we get the formula of the statement, with:

$$2\,\mathrm{div}\,\Omega = G\times D^{\Phi}F + \sum_{j=0}^{d}\mathcal{L}_jG\times\mathcal{L}_jF \quad \text{and} \quad \mathrm{d}\big\langle M_s^{\Omega}\big\rangle = \sum_{j=0}^{d}(G\times\mathcal{L}_jF)^2\big(\xi_s^{\Phi}\big)\,\mathrm{d}s.$$

This gives the wanted formula, since $\Omega = G\,\mathrm{d}F$ and $D^{\Phi} = \sum_{j=0}^{d}\mathcal{L}_j^2 + 2\sum_{j=0}^{d}h_j\mathcal{L}_j$ imply:

$$\Omega(\mathcal{L}_j) = G\mathcal{L}_jF \quad \text{and}$$
$$\sum_{j=0}^{d}\mathcal{L}_j\big(\Omega(\mathcal{L}_j)\big) = G\sum_{j=0}^{d}\mathcal{L}_j^2F + \sum_{j=0}^{d}\mathcal{L}_jG\mathcal{L}_jF = 2\,\mathrm{div}\,\Omega - 2\sum_{j=0}^{d}h_j\Omega(\mathcal{L}_j). \qquad \square$$

Let us apply this Lemma 5 to ω (see Definition 4(iv)). Observe that this is licit by Lemma 4. We get

$$\text{(F)} \qquad \int\limits_{\xi^{\Phi}[0,t]} \omega = M_t^{\omega} + \int_0^t Kf\big(\xi_s^{\Phi}\big)\,\mathrm{d}s,$$

with

$$\text{(F}'\text{)} \quad Kf := \frac{1}{2}\sum_{j=0}^{d}\mathcal{L}_jf_j + \sum_{j=0}^{d}(\mathcal{L}_j\log\Phi\circ\pi_2)f_j - \frac{d}{2}f = \left(\frac{d}{2}-\delta\right)f - \frac{1}{2}\sum_{j=0}^{d}\mathcal{L}_jf_j - \sum_{j=0}^{d}\mathcal{L}_j^*f_j,$$

where $\mathcal{L}_j^* := -\mathcal{L}_j - \mathcal{L}_j(\log\Phi\circ\pi_2) + 1_{\{j=0\}}(d-\delta)$ is the adjoint of $\mathcal{L}_j$ with respect to ν (this is the infinitesimal version of Proposition 1), and where M_t^{ω} is a continuous martingale having its increasing process given by

$$\text{(F}''\text{)} \qquad \big\langle M_t^{\omega}\big\rangle = \int_0^t \sum_{j=0}^{d} f_j^2\big(\xi_s^{\Phi}\big)\,\mathrm{d}s.$$

LEMMA 6. – *The function Kf is SO_d-invariant, bounded, and Hölderian.*

Proof. – We already observed in Lemma 3 that the f_j and the $\mathcal{L}_jf_j$ are bounded. Moreover we have also that the $\mathcal{L}_k\log\Phi\circ\pi_2$ and $\mathcal{L}_{0,j}\mathcal{L}_k\log\Phi\circ\pi_2$ are bounded, showing the boundedness of Kf. Indeed, by (4) and since $\mathcal{L}_{0,j}\log\Phi\circ\pi_2 = 0$, it is sufficient to verify the first assertion.

Now it is straightforward to see that $|y\frac{\partial}{\partial y}\log p(z,v)| \leqslant 1$ and $|y\frac{\partial}{\partial x^j}\log p(z,v)| \leqslant 1$, which in turn implies that $|\mathcal{L}_j \log \Phi \circ \pi_2| \leqslant \delta$, using that by (5)

$$\mathcal{L}_j \log \Phi \circ \pi_2(\xi) = \frac{\partial_0}{\partial x^j} \int \frac{p^\delta(\xi((x,1)),u)}{p^\delta(e_0,u)}\, \mathrm{d}\mu_{e_0}(u) = \frac{\partial_0}{\partial x^j} \int \frac{p^\delta((x,1),\xi^{-1}(u))}{p^\delta(\xi^{-1}(e_0),\xi^{-1}(u))}\, \mathrm{d}\mu_{e_0}(u).$$

Then the SO_d-invariance follows from the observation that ξ_t^Φ is SO_d-invariant, and that the form ω is equivariant with respect to the SO_d-action on $O\mathcal{M}$.

It remains to show that Kf is Hölderian. Now, by the beginning of this proof, we only have to ensure that $\mathcal{U}^2\mathcal{L}_j^2 f$ and $\mathcal{U}^1\mathcal{L}_j f$ are Hölderian. But this follows immediately from Lemma 4. □

PROPOSITION 6. – *We have* $\int Kf\, d\nu = 0$.

Proof. – By the very definition (4(iii)) of f_j, we have:

$$\begin{aligned} f_j(\xi) &= -\lim_{S\to\infty}\int_0^S \mathrm{e}^{-s}\mathcal{L}_j f(\xi\theta_s)\,\mathrm{d}s = -\lim_{S\to\infty}\int_0^S \mathrm{e}^{-s}\frac{\mathrm{d}_0}{\mathrm{d}t} f\big(\xi\theta_s\theta_t^j\big)\,\mathrm{d}s \\ &= -\lim_{S\to\infty}\int_0^S \mathrm{e}^{-s}\frac{\mathrm{d}_0}{\mathrm{d}t} f\big(\xi\theta^j_{t\mathrm{e}^s}\theta_s\big)\,\mathrm{d}s = -\lim_{S\to\infty}\int_0^S \frac{\mathrm{d}_0}{\mathrm{d}t} f\big(\xi\theta_t^j\theta_s\big)\,\mathrm{d}s \\ &= -\lim_{S\to\infty}\mathcal{L}_j\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right)(\xi). \end{aligned}$$

Whence by formula (6) (defining D^δ, in Section 4.2)

$$\begin{aligned} \sum_{j=1}^d \mathcal{L}_j f_j &= -\lim_{S\to\infty}\sum_{j=1}^d \mathcal{L}_j^2\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right) = \lim_{S\to\infty}\big[-D^\delta + \mathcal{L}_0^2 + (2\delta-d)\mathcal{L}_0\big]\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right) \\ &= \lim_{S\to\infty}\left(-D^\delta\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right) + \int_0^S\big[\mathcal{L}_0^2 + (2\delta-d)\mathcal{L}_0\big] f(\cdot\,\theta_s)\,\mathrm{d}s\right) \\ &= \lim_{S\to\infty}\left(-D^\delta\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right) + \mathcal{L}_0 f(\cdot\,\theta_S) - \mathcal{L}_0 f + (2\delta-d)f(\cdot\,\theta_S) - (2\delta-d)f\right). \end{aligned}$$

Therefore:

$$\left(\frac{d}{2}-\delta\right)f - \frac{1}{2}\sum_{j=0}^d \mathcal{L}_j f_j = \lim_{S\to\infty}\left(\frac{1}{2}D^\delta\left(\int_0^S f(\cdot\,\theta_s)\,\mathrm{d}s\right) - \frac{1}{2}\mathcal{L}_0 f(\cdot\,\theta_S) + \left(\frac{d}{2}-\delta\right)f(\cdot\,\theta_S)\right).$$

Finally, using the duality with respect to ν, that is to say Proposition 2, the fact that $D^\Phi 1 = \mathcal{L}_j 1 = 0$, Corollary 2, and the hypothesis (H) on f, we deduce that:

$$\int Kf\,\mathrm{d}\nu = \lim_{S\to\infty}\left(\left(\frac{d}{2}-\delta\right)\int f\,\mathrm{d}\theta_S^*\nu - \frac{1}{2}\int \mathcal{L}_0 f\,\mathrm{d}\theta_S^*\nu\right)$$

$$= \left(\frac{\mathrm{d}}{2} - \delta\right) \int f\,\mathrm{d}m - \frac{1}{2} \int \mathcal{L}_0 f\,\mathrm{d}m = 0. \qquad \square$$

7.2. End of proof of the main result (the theorem in Section 1)

Let us fix a sequence $\{F_n \mid n \in \mathbb{N}\}$ of compactly supported and smooth functions on $T^1\mathcal{M}$, such that:

(i) F_n converges in $L^2(\nu)$ towards Kf as $n \to \infty$;
(ii) each F_n is Hölderian, with the same constants as Kf;
(iii) the F_n are uniformly bounded and have zero mean with respect to ν.

Note that this is possible by using Lemma 6 and Proposition 6, and by using a partition of unity and some convolution. Observe then that by Theorem 2 and its proof (see Section 5), there exists some $\varrho > 0$ such that

$$\|Q_t^{\Phi} F\|_2 \leqslant \varrho^{-1} \mathrm{e}^{-\varrho t} \quad \text{for } t \geqslant 0 \text{ and } F = (Kf \text{ or any } F_n). \tag{7}$$

Definition 5. – *Set* $V_b F := \int_0^b Q_t^{\Phi} F\,\mathrm{d}t$, *for* $0 < b < \infty$ *and* $F = (Kf$ *or any* $F_n)$, *and write* VF *for* $V_\infty F$.

Formula (7) shows that the convergence of VF holds in $L^2(\nu)$. Moreover, it shows that for $b > 0$ and $F = (Kf$ or $F_n)$, we have:

$$\|(V - V_b)F\|_2 \leqslant \int_b^\infty \|Q_t^{\Phi} F\|_2\,\mathrm{d}t \leqslant \varrho^{-2} \mathrm{e}^{-\varrho b}$$

and in the same vein:

$$\|VKf - VF_n\|_2 \leqslant \int_0^\infty \|Q_t^{\Phi}(Kf - F_n)\|_2\,\mathrm{d}t \leqslant \int_0^\infty \min\{\|Kf - F_n\|_2, 2\varrho^{-1}\mathrm{e}^{-\varrho t}\}\,\mathrm{d}t$$
$$= \varrho^{-1}\|Kf - F_n\|_2 \log\bigl(2\,\mathrm{e}\,\varrho^{-1}\|Kf - F_n\|_2^{-1}\bigr).$$

This shows that $V_b F_n$ converges towards VKf in $L^2(\nu)$ as $b, n \uparrow \infty$.

Lemma 7. – *For any smooth bounded function F with bounded derivatives on $O\mathcal{M}$ and for any $t \geqslant 0$, $Q_t^{\Phi} F$ is also smooth on $O\mathcal{M}$, with bounded $\mathcal{L}_j$-derivatives.*

Proof. – Let us first observe that for any $n, k \in \mathbb{N}$, $\xi \in O\mathcal{M}$ and $t \geqslant 0$, we have $\mathbb{E}(|x_t^0|^n (y_t^0)^{-k} \Phi \circ \pi_2(\xi_t^0)) < \infty$. (Recall from Section 4.1 that $Z_t^0 = (x_t^0, y_t^0)$ and from Remark 4 that $\xi_t^0 = \xi T_{Z_t^0}$.) Indeed we use Schwarz inequality, and the two following facts:

- on one hand $\mathbb{E}(\Phi^2 \circ \pi_2(\xi_t^0)) = P_t \Phi^2 \circ \pi_2(\xi)$ is continuous and λ'-integrable, by invariance of the Brownian semi-group P_t, and thus finite;
- on the other hand it is clear from the expression of y_t^0 that $\mathbb{E}((y_t^0)^k)$ is finite for any $k \in \mathbb{Z}$, and for any $n \in \mathbb{N}$, we have using Doob inequality:

$$\mathbb{E}\bigl(|x_t^0|^{2n}\bigr) \leqslant c_n \mathbb{E}\left[\left(\int_0^t \exp[2w_s - ds]\,\mathrm{d}s\right)^n\right] \leqslant c_n t^n \mathbb{E}\bigl[\sup_{0\leqslant s\leqslant t} \exp[2nw_s - nds]\bigr]$$
$$\leqslant c_n(t)\mathbb{E}\Bigl[\Bigl(\sup_{0\leqslant s\leqslant t} \exp\bigl[nw_s - n^2 s/2\bigr]\Bigr)^2\Bigr] \leqslant c_n'(t)\mathbb{E}\Bigl[\exp\bigl[2nw_t - n^2 t\bigr]\Bigr] < \infty.$$

Then, recall from the proof of Lemma 6 that the $\mathcal{L}_j$-derivatives of $\log \Phi \circ \pi_2$ are bounded by δ. Thus, we see that $\mathcal{L}_j(\Phi \circ \pi_2 \times F)/\Phi \circ \pi_2$ is smooth bounded for $0 \leqslant j \leqslant d$, and that $\Phi \circ \pi_2(\xi\theta_s^j) \leqslant e^{\delta}\Phi \circ \pi_2(\xi)$ for $0 \leqslant j \leqslant d$ (here $\theta_s^0 := \theta_s$), $|s| \leqslant 1$ and $\xi \in O\mathcal{M}$.

Observe yet that by the commutation formula (2) we have: $\theta_s^j T_{Z_t^0} = T_{Z_t^0}\theta^j_{s/y_t^0}$ and $\theta_s T_{Z_t^0} = T_{Z_t^0}\theta^+_{(e^s-1)x_t^0/y_t^0}\theta_s$. Therefore we can differentiate under the expectation and get by using Remark 4 and the above:

$$\mathcal{L}_0 Q_t^{\Phi} F(\xi) = \frac{e^{-\lambda_0 t}}{\Phi(\xi)} \times \Bigg(\mathbb{E}\Bigg[\Bigg(\mathcal{L}_0 + \big(y_t^0\big)^{-1}\sum_{j=1}^{d}\big(x_t^0\big)^j \mathcal{L}_j\Bigg)(\Phi F)\big(\xi_t^0\big)\Bigg] - \mathbb{E}\big[(\Phi F)\big(\xi_t^0\big)\big]\mathcal{L}_0 \log\Phi(\xi)\Bigg)$$

(we write here Φ for $\Phi \circ \pi_2$) and for $1 \leqslant j \leqslant d$:

$$\mathcal{L}_j Q_t^{\Phi} F(\xi) = e^{-\lambda_0 t}\Phi(\xi)^{-1} \times \big(\mathbb{E}\big[\big(y_t^0\big)^{-1}\mathcal{L}_j(\Phi F)\big(\xi_t^0\big)\big] - \mathbb{E}\big[(\Phi F)\big(\xi_t^0\big)\big]\mathcal{L}_j \log\Phi(\xi)\big).$$

This proves the existence of the first-order derivatives, and also of the higher-order derivatives, since they will have the same form as the first-order ones above, but with powers of $(x_t^0)^j$ and $(y_t^0)^{-1}$.

To prove that $\mathcal{L}_j Q_t^{\Phi} F$ is bounded, we use again that $\mathcal{L}_0 \log \Phi \circ \pi_2$ is bounded by δ, to get the uniform estimate: $\Phi \circ \pi_2(\xi_t^0)/\Phi \circ \pi_2(\xi) \leqslant \exp[\delta \times \mathrm{dist}_{\mathcal{M}}(\pi_2(\xi_t^0), \pi_2(\xi))]$. Now recall from Section 4.1 that $\pi_2(\xi_t^0) = \xi(Z_t^0)$ is a Brownian motion on $\mathcal{M}$ starting from $\pi_2(\xi) = \xi(e_0)$, so using that ξ is an isometry, we get:

$$\Phi \circ \pi_2\big(\xi_t^0\big)/\Phi \circ \pi_2(\xi) \leqslant \exp\big[\delta \times \mathrm{dist}_{\mathbb{H}}\big(e_0, Z_t^0\big)\big].$$

Hence we see, using the above expressions for $\mathcal{L}_j Q_t^{\Phi} F$ and the Schwarz inequality, that the proof will be complete if we show that $\mathbb{E}(\exp[2\delta \times \mathrm{dist}_{\mathbb{H}}(e_0, Z_t^0)])$ is finite for any t. Now, we have (using the classical formula for the distance, see [13]):

$$\exp\big[2\delta \times \mathrm{dist}_{\mathbb{H}}\big(e_0, Z_t^0\big)\big] \leqslant 4^{\delta} ch^{2\delta}\big[\mathrm{dist}_{\mathbb{H}}\big(e_0, Z_t^0\big)\big] = \big(\big|x_t^0\big|^2 + \big(y_t^0\big)^2 + 1\big)^{2\delta} \times \big(y_t^0\big)^{-2\delta}$$

and thus we only have to use again that $\mathbb{E}(|x_t^0|^n (y_t^0)^k)$ is finite, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. □

This Lemma 7 allows to write for any $b > 0$ and $n \in \mathbb{N}$:

$$Q_b^{\Phi} F_n - F_n = \int_0^b \frac{d}{dt} Q_t^{\Phi} F_n \, dt = \frac{1}{2}\int_0^b D^{\Phi} Q_t^{\Phi} F_n \, dt = \frac{1}{2} D^{\Phi} V_b F_n,$$

whence:

$$D^{\Phi} V_b F_n = 2\big(Q_b^{\Phi} F_n - F_n\big). \tag{8}$$

Thus replacing ω by $\omega + d(V_b F_n)$ in the formula (F) of Section 7.1, we get:

$$\int_{\xi^{\Phi}[0,t]} \omega + V_b F_n\big(\xi_t^{\Phi}\big) - V_b F_n(\xi) = M_t^{b,n} + \int_0^t \big(Kf - F_n + Q_b^{\Phi} F_n\big)\big(\xi_s^{\Phi}\big)\, ds, \tag{9}$$

where $M_t^{b,n}$ is a continuous martingale having as increasing process:

$$\langle M_t^{b,n}\rangle = \int_0^t \sum_{j=0}^d (f_j + \mathcal{L}_j V_b F_n)^2(\xi_s^\Phi)\,\mathrm{d}s. \tag{10}$$

We want to go to the limit in the formula (9) above, as $b, n \uparrow \infty$. But for that, we need to control also the $\mathcal{L}_j$-derivatives, in the convergence of $V_b F_n$ to VKf.

LEMMA 8. – *The potential VKf admits $\mathcal{L}_j$-derivatives in $L^2(\nu')$, and $\mathcal{L}_j VKf$ is the limit in $L^2(\nu')$ of $\mathcal{L}_j V_b F_n$ as $b, n \uparrow \infty$, for $0 \leqslant j \leqslant d$.*

Proof. – Let us first observe that by using the adjoint $\mathcal{L}_j^*$ mentioned in the formula (F′) of Section 7.1, we have: $\sum_{j=0}^d \mathcal{L}_j^* \mathcal{L}_j = -(D^\Phi + D^\delta)/2$, and that by using Proposition 2 this implies: $\int \sum_{j=0}^d |\mathcal{L}_j F|^2\,\mathrm{d}\nu' = -\int F \times D^\Phi F\,\mathrm{d}\nu'$.

Using this and formulas (8), (7), we get on one hand, for any $b, c > 0$ and $n \in \mathbb{N}$:

$$\begin{aligned}\int \sum_{j=0}^d \big|\mathcal{L}_j (V_b - V_c) F_n\big|^2\,\mathrm{d}\nu' &= -\int (V_b - V_c) F_n \times D^\Phi (V_b - V_c) F_n\,\mathrm{d}\nu' \\ &= \int (V_c - V_b) F_n \times 2\big(Q_b^\Phi - Q_c^\Phi\big) F_n\,\mathrm{d}\nu' \\ &\leqslant \big\|(V_b - V_c) F_n\big\|_2 \times 2\big(\big\|Q_b^\Phi F_n\big\|_2 + \big\|Q_c^\Phi F_n\big\|_2\big) \\ &\leqslant \|V F_n\|_2 \times 2\varrho^{-1}\big(\mathrm{e}^{-\varrho b} + \mathrm{e}^{-\varrho c}\big) \leqslant 4\varrho^{-3}\big(\mathrm{e}^{-\varrho b} + \mathrm{e}^{-\varrho c}\big);\end{aligned}$$

and on the other hand, for any $b > 0$ and $n, p \in \mathbb{N}$:

$$\begin{aligned}\int \sum_{j=0}^d \big|\mathcal{L}_j V_b (F_n - F_p)\big|^2\,\mathrm{d}\nu' &= -\int V_b (F_n - F_p) \times D^\Phi V_b (F_n - F_p)\,\mathrm{d}\nu' \\ &= -\int V_b (F_n - F_p) \times 2\big(Q_b^\Phi (F_n - F_p) - (F_n - F_p)\big)\,\mathrm{d}\nu' \\ &\leqslant 2\big(\|V_b F_n\|_2 + \|V_b F_p\|_2\big) \times \big(\big\|Q_b^\Phi (F_n - F_p)\big\|_2 + \big\|(F_n - F_p)\big\|_2\big) \\ &\leqslant 8\varrho^{-2} \times \big\|(F_n - F_p)\big\|_2.\end{aligned}$$

This shows that $\mathcal{L}_j V_b F_n$ is Cauchy in $L^2(\nu')$, and then proves the result. □

We can now go to the limit in formulas (9) and (10), thereby showing the following:

PROPOSITION 7. – *We have $\int_{\xi^\Phi[0,t]} \omega = VKf(\xi) - VKf(\xi_t^\Phi) + M_t$, where M_t is a continuous martingale, with as increasing process*

$$\langle M_t\rangle = \int_0^t \sum_{j=0}^d (f_j + \mathcal{L}_j VKf)^2(\xi_s^\Phi)\,\mathrm{d}s.$$

COROLLARY 6. – *As $t \to \infty$, the law of $t^{-1/2} \int_{\xi^\Phi[0,t]} \omega$ converges towards the centered Gaussian law with variance $\mathcal{V}(f) := \int \sum_{j=0}^d (f_j + \mathcal{L}_j VKf)^2\,\mathrm{d}\nu'$, which vanishes if and only if f equals $\mathcal{L}_0 h$, for some $h \in L^2(T^1\mathcal{M}, \nu)$.*

Proof. – Firstly, $t^{-1/2}(VKf(\xi) - VKf(\xi_t^\Phi))$ goes to zero in $L^2(\nu \otimes \mathbb{P})$-probability as $t \to \infty$, since $VKf(\xi_t^\Phi)$ is stationary. Hence, applying Proposition 7, we see that we have to deal with $\lim_{t\to\infty} \nu' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t/\sqrt{t})]$. Now, observe that by Lemma 8 we have $\lim_{\{b,n\uparrow\infty\}} \|(M_t - M_t^{b,n})/\sqrt{t}\|^2_{L^2(\nu'\otimes\mathbb{P})} = \lim_{\{b,n\uparrow\infty\}} \sum_{j=0}^d \|\mathcal{L}_j VKf - \mathcal{L}_j V_b F_n\|^2_{L^2(\nu')} = 0$. Thus

$$
\begin{aligned}
&\lim_{t\to\infty} \nu' \otimes \mathbb{E}\big[\exp\big(a\sqrt{-1}M_t/\sqrt{t}\big)\big] \\
&\quad = \lim_{b,n\uparrow\infty} \lim_{t\to\infty} \nu' \otimes \mathbb{E}\big[\exp\big(a\sqrt{-1}M_t^{b,n}/\sqrt{t}\big)\big] \qquad (11)\\
&\quad = \lim_{b,n\uparrow\infty} \lim_{t\to\infty} \nu' \otimes \mathbb{E}\big[\exp\big(a\sqrt{-1}M_t^{b,n}/\sqrt{t} + a^2\langle M_t^{b,n}\rangle/2t\big)\exp\big(-a^2\langle M_t^{b,n}\rangle/2t\big)\big].
\end{aligned}
$$

We now need some ergodic property for the degenerate diffusion ξ_t^Φ, which is not clear in the present context. So let us show that $\langle M_t^{b,n}\rangle/t$ converges in $L^2(\nu' \otimes \mathbb{P})$ as $t \to \infty$, towards $\mathcal{V}_{b,n}(f) := \int \sum_{j=0}^d (f_j + \mathcal{L}_j V_b F_n)^2 \,\mathrm{d}\nu'$, for every $b > 0$ and $n \in \mathbb{N}$.

Indeed, we see from formula (10) that $\langle M_t^{b,n}\rangle/t - \mathcal{V}_{b,n}(f) = t^{-1}\int_0^t H_{b,n}(\xi_s^\Phi)\,\mathrm{d}s$, with $H_{b,n}$ smooth bounded on $O\mathcal{M}$ by Lemma 7, and such that $\int H_{b,n}\,\mathrm{d}\nu' = 0$.

Then applying Itô's Formula to $V_c H_{b,n}(\xi_t^\Phi)$, and going to the limit as $c \to \infty$, we get as for the proof of Proposition 7:

$$
\int_0^t H_{b,n}\big(\xi_s^\Phi\big)\,\mathrm{d}s = VH_{b,n}(\xi) - VH_{b,n}\big(\xi_t^\Phi\big) + M_t^{b,n,\infty},
$$

where $M_t^{b,n,\infty}$ is a continuous martingale having as increasing process

$$
\big\langle M_t^{b,n,\infty}\big\rangle = \int_0^t \sum_{j=0}^d (\mathcal{L}_j VH_{b,n})^2\big(\xi_s^\Phi\big)\,\mathrm{d}s.
$$

Therefore

$$
\left\| t^{-1}\int_0^t H_{b,n}\big(\xi_s^\Phi\big)\,\mathrm{d}s \right\|^2_{L^2(\nu'\otimes\mathbb{P})} \leqslant 4t^{-2}\|VH_{b,n}\|_2^2 + 2t^{-1}\sum_{j=0}^d \|\mathcal{L}_j VH_{b,n}\|_2^2 = \mathcal{O}\big(t^{-1}\big).
$$

As a consequence, we get the convergence in $L^1(\nu' \otimes \mathbb{P})$ of $\exp(-a^2\langle M_t^{b,n}\rangle/2t)$ towards $\exp(-a^2\mathcal{V}_{b,n}(f)/2)$. Finally, we know from Lemma 7 that for each $b > 0, n \in \mathbb{N}$, $\langle M_t^{b,n}\rangle/t$ is uniformly bounded, and thus that the continuous martingale $\exp(a\sqrt{-1}M_t^{b,n}/\sqrt{t} + a^2\langle M_t^{b,n}\rangle/2t)$ is bounded and has expected value one.

Hence we conclude from formula (11) and Lemma 8, by:

$$
\lim_{t\to\infty} \nu' \otimes \mathbb{E}\big[\exp\big(a\sqrt{-1}M_t/\sqrt{t}\big)\big] = \lim_{b,n\uparrow\infty} \exp\big(-a^2\mathcal{V}_{b,n}(f)/2\big) = \exp\big(-a^2\mathcal{V}(f)/2\big). \qquad \square
$$

Finally, the central limit theorem stated in the introduction follows from Corollaries 1, 5 and 6, even under the slightly weaker assumption (H) of Remark 7: f and $\mathcal{L}_0 f$ are bounded, rotationally Hölderian on $T^1\mathcal{M}$, and continuous along the stable leaves, and, for $1 \leqslant j \leqslant d$, $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded and Hölderian on $O\mathcal{M}$.

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