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Stable windings on hyperbolic surfaces

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Abstract. Let $\mathscr{M}$ be a geometrically finite hyperbolic surface with infinite volume, having at least one cusp. We obtain the limit law under the Patterson-Sullivan measure on $T^1\mathscr{M}$ of the windings of the geodesics of $\mathscr{M}$ around the cusps. This limit law is stable with parameter $2\delta - 1$, where δ is the Hausdorff dimension of the limit set of the subgroup Γ of Möbius isometries associated with $\mathscr{M}$. The normalization is $t^{-1/(2\delta-1)}$, for geodesics of length t. Our method relies on a precise comparison between geodesics and diffusion paths, for which we need to approach the Patterson-Sullivan measure mentioned above by measures that are regular along the stable leaves.

1. Introduction

The study of the geodesic flow in constant negative curvature and finite volume started with the proof of its chaotic behavior by Hadamard. Later Hopf proved its ergodicity, starting a long line of results on the stochastic properties of hyperbolic dynamical systems: a central limit theorem was proved by [Rat] and [Si] in the compact case, was then extended in particular in [D-P], and a large deviation extension of the central limit theorem was also obtained more recently in [W].

A first result on geodesic windings was established in [G-LJ], by means of a coding method. Such coding method is also applied to counting closed geodesics in homology classes, for example by [K-S], [La], [P-P], and recently by [B-P] in the case of Schottky groups. See also [A-D] for related results.

A series of works ([LJ1], [LJ2], [Le], [E], [E-LJ], [F2]) uses another approach, based on comparison between geodesics and Brownian paths. These works mostly deal with a hyperbolic manifold $\mathscr{M}$ of finite volume, that can be represented by the quotient of the hyperbolic space $\mathbb{H}$ under the action of a geometrically finite discrete subgroup Γ of isometries.

When Γ is not cofinite, an important role is played by the Hausdorff dimension δ of the limit set (see [P], [Su1] or [Su2]). When the manifold is a surface

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admitting at least one cusp (this will be our fundamental assumption), δ is known to be larger than $1/2$ and $\delta(\delta-1)$ is the highest eigenvalue of the Laplacian. The associated eigenstate Φ admits a representation by a measure μ on the boundary which can be used to construct a probability measure on $T^1\mathscr{M}$ (see [P], [Su1] or [Su2]). This probability measure, called Patterson-Sullivan measure (or sometimes Bowen-Margulis measure), is carried by the geodesics that do not end in the funnels, and is invariant under the geodesic flow.

Our goal in this article is to extend the methods and the results of the series of articles mentioned above to the infinite volume situation, where it is natural to substitute the Patterson-Sullivan measure for the Liouville measure.

We obtain the limit law under the Patterson-Sullivan measure on $T^1\mathscr{M}$ of the normalized integral along the geodesics of $\mathscr{M}$ of any differential 1-form, closed near the cusps. This limit law is stable with parameter $2\delta-1$, and with a rate that we compute explicitly. The normalization is $t^{\frac{-1}{2\delta-1}}$, for geodesics of length t.

The basic idea of the method is to transfer the problem from the geodesics to the paths of some diffusion. This diffusion can be obtained by lifting the diffusion associated with the fundamental state of $\mathscr{M}$ to the stable foliation. We then estimate the windings of the diffusion paths, continuing thereby a long series of works on stochastic windings (see in particular [Sp], [P-Y], [F1], [E]). This method also requires a new approximation procedure for the Patterson-Sullivan measure on $T^1\mathscr{M}$, as a limit of measures which are absolutely continuous and quasi-invariant along the stable leaves.

This strategy globally resembles the one which was already employed in [LJ1], [LJ2], [L], [E], [E-LJ], and [F2] for dealing with the finite volume case, but its implementation is here much more complicated. For example, the Brownian motion has to be replaced by the ground state diffusion, and, even after that, the geodesic flow and the diffusion lifted to the stable foliation do not have the same invariant measure anymore. Also, we need to look at excursions in each cusp relative to an approximating diffusion for which the calculations are possible, and that above a level which increases ultimately to infinity, in order to use the asymptotic form of the generator. Moreover, the different excursions do not contribute any longer independently. These difficulties lead to a lot of estimations, for which the half-plane model is convenient.

Let us describe in some extent how we proceed.

Firstly, since the Patterson-Sullivan measure m is too singular to be related to any good diffusion, we smoothen it, by some convolution along he stable horocycles, into a less singular probability measure ν, we prove to be quasi-invariant with respect to the geodesic and the stable horocycle flows. Moreover, we can recover the measure m as limit of the images of ν by the positive geodesic flow.

Then we introduce Z^δ, a Brownian motion with a constant drift on the Poincaré half-plane, which is a model for the generic stable leaf. Z^δ induces a diffusion ξ^δ on the tangent bundle $T^1\mathscr{M}$, which admits ν as an invariant measure, and projects onto the Φ-diffusion Z^Φ, associated with the fundamental eigenstate Φ of the Laplacian Δ on $\mathscr{M}$, and also called the infinite Brownian loop on $\mathscr{M}$ (see [A-B-J]).

Then we have to perform the substitution of the geodesics by the paths of the diffusion Z^Φ. This we do by means of a contour deformation, for which we have to use a first passage time τ_t for Z^δ, and to control two residual terms. In order to allow this contour deformation, we need to lift the given 1-form ω on $\mathscr{M}$ to a closed 1-form on each stable leaf.

So far, the problem relative to the geodesics is reduced to the problem relative to the Φ-diffusion. And this amounts to calculate the asymptotic law of windings of Z^Φ in the cusps of $\mathscr{M}$. In order to estimate those windings, we compute an asymptotic of Φ in each cusp, which depends only on the height in this cusp. We then replace the Φ-diffusion Z^Φ by a modified diffusion $\tilde{Z}$, which behaves in the cusps according to the computed asymptotic of Φ.

In the cusps, the modified diffusion $\tilde{Z}$ is a skew-product, hence we are able to compute the winding law of its excursions. However, to get independence between those excursions, we must show that the conditioning by the endpoints can be neglected. It remains mainly to estimate the frequency of excursions of Z^Φ in the cusps, and to show we can replace the numbers of achieved excursions at time τ_t by their deterministic equivalents.

The results of the present article have been partially announced in [E-F-LJ-1] and [E-F-LJ-2].

2. Notations, basic data, and main result

Let $\mathbb{H}$ denote the hyperbolic plane, with boundary $\partial\mathbb{H}$, unitary tangent bundle $T^1\mathbb{H}$, Riemannian area $d\tilde{V}$, and (hyperbolic) Laplacian Δ.

Given (z, z', u) in $\mathbb{H}\times\mathbb{H}\times\partial\mathbb{H}$, denote by $\log[B_u(z, z')]$ the Busemann function, that is to say the algebraic hyperbolic distance, on any geodesic ending at u, from the stable horocycle $H(z, u)$ determined by z to the stable horocycle $H(z', u)$. In the Poincaré half-plane model, we have $B_u(z, z') = p(z', u)/p(z, u)$, $p(z, u)$ denoting the Poisson kernel: $p(z, u) = \mathscr{I}m(z) \times |z-u|^{-2}$ if $u \neq \infty$ and $p(z, \infty) = \mathscr{I}m(z)$. We have the cocycle property:

$$B_u(z, z'') = B_u(z, z') \times B_u(z', z''). \tag{2.1}$$

Let Γ be a discrete torsion-free (non-elementary) group of Möbius isometries of $\mathbb{H}$, that we suppose geometrically finite. Let $\Lambda = \Lambda(\Gamma)$ denote its limit set, with Hausdorff dimension say δ. Recall that δ is also the critical convergence exponent of the Poincaré series relative to Γ; (see for example ([Su2], theorem 1)). Obviously $\delta \le 1$.

Let $\{\mu_z \mid z \in \mathbb{H}\}$ denote the family of Patterson (finite) measures on Λ associated with Γ. It can be defined, up to a multiplicative constant (that we definitively fix), as the only family of measures on Λ satisfying the following geometric "conformal density" property:

$$d\mu_{z'}(u) = B_u^\delta(z, z')\, d\mu_z(u) \qquad \text{for any } z, z' \text{ in } \mathbb{H} \tag{2.2}$$

together with the invariance property by the group Γ, in the sense that:

$$\gamma^* \mu_z = \mu_{\gamma z} \quad \text{for any } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \mathbb{H}, \tag{2.3}$$

with the convention $\gamma^* \mu := \mu \circ \gamma^{-1}$. See for example ([P], lecture 2), [Su2], or ([Ni], sections 3.4 and 4.7).

We shall be concerned with the hyperbolic quotient manifold $\mathscr{M} := \Gamma \backslash \mathbb{H}$, for which we make the additional *hypothesis*: $\mathscr{M}$ possesses at least one cusp.

It is known, due to Beardon, see for example ([Su2], page 265), that this implies: $\delta > 1/2$.
In particular

$$c(\delta) := \int_{\mathbb{R}} (1 + v^2)^{-\delta} \, dv = B(\delta - 1/2, 1/2)/2 \quad \text{is finite.} \tag{2.4}$$

We shall identify the functions on $\mathscr{M}$ (respectively on $T^1\mathscr{M}$) with the Γ-invariant functions on $\mathbb{H}$ (respectively on $T^1\mathbb{H}$).

Set

$$\Phi(z) := \int d\mu_z = \mu_z(\partial\mathbb{H}) = \mu_z(\Lambda), \quad \text{and} \quad \lambda_o := \delta(\delta - 1)/2. \tag{2.5}$$

This is a function on $\mathbb{H}$ which satisfies $\Delta\Phi = 2\lambda_o\Phi$. See ([P], theorem 1 page 301).
Note that for every γ in Γ we have

$$\Phi(\gamma z) = \gamma^* \mu_z(\partial\mathbb{H}) = \mu_z(\gamma^{-1}(\partial\mathbb{H})) = \Phi(z), \tag{2.6}$$

which shows that Φ is also a function on $\mathscr{M}$. Moreover, it is the fundamental eigenstate on $\mathscr{M}$. See ([P], theorem 1 page 301), or ([P-S], page 177). Note that an obvious corollary is that the volume of $\mathscr{M}$ is finite if and only if $\delta = 1$.
Let π denote the canonical projection from $T^1\mathbb{H}$ onto $\mathbb{H}$ as well as from $T^1\mathscr{M}$ onto $\mathscr{M}$. We shall use on the unitary tangent bundle $T^1\mathbb{H}$ the two following systems of coordinates:

- firstly, $(z, u) \in \mathbb{H} \times \partial\mathbb{H}$, the geodesic running from z to u determining the unitary tangent vector at the base point z; this identifies $T^1\mathbb{H}$ with $\mathbb{H} \times \partial\mathbb{H}$;
- secondly, given a reference point $z_0 \in \mathbb{H}$, the point (z, u) of $T^1\mathbb{H}$ (just defined above) can be represented by the triple $(u, v, s) \in \partial\mathbb{H} \times \partial\mathbb{H} \times \mathbb{R}$, where
  - v is the starting point of the geodesic ending at u and running through z;
  - s is the algebraic hyperbolic distance from z to the orthogonal projection z_1 of z_0 onto the geodesic $\overrightarrow{vu}$.

Denote by $dist(\zeta, uv)$ the hyperbolic distance from $\zeta \in \mathbb{H}$ to the geodesic $\overrightarrow{vu}$. The following well-known identity is valid for any ζ in $\mathbb{H}$, any distinct u, v in $\partial\mathbb{H}$, and any z on the geodesic $\overrightarrow{vu}$ running from v to u.

$$\cosh^2(dist(\zeta, uv)) = B_u(\zeta, z)\, B_v(\zeta, z). \tag{2.7}$$

(Indeed, since this is an intrinsic formula, we may consider the half-plane model with $u = \infty$ and $v = 0$. Denoting then by (X, Y) the Euclidean coordinates of ζ in this model, and by $(0, y)$ those of z, it is elementary that $B_u(\zeta, z) = y/Y$, $B_v(\zeta, z) = (X^2 + Y^2)/(yY)$, and, using the classical formula for the distance (see [P]), that $\cosh^2(dist(\zeta, uv)) = \cosh^2 dist(\zeta, (0, |\zeta|)) = (X^2 + Y^2)^2/(Y|\zeta|)^2 = 1 + X^2/Y^2 = B_u(\zeta, z)\, B_v(\zeta, z)$.)

Define the normalized Patterson-Sullivan measure $\tilde{m}$ on $T^1\mathbb{H}$ by

$$d\tilde{m}(u, v, s) := \|\Phi\|_2^{-2}\, c(\delta) \times \cosh^{2\delta}\big(dist(z_0, uv)\big)\, d\mu_{z_0}(u)\, d\mu_{z_0}(v)\, ds. \quad (2.8)$$

Note that by the geometric property (2.2) for (μ_z) and by the identity (2.7) above, $\tilde{m}$ does not depend on the choice of the reference point z_0. Hence it is intrinsic (it depends only on the subgroup Γ), and then it is Γ-invariant. Moreover it is plainly invariant with respect to the geodesic flow. It is sometimes also called (normalized) Bowen-Margulis measure.

The Liouville measure $\tilde{\lambda}$ on $T^1\mathbb{H}$ can be expressed in a similar way, for any reference point z_0, by:

$$d\tilde{\lambda}(u, v, s) = \cosh^2\big(dist(z_0, uv)\big)\, d\mu^h_{z_0}(u)\, d\mu^h_{z_0}(v)\, ds, \quad (2.9)$$

where μ^h_z denotes the harmonic measure at z. Recall that we have in the half-plane model: $d\mu^h_z(u) = p(z, u)\, du$. Note that the above geometric property for harmonic measures holds, by changing δ into 1: $d\mu^h_{z'}(u) = B_u(z, z')\, d\mu^h_z(u)$ for any z, z' in $\mathbb{H}$. As can be verified by a direct elementary computation, the expression of $\tilde{\lambda}$ in the (z, u) coordinates is: $d\tilde{\lambda}(z, u) = d\mu^h_z(u) d\tilde{V}(z)$.

Recall that any Γ-invariant measure $\tilde{n}$ on $T^1\mathbb{H}$ induces a measure n on $T^1\mathscr{M}$. In particular, denote by λ the Liouville measure induced by $\tilde{\lambda}$, and by m the Patterson-Sullivan measure induced by $\tilde{m}$. Similarly, denote by $dV = \pi^*\lambda$ the area measure on $\mathscr{M}$.

Recall that the Liouville measures are invariant by the horocycle and geodesic flows. But λ is finite only when $\delta = 1$, whereas m is always finite (see for example ([P], th.1 p. 309); see also corollary 1 below).

Observe from the two expressions above for the Liouville measure the following formula

$$\int F\, d\tilde{V} = \int F \circ \pi(u, v, s)\, B_v(z_0, \pi(u, v, s))\, d\mu^h_{z_0}(v)\, ds, \quad (2.10)$$

valid for any $u \in \partial\mathbb{H}$, any $z_0 \in \mathbb{H}$, and any test function F on $\mathbb{H}$.

Let θ_t and θ^+_t denote respectively the geodesic and the positive horocyclic flows, on $T^1\mathbb{H}$ and $T^1\mathscr{M}$ as well. Moreover for any $z = (x, y) \in \mathbb{R} \times \mathbb{R}^*_+$, set

$$T_z := \theta^+_x\, \theta_{Log\, y}. \quad (2.11)$$

Observe the following important classical relation:

$$T_{(x,y)}\, T_{z'} = T_{(x,0)\,+\,y\,z'}. \quad (2.12)$$

This means in particular that the set $\{T_z \,|\, z \in \mathbb{R} \times \mathbb{R}_+^*\}$ constitutes a group, isomorphic to the orientation preserving affine group of $\mathbb{R}$.

Let us introduce the Lie derivatives: for any smooth function F on $T^1\mathscr{M}$, any ξ in $T^1\mathscr{M}$, we set:

$$\mathscr{L}_0 F(\xi) := \frac{d_o}{dt} F\big(\xi\,\theta_t\big), \quad \mathscr{L}_1 F(\xi) := \frac{d_o}{dt} F\big(\xi\,\theta_t^+\big). \tag{2.13}$$

$\frac{d_o}{dt}$ means and will mean the derivative at $t = 0$ with respect to t. We immediately see that:

$$[\mathscr{L}_0, \mathscr{L}_1] = \mathscr{L}_1, \tag{2.14}$$

and

$$\mathscr{L}_0 F(\xi T_{(x,y)}) = y\,\frac{\partial}{\partial y} F(\xi T_{(x,y)}), \quad \mathscr{L}_1 F(\xi T_{(x,y)}) = y\,\frac{\partial}{\partial x} F(\xi T_{(x,y)}). \tag{2.15}$$

Since the flows act on the right hand side, while the quotient by Γ takes place on the left hand side, these two operations commute. Thus the flows indeed make sense on $T^1\mathscr{M}$, and the translated measures $T_z^*\tilde{m}$ and $T_z^*\tilde{\lambda}$ induce respectively T_z^*m and $T_z^*\lambda$ on $T^1\mathscr{M}$. In particular, to prove some formula relative to $T^1\mathscr{M}$, m, λ, V and to the flows, it will be sufficient to establish it (for any test-function) at the level of $T^1\mathbb{H}$, $\tilde{m}$, $\tilde{\lambda}$, $\tilde{V}$.

We are now able to state our main result:

Theorem. *For any 1-form ω on $\mathscr{M}$, closed in some neighborhood $\mathscr{N}$ of the cusps of $\mathscr{M}$ and m-integrable in $\mathscr{M} \setminus \mathscr{N}$, having residue $r(\mathscr{P}_i, \omega)$ at the cusp $\mathscr{P}_i$, and for any real α, we have:*

$$\lim_{t\to\infty} \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\,\alpha\, t^{\frac{-1}{2\delta-1}} \int_{\xi[0,t]} \omega \right] m(d\xi) = \exp\left[-|\alpha|^{(2\delta-1)}\, C'(\Gamma,\omega)\right],$$

where $C'(\Gamma,\omega) := \|\Phi\|_2^{-2} \times c(\delta)^2 \times \frac{\Gamma(3/2-\delta)}{\Gamma(\delta+1/2)} \times \sum_{i=1}^{N} |r(\mathscr{P}_i,\omega)/2|^{(2\delta-1)}\, \lambda(\mathscr{P}_i)^2.$

Equivalently: the law under the probability measure m of the normalized integral $t^{\frac{-1}{2\delta-1}} \int_{\xi[0,t]} \omega$ of ω along the geodesics of length t converges towards the two-sided stable law with exponent $(2\delta - 1)$ and rate $C'(\Gamma,\omega)$.

Here $\mathscr{P}_1, \ldots, \mathscr{P}_N$ are the cusps of $\mathscr{M}$, and the intrinsic parameter $\lambda(\mathscr{P}_i)$ is defined in definition 2 of section 7.1.

Observe that the expression of the rate $C'(\Gamma,\omega)$ of our limit stable law shows that the form takes place only through its residues, and that the different cusps asymptotically contribute independently.

3. A new intrinsic measure on $T^1\mathscr{M}$

We introduce a new intrinsic measure ν on $T^1\mathscr{M}$. Its first interest is to be smooth along the stable leaves and quasi-invariant under the geodesic and positive horocyclic flows, and to have its pullback measures $\theta_S{}^*\,\nu$ regularized from m by some convolution along the stable horocycles of $T^1\mathscr{M}$, and then converging towards m as S increases to infinity.

Definition 1. *Let $\tilde{\nu}$ be the measure on $T^1\mathbb{H}$ defined by:*

$$d\tilde{\nu}(z,u) = \|\Phi\|_2^{-2}\ \Phi(z)\, d\mu_z(u)\, d\tilde{V}(z). \tag{3.1}$$

The L^2-norm is relative to $(\mathscr{M}, dV)$. Let ν be the corresponding measure on $T^1\mathscr{M}$, and denote also

$$dV^{\Phi}(z) := \|\Phi\|_2^{-2}\ \Phi^2(z)\, dV(z). \tag{3.2}$$

Remark 1. Observe that the definition of ν is consistent, since the Γ-invariance of Φ, $\tilde{V}$ and the geometric property (2.2) of (μ_z) imply the Γ-invariance of $\tilde{\nu}$. Observe also that by definition of Φ we have $\pi^*\nu = V^{\Phi}$, and then that ν is a probability measure on $T^1\mathscr{M}$.

Remark 2. In the finite volume case, we have $\delta = 1$, Φ constant, $d\mu(u)$ is proportional to the uniform measure du, and then our measure ν is proportional to the measure m and to the Liouville measure λ.

Proposition 1. *The measure ν is quasi-invariant under the geodesic and positive horocycle flows*:

$$\frac{d(T_z{}^*\,\nu)}{d\nu}(\xi) = y^{(1-\delta)} \times \frac{\Phi\circ\pi(\xi T_z^{-1})}{\Phi\circ\pi(\xi)} \quad \textit{for any } \xi \textit{ in } T^1\mathscr{M} \textit{ and}$$
$$z = (x,y) \textit{ in } \mathbb{R}\times\mathbb{R}_+^*.$$

Note that this quasi-invariance property is what remains from the invariance of the Liouville measure λ under the flows, in the finite volume case.

Proof. As observed at the end of section 2, it is enough to prove the identity for $\tilde{\nu}$. Let us use the invariance of the Liouville measure and of the coordinate u under the flows, and the expression of the Liouville measure in the coordinates system $\xi = \xi(z,u) \in T^1\mathbb{H}$. We get for any $\zeta \in \mathbb{R}\times\mathbb{R}_+^*$ and any test functions H on $\partial\mathbb{H}$ and G on $T^1\mathbb{H}$:

$$\int G(\xi(z,u)T_\zeta)\, H(u)\, d\mu_z^h(u)\, d\tilde{V}(z) = \int G(\xi(z,u))\, H(u)\, d\mu_z^h(u)\, d\tilde{V}(z).$$

Thus we obtain for any $u \in \partial\mathbb{H}$, $z_0 \in \mathbb{H}$, and any test function G on $T^1\mathbb{H}$:

$$\int G(\xi(z,u)T_\zeta)\, B_u(z_0,z)\, d\tilde{V}(z) = \int G(\xi(z,u))\, B_u(z_0,z)\, d\tilde{V}(z). \tag{3.3}$$

Whence using the definition 1 of $\tilde{\nu}$, a reference point $z_0 \in \mathbb{H}$, the geometric property (2.2) of (μ_z), and the (z, u)-coordinates on $T^1\mathbb{H}$, we get:

$$\begin{aligned}\int G(\xi T_\zeta)\, d\tilde{\nu}(\xi) &= \|\Phi\|_2^{-2} \int G(\xi(z,u)T_\zeta)\, \Phi(z)\, B_u^\delta(z_0, z)\, d\mu_{z_0}(u)\, d\tilde{V}(z)\\ &= \|\Phi\|_2^{-2} \int G(\xi(z,u))\, \Phi\circ\pi(\xi(z,u)T_\zeta^{-1})\\ &\quad \times B_u^{\delta-1}(z_0, \pi(\xi(z,u)T_\zeta^{-1}))\, B_u(z_0, z)\, d\mu_{z_0}(u)\, d\tilde{V}(z)\\ &\quad \text{(we used here the identity (3.3))}\\ &= \int G(\xi) \times \frac{\Phi\circ\pi(\xi T_\zeta^{-1})}{\Phi\circ\pi(\xi)} \times \frac{B_u^{\delta-1}(z_0, \pi(\xi T_\zeta^{-1}))}{B_u^{\delta-1}(z_0, \pi(\xi))}\, d\nu(\xi)\\ &= \int G(\xi) \times \frac{\Phi\circ\pi(\xi T_\zeta^{-1})}{\Phi\circ\pi(\xi)} \times B_{\xi\theta_\infty}^{\delta-1}(\pi(\xi), \pi(\xi T_\zeta^{-1}))\, d\nu(\xi).\end{aligned}$$

The result follows, since writing $\zeta = (x, y)$, we clearly have by the very definition of B_u:

$$B_{\xi\theta_\infty}^{\delta-1}(\pi(\xi), \pi(\xi T_\zeta^{-1})) = B_{\xi\theta_\infty}^{1-\delta}(\pi(\xi\theta_{-\log y}), \pi(\xi)) = y^{1-\delta}. \qquad \square$$

Theorem 1. *The measure ν can be expressed by a convolution of m along the stable horocycles of $\mathscr{M}$. Precisely, we have:*

$$\nu = c(\delta)^{-1} \int_{\mathbb{R}} (\theta_x^+)^*\, m(1+x^2)^{-\delta}\, dx. \tag{3.4}$$

Proof. As observed at the end of section 2, it is enough to prove the identity for $\tilde{\nu}$ and $\tilde{m}$. Fix a test function F on $T^1\mathbb{H}$ and a reference point $z_0 \in \mathbb{H}$. Recall the notation $z = \pi(z, u) = \pi(u, v, s)$ in the (u, v, s) coordinates on $T^1\mathbb{H}$. We have by definition (2.5) of Φ, (3.1) in definition 1, the geometric property (2.2) of (μ_z), and formula (2.10):

$$\begin{aligned}&\|\Phi\|_2^2 \int F\, d\tilde{\nu}\\ &= \int F(z,u)\, d\mu_z(w)\, d\mu_z(u)\, d\tilde{V}(z)\\ &= \int F(z,u)\, B_u^\delta(z_0, z)\, B_w^\delta(z_0, z)\, d\mu_{z_0}(u)\, d\mu_{z_0}(w)\, d\tilde{V}(z)\\ &= \int F(u,v,s)\, B_u^\delta(z_0, z)\, B_w^\delta(z_0, z)\, B_v(z_0, z)\, d\mu_{z_0}^h(v)\, ds\, d\mu_{z_0}(u)\, d\mu_{z_0}(w).\end{aligned}$$

Consider now the intersection z' of the stable horocycle $H(z, u)$ with the geodesic $\overrightarrow{wu}$. So we have, using both coordinate systems on $T^1\mathbb{H}$: $\xi' := \xi(z', u) = \xi(z, u)\theta_{-x}^+ = (u, w, s')$ for some real x and s'. Let us perform the change of variable: $s \longmapsto s'$, keeping the other variables $u, v, w,$ fixed. Observe that the map

$\xi = \xi(z, u) \longmapsto \xi'$ commutes with the geodesic flow, and therefore that $ds = ds'$. Thus using (2.1), (2.7) and (2.8) we have:

$$\begin{aligned}
\int F\, d\tilde{\nu} &= \|\Phi\|_2^{-2} \int F(\xi'\theta_x^+)\, B_u^\delta(z_0, z)\, B_w^\delta(z_0, z)\, B_v(z_0, z)\, d\mu_{z_0}^h(v) \\
&\quad \times d\mu_{z_0}(u)\, d\mu_{z_0}(w)\, ds' \\
&= c(\delta)^{-1} \int F(\xi'\theta_x^+)\, B_u^\delta(z_0, z)\, B_w^\delta(z_0, z)\, B_u^\delta(z', z_0)\, B_w^\delta(z', z_0) \\
&\quad \times d\mu_z^h(v)\, d\tilde{m}(u, w, s') \\
&= c(\delta)^{-1} \int F(\xi'\theta_x^+)\, B_u^\delta(z', z)\, B_w^\delta(z', z)\, d\mu_z^h(v)\, d\tilde{m}(u, w, s') \\
&= c(\delta)^{-1} \int F(\xi'\theta_x^+)\, B_w^\delta(z', z)\, d\mu_z^h(v)\, d\tilde{m}(\xi') \\
&= c(\delta)^{-1} \int F(\xi\theta_x^+)\, B_{\xi\theta_{-\infty}}^\delta(\pi(\xi), z)\, B_v(\pi(\xi), z)\, d\mu_{\pi(\xi)}^h(v)\, d\tilde{m}(\xi),
\end{aligned}$$

where $z = \pi(\xi\theta_x^+)$ and $v = \xi\theta_x^+\theta_{-\infty}$.

It remains to show that $B_{\xi\theta_{-\infty}}^\delta(\pi(\xi), z)\, B_v(\pi(\xi), z)\, d\mu_{\pi(\xi)}^h(v) = (1+x^2)^{-\delta} dx$. Now, this is an intrinsic formula (x is a distance along the horocycle $H(\xi)$), thus it is enough to prove it in a particular model for $\mathbb{H}$. Let us take the half-plane model, with $\xi\theta_\infty = u = \infty$ and $\xi\theta_{-\infty} = w = 0$.
Then we easily see that $v = yx$, $B_0(z, \pi(\xi)) = B_v(\pi(\xi), z) = 1 + x^2$, and that $d\mu_{\pi(\xi)}^h(v) = p(\pi(\xi), v)\, dv = (y(1 + x^2))^{-1} y\, dx$, where $y = \mathscr{I}m(\pi(\xi))$. The result follows. □

Taking $F = 1$ and using the definition (2.4) of $c(\delta)$ and remark 1, we immediately deduce

Corollary 1. *m is a probability measure on $T^1\mathscr{M}$.*

Remark 3. In particular we recover for $\delta > 1/2$ that m is finite on $T^1\mathscr{M}$, then that $\mu_z \otimes \mu_z$ does not charge the diagonal, and then the classical result that μ_z is atomless.

We also deduce the following approximation result on the measure m:

Corollary 2. *The measure $\theta_S^*\nu$ converges as $S \to +\infty$, in the sense of the evaluation on each bounded continuous function on $T^1\mathscr{M}$, towards the normalized Patterson-Sullivan measure m.*

Proof. This is a straightforward consequence of the following formula, valid for any real S:

$$(\theta_S)^*\nu = c(\delta)^{-1} \int_{\mathbb{R}} (\theta_{e^{-S}x}^+)^*\, m \;\; (1 + x^2)^{-\delta}\, dx. \tag{3.5}$$

Indeed, using theorem 1, the commutation formula (2.12), and the geodesic invariance of m, we get:

$$\begin{aligned} c(\delta) \int F \, d((\theta_S)^* \nu) &= c(\delta) \int F(\xi \theta_S) \, d\nu(\xi) \\ &= \int F(\xi \theta_x^+ \theta_S) \, dm(\xi) \, (1+x^2)^{-\delta} \, dx \\ &= \int F(\xi \theta_S \theta_{e^{-S}x}^+) \, dm(\xi) \, (1+x^2)^{-\delta} \, dx \\ &= \int F(\xi \theta_{e^{-S}x}^+) \, dm(\xi) \, (1+x^2)^{-\delta} \, dx. \qquad \square \end{aligned}$$

4. Diffusions on $\mathbb{H}$, on $\mathscr{M}$, and on $T^1\mathscr{M}$

Let us from now on identify $\mathbb{H}$ with its Poincaré half-plane model $\mathbb{R} \times \mathbb{R}_+^*$, and denote by $z = (x, y)$ the current point. Recall that $\Delta = y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right)$.

4.1. The diffusions Z_t^d

Let w_t and W_t denote two independent standard real Brownian motions, defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Set

$$y_t := \exp\left[w_t + (\delta - \tfrac{1}{2})\, t\right], \; Y_t := \int_0^t y_s^2 \, ds, \; x_t := \int_0^t y_s \, dW_s, \; Z_t^\delta := (x_t, y_t) \in \mathbb{H}. \tag{4.1}$$

For all δ, Z_t^δ is the diffusion on $\mathbb{H}$ starting from $e_o = (0, 1)$, with invariant measure $y^{2\delta-2} dx dy$, and generator

$$\frac{1}{2} \Delta^\delta := \frac{1}{2} \Delta + \delta \, y \frac{\partial}{\partial y} = \frac{y^2}{2} \left(\frac{\partial^2}{\partial y^2} + \frac{2\delta}{y} \times \frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} \right). \tag{4.2}$$

Similarly, denote by Z_t^d the analogous process with δ replaced by $d \in [0, 1]$. In particular, Z_t^0 is the Brownian motion on $\mathbb{H}$.

We gather in the following lemma the facts we shall need, relating to these diffusions.

Lemma 1. (i) *Denoting by P_t the Brownian semi-group on $\mathscr{M}$, we have for any $\xi \in T^1\mathscr{M}$, test-function f and $t \geq 0$: $\mathbb{E}\big(f \circ \pi\, (\xi T_{Z_t^0})\big) = P_t f(\pi(\xi))$.*

(ii) *$T_{Z_t^d}$ is a right diffusion on the orientation preserving affine group of $\mathbb{R}$.*

(iii) *$\alpha_t := (x + x_t)/y_t$ is a symmetrical ergodic diffusion process on $\mathbb{R}$, starting from x, and with invariant measure having density $x \longmapsto (1 + x^2)^{-\delta}$.*

(iv) *For each $t \geq 0$, $T_{Z_t^1}^{-1}$ has the same law as $T_{Z_t^0}$.*

Proof. (i) Recall the classical identification between $T^1\mathbb{H}$ and the set of Möbius isometries of $\mathbb{H}$, and that in this identification we have for any $\xi \in T^1\mathscr{M}$ and any $z \in \mathbb{H}$: $\xi(z) = \pi(\xi T_z)$, which remains valid with $\xi \in T^1\mathscr{M}$ and $z \in \mathscr{M}$. In particular, we see that $\pi(\xi T_{Z_t^0}) = \xi(Z_t^0)$ is a Brownian motion on $\mathscr{M}$, started from $\pi(\xi)$.

(ii) We have indeed for any $d \in [0, 1]$: $T_{Z_t^d}^{-1} T_{Z_{t+s}^d} = T_{\hat{Z}_{s,t}^d} = T_{Z_s^d} \circ \Theta_t$, which is independent of the sub-σ-field $\mathscr{F}_t$ generated by the coordinates until time t.

Here $\hat{Z}^d_{s,t}$ stands for $\left(\frac{x^d_{t+s}-x^d_t}{y^d_t}, \frac{y^d_{t+s}}{y^d_t}\right)$, Θ_t denotes the standard shift on time, and we naturally write $Z^d_t = (x^d_t, y^d_t)$.

(iii) Itô's formula shows that $d\alpha_t = \sqrt{1+\alpha_t^2} d\tilde{w}_t + (1-\delta)\alpha_t \, dt$, for some Brownian motion $\tilde{w}_t$, and thus that α_t is a diffusion process with generator $G := (\frac{1+\alpha^2}{2}) \frac{d^2}{d\alpha^2} + (1-\delta)\alpha \frac{d}{d\alpha}$.Then we have for any test-functions f and g:

$$\int_{\mathbb{R}} Gf(\alpha)\, g(\alpha)\, (1+\alpha^2)^{-\delta} d\alpha = -\frac{1}{2} \int_{\mathbb{R}} f'(\alpha)\, g'(\alpha)\, (1+\alpha^2)^{1-\delta} d\alpha.$$

Therefore G and the semi-group of α_t are selfadjoint with respect to the finite measure $(1+x^2)^{-\delta}\, dx$.

(iv) Since $Z^1_t = (x^1_t, y^1_t)$ with $y^1_t = \exp(w_t + t/2)$ and $x^1_t = \int_0^t y^1_s \, dW_s$, we have: $1/y^1_t = \exp[-w_t - t/2]$ and

$$\begin{aligned} -x^1_t/y^1_t &= -\int_0^t \exp[w_s - w_t + (s-t)/2]\, dW_s \\ &= -\int_0^t \exp[w_{t-s} - w_t - s/2]\, dW_{t-s}, \end{aligned}$$

so that for each fixed t:

$$\hat{Z}^1_t := \left(\frac{-x^1_t}{y^1_t}, \frac{1}{y^1_t}\right) = \left(\int_0^t \exp[w'_s - s/2]\, dW'_s, \exp[w'_t - t/2]\right) \overset{(law)}{=} (x^0_t, y^0_t),$$

where $w'_s := w_{t-s} - w_t$ and $W'_s := W_t - W_{t-s}$. This shows that $T^{-1}_{Z^1_t} = T_{\hat{Z}^1_t} \overset{(law)}{=} T_{(x^0_t, y^0_t)} = T_{Z^0_t}$, for each fixed $t \geq 0$. □

4.2. The stopping time τ_t

Our contour deformation in section 6 will use the following stopping time: set for $t \geq 0$

$$\tau_t := \inf\{s > 0 |\ y_s = y_o e^t\}. \tag{4.3}$$

It is clear from the formula (4.1) defining y_s that τ_t is merely the first passage time of some real Brownian motion with constant drift.
We collect in the following technical lemma all we shall need to handle τ_t.

Lemma 2. (i) *Let I_r be the usual modified Bessel function. The joint law of τ_t and x_{τ_t} is given by: for any real w and any $a \geq -(\delta - 1/2)^2/2$*

$$\begin{aligned} &\mathbb{E}\left[\exp(\sqrt{-1} w\, x_{\tau_t} - a\, \tau_t)\right] \\ &= e^{(\delta-1/2)\,t} \times I_{\sqrt{2a+(\delta-1/2)^2}}(|w|) \Big/ I_{\sqrt{2a+(\delta-1/2)^2}}(|w|\, e^t). \end{aligned}$$

(ii) The law of $e^{-t}x_{\tau_t}$ *converges as* $t \to \infty$.

(iii) We have $\mathbb{E}\left(e^{-a\,\tau_t}\right) = \exp\left[-\left(1/2 - \delta + \sqrt{(\delta - 1/2)^2 + 2a}\right)\,t\,\right]$, *for any* $a \geq -(\delta - 1/2)^2/2$, *and* $\mathbb{E}(\tau_t) = t/(\delta - 1/2)$, $\mathbb{E}(\tau_t^2) = (\delta - 1/2)^{-2}t^2 + (\delta - 1/2)^{-3}t$.

(iv) For any $q \in]1/2, 1[$ $\tau_t - t/(\delta - 1/2) = o(t^q)$, *almost surely as* $t \to +\infty$.

Proof. Set $U_t := y(Y^{-1}(t))$, $A_t := Y^{-1}(t)$ and $\sigma_t := inf\,\{s > 0 \,|\, U_s = e^t\}$. We have then $\langle x_s \rangle = Y_s$, $y_s = U(Y_s)$, $\sigma_s = Y(\tau_s)$, $A_s = \int_0^s U_r^{-2} dr$, $A_{\sigma_s} = \tau_s$. Moreover the generator of U_s is $\frac{1}{2}\frac{d^2}{du^2} + \frac{\delta}{u}\frac{d}{du}$, whence $dU_s = d\beta_s + (\delta/U_s)\,ds$: U_s is the Bessel process with index $(2\delta + 1)$. Let us look for a C^2 function g on $\mathbb{R}_+$ such that $M_s := \exp(-a\,A_s - \varrho\,s) g(U_s)$ be a martingale, for fixed positive a and ϱ. We have

$$M_s = g(1) + \int_0^s \exp(-aA_r - \varrho r)\, g'(U_r)\, d\beta_r + \tfrac{1}{2}\int_0^s \exp(-aA_r - \varrho r)$$
$$\times\Big[g''(U_r) + \frac{2\delta}{U_r}\, g'(U_r) - 2\Big(\varrho + \frac{a}{U_r^2}\Big)\, g(U_r)\Big] dr.$$

We must then have $g''(u) + 2\delta g'(u)/u - 2(au^{-2} + \varrho)\, g(u) = 0$. Set $h(u) := u^{\delta - 1/2}\, g(u)$; this is a C^2 function on $\mathbb{R}_+$ which satisfies:

$$h''(u) + h'(u)/u - \Big(2\varrho + (2a + (\delta - 1/2)^2)u^{-2}\Big)\, h(u) = 0.$$

Therefore we have $h(u) = I_{b(a)}(u\sqrt{2\varrho})$ and $g(u) = u^{1/2-\delta}\, I_{b(a)}(u\sqrt{2\varrho})$, where $b(a) := \sqrt{(\delta - 1/2)^2 + 2a}$ and $I_r(z) = z^r \times \sum_{k\geq 0} \frac{z^{2k}}{2^{r+2k} k!\Gamma(r + 2k + 1)}$.

Finally we have by the optional sampling theorem

$$\mathbb{E}\left[\exp\left(\sqrt{-1}\, w\, x_{\tau_t} - a\,\tau_t\right)\right] = \mathbb{E}\left[\exp\left(-\frac{|w|^2}{2}\sigma_t - A_{\sigma_t}\right)\right] = g(1)/g(e^t)$$
$$= e^{(\delta - 1/2)t}\; I_{b(a)}(|w|) \Big/ I_{b(a)}(|w|e^t).$$

This proves (i) for $a \geq 0$.

For getting (ii), it is enough to change a into 0 and w into we^{-t}. Indeed, this gives

$$\mathbb{E}\Big[\exp\left(\sqrt{-1}\, w\, e^{-t}\, x_{\tau_t}\right)\Big] = e^{(\delta - 1/2)t}\; I_{(\delta - 1/2)}(|w|e^{-t}) \Big/ I_{(\delta - 1/2)}(|w|)$$
$$\stackrel{t\to\infty}{\longrightarrow} \left(\sum_{k\geq 0} \frac{\Gamma(\delta + 1/2)\; w^{2k}}{4^k k!\Gamma(\delta + 2k + 1/2)}\right)^{-1} \in L^2(\mathbb{R}).$$

Since $\tau_t = \inf\{s > 0 \,|\, w_s + (\delta - 1/2)s > t\}$, (iii) follows easily from the Cameron-Martin formula. And this allows to complete the proof of (i) for negative a, by analytic continuation. Finally, (iv) is straightforward from the following observation:

$$t = \log\, y_{\tau_t} = (\delta - 1/2)\,\tau_t + w_{\tau_t} = (\delta - 1/2)\,\tau_t + o(\tau_t^q). \qquad \square$$

4.3. *The diffusion ξ_t^δ and its invariant measure ν*

For any $\xi \in T^1\mathscr{M}$, set

$$\xi_t^\delta := \xi\, T_{Z_t^\delta}, \quad \text{and} \tag{4.4}$$

$$D^\delta := \mathscr{L}_0^2 + \mathscr{L}_1^2 + (2\delta - 1)\,\mathscr{L}_0 = D^0 + 2\,\delta\,\mathscr{L}_0. \tag{4.5}$$

Remark 4. We have on $T^1\mathbb{H}$:

$$D^0(F\circ\pi)(\xi\, T.) = \Delta[F\circ\pi(\xi\, T.)] = \Delta(F\circ\xi) = (\Delta F)\circ\xi = (\Delta F)\circ\pi(\xi\, T.),$$

whence $D^0(F\circ\pi) = (\Delta F)\circ\pi$, which remains valid on $T^1\mathscr{M}$. But if $\delta \neq 0$ then $D^\delta(F\circ\pi) \neq (\Delta^\delta F)\circ\pi$.

Lemma 3. *The process ξ_t^δ is the diffusion on $T^1\mathscr{M}$ starting from ξ and having generator $\frac{1}{2}\,D^\delta$. Moreover it admits ν as an invariant measure.*

Proof. The diffusion property is straightforward from lemma (1, ii). Let us fix a test-function F on $T^1\mathscr{M}$, and a positive t. From (2.15) we get $\Delta^\delta[F(\xi\, T_z)] = (D^\delta F)(\xi\, T_z)$. Finally using theorem 1, the commutation formula (2.12), the geodesic invariance of m, and lemma (1, iii), we have:

$$\begin{aligned}
\int \mathbb{E}\Big(F(\xi_t^\delta)\Big)\, d\nu(\xi) &= \|\Phi\|_2^{-2} \int_{\mathbb{R}}\int \mathbb{E}\Big(F\Big(\xi\theta_x^+ T_{Z_t^\delta}\Big)\Big)\, dm(\xi)\,(1+x^2)^{-\delta}\, dx \\
&= \|\Phi\|_2^{-2} \int_{\mathbb{R}}\int \mathbb{E}\Big(F\Big(\xi\theta^+_{\left(\frac{x+x_t}{y_t}\right)}\Big)\Big)\, dm(\xi)\,(1+x^2)^{-\delta}\, dx \\
&= \|\Phi\|_2^{-2} \int\int_{\mathbb{R}} F\Big(\xi\theta_x^+\Big)\,(1+x^2)^{-\delta}\, dx\, dm(\xi) = \int F\, d\nu.
\end{aligned}$$

□

4.4. *$\pi(\xi_t^\delta)$ is the Φ-diffusion*

We construct here, by projection of the stationary diffusion ξ_t^δ, a stationary diffusion on $\mathscr{M}$, related to the fundamental function Φ, that we call "Φ-diffusion" and denote by Z_t^Φ, and that we shall study in section 8 and use in section 9 to perform the probabilistic approach of the geodesic behavior. This role is played by the Brownian motion in the finite volume case. Note that the lift to $\mathbb{H}$ of this Φ-diffusion happens to be the diffusion considered in [Su1], where the path measure is constructed from the semi-group via Kolmogorov extension theorem. The major reason of the relevance of this Φ-diffusion in our problem is that it is a Brownian motion conditioned to converge to the limit set $\Lambda(\Gamma)$, according to the Patterson measure μ_z when started from z. (See [E-F-LJ-3]. We shall however not need this fact here).

Proposition 2. *Under the probability law $\nu \otimes \mathbb{P}$, the projection $\pi(\xi_t^\delta)$ of the stationary diffusion ξ_t^δ on $T^1\mathscr{M}$ is Z_t^Φ, the stationary Φ-diffusion on $\mathscr{M}$. It is symmetrical with invariant measure V^Φ, and generator $\frac{1}{2}\Delta^\Phi := \frac{1}{2}\Phi^{-1}\Delta \circ \Phi - \lambda_o = \frac{1}{2}\Delta + (\nabla \log \Phi)\cdot\nabla$. (We identify the function Φ with the corresponding multiplication operator, and recall that $\lambda_o := \delta(\delta-1)/2$, see* (2.5)).

Proof. A straightforward computation shows that $y^{\delta-1}\Delta^\delta \circ y^{1-\delta} = \Delta^1 - 2\lambda_o$. Hence we have for each test-function φ and each t: $\mathbb{E}\Big(y_t^{1-\delta}\,\varphi(T^{-1}_{Z_t^\delta})\Big) = e^{-\lambda_o t}$ $\mathbb{E}\Big(\varphi(T^{-1}_{Z_t^1})\Big)$.

Thus using lemma (1, iv) we get: $\mathbb{E}\Big(y_t^{1-\delta}\,\varphi(T^{-1}_{Z_t^\delta})\Big) = e^{-\lambda_o t}\,\mathbb{E}\Big(\varphi(T_{Z_t^0})\Big)$.
In particular, using lemma (1, i), we have for any test-function f on $\mathscr{M}$, any t and any $\xi \in T^1\mathscr{M}$:

$$\mathbb{E}\Big(y_t^{1-\delta}\, f\circ\pi(\xi T^{-1}_{Z_t^\delta})\Big) = e^{-\lambda_o t}\,\mathbb{E}\Big(f\circ\pi(\xi T_{Z_t^0})\Big) = e^{-\lambda_o t}\, P_t f(\pi(\xi)). \tag{4.6}$$

Consider now $0 = t_0 < t_1 < \cdots < t_n$ and test-functions $f_0, \ldots, f_n$ on $\mathscr{M}$. We have, using proposition 1:

$$\begin{aligned}
&\int\int \prod_{i=0}^n f_i \circ \pi(\xi_{t_i}^\delta) d\nu(\xi)\, d\mathbb{P} \\
&= \int\int f_0\circ\pi\Big(\xi T^{-1}_{Z^\delta_{t_1}}\Big) \times \prod_{i=1}^n f_i\circ\pi\Big(\xi T^{-1}_{Z^\delta_{t_1}} T_{Z^\delta_{t_i}}\Big) d\left(\Big(T_{Z^\delta_{t_1}}\Big)^* \nu\right)(\xi)\, d\mathbb{P} \\
&= \int\int (y_{t_1})^{1-\delta} \times (\Phi f_0)\circ\pi\Big(\xi T^{-1}_{Z^\delta_{t_1}}\Big) \times (f_1/\Phi)\circ\pi(\xi) \\
&\quad \times \prod_{i=2}^n f_i\circ\pi\Big(\xi^\delta_{(t_i-t_1)}\circ\Theta_{t_1}\Big) d\nu(\xi)\, d\mathbb{P} \\
&= \int \mathbb{E}\Big[(y_{t_1})^{1-\delta} \times (\Phi f_0)\circ\pi\Big(\xi T^{-1}_{Z^\delta_{t_1}}\Big)\Big] \times (f_1/\Phi)\circ\pi(\xi) \\
&\quad \times \mathbb{E}\left[\prod_{i=2}^n f_i\circ\pi(\xi^\delta_{(t_i-t_1)})\right] d\nu(\xi),
\end{aligned}$$

since the increments of $T_{Z_t^\delta}$ are independent, as already noticed for lemma (1, ii). Thus using (4.6) above, we get:

$$\begin{aligned}
&\int\int \prod_{i=0}^n f_i\circ\pi(\xi^\delta_{t_i}) d\nu(\xi)\, d\mathbb{P} \\
&= e^{-\lambda_o t_1} \int \left(\frac{f_1}{\Phi} \times P_{t_1}(\Phi f_0)\right)(\pi(\xi)) \times \mathbb{E}\left(\prod_{i=2}^n f_i\circ\pi(\xi^\delta_{(t_i-t_1)})\right) d\nu(\xi)
\end{aligned}$$

$$= e^{-\lambda_o t_n} \int \left(\frac{f_n}{\Phi} \times P_{(t_n - t_{n-1})}(f_{n-1} \times \cdots \times P_{(t_2 - t_1)}(f_1 \times P_{t_1}(\Phi f_0)) \cdots) \right) (\pi(\xi))$$
$$d\nu(\xi)$$

(by induction on $n \in \mathbb{N}$)

$$= e^{-\lambda_o t_n} \int \frac{f_0}{\Phi} \times P_{t_1}\Big(f_1 \times P_{(t_2 - t_1)}(f_2 \times \cdots \times P_{(t_n - t_{n-1})}(\Phi f_n) \cdots) \Big) dV^{\Phi}$$

(since $\pi^* \nu = V^{\Phi}$ and by symmetry of Δ^{Φ} with respect to V^{Φ})

$$= \mathbb{E} \left(\prod_{i=0}^{n} f_i(Z_{t_i}^{\Phi}) \right). \qquad \square$$

Remark 5. The preceding statements are sufficient for our purpose. However, it is possible to show some more relation between the diffusions considered here. Indeed, by using the infinitesimal version of the quasi-invariance of ν and the observation that $D^0(\Phi \circ \pi) = (\Delta \Phi) \circ \pi = 2\lambda_o \Phi \circ \pi$, we can compute the adjoint of D^{δ} with respect to its invariant measure ν. We find that this adjoint equals $D^{\Phi} := (\Phi \circ \pi)^{-1} D^0 \circ (\Phi \circ \pi) - 2\lambda_o$. See [E-F-LJ-3]. Moreover, we see that $D^{\Phi}(f \circ \pi) = (\Delta^{\Phi} f) \circ \pi$ for any test-function f on $\mathscr{M}$, although $D^{\delta}(f \circ \pi) \neq (\Delta^{\delta} f) \circ \pi$.

5. Lift of a 1-form to a closed 1-form on each leaf

5.1. Contribution of a 1-form

Fix a 1-form ω on $\mathscr{M}$, closed in some neighborhood $\mathscr{N}$ of the cusps of $\mathscr{M}$ and m-integrable in $\mathscr{M} \setminus \mathscr{N}$.

We are interested in the asymptotic behavior of the following quantity:

$$J_t(\omega) := \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\, r_t \int_{\xi[0,t]} \omega \right] dm(\xi)$$
$$= \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\, r_t \int_0^t f_\omega(\xi\theta_s)\, ds \right] dm(\xi), \tag{5.1}$$

where t is positive, $r_t := t^{\frac{-1}{2\delta - 1}}$, and where the function f_ω is defined on $T^1\mathscr{M}$ by:

$$f_\omega(\xi) := \langle \omega(\pi(\xi)), \vec{\xi} \rangle, \quad \text{where} \quad \xi = (\pi(\xi), \vec{\xi}), \;\; \vec{\xi} \in T^1_{\pi(\xi)}\mathscr{M}. \tag{5.2}$$

Recall that $\xi[0,t]$ means $\{\pi(\xi\theta_s) | 0 \le s \le t\}$, that is to say the geodesic of length t starting from ξ. Note that we have $r_t \le 1/t$.

Let us decompose ω in $\omega = \tilde{\omega} + \omega' + dF$, with F of class C^1 on $\mathscr{M}$, $\tilde{\omega}$ smooth on $\mathscr{M}$, harmonic in some neighborhood of each cusp and vanishing out of another neighborhood of each cusp, and ω' m-integrable, vanishing in some neighborhood of the cusps. Note that $\tilde{\omega}$ is defined up to some smooth form null far from the cusps and near the cusps.

Lemma 4. $|J_t(\omega) - J_t(\tilde{\omega})|$ *converges to zero as t goes to infinity. Thus we have*

$$\lim_{t\to\infty} J_t(\omega) = \lim_{t\to\infty} J_t^f, \quad where$$

$$J_t^f := \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\, t^{\frac{-1}{2\delta-1}} \int_0^t f(\xi\theta_s)\, ds\right] dm(\xi), \quad and \quad f := f_{\tilde{\omega}}. \tag{5.3}$$

Proof. We have

$$|J_t(\omega) - J_t(\tilde{\omega})| \leq \int \left|1 - \exp\left[\sqrt{-1}\, r_t \int_0^t f_{dF}(\xi\theta_s) ds\right]\right| m(d\xi) + \int \left|1 - \exp\left[\sqrt{-1}\, r_t \int_0^t f_{\omega'}(\xi\theta_s) ds\right]\right| m(d\xi).$$

Now $\int_0^t f_{dF}(\xi\theta_s)ds = F(\xi\theta_t) - F(\xi)$, and $F(\xi\theta_t)$ is stationary by the geodesic invariance of m. Thus the first term on the right hand side above goes to zero.

To deal with the second term, we observe that $f_{\omega'}$, as ω', is m-integrable on $\mathscr{M}$. Thus the ergodic theorem applies, to ensure the almost sure convergence of $t^{-1}\int_0^t f_{\omega'}(\xi\theta_s)ds$ towards $\int f_{\omega'}\, dm$. Now this last integral is null by the invariance of m under the symmetry $\vec{\xi} \longmapsto -\vec{\xi}$ (which amounts to exchange the coordinates u and v), while $f_{\omega'}(\xi)$ becomes $-f_{\omega'}(\xi)$ under this same symmetry.
This concludes the proof, since $r_t \leq 1/t$. □

5.2. Coordinates in the cusps

Denote by $\mathscr{P}_1, \ldots, \mathscr{P}_N$ the cusps of $\mathscr{M}$.
For each fixed cusp $\mathscr{P}_i$, consider the half-plane model for $\mathbb{H}$ adapted to $\mathscr{P}_i$, that is to say such that $\mathscr{P}_i = \infty$ and such that the maximal parabolic subgroup of Γ fixing $\mathscr{P}_i$ is precisely $\mathbb{Z}$, that is to say the isometry subgroup of horizontal translations of integer lengths. Denote by $(x^i, y^i) \in \mathbb{R} \times \mathbb{R}_+^*$ the canonical coordinates in this particular model adapted to the cusp $\mathscr{P}_i$. Then consider the horocyclic neighborhood of $\mathscr{P}_i$ at height a, that is to say $\mathscr{N}_i(a) := \Gamma\backslash\{y^i \geq a\}$, for positive a.

Let us fix a large enough, so that $\mathscr{N}_i(a) = \mathbb{Z}\backslash\{y^i \geq a\}$, and that the distinct $\mathscr{N}_i(a)$ be disjoint. Finally Set $\mathscr{N}(a) := \cup_{i=1}^N \mathscr{N}_i(a)$.
So $\mathscr{D}_i := \{0 \leq x^i < 1\} \cap \{y^i \geq a\}$ is a fundamental domain for $\mathscr{N}_i(a)$, and the metric in $\mathscr{D}_i$ is given by

$$d\ell^2 = (y^i)^{-2} \left((dx^i)^2 + (dy^i)^2\right). \tag{5.4}$$

Let us specify the $\tilde{\omega}$ we choose in the decomposition of ω in section 5.1: we fix $b_o > a$ and a smooth function h on $\mathbb{R}_+$ such that $h = 0$ on $[0, b_o]$ and $h = 1$ on $[b_o + 1, \infty[$, and we take

$$\tilde{\omega} = \sum_{i=1}^{N} (h \circ y^i)\, r^i\, dx^i . \tag{5.5}$$

The coefficients $r^i = r(\mathscr{P}_i, \omega)$ are the residues of $\tilde{\omega}$ and of ω (near the cusps). Note that (dx^i) is a basis of the one-dimensional cohomology $H^1(\mathscr{N}_i(a))$.
Note further that by definition of $\mathscr{L}_0$ and f (see (2.13) and (5.3)) we get from (5.5):

$$f = \sum_{i=1}^{N} (h \circ y^i)\, r^i\, \mathscr{L}_0\, x^i . \tag{5.6}$$

Of course we may extend any function on $\mathbb{H}$ (respectively on $\mathscr{M}$) to a function on $T^1\mathbb{H}$ (respectively on $T^1\mathscr{M}$), by identifying for example x^i and $x^i \circ \pi$.

For any $\xi \in T^1\mathscr{M}$ let π_ξ denote the covering map from $\mathbb{H} \equiv \mathbb{R} \times \mathbb{R}_+^*$ onto $\mathscr{M}$ defined by $\pi_\xi(z) := \pi(\xi T_z)$. So $\pi_\xi^* \tilde{\omega}$ is naturally a 1-form on $\mathbb{H}$. Moreover using (2.15) we get:

$$\pi_\xi^* dx^i = d(x^i \circ \pi_\xi) = y^{-1}\, (\mathscr{L}_1\, x^i)(\xi T_\cdot)\, dx + y^{-1}\, (\mathscr{L}_0\, x^i)(\xi T_\cdot)\, dy,$$

and then

$$\pi_\xi^* \tilde{\omega}\,(z) = y^{-1}\, f_1(\xi T_z)\, dx + y^{-1}\, f(\xi T_z)\, dy, \tag{5.7}$$

with

$$f_1 := \sum_{i=1}^{N} (h \circ y^i)\, r^i\, \mathscr{L}_1\, x^i . \tag{5.8}$$

5.3. The conjugate function

We construct here a function g on $T^1\mathscr{M}$ which is conjugate to the windings function f. That is to say: we complete the winding form $f(\xi T_z)\, dy/y$ into a closed form ω_ξ on the leaf through ξ, by adding a term $g(\xi T_z)\, dx/y$. We also check that this new form is close to the pull-back by π_ξ of the winding form $\tilde{\omega}$ on $\mathscr{M}$. This will be crucial in section 6 in order to replace geodesics by diffusion paths.

Lemma 5. *Set*

$$\tilde{f}_1 := (\mathscr{L}_0 - 1)\, f_1 - \mathscr{L}_1 f. \tag{5.9}$$

Then this is a compactly supported smooth function on $T^1\mathscr{M}$.

Proof. We use the commutation relation (2.14) between $\mathscr{L}_0$ and $\mathscr{L}_1$, (5.6) and (5.8) to get:

$$\tilde{f}_1 = \sum_{i=1}^{N} h' \circ y^i \times r^i \left(\mathscr{L}_0\, y^i\, \mathscr{L}_1\, x^i - \mathscr{L}_1\, y^i\, \mathscr{L}_0\, x^i \right),$$

which is smooth and vanishes where $h' \circ y^i$ vanishes, namely outside the compact $\pi^{-1}\Big(\bigcup_{i=1}^N \Gamma \backslash \{a \le y^i \le b\} \Big)$. □

Lemma 6. *Set*

$$\mathcal{U}_A\phi(\xi) := -\int_0^A e^{-s}\,\phi(\xi\theta_s)\,ds, \tag{5.10}$$

for A in $[0,\infty[$, ξ *in* $T^1\mathcal{M}$, *and* ϕ *a continuous function on* $T^1\mathcal{M}$. *We have*
(i)

$$g := \lim_{A\to\infty} \mathcal{U}_A\,\mathcal{L}_1 f \quad \textit{exists in m-probability}. \tag{5.11}$$

(ii) $g - f_1 = -\lim_{A\to\infty} \mathcal{U}_A\tilde{f}_1$ *is bounded, with bounded* $\mathcal{L}_0$ *and* $\mathcal{L}_1$ *iterated derivatives.*
(iii) g *is smooth along the stable leaves, and decreases exponentially fast above the funnels.*
(iv) $(\mathcal{L}_0 - 1)g = \mathcal{L}_1 f$.

Proof. (i) $\mathcal{U}_A\mathcal{L}_1 f(\xi) = \mathcal{U}_A(\mathcal{L}_0 - 1)f_1(\xi) - \mathcal{U}_A\tilde{f}_1(\xi)$ (by Lemma 5)

$$= -\int_0^A e^{-s}\frac{d}{ds}f_1(\xi\theta_s)\,ds + \int_0^A e^{-s} f_1(\xi\theta_s)\,ds - \mathcal{U}_A\tilde{f}_1(\xi)$$
$$= f_1(\xi) - e^{-A}f_1(\xi\theta_A) - \mathcal{U}_A\tilde{f}_1(\xi)$$

converges as $A\to\infty$ towards $f_1(\xi) - \mathcal{U}_\infty\tilde{f}_1(\xi)$ in m-probability, by lemma 5.

(ii) By the above we have $g - f_1 = -\mathcal{U}_\infty\tilde{f}_1$, which is bounded by lemma 5 again. Moreover since $\mathcal{L}_0$ and $\mathcal{U}_A$ commute, the $\mathcal{L}_0$-derivatives of $g - f_1$ are bounded as well. Finally we uniformly have:

$$\mathcal{L}_1(\mathcal{U}_A - \mathcal{U}_\infty)\tilde{f}_1(\xi) = \mathcal{L}_1\int_A^\infty e^{-s}\tilde{f}_1(\xi\theta_s)\,ds$$
$$= \int_A^\infty e^{-2s}\mathcal{L}_1\tilde{f}_1(\xi\theta_s)\,ds \longrightarrow_{\{A\to\infty\}} 0,$$

whence $\mathcal{L}_1(f_1 - g) = \mathcal{L}_1\mathcal{U}_\infty\tilde{f}_1 = \lim_{A\to\infty}\mathcal{L}_1\mathcal{U}_A\tilde{f}_1$ is bounded by $\|\mathcal{L}_1\tilde{f}_1\|_\infty$. The same argument works for higher order derivatives.

(iii) Firstly, the smoothness is clear from (ii) and the smoothness of f_1. Now for ξ in a funnel, denote by $t(\xi)$ the first time t at which $\xi\theta_t$ hits $\mathcal{N}(a)$. Since f is null on $\xi\theta_{[0,t(\xi)]}$, we have

$$-g(\xi) = \int_{t(\xi)}^\infty e^{-s}\mathcal{L}_1 f(\xi\theta_s)\,ds = e^{-t(\xi)}\,g(\xi\theta_{t(\xi)}),$$

and thus we get

$$|g(\xi)| \le \exp[-dist\,(\pi(\xi),\mathcal{N}(a))] \times \sup\,|g(\pi^{-1}(\partial\mathcal{N}(a)))|.$$

(iv) Then on one hand $(\mathcal{L}_0 - 1)$ and $\mathcal{U}_A$ commute and we see as in (i) above:

$$(\mathscr{L}_0 - 1)\mathscr{U}_A \mathscr{L}_1 f(\xi) = \mathscr{L}_1 f(\xi) - e^{-A} \mathscr{L}_1 f(\xi\theta_A) \longrightarrow_{\{A\to\infty\}} \mathscr{L}_1 f(\xi)$$

in m-probability, and on the other hand:

$$(\mathscr{L}_0-1)\mathscr{U}_A \mathscr{L}_1 f(\xi) = (\mathscr{L}_0-1) f_1(\xi) - e^{-A}(\mathscr{L}_0-1) f_1(\xi\theta_A) - (\mathscr{L}_0-1)\mathscr{U}_A \tilde{f}_1(\xi)$$

$$\longrightarrow_{\{A\to\infty\}} (\mathscr{L}_0 - 1)(f_1 - \mathscr{U}_\infty \tilde{f}_1)(\xi) \text{ in } m\text{−probability}. \qquad \square$$

Corollary 3. *Set*

$$\omega_\xi(z) := g(\xi T_z)\, y^{-1}\, dx + f(\xi T_z)\, y^{-1}\, dy. \tag{5.12}$$

This is a smooth closed 1-form on $\mathbb{H}$, for any ξ in $T^1\mathscr{M}$. Moreover

$$(\pi_\xi^*\tilde{\omega} - \omega_\xi)(z) = (f_1 - g)(\xi T_z)\, y^{-1}\, dx \quad \textit{is a bounded 1-form on } \mathbb{H}.$$

Proof. The relation of lemma $(6, iv)$ exactly means that the coefficient of $d\omega_\xi$ vanishes. The formula for $\pi_\xi^*\tilde{\omega} - \omega_\xi$ is clear from (5.7), and the boundedness is clear from lemma $(6, ii)$. $\square$

6. From geodesic flow to stochastic flow

We take here advantage of the closedness of the form ω_ξ to change the integration path in J_t^f (introduced in (5.3), Lemma 4): we substitute the diffusion path $\{\xi T_{Z_s^\delta} \mid 0 \le s \le \tau_t\}$ for the geodesic $\xi[0, t]$. In this contour deformation two residual terms appear, that we prove to be negligible. In the last part of this section, we use the proximity between ω_ξ and $\pi_\xi^*\tilde{\omega}$ to come back to the form $\tilde{\omega}$ (introduced in section 5.1, formula (5.5)) on $\mathscr{M}$.

6.1. From geodesics to diffusion paths

Using corollary 2, we get from (5.3): almost surely

$$\begin{aligned} J_t^f &= \lim_{S\to\infty} \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\, t^{\frac{-1}{2\delta-1}} \int_{\log\, y_S}^{t+\log\, y_S} f(\xi\theta_s)\, ds\right] d\nu(\xi) \\ &= \lim_{S\to\infty} \int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}\, t^{\frac{-1}{2\delta-1}} \int_{(0,y_S)}^{(0,y_S e^t)} \omega_\xi\right] d\nu(\xi), \end{aligned}$$

where we recall from lemma 6 and corollary 3 that there is a function g on $T^1\mathscr{M}$, bounded in $\pi^{-1}\Big(\mathscr{M} \setminus \mathscr{N}(a)\Big)$, such that
$\omega_\xi(x, y) := g(\xi T_{(x,y)})\, \frac{dx}{y} + f(\xi T_{(x,y)})\frac{dy}{y}$ is a closed smooth 1-form on $\mathbb{H}$.

Denote by Θ_t the shift on $(\Omega, \mathscr{F}, \mathbb{P})$.

Note that we have in particular (by definitions (4.1) and (4.3) of y_t and τ_t): $\tau_t \circ \Theta_S + S = inf\, \{r > S \mid y_r = y_S\, e^t\}$ and $y_{\tau_t} \circ \Theta_S = y_{\tau_t\circ\Theta_S+S} = y_S\, e^t$. We write:

$$\int_{(0,y_S)}^{(0,y_S e^t)} \omega_\xi = \int_{Z_S^\delta}^{Z_{\tau_t}^\delta \circ \Theta_S} \omega_\xi + \int_{(0,y_S)}^{(x_S,y_S)} \omega_\xi - \int_{(0,y_S e^t)}^{Z_{\tau_t}^\delta \circ \Theta_S} \omega_\xi$$
$$= \left(\int_{Z_0^\delta}^{Z_{\tau_t}^\delta} \omega_\xi \right) \circ \Theta_S + \int_{(0,y_S)}^{(x_S,y_S)} \omega_\xi - \left(\int_{(0,y_{\tau_t})}^{Z_{\tau_t}^\delta} \omega_\xi \right) \circ \Theta_S.$$

Then we have (using lemma 1):

$$\left| \int_{(0,y_S)}^{(x_S,y_S)} \omega_\xi \right| = \left| \int_0^{x_S} g(\xi T_{(x,y_S)})\, dx \,/ y_S \right| \le |x_S/y_S| \times \sup \left| g\left(\xi T_{([0,x_S],y_S)}\right) \right|$$
$$= |\alpha_S| \times \sup \left| g\left(\xi \theta_{\log\, y_S} \theta^+_{\pm[0,|\alpha_S|]} \right) \right|.$$

Fix now some strictly positive ε.

We know from lemma 1 that the laws of $\{\alpha_S \mid S \in \mathbb{R}_+\}$ form a tight family. Thus there exists $A = A_\varepsilon > 0$ such that for all $S \in \mathbb{R}_+$ we have: $\mathbb{P}(\alpha_S \in [-A, A]) > 1 - \varepsilon$.

We also know from corollary 2 that the family $\{\theta_S^* \nu \mid S \in \mathbb{R}_+\}$ is tight. Thus there exists $h = h_\varepsilon > A$ such that for all $S \in \mathbb{R}_+$ we have: $\theta_S^* \nu \left(\pi^{-1}(\mathscr{N}(h - A)) \right) < \varepsilon$.

Here we slightly modify the definition (see section 5.2) of the horocyclic neighborhood $\mathscr{N}(h)$: we fix it such that (for any positive h') for a given reference point $z_0 \in \mathscr{M}$ we have: $h' = dist\,(z_0, \mathscr{N}(h'))$.

Set $B_\varepsilon := \sup \left| g\left(\pi^{-1}[\mathscr{M} \setminus \mathscr{N}(h)] \right) \right| < \infty$. We have for all $B > B_\varepsilon$:

$$\begin{aligned}
&\nu \otimes \mathbb{P}\left[\sup \left| g\left(\xi \theta_{\log\, y_S} \theta^+_{\pm[0,|\alpha_S|]} \right) \right| > B \right] \\
&< \varepsilon + \nu \otimes \mathbb{P}\left[\sup \left| g\left(\xi \theta_{\log\, y_S} \theta^+_{[-A,A]} \right) \right| > B \right] \\
&\le \varepsilon + \nu \otimes \mathbb{P}\left[\pi \left(\xi \theta_{\log\, y_S} \theta^+_{[-A,A]} \right) \cap \mathscr{N}(h) \ne \emptyset \right] \\
&\le \varepsilon + \nu \otimes \mathbb{P}\left[\pi \left(\xi \theta_{\log\, y_S} \right) \in \mathscr{N}(h - A) \right] \\
&\le \varepsilon + \mathbb{P}(y_S < 1) + \sup_{S \ge 0} \theta_S^* \nu \left(\pi^{-1}(\mathscr{N}(h - A)) \right) \\
&< 3\varepsilon \quad \text{for } S > S_0.
\end{aligned}$$

On the other hand, we know from lemma 1 that ν is an invariant measure for $\xi_t^\delta = \xi T_{Z_t^\delta}$, and thus we have:

$$\left(\int_{(0,y_{\tau_t})}^{Z_{\tau_t}^\delta} \omega_\xi \right) \circ \Theta_S = \left(\int_{(0,e^t)}^{(x_{\tau_t},e^t)} \omega_\xi \right) \circ \Theta_S \overset{(law)}{=} \int_{(0,e^t)}^{(x_{\tau_t},e^t)} \omega_\xi .$$

Then

$$\left| \int_{(0,e^t)}^{(x_{\tau_t},e^t)} \omega_\xi \right| = e^{-t} \left| \int_0^{x_{\tau_t}} g(\xi T_{(x,e^t)})\, dx \right| \le |\alpha_{\tau_t}| \times \sup \left| g\left(\xi \theta_t \theta^+_{\pm[0,|\alpha_{\tau_t}|]} \right) \right|.$$

Now lemma (2, (ii)) shows that the laws of $\{\alpha_{\tau_t} \mid t \geq 0\}$ again form a tight family. Thus the preceding argument applies again, and so we have just proved that the two extra terms in our contour deformation (multiplied by r_t) asymptotically vanish in $\nu \otimes \mathbb{P}$-probability. Therefore we obtain that
$J_t^f - \left(\int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}t^{\frac{-1}{2\delta-1}} \int_{Z^\delta_{[0,\tau_t]}} \omega_\xi \right] d\nu(\xi) \right) \circ \Theta_S$ goes to zero in probability.
Since ν is an invariant measure for $\xi_t^\delta = \xi T_{Z_t^\delta}$, we see that the law of

$$\left(\int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}t^{\frac{-1}{2\delta-1}} \int_{Z^\delta_{[0,\tau_t]}} \omega_\xi \right] d\nu(\xi) \right) \circ \Theta_S \text{ does not depend on } S.$$

Thus we deduce in particular the following:

Proposition 3. $J_t^f - \mathbb{E}\left(\int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}t^{\frac{-1}{2\delta-1}} \int_{Z^\delta_{[0,\tau_t]}} \omega_\xi \right] d\nu(\xi) \right)$ *goes to zero as* $t \to \infty$.

6.2. *Back to windings on* $\mathscr{M}$

We now take advantage of the proximity between ω_ξ and $\pi_\xi^*\tilde{\omega}$ to come back to the form $\tilde{\omega}$ of section 5 (see (5.5), (5.7) and (5.12)).

Lemma 7. $t^{\frac{-1}{2\delta-1}} \times \int_{Z^\delta[0,\tau_t]} (\omega_\xi - \pi_\xi^*\tilde{\omega})$ *goes to zero in* $L^2(\nu \otimes \mathbb{P})$.

proof. We deduce from corollary 3 that $\int_{Z^\delta[0,\tau_t]} (\omega_\xi - \pi_\xi^*\tilde{\omega}) = \int_0^{\tau_t} (f_1 - g)(\xi_s^\delta)\, dW_s'$ is a square-integrable martingale, which using lemmas 2 and (6, ii) implies

$$\begin{aligned}
\left\| r_t \times \int_{Z^\delta[0,\tau_t]} (\omega_\xi - \pi_\xi^*\tilde{\omega}) \right\|^2_{L^2(\nu\otimes\mathbb{P})} &= r_t^2 \int\int\int_0^{\tau_t} |f_1 - g|^2(\xi_s^\delta)\, ds\, d\mathbb{P}\, d\nu(\xi) \\
&\leq t^{\frac{-1}{\delta-1/2}} \times \mathbb{E}(\tau_t) \times \|f_1 - g\|^2_\infty \\
&= \mathcal{O}\left(t^{\frac{\delta-3/2}{\delta-1/2}} \right) \longrightarrow 0.
\end{aligned}$$

□

We deduce from proposition 3, from lemma 7 above, and by definition of π_ξ (in section 5.2):

Corollary 4. $J_t^f - \mathbb{E}\left(\int_{T^1\mathscr{M}} \exp\left[\sqrt{-1}t^{\frac{-1}{2\delta-1}} \int_{\pi\circ\xi^\delta[0,\tau_t]} \tilde{\omega} \right] d\nu(\xi) \right)$ *goes to zero as* $t \to \infty$.

7. Asymptotics for Φ near the cusps

We shall need estimates on Φ and $\nabla\Phi$ at two crucial steps. We establish them now. In the estimate of Φ, a new intrinsic parameter of each cusp $\mathscr{P}$ appears, that we denote by $\lambda(\mathscr{P})$. We introduce it in two ways, before performing the estimates.

7.1. The intrinsic parameter $\lambda(\mathscr{P})$

Let us introduce some intrinsic objects relating to some fixed cusp $\mathscr{P}$. We fix first some horocyclic neighborhood $\mathscr{N}(\mathscr{P})$ of $\mathscr{P}$, small enough to be homeomorphic to $\mathbb{T} \times \mathbb{R}_+$, and to each $z \in \mathscr{N}(\mathscr{P})$, we associate the line element, say $\xi(\mathscr{P}, z) \in T_z^1\mathscr{M}$, based at z, orthogonal to the $\mathscr{P}$-horocycle passing through z, say $H(\mathscr{P}, z) = H(\xi(\mathscr{P}, z))$, and pointing at $\mathscr{P}$. $H(\mathscr{P}, z)$ has length

$$\ell(\mathscr{P}, z) := \inf\{x > 0 \mid \xi(\mathscr{P}, z)\theta_x^+ = \xi(\mathscr{P}, z)\}. \tag{7.1}$$

Then we observe that there exists a unique continuous map $x \longmapsto \xi_z(x)$ from $[0, \ell(\mathscr{P}, z)]$ into $T_z^1\mathscr{M} = \pi^{-1}(\{z\})$, such that $\xi_z(0)\theta_{\mathbb{R}_+} = \xi(\mathscr{P}, z)\theta_{\mathbb{R}_-}$ and such that for each x the half-geodesics $\xi_z(x)\theta_{\mathbb{R}_+}$ and $\xi(\mathscr{P}, z)\theta_x^+\theta_{\mathbb{R}_-}$ are asymptotic.
Observe also that, due to the geometric property (2.2), the Patterson measure μ_z, seen as a measure on $T_z^1\mathbb{H}$ by means of the coordinate system (z, u), makes sense on $T_z^1\mathscr{M}$ as well. Then we set for any $z \in \mathscr{N}(\mathscr{P})$ and any $x \in [0, \ell(\mathscr{P}, z)]$:

$$M_{\mathscr{P},z}\Big(\xi(\mathscr{P}, z)\theta_{[0,x]}^+\Big) := \ell(\mathscr{P}, z)^{-\delta} \times \mu_z\Big(\xi_z([0, x])\Big). \tag{7.2}$$

This defines a positive measure $M_{\mathscr{P},z}$ on the horocycle $H(\mathscr{P}, z)$.
By identifying $\xi(\mathscr{P}, z)$ with the geodesic it generates, we may see $M_{\mathscr{P},z}$ as a measure on the set $\mathscr{G}_{\mathscr{P}}$ of all geodesics pointing at $\mathscr{P}$.

Lemma 8. *The measure $M_{\mathscr{P},z}$ depends only on the horocycle $H(\mathscr{P}, z)$, and, as z goes to $\mathscr{P}$, $M_{\mathscr{P},z}$ converges weakly, on the compact set $\mathscr{G}_{\mathscr{P}}$ of all geodesics pointing at $\mathscr{P}$, to some intrinsic positive measure $M_{\mathscr{P}}$.*

Proof. Let us consider as in section 5.2 the half-plane model for $\mathbb{H}$ adapted to $\mathscr{P}$, that is to say such that $\mathscr{P} = \infty$ and such that the maximal parabolic subgroup of Γ fixing $\mathscr{P}$ is precisely $\mathbb{Z}$ (that is to say the isometry subgroup of horizontal translations of integer lengths).
Consider also the measure $\mu_{\mathscr{P}}$ on $\mathbb{R}$, relative to this choice of model, defined by:

$$d\mu_{\mathscr{P}}(u) := p^{-\delta}(z, u)\, d\mu_z(u). \tag{7.3}$$

Note that it does not depend on the point $z \in \mathbb{H}$.
Let us use the following elementary identity on Möbius transforms:

$$p(gz, gu) \times |g'(u)| = p(z, u), \tag{7.4}$$

for all z in $\mathbb{H}$, u in $\partial\mathbb{H}$ and g isometry of $\mathbb{H}$.
The norm is the Euclidean one in our half-plane model. This identity is given by ([P], page 282), and can be obtained by a direct computation. As a consequence,

we obtain the following translation of the geometric property (2.2) of (μ_z) in terms of $\mu_{\mathscr{P}}$: we have for any $\gamma \in \Gamma$

$$\begin{aligned} d(\gamma^* \mu_{\mathscr{P}}) = p^{-\delta}(z, \gamma^{-1} \cdot)\, d(\gamma^* \mu_z) = p^{-\delta}(\gamma z, \cdot)\, |\gamma'(\gamma^{-1} \cdot)|^{-\delta}\, d\mu_{\gamma z} \\ = |(\gamma^{-1})'(\cdot)|^{\delta}\, d\mu_{\mathscr{P}}. \end{aligned}$$

In particular we see that $\mu_{\mathscr{P}}$ is 1-periodic.
Now in the model adapted to $\mathscr{P}$, we merely have, for $0 \le r \le 1$:

$$\begin{aligned} M_{\mathscr{P},z}\Big(\xi(\mathscr{P}, z)\theta^+_{[0, r \times \ell(\mathscr{P},z)]}\Big) &= \ell(\mathscr{P}, z)^{-\delta} \times \mu_z([x, x+r]) \\ &= \ell(\mathscr{P}, z)^{-\delta} \times \int_x^{x+r} p^{\delta}(z, u)\, d\mu_{\mathscr{P}}(u) \\ &= y^{\delta} \int_x^{x+r} y^{\delta}(y^2 + u^2)^{-\delta}\, d\mu_{\mathscr{P}}(u) \\ &= \int_x^{x+r} (1 + \ell(\mathscr{P}, z)^2 u^2)^{-\delta}\, d\mu_{\mathscr{P}}(u). \end{aligned}$$

Thus we see that $M_{\mathscr{P},z}$ indeed depends only on $H(\mathscr{P}, z)$.
Moreover, seen as a subset of $\mathscr{G}_{\mathscr{P}}$, $\mathscr{G}^{x,r}_{\mathscr{P}} := \xi(\mathscr{P}, z)\theta^+_{[0, r \times \ell(\mathscr{P},z)]}$ clearly does not depend on y, and as z goes to $\mathscr{P}$, we see that $M_{\mathscr{P},z}\Big(\mathscr{G}^{x,r}_{\mathscr{P}}\Big)$ goes to $\mu_{\mathscr{P}}([x, x+r]) =: M_{\mathscr{P}}\Big(\mathscr{G}^{x,r}_{\mathscr{P}}\Big)$. □

We can thus define the following intrinsic parameter in two ways:

Definition 2. *Set*

$$\lambda(\mathscr{P}) := M_{\mathscr{P}}\Big(\mathscr{G}_{\mathscr{P}}\Big) = \mu_{\mathscr{P}}([0, 1]). \tag{7.5}$$

7.2. The estimates relating to Φ

The following proposition gives the necessary estimates of Φ and of $\nabla\Phi$ in the cusps. The notations $\mathscr{N}(\mathscr{P})$, $\ell(\mathscr{P}, z)$, $\xi(\mathscr{P}, z)$ were introduced in section 7.1 above.

Proposition 4. *For each cusp $\mathscr{P}$ of $\mathscr{M}$ and for $z \in \mathscr{N}(\mathscr{P})$, we have:*
(i) $\Phi(z) = c(\delta)\, \lambda(\mathscr{P})\, \ell(\mathscr{P}, z)^{\delta - 1} + \mathscr{O}\Big(\ell(\mathscr{P}, z)^{\delta}\Big)$;
(ii) $\mathscr{L}_1 \log \Phi \circ \pi \Big(\xi(\mathscr{P}, z)\Big) = \mathscr{O}\Big(\ell(\mathscr{P}, z)\Big)$;
(iii) $\mathscr{L}_0 \log \Phi \circ \pi \Big(\xi(\mathscr{P}, z)\Big) = 1 - \delta + \mathscr{O}\Big(\ell(\mathscr{P}, z)\Big)$.

Proof. Recall from section 5.2 that $(x^i, y^i) \in \mathbb{R} \times \mathbb{R}^*_+$ denote the canonical coordinates in the particular model adapted to the cusp $\mathscr{P}_i$. Since we consider a single cusp, let us forget the index i of this cusp. We have by periodicity of $\mu := \mu_{\mathscr{P}}$ (see section 7.1):

$$\begin{aligned} \Phi(z) &= \int_{[0,1[} \sum_{n \in \mathbb{Z}} p^{\delta}(z, u + n)\, d\mu(u) \\ &= y^{-\delta} \int_{[0,1[} \sum_{n \in \mathbb{Z}} \Big(1 + y^{-2}(x - u - n)^2\Big)^{-\delta} d\mu(u). \end{aligned}$$

Now the function $v \longmapsto (1+y^{-2}v^2)^{-\delta}$ is monotone on $\mathbb{R}_-$ and on $\mathbb{R}_+$, and thus its Riemannian sum in the last integral above cannot differ from its integral by more than twice its maximum:

$$\left|\sum_{n\in\mathbb{Z}} \left(1 + y^{-2}(x-u-n)^2\right)^{-\delta} - \int_{\mathbb{R}} (1+y^{-2}v^2)^{-\delta}\,dv\right| \le 2.$$

Whence
$\Phi(z) = y^{-\delta}\,\mu([0,1[)\int_{\mathbb{R}}(1+y^{-2}v^2)^{-\delta}\,dv + \mathcal{O}(y^{-\delta})$, which gives (i), since $y^{-1} = \ell(\mathscr{P}, z)$.
In the same way

$$\begin{aligned} y\frac{\partial}{\partial x}\Phi(z) &= -2\delta\, y^{-\delta-1}\int_{[0,1[}\sum_{n\in\mathbb{Z}}(x-u-n)\left(1+y^{-2}(x-u-n)^2\right)^{-\delta-1} d\mu(u) \\ &= -2\delta\,\mu([0,1[)\, y^{1-\delta}\int_{\mathbb{R}}(1+v^2)^{-\delta-1}\,v\,dv + \mathcal{O}(y^{-\delta}) = \mathcal{O}(y^{-\delta}), \end{aligned}$$

which with (i) gives (ii), and also

$$\begin{aligned} y\frac{\partial}{\partial y}\Phi(z) &= -\delta\,\Phi(z) + 2\delta\, y^{-\delta}\int_{[0,1[}\sum_{n\in\mathbb{Z}} y^{-2}\,(x-u-n)^2 \\ &\quad\times\left(1+y^{-2}(x-u-n)^2\right)^{-\delta-1} d\mu(u) \\ &= -\delta\,\Phi(z) + 2\delta\,\mu([0,1[)\, y^{-\delta} \\ &\quad\times\int_{\mathbb{R}}(1+y^{-2}v^2)^{-\delta-1}\,y^{-2}v^2\,dv + \mathcal{O}(y^{-\delta}) \\ &= -\delta\,\Phi(z) + 2\delta\,\mu([0,1[)\, y^{1-\delta}\,(c(\delta)-c(\delta+1)) + \mathcal{O}(y^{-\delta}) \\ &= -\delta\Phi(z) + c(\delta)\,\mu([0,1[)\; y^{1-\delta} + \mathcal{O}(y^{-\delta}), \end{aligned}$$

which with (i) gives (iii). □

8. Windings of the Φ-diffusion

In this section we shall calculate the limit law of $t^{\frac{-1}{2\delta-1}}\int_{Z^\Phi[0,t]}\tilde{\omega}$, that is to say of the joint windings of the stationary Φ-diffusion Z_t^Φ, by using discrete excursions in the cusps.

The asymptotics of Φ (in section 7 above) lead to the study of an approximating diffusion $\tilde{Z}_t$ on $\mathscr{M}$, whose generator is simpler in the cusps, thereby allowing the precise calculation of its windings in the cusps.

The passage from Z^Φ to $\tilde{Z}$ results from Girsanov's theorem, and is close to be given by a Doob's h-transform. This allows to deduce the law of the excursions of Z^Φ from the law of the excursions of $\tilde{Z}$, however necessarily conditioned by their end-point.

We must then show that this conditioning does not contribute asymptotically. For that, we have to estimate its influence and ultimately let go to infinity the level in the cusps above which we consider the excursions.

We perform then the passage from the number N_t of excursions achieved at time t to a deterministic number by means of a precise estimation of N_t.

A key argument is that the Φ-diffusion admits a spectral gap. Indeed it is known that the spectrum of $-\Delta$ in $\mathscr{M}$ is discrete up to the value $1/4$, see for example ([P-S], theorem 2.4). Hence by the definition of Δ^Φ (in section 4.4, proposition 2), we see that it admits a spectral gap and therefore that there exists $\lambda_1 > 0$ such that

$$\left\| P_t^\Phi \varphi \right\|_2 \leq e^{-\lambda_1 t} \left\| \varphi \right\|_2 \tag{8.1}$$

for any positive t and any centered $\varphi \in L^2(V^\Phi)$, where P_t^Φ denotes the semi-group associated with Z_t^Φ.

8.1. *The modified diffusion $\tilde{Z}$*

Since the value of $\nabla \log\, \Phi$ in the cusps can be known only asymptotically (see section 7), we are led to consider the winding contributions of the diffusion paths above a sufficiently high level b, and to modify the Φ-diffusion Z_t^Φ above this level b, in order to get a simplified generator allowing the calculation of windings.

More precisely, proposition 4 leads to replace in the cusps the generator Δ^Φ of Z_t^Φ by the simplified generator $\Delta^{1-\delta}$ (notation (4.2)). Now we observe that

$$\Delta^{1-\delta}(y^{\delta-1}\Phi) = \Big(y^{\delta-1}\Delta \circ y^{1-\delta} - 2\lambda_o\Big)(y^{\delta-1}\Phi) = 0,$$

which means that the above modification can be performed by a Doob's h-transform, by means of the harmonic function $y^{\delta-1}\Phi$. But this is the case only in the cusps, and we have to take care of the necessary transition between the cusps and the body of $\mathscr{M}$, and to use a Girsanov's transform. Of course we shall verify that this Girsanov's transform is close to a h-transform.

Recall from section 5.2 that we set $\mathscr{N}_i(a) := \{y^i \geq a\}$ and $\mathscr{N}(a) := \cup_{i=1}^N \mathscr{N}_i(a)$. These are horocyclic neighborhoods of the cusps of $\mathscr{M}$. Fix some smooth increasing function h^o from $\mathbb{R}$ onto $[0,1]$, null on $\mathbb{R}_-$ and equal to 1 on $[1,\infty[$.

Let us take b in $]2,\infty[$, larger than b_o used for the definition (5.5) of $\tilde{\omega}$.
For each $i \in \{1,\dots,N\}$, set $h_i := 1_{\mathscr{N}_i(b)} \times h^o \circ (y^i - b)$.
So $h_i \in C^\infty(\mathscr{M},[0,1])$, $h_i = 0$ in $\mathscr{M} \setminus \mathscr{N}_i(b)$, and $h_i = 1$ in $\mathscr{N}_i(b+1)$.

Definition 3. *Set in $\mathscr{N}_i(b)$: $\Phi_i := \Big(c(\delta)\lambda(\mathscr{P}_i)\Big)^{-1} \times (y^i)^{\delta-1} \times \Phi - 1$.*
Set then $H := 1 + \sum_{i=1}^N h_i \Phi_i$, and $G := \frac{1}{2}\Delta^\Phi - (\nabla \log H)\cdot\nabla = \frac{1}{2}\Delta + (\nabla \log(\Phi/H))\cdot\nabla$.
Let $\tilde{Z}_t$ be the diffusion on $\mathscr{M}$ with generator G and initial law V^Φ, whose law is given by the following Girsanov's transform: for each $t \geq 0$ and for any test-

functional F:

$$\mathbb{E}\Big[F(\tilde{Z}[0,t])\Big] = \mathbb{E}\Big[F(Z^{\Phi}[0,t])\ \exp\Big(-\int_0^t \nabla \log H(Z_s^{\Phi}) \cdot dZ_s^0 - \tfrac{1}{2}\int_0^t |\nabla \log H|^2(Z_s^{\Phi})\, ds\Big)\Big].$$

Remark 6. (i) Proposition 4 shows that $\Phi_i = \mathcal{O}(1/y^i)$ and $\nabla \log(1+\Phi_i) = \mathcal{O}(1/y^i)$.

(ii) G equals $\frac{1}{2}\Delta^{\Phi}$ in $\mathcal{M} \setminus \mathcal{N}(b)$ and equals $\frac{1}{2}\Delta + (1-\delta) y^i \frac{\partial}{\partial y^i}$ in $\mathcal{N}_i(b+1)$.

(iii) $H = 1$ in $\mathcal{M} \setminus \mathcal{N}(b)$, $H \longrightarrow 1$ at each cusp, and $H > 0$ on $\mathcal{M}$.

As a consequence, there are two constants c_1 and c_2 such that $0 < c_1 \le H \le c_2 < \infty$ on $\mathcal{M}$. Moreover $\nabla \log H = \sum_{i=1}^N 1_{\mathcal{N}_i(b)} \mathcal{O}(1/y^i) = \mathcal{O}(1/b)$.

Let us now apply the Girsanov's theorem: we have for each $t \ge 0$ and for any test-functional F:

$$\mathbb{E}\Big[F(Z^{\Phi}[0,t])\Big] = \mathbb{E}\Big[F(\tilde{Z}[0,t]) \times M_t\Big], \tag{8.2}$$

where the martingale M_t is defined (for some Brownian motion Z_s^o on $\mathcal{M}$) by

$$M_t := \exp\Big[\int_0^t \nabla \log H(\tilde{Z}_s) \cdot dZ_s^o - \tfrac{1}{2}\int_0^t |\nabla \log H|^2(\tilde{Z}_s)\, ds\Big]. \tag{8.3}$$

Observe that M_t indeed is a martingale, since by remark (6, iii) above $\nabla \log H$ is bounded.

Lemma 9. *Set $\varphi := -G \log H - \frac{1}{2}|\nabla \log H|^2$. Then φ vanishes outside $\mathcal{N}(b) \setminus \mathcal{N}(b+1)$, is $\mathcal{O}(1/b)$, and we have*
$M_t = H(\tilde{Z}_0)^{-1} \times H(\tilde{Z}_t) \times \exp\Big[\int_0^t \varphi(\tilde{Z}_s)\, ds\Big].$

Proof. Using Itô's formula we get

$$\int_0^t \nabla \log H(\tilde{Z}_s) \cdot dZ_s^o = \log H(\tilde{Z}_t) - \log H(\tilde{Z}_0) - \int_0^t G(\log H)(\tilde{Z}_s)\, ds,$$

whence the alternative formula for M_t, from the above definitions of M_t and of φ.

Then we have

$$\begin{aligned}\varphi &= -\frac{1}{2}\Big(\Delta \log H + |\nabla H|^2 \times H^{-2}\Big) - \Big(\nabla \log(\Phi/H)\Big) \cdot \nabla \log H \\ &= -\frac{1}{2}\Big(div((\nabla H)/H) + H^{-2} \times |\nabla H|^2\Big) + \Big(\nabla \log(H/\Phi)\Big) \cdot \nabla \log H \\ &= \Big(\nabla \log(H/\Phi)\Big) \cdot \nabla \log H - (2H)^{-1}\Delta H.\end{aligned}$$

As a consequence, we deduce as expected that $\varphi = 0$ on $\mathscr{M} \setminus \mathscr{N}(b)$, and also in $\mathscr{N}(b+1)$, since we have in each $\mathscr{N}_i(b+1)$:

$$\begin{aligned}\varphi &= \nabla \log (y^i)^{\delta-1} \cdot \nabla \log ((y^i)^{\delta-1}\Phi) - (2(y^i)^{\delta-1}\Phi)^{-1}\Delta((y^i)^{\delta-1}\Phi)\\ &= |\nabla \log (y^i)^{\delta-1}|^2 - (2(y^i)^{\delta-1})^{-1}\Delta((y^i)^{\delta-1}) - (2\Phi)^{-1}\Delta\Phi\\ &= (\delta-1)^2 - (\delta-1)(\delta-2)/2 - \delta(\delta-1)/2 = 0.\end{aligned}$$

Finally remark 6 and proposition 4 show that we have equivalence between $\varphi = \mathscr{O}(1/b)$ and $\Delta H = \mathscr{O}(1/b)$, and then that it is enough to verify that $\Delta\Phi_i = \mathscr{O}(1/y^i)$ for each i. Now we have by definition 3, up to a constant and by application of proposition (4, iii, i):

$$\begin{aligned}\Delta\Phi_i &= (y^i)^{\delta-1}\Delta\Phi + 2\nabla(y^i)^{\delta-1}\cdot\nabla\Phi + \Phi\Delta(y^i)^{\delta-1}\\ &= (y^i)^{\delta-1}\Phi \times \Big(\delta(\delta-1) + 2(\delta-1)((1-\delta) + \mathscr{O}(1/y^i)) + (\delta-1)(\delta-2)\Big)\\ &= (y^i)^{\delta-1}\Phi \times \mathscr{O}(1/y^i) \;=\; \mathscr{O}(1/y^i).\end{aligned}$$ □

8.2. *Discrete excursions and uniform integrability of M_t*

Denote by $[\tau_n, \zeta_n]$ the n-th time interval of excursion in the cusps, that is to say precisely: $\zeta_0 := 0$, $\tau_1 = \tau$ is the first hitting time of $\mathscr{N}(b+\sqrt{b})$, ζ is the first exit time of $\mathscr{N}(b+1)$, and for each $n \in \mathbb{N}^*$:

$$\zeta_n := \zeta \circ \Theta_{\tau_n} + \tau_n \quad \text{and} \quad \tau_{n+1} := \tau \circ \Theta_{\zeta_n} + \zeta_n. \tag{8.4}$$

We shall need the Girsanov's formula until the stopping time ζ_n. The following lemma shows that this formula is valid, for any large enough fixed b.

Lemma 10. *The martingale M_t is uniformly integrable on $[0, \zeta_n]$, for each $n \in \mathbb{N}$.*

Proof. 1) Observe first that by remark (6, iii) and lemma 9

$$\begin{aligned}\sup\{M_t \mid 0 \le t \le \zeta_n\} &\le \frac{c_2}{c_1} \times \exp\left[\int_0^{\zeta_n} |\varphi(\tilde{Z}_s)|\, ds\right]\\ &\le c\,\exp\left[\mathscr{O}(1/b)\int_0^{\zeta_n} 1_{\mathscr{N}(b)\setminus\mathscr{N}(b+1)}(\tilde{Z}_s)\, ds\right].\end{aligned}$$

We have then

$$\int_0^{\zeta_n} 1_{\mathscr{N}(b)\setminus\mathscr{N}(b+1)}(\tilde{Z}_s)\, ds \le \sum_{k=0}^{n-1}\left(\int_0^{\tau} 1_{\mathscr{N}(b)}(\tilde{Z}_s)\, ds\right)\circ\Theta_{\zeta_k},$$

whence

$$\begin{aligned}&\mathbb{E}\Big(\sup\{M_t \mid 0 \le t \le \zeta_n\}\Big)\\ &\quad\le c\left(\sup_{z\in\partial\mathscr{N}(b)} \mathbb{E}_z\left(\exp\left[\mathscr{O}(1/b)\int_0^{\tau} 1_{\mathscr{N}(b)}(\tilde{Z}_s)\, ds\right]\right)\right)^n.\end{aligned}$$

2) Let us consider for some $\lambda > -1$ the function f_λ defined on $[b, b+\sqrt{b}\,]$ such that $f'_\lambda(b) = 0$ and $y^2 f''_\lambda(y) - y f'_\lambda(y) - \lambda f_\lambda(y) = 0$. Up to a multiplicative constant, this implies

$$f_\lambda(y) = y \times \Big[\sqrt{\lambda+1} \times \Big((y/b)^{\sqrt{\lambda+1}} + (y/b)^{-\sqrt{\lambda+1}}\Big) \\ -\Big((y/b)^{\sqrt{\lambda+1}} - (y/b)^{-\sqrt{\lambda+1}}\Big)\Big].$$

Note that f_λ is positive decreasing for $-1 < \lambda < 0$ and large enough b.
Fix now $\lambda = -2/3$, and consider the following continuous function g on $\mathbb{R}$: g is constant on $]-\infty, b]$ and on $[b+\sqrt{b}, \infty[$, and equals $f_{-2/3}/f_{-2/3}(b+\sqrt{b}\,)$ on $[b, b+\sqrt{b}]$.

Observe that g decreases, has derivative zero at b, and equals 1 on $[b+\sqrt{b}, \infty[$.

Extend g to a continuous function $\tilde{g}$ on $\mathscr{M}$, by taking $\tilde{g}$ constant on $\mathscr{M} \setminus \mathscr{N}(b)$ and equal to $g(y)$ on $\partial\mathscr{N}(y)$, for $y \geq b$.
Now, by proposition 4 and remark $(6, iii)$, we have in $\mathscr{N}(b)$, for large enough fixed b:

$$y\,\frac{\partial}{\partial y}\log(\Phi/H) = 1 - \delta + \mathscr{O}(1/b) \geq -1/2,$$

and thus for $z \in \mathscr{N}(b) \setminus \mathscr{N}(b+\sqrt{b}\,)$: $G\tilde{g}(z) \leq \frac{1}{2}\,(y^2 g''(y) - y g'(y)) = -\frac{1}{3}\,g(y)$.
Therefore we have $\quad G\tilde{g} + \frac{1}{3}\,1_{\mathscr{N}(b)}\,\tilde{g} \leq 0$ on $\mathscr{M} \setminus \mathscr{N}(b+\sqrt{b}\,)$.
As a consequence, using Itô's formula (valid since $g'(b) = 0$) we get for any $z \in \mathscr{N}(b)$:

$$\mathbb{E}_z\left[\tilde{g}\Big(\tilde{Z}_{t\wedge\tau}\Big)\exp\left(\tfrac{1}{3}\int_0^{t\wedge\tau} 1_{\mathscr{N}(b)}(\tilde{Z}_s)\,ds\right)\right] - g(b) \\ = \mathbb{E}_z\left[\int_0^{t\wedge\tau}\Big(G\tilde{g} + \tfrac{1}{3}\,1_{\mathscr{N}(b)}\,\tilde{g}\Big)(\tilde{Z}_s)\,\exp\left(\tfrac{1}{3}\int_0^{s} 1_{\mathscr{N}(b)}(\tilde{Z}_u)\,du\right)ds\right] \leq 0,$$

whence by letting t go to infinity:

$$\sup_{z\in\mathscr{N}(b)} \mathbb{E}_z\left[\exp\left(\tfrac{1}{3}\int_0^{\tau} 1_{\mathscr{N}(b)}(\tilde{Z}_s)\,ds\right)\right] \leq g(b) < \infty.$$

With 1) above, this concludes the proof. □

Set

$$\psi_n := \int_{Z^\Phi[\tau_n,\zeta_n]} \tilde{\omega}, \quad \psi'_n := \int_{\tilde{Z}[\tau_n,\zeta_n]} \tilde{\omega}, \quad \text{and} \quad \psi' := \int_{\tilde{Z}[0,\zeta]} \tilde{\omega}. \tag{8.5}$$

By lemmas 9 and 10 and the strong Markov property, we have for any real r and $n \in \mathbb{N}^*$:

$$\mathbb{E}\left(\exp\Big[\sum_{j=1}^{n}\sqrt{-1}\;r\,\psi_j\Big]\right) = \mathbb{E}\left(\exp\Big[\sum_{j=1}^{n}\sqrt{-1}\;r\,\psi'_j\Big] \times M_{\zeta_n}\right)$$

$$= \mathbb{E}\left(H(\tilde{Z}_0)^{-1} \times \exp\Big[\sum_{j=1}^{n} \sqrt{-1}\, r\, \psi'_j\Big] \right.$$
$$\left. \times \exp\Big[\sum_{j=1}^{n} \int_{\zeta_{j-1}}^{\tau_j} \varphi(\tilde{Z}_s)\, ds\Big] \times H(\tilde{Z}_{\zeta_n}) \right)$$
$$= \mathbb{E}\left(H(\tilde{Z}_0)^{-1} \times \prod_{j=1}^{n} \exp\Big[\int_{\zeta_{j-1}}^{\tau_j} \varphi(\tilde{Z}_s)\, ds\Big] \right.$$
$$\left. \times \prod_{j=1}^{n} \mathbb{E}_{\tilde{Z}_{\tau_j}} \Big[e^{\sqrt{-1}\, r\, \psi'} \Big| \tilde{Z}_{\zeta_j} \Big] \times H(\tilde{Z}_{\zeta_n}) \right)$$
$$= \mathbb{E}\left(M_{\zeta_n} \times \prod_{j=1}^{n} \mathbb{E}_{\tilde{Z}_{\tau_j}} \Big[e^{\sqrt{-1}\, r\, \psi'} \Big| \tilde{Z}_{\zeta_j} \Big] \right). \qquad (8.6)$$

8.3. Approximation of the excursion law

Let us consider an excursion path $\tilde{Z}[\tau_n, \zeta_n]$. It occurs near some cusp $\mathscr{P}$, and we may lift it to $\mathbb{H}$, in a model adapted to $\mathscr{P}$. So we get a diffusion path $\{\tilde{Z}_s = (\tilde{x}_s, \tilde{y}_s) \mid \tau_n \leq s \leq \zeta_n\}$, taking its values in $\mathbb{R} \times [b+1, \infty[$, and such that $\tilde{Z}_{\tau_n} = (0, b + \sqrt{b})$ and $\tilde{Z}_{\zeta_n} = (\tilde{x}_{\zeta_n}, b+1)$. Moreover $\tilde{Z}_s = (\tilde{x}_s, \tilde{y}_s)$ has generator $\Delta^{1-\delta}$, and thus there exist two independent standard real Brownian motions w_t and W_t such that $\tilde{x}_{\zeta_n} = W(Y)$ with $Y := \int_0^{\zeta} (\tilde{y}_s)^2\, ds$ and $d\tilde{y}_s = \tilde{y}_s\, dw_s + (1-\delta)\, \tilde{y}_s\, ds$.
Recall that by (5.5) and (8.5) $\psi'_n = r(\mathscr{P}, \omega) \times \tilde{x}_{\zeta_n}$.

We compute the law of the variable Y in the following lemma.

Lemma 11. *For any positive r we have*

$$\mathbb{E}\Big(e^{-r^2 Y/2}\Big) = \tilde{K}_\delta\Big((b+\sqrt{b}\,)r\Big) \Big/ \tilde{K}_\delta\Big((b+1)\, r\Big),$$

where $\tilde{K}_\delta(r) := r^{(\delta-1/2)} K_{(\delta-1/2)}(r)$ and $K_{(\delta-1/2)}$ denotes the usual modified Bessel function of index $(\delta - 1/2)$. As a consequence, we have $\mathbb{E}(Y^{-1/2}) = \mathscr{O}(b^{-1/2})$.

Proof. 1) $\exp\Big[-\frac{r^2}{2}\int_0^t (\tilde{y}_s)^2\, ds\Big] \times \tilde{K}(\tilde{y}_t)$ is equal to martingale $+ \frac{1}{2}\int_0^t \exp\Big[-\frac{r^2}{2}\int_0^s (\tilde{y}_u)^2 du\Big] \Big((\tilde{y}_s)^2 \tilde{K}''(\tilde{y}_s) + 2(1-\delta)\, \tilde{y}_s \tilde{K}'(\tilde{y}_s) - r^2\, (\tilde{y}_s)^2 \tilde{K}(\tilde{y}_s)\Big)\, ds$
and then is a bounded martingale for $\tilde{K}$ bounded near $+\infty$ satisfying
$\tilde{K}''(y) + \frac{2(1-\delta)}{y}\, \tilde{K}'(y) - r^2\, \tilde{K}(y) = 0$.
Setting $f(y) := y^{(1/2-\delta)}\, \tilde{K}(y)$, we get f null at $+\infty$ satisfying
$f''(y) + f'(y)/y - (r^2 + (\delta - 1/2)^2 y^{-2})\, f(y) = 0$, whence $f(y) = c\, K_{(\delta-1/2)}(ry)$.
Hence $\tilde{K}(y) = c'\, \tilde{K}_\delta(ry)$, and finally the optional sampling theorem gives the result.

2) We deduce from 1) above that

$$\mathbb{E}(Y^{-1/2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathbb{E}\left(e^{-r^2 Y/2}\right) dr$$
$$= (b+1)^{-1}\sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{K}_\delta\left(\frac{b+\sqrt{b}}{b+1} \times r\right) \Big/ \tilde{K}_\delta(r)\, dr.$$

Now there exist two positive constants c_1 and c_2 such that $0 < c_1 \leq \tilde{K}_\delta \leq c_2$ on $[0,2]$ and $c_1\, r^{\delta-1} e^{-r} \leq \tilde{K}_\delta(r) \leq c_2\, r^{\delta-1} e^{-r}$ on $\{1 \leq r < \infty\}$. Therefore

$$\int_0^\infty \tilde{K}_\delta\left(\frac{b+\sqrt{b}}{b+1} \times r\right) \Big/ \tilde{K}_\delta(r)\, dr \leq \frac{c_2}{c_1} \times \left(1 + \int_1^\infty e^{(1-\sqrt{b})(b+1)^{-1} r}\, dr\right)$$
$$= \mathcal{O}\Big((b+1)b^{-1/2}\Big),$$

whence the result. □

We see from expression (8.6) in section 8.2 that we need to estimate the conditioned winding law by means of the unconditioned one. This we do in the following lemma, where our notations remain as above.

Lemma 12. *There exists a real measurable function a on $\partial\mathcal{N}(b+\sqrt{b}) \times \partial\mathcal{N}(b+1)$ such that $\|a\|_\infty = \mathcal{O}(b^{-1/2})$ and such that for any integer $n \geq 2$ and any real r we have:*

$$\mathbb{E}_{\tilde{Z}_{\tau_n}}\left(e^{\sqrt{-1}\, r\, \tilde{x}_\zeta} \,\Big|\, \tilde{Z}_{\zeta_n}\right) = 1 - \Big(1 + a(\tilde{Z}_{\tau_n}, \tilde{Z}_{\zeta_n})\Big) \times \Big(1 - \mathbb{E}\Big[e^{-r^2 Y/2}\Big] + \mathcal{O}(|r|)\Big).$$

Proof. 1) We have: $\mathbb{E}_{\tilde{Z}_{\tau_n}}\left(e^{\sqrt{-1}\, r\, \tilde{x}_\zeta} \,\Big|\, \tilde{Z}_{\zeta_n}\right) = \mathbb{E}\Big(e^{\sqrt{-1}\, r\, W(Y)} \,\Big|\, W(Y) \text{ modulo } 1\Big)$. Let us fix some real η and some positive ε. We have firstly

$$\mathbb{E}\Big(e^{\sqrt{-1}\, r\, W(Y)} \,\Big|\, W(Y) \in]\eta, \eta+\varepsilon[+ \mathbb{Z}\Big) - 1$$
$$= \frac{\mathbb{E}\left[\left(e^{\sqrt{-1}\, r\, W(Y)} - 1\right) \times \sum_{k\in\mathbb{Z}} 1_{\{\eta < W(Y)-k < \eta+\varepsilon\}}\right]}{\mathbb{E}\left[\sum_{k\in\mathbb{Z}} 1_{\{\eta < W(Y)-k < \eta+\varepsilon\}}\right]}$$
$$= \frac{\mathbb{E}\left[(2\pi Y)^{-1/2} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \left(e^{\sqrt{-1}\, r\, x} - 1\right) e^{-x^2/(2Y)}\, 1_{\{\eta < x-k < \eta+\varepsilon\}}\, dx\right]}{\mathbb{E}\left[(2\pi Y)^{-1/2} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} e^{-x^2/(2Y)}\, 1_{\{\eta < x-k < \eta+\varepsilon\}}\, dx\right]}$$
$$= \frac{\mathbb{E}\left[(2\pi Y)^{-1/2} \int_\eta^{\eta+\varepsilon} \sum_{k\in\mathbb{Z}} \left(e^{\sqrt{-1}\, r\, (x+k)} - 1\right) e^{-(x+k)^2/(2Y)}\, dx\right]}{\mathbb{E}\left[(2\pi Y)^{-1/2} \int_\eta^{\eta+\varepsilon} \sum_{k\in\mathbb{Z}} e^{-(x+k)^2/(2Y)}\, dx\right]}.$$

2) Then

$$\begin{aligned}
&\sup\left\{|e^{\sqrt{-1}\,rk}-1|\times e^{-k^2/(2Y)}\,\Big|\,k\in\mathbb{R}\right\}\\
&\quad\le \sup\left\{\min\{2,|rk|\}\times e^{-k^2/(2Y)}\,\Big|\,k\in\mathbb{R}\right\}\\
&\quad= \max\left\{\max\{e^{-k^2/(2Y)}\,|\,k\ge 2/r\};\ \max\{|rk|\times e^{-k^2/(2Y)}\,|\,0\le k\le 2/r\}\right\}\\
&\quad= \max\left\{e^{-2(r^2Y)^{-1}};\ \min\{2,|r|\sqrt{Y}\}\times\exp[-\min\{1,4r^{-2}/Y\}/2]\right\}\\
&\quad= 2\,e^{-2(r^2Y)^{-1}}\,1_{\{Y>4r^{-2}\}}+|r|\sqrt{Y}\times e^{-1/2}\,1_{\{Y\le 4r^{-2}\}}\le |r|\sqrt{Y}.
\end{aligned}$$

Hence we can replace the Riemannian sum in 1) above by a Riemannian integral + an error term:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(e^{\sqrt{-1}\,r\,(x+k)}-1\right)e^{-(x+k)^2/(2Y)}\\
&\quad=\int_{\mathbb{R}}\left(e^{\sqrt{-1}\,r\,(x+k)}-1\right)e^{-(x+k)^2/(2Y)}\,dk+\mathcal{O}(|r|\sqrt{Y})\\
&\quad=\left(e^{-r^2Y/2}-1\right)\sqrt{2\pi Y}\ +\mathcal{O}(|r|\sqrt{Y}).
\end{aligned}$$

Therefore we obtain for all η and ε (with a uniform $\mathcal{O}$):

$$\begin{aligned}
&\mathbb{E}\left(e^{\sqrt{-1}\,r\,W(Y)}\,\Big|\,W(Y)\in]\eta,\eta+\varepsilon[+\mathbb{Z}\right)-1\\
&\quad=\frac{\varepsilon\times\mathbb{E}\left[\left(e^{-r^2Y/2}-1\right)+\mathcal{O}(|r|)\right]}{\varepsilon\times\left(1+\mathcal{O}(\mathbb{E}(Y^{-1/2}))\right)}\\
&\quad=\left[\mathbb{E}\left(e^{-r^2Y/2}\right)-1+\mathcal{O}(|r|)\right]\times\left(1+\mathcal{O}(\mathbb{E}(Y^{-1/2}))\right).
\end{aligned}$$

This proves

$$\begin{aligned}
&\mathbb{E}\left(e^{\sqrt{-1}\,r\,W(Y)}\,\Big|\,W(Y)\ modulo\ 1\right)\\
&\quad=1-\left(1+\mathcal{O}(\mathbb{E}(Y^{-1/2}))\right)\times\left[1-\mathbb{E}\left(e^{-r^2Y/2}\right)+\mathcal{O}(|r|)\right],
\end{aligned}$$

and then the result, with the help of lemma 11. □

8.4. *Asymptotic law of the excursions*

We shall now let the parameters r and n depend on the time t. Precisely, we take $r=\beta\,r_t=\beta\,t^{\frac{-1}{2\delta-1}}$ and $n=[\varrho\,t]:=\max\{j\in\mathbb{N}\,|\,j\le\varrho\,t\}$.
Here β is a real parameter and ϱ is some positive parameter to be specified later. Set

$$\gamma(\beta):=2^{1-2\delta}\times\frac{\Gamma(3/2-\delta)}{\Gamma(\delta+1/2)}\times\left((b+\sqrt{b})^{2\delta-1}-(b+1)^{2\delta-1}\right)\times|\beta|^{2\delta-1}. \tag{8.7}$$

Since $\tilde{K}_\delta(y) = 2^{\delta-3/2}\,\Gamma(\delta-1/2) - 2^{1/2-\delta}\,(2\delta-1)^{-1}\Gamma(3/2-\delta)\,y^{2\delta-1} + o(y^{2\delta-1})$, we get from lemma 11 that

$$1-\mathbb{E}\Big(e^{-r^2\,Y/2}\Big) = \frac{\tilde{K}_\delta\Big((b+1)\,|r|\Big)-\tilde{K}_\delta\Big((b+\sqrt{b}\,)\,|r|\Big)}{\tilde{K}_\delta\Big((b+1)\,|r|\Big)}$$
$$= \gamma(r) + o(|r|^{2\delta-1}).$$

Let us first get rid of the somewhat particular case $\delta = 1$. In this cofinite case, we merely have: $\Phi = H = 1$, $G = \Delta^\Phi = \Delta$, $M_t \equiv 1$, $\tilde{Z}_t = Z_t^\Phi = Z_t^o$.

Moreover $\mathbb{E}_{Z^o_{\tau_n}}\left(e^{\sqrt{-1}\,r\,\psi}\right)$ does not depend upon $Z^o_{\tau_n}$, and then, denoting by r^j the residue corresponding to ψ_j, we have:

$$\mathbb{E}\left(\prod_{j=1}^{n} e^{\sqrt{-1}\,r\,\psi_j}\right) = \mathbb{E}\left(\left[\prod_{j=1}^{n-1} e^{\sqrt{-1}\,r\,\psi_j}\right]\times \mathbb{E}_{Z^o_{\tau_n}}\left(e^{\sqrt{-1}\,r\,\psi}\right)\right)$$
$$= \mathbb{E}\left(\prod_{j=1}^{n-1} e^{\sqrt{-1}\,r\,\psi_j}\right)\times \mathbb{E}\left(e^{\sqrt{-1}\,rr^n\,W(Y)}\right)$$
$$= \prod_{j=1}^{n} \mathbb{E}\left(e^{-(rr^j)^2\,Y/2}\right).$$

This means that the windings of the different excursions are independent, and thus we get for any real β and positive ϱ, as $t \to \infty$ (assuming for a while that all residues equal r):

$$\mathbb{E}\left(\prod_{j=1}^{[\varrho t]} e^{\sqrt{-1}\,\beta\,r_t\,\psi_j}\right) = \left[\mathbb{E}\Big(e^{\sqrt{-1}\,\beta\,t^{-1}\,\psi}\Big)\right]^{[\varrho\,t]}$$
$$= \Big(1-\gamma(\beta\,r)/t\Big)^{[\varrho\,t]} \longrightarrow e^{-\gamma(\beta\,r)\,\varrho}.$$

Let us now come to the main (infinite area) case $\delta < 1$.

Then $r_t = o\Big(r_t^{2\delta-1}\Big) = o(1/t)$, and assuming for a while and for convenience that all residues of ω equal some r, we get from lemma 12 and (8.7):

$$\prod_{j=1}^{[\varrho t]} \mathbb{E}_{\tilde{Z}_{\tau_j}}\left(e^{\sqrt{-1}\,\beta\,r_t\,\psi'}\;\Big|\;\tilde{Z}_{\zeta_j}\right)$$
$$= \prod_{j=1}^{[\varrho t]}\left[1-\Big(1+a(\tilde{Z}_{\tau_j},\tilde{Z}_{\zeta_j})\Big)\times\Big(\gamma(\beta\,r)/t+o(1/t)\Big)\right]$$
$$= \exp\left(\sum_{j=1}^{[\varrho t]}\log\left[1-\Big(1+a(\tilde{Z}_{\tau_j},\tilde{Z}_{\zeta_j})\Big)\times\Big(\gamma(\beta\,r)/t+o(1/t)\Big)\right]\right)$$

$$= \exp\left(-\gamma(\beta\, r)\, t^{-1} \sum_{j=1}^{[\varrho t]} \Big(1 + a(\tilde Z_{\tau_j}, \tilde Z_{\zeta_j})\Big) + o(1)\right).$$

Hence coming back to the Markovian expression (8.6) of section 8.2 and using (8.2) and lemma 10, we see that:

$$\begin{aligned}
&\mathbb{E}\left[\prod_{j=1}^{[\varrho t]} e^{\sqrt{-1}\,\beta\, r_t\, \psi_j}\right] \\
&= \mathbb{E}\left[M_{\zeta_{[\varrho t]}} \times \prod_{j=1}^{[\varrho t]} \mathbb{E}_{\tilde Z_{\tau_j}}\Big(e^{\sqrt{-1}\,\beta\, r_t\, \psi'} \,\Big|\, \tilde Z_{\zeta_j}\Big)\right] \\
&= \mathbb{E}\left[M_{\zeta_{[\varrho t]}} \times \exp\left(-\gamma(\beta\, r)\, t^{-1} \sum_{j=1}^{[\varrho t]} \Big(1 + a(\tilde Z_{\tau_j}, \tilde Z_{\zeta_j})\Big) + o(1)\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\gamma(\beta\, r)\, t^{-1} \sum_{j=1}^{[\varrho t]} \Big(1 + a(Z^\Phi_{\tau_j}, Z^\Phi_{\zeta_j})\Big) + o(1)\right)\right] \\
&\longrightarrow \exp\left[-\gamma(\beta\, r) \times \varrho \times \left(1 + \int a\, d\chi\right)\right],
\end{aligned}$$

as $t \to \infty$, where χ is the normalized Palm invariant probability measure of the induced Markov chain $(Z^\Phi_{\tau_j}, Z^\Phi_{\zeta_j})$. Indeed we may view the ergodic stationary historical process $(Z^\Phi_s, -\infty < s \le t)$ as a suspended flow under the function ζ_1. This allows to apply Ambrose's theorem and then to deduce that the induced Markov chain $(Z^\Phi_{\tau_j}, Z^\Phi_{\zeta_j})$ is stationary and ergodic, with invariant Palm law χ. See ([SL-M], Exposés I and II).

The extension to different residues is straightforward.

Observe that the finite area case $\delta = 1$ can again be handled as the general case by merely considering that $\int a\, d\chi = 0$ for $\delta = 1$.

8.5. *Counting the excursions*

We need some information on the duration of the excursions of Z^Φ_t in the cusps. This is the aim of the following lemma.

Lemma 13. *There exists $\lambda > 0$ such that $\mathbb{E}\Big(\exp\big[\lambda(\zeta_n - \tau_n)\big]\Big)$ is finite for any $n \in \mathbb{N}^*$. Moreover, $\mathbb{E}(\zeta_n - \tau_n) = (\delta - 1/2)^{-1} \times b^{-1/2} \times \Big(1 + \mathcal{O}(b^{-1/2})\Big)$ for all $n \ge 2$.*

Proof. Fix $0 < \delta_1 < \delta - 1/2 < \delta_2 < 1$, and $n \ge 2$.

By proposition 2, the drift term of Z^Φ_t is $\nabla \log \Phi$, and thus by using proposition 4 we have on each $[\tau_n, \zeta_n]$ in adapted coordinates: $Z^\Phi_t = (x^\Phi_t, y^\Phi_t)$, with

$(y_t^\Phi)^{-1}\, dy_t^\Phi = dw'_t + ((1-\delta) + \mathcal{O}(1/b))\, dt.$
Let y_t^i denote the diffusion on $[b+1, \infty[$, starting from $b+\sqrt{b}$, and solving $(y_t^i)^{-1}\, dy_t^i = dw'_t + (1/2 - \delta_i)\, dt.$ Set $\zeta^i := \inf\{t > 0 \mid y_t^i = b+1\}$.
The classical comparison lemma (see [I-W]) applies for large enough fixed b, and gives almost surely: $\zeta^2 \le \zeta_n - \tau_n \le \zeta^1$. Now, we have $y_t^i = (b+\sqrt{b}\,) \times \exp[w'_t - \delta_i t]$ on $[0, \zeta^i]$. Hence we have by the Cameron-Martin formula, for $\lambda \le \delta_1^2/2$:

$$\begin{aligned}\mathbb{E}\Big(e^{\lambda\,\zeta^i}\Big) &= \mathbb{E}\Big(\exp\Big[\lambda\ \inf\{t \mid -\ w'_t + \delta_i\, t = \log((b+\sqrt{b}\,)/(b+1))\}\Big]\Big)\\ &= \exp\Big[\delta_i\ \log((b+\sqrt{b}\,)/(b+1))\Big]\\ &\quad \times \mathbb{E}\Big(\exp\Big[(\lambda - \delta_i^2/2)\ \inf\{t \mid w_t = \log((b+\sqrt{b}\,)/(b+1))\}\Big]\Big)\\ &= \Big((b+\sqrt{b}\,)/(b+1)\Big)^{\delta_i - \sqrt{\delta_i^2 - 2\lambda}}.\end{aligned}$$

The first assertion follows. Then we deduce that

$$\mathbb{E}(\zeta^i) = (\delta_i)^{-1} \times \log((b+\sqrt{b}\,)/(b+1)) = (\delta_i)^{-1} \times b^{-1/2} + \mathcal{O}(1/b).$$

The proof is achieved for $n \ge 2$, since the only constraint on $|\delta - 1/2 - \delta_i|$ was to dominate some $\mathcal{O}(1/b)$. For $n = 1$, we only have to modify the above estimate by using the asymptotic form of V^Φ in the cusps (following from proposition 4). This gives:

$$\begin{aligned}\mathbb{E}\left(e^{\lambda\,(\zeta_1 - \tau_1)}\right) &\le \left(\frac{b+\sqrt{b}}{b+1}\right)^{\delta_1 - \sqrt{\delta_1^2 - 2\lambda}} + c \int_{b+\sqrt{b}}^{\infty} \left(\frac{y}{b+1}\right)^{\delta_1 - \sqrt{\delta_1^2 - 2\lambda}} y^{-2\delta}\, dy\\ &< \infty.\end{aligned}$$

□

This lemma allows us to compute now the equivalent of the number N_t of excursions achieved at time t by the Φ-diffusion. Since we need to distinguish the different cusps, we let an exponent i denote the excursions that occur in $\mathcal{N}_i(b)$.

Lemma 14. *Denote by $[\tau_n^i, \zeta_n^i]$ the sequence of intervals of those excursions which occur in $\mathcal{N}_i(b)$, for $1 \le i \le N$. So that the previous sequence $[\tau_n, \zeta_n]$ is just the reordering of the disjoint union of the present ones. For each i, set $N_t^i := \max\{n \in \mathbb{N} \mid \zeta_n^i \le t\}$. Then for each i N_t^i/t converges almost surely as $t \to \infty$ towards some positive real ϱ^i. Moreover we have: $\varrho^i = \frac{1}{2}\, \|\Phi\|_2^{-2} \times \Big(c(\delta)\, \lambda(\mathcal{P}_i)\Big)^2 \times b^{3/2 - 2\delta} \times \Big(1 + \mathcal{O}(b^{-1/2})\Big)$.*

Proof. Observe first the obvious inequalities:

$$\int_0^t 1_{\mathcal{N}_i(b+\sqrt{b}\,)}(Z_s^\Phi)\, ds < \sum_{n=0}^{1+N_t^i} (\zeta_n^i - \tau_n^i) \quad \text{and}$$

$$\sum_{n=0}^{N_t^i} (\zeta_n^i - \tau_n^i) < \int_0^t 1_{\mathcal{N}_i(b+1)}(Z_s^\Phi)\, ds.$$

Dividing by t and applying the ergodic theorem twice gives almost surely:

$$V^{\Phi}(\mathscr{N}_i(b+\sqrt{b})) \le \liminf_{t\to\infty} \frac{N_t^i}{t} \times \mathbb{E}(\zeta_2^i - \tau_2^i)$$
$$\le \limsup_{t\to\infty} \frac{N_t^i}{t} \times \mathbb{E}(\zeta_2^i - \tau_2^i) \le V^{\Phi}(\mathscr{N}_i(b+1)).$$

On the other hand, the almost sure convergence of N_t^i/t follows from the additive functional property of $\max\{n \in \mathbb{N} \mid \max\{s < \zeta_n^i \mid Z_s^{\Phi} \in \mathscr{N}_i(b+\sqrt{b})\} \le t\}$.

Now, we get by definition of V^{Φ} and by proposition 4:

$$V^{\Phi}(\mathscr{N}_i(b)) = \|\Phi\|_2^{-2} \int_0^1 \int_b^{\infty} \Phi^2(z)\, y^{-2}\, dx\, dy$$
$$= \|\Phi\|_2^{-2} \times \Big(c(\delta)\, \lambda(\mathscr{P}_i)\Big)^2 \int_b^{\infty} (1+\mathcal{O}(1/y))\, y^{-2\delta}\, dy$$
$$= \|\Phi\|_2^{-2} \times \Big(c(\delta)\, \lambda(\mathscr{P}_i)\Big)^2 \times (2\delta-1)^{-1} \times b^{1-2\delta} \times (1+\mathcal{O}(1/b)).$$

The result is now straightforward from lemma 13 and from the above. □

The following lemma will be useful to replace N_t^i by its deterministic approximate $\varrho^i\, t$.

Lemma 15. *For each i and any $q > 1/2$, $t^{-q}\mid N_t^i - \varrho^i\, t\mid$ goes to zero in probability.*

Proof. Let us drop the index i, useless in this proof.
We essentially need to prove that the variables $\zeta_j - \zeta_{j-1}$ are square-integrable and that their correlations decay exponentially. To proceed, let us first show that the variables ζ_j have an exponential moment, by using the spectral gap of Z_t^{Φ}.

1) Consider some Borelian function f from $\mathscr{M}$ into $[0,1]$, and for any positive ε set $G_\varepsilon := \varepsilon f - \Delta^{\Phi}/2$ and denote by $\lambda(\varepsilon)$ the bottom of its spectrum. We know that $\lambda(0) = 0$ is an isolated eigenvalue, with eigenstate 1, and then the Kato-Rellich theorem ensures that for small enough ε $\lambda(\varepsilon)$ is still isolated, with eigenstate say φ_ε, and that $\varepsilon \mapsto (\lambda(\varepsilon), \varphi_\varepsilon)$ is analytical at 0. We deduce then from the variational expression for $\lambda(\varepsilon)$ that we have $\lambda(\varepsilon) = \varepsilon \int f\, dV^{\Phi} + o(\varepsilon)$.

Since $\varepsilon \mapsto \lambda(\varepsilon)$ clearly cannot decrease, we obtain that $\lambda(1) > 0$ if $\int f\, dV^{\Phi} > 0$.

Since $-G_1$ is the infinitesimal generator of the semi-group P_t^f defined by $P_t^f \varphi(z) := \mathbb{E}_z\Big(\varphi(Z_t^{\Phi}) \times \exp\Big[-\int_0^t f(Z_s^{\Phi})\, ds\Big]\Big)$, we deduce from the above that for any positive t we have

$$\mathbb{E}\left(\exp\left[-\int_0^t f(Z_s^{\Phi})\, ds\right]\right) = \langle P_t^f 1, 1\rangle_{L^2(V^{\Phi})} \le e^{-\lambda(1)\, t}.$$

Consider now some Borel subset B of $\mathscr{M}$ such that $V^{\Phi}(B) > 0$, and its hitting time by Z_t^{Φ}, say S. Applying the preceding to $f = 1_B$, we get some strictly positive λ such that for any positive t we have

$$\mathbb{P}(S > t) \le \mathbb{P}\left(\int_0^t 1_B(Z_s^{\Phi})\, ds = 0\right) \le \mathbb{E}\left(\exp\left[-\int_0^t 1_B(Z_s^{\Phi})\, ds\right]\right) \le e^{-\lambda t},$$

whence $\mathbb{E}\Big(\exp[\lambda S/2]\Big) < \infty$.

Applying this successively to $B = \mathscr{N}(b+\sqrt{b})$ and to $B = \mathscr{M} \setminus \mathscr{N}(b+1)$, we get the exponential integrability of the variables ζ_j.

2) We now establish the exponential decay of the correlations between the variables $\zeta_j - \zeta_{j-1}$ by using Dœblin's method.

Denote by $q(z,\cdot)$ the density of $Z_{\zeta_1}^{\Phi}$ starting from $z \in \partial\mathscr{N}(b+1)$, with respect to the Palm measure χ', which denotes the projection on $\partial\mathscr{N}(b+1)$ of the (normalized) Palm measure χ. See the end of section 8.4. This is then the transition density of the symmetrical stationary Markov chain $(Z_{\zeta_j}^{\Phi})$.

Now by ellipticity of Δ^{Φ} and compactness of $\partial\mathscr{N}(b+1)$, there exist $0 < c < c' < \infty$ such that $c \le q(z,z') \le c'$ for any $z, z' \in \partial\mathscr{N}(b+1)$.

Setting q_n^z for $q_n(z,\cdot)$, using the symmetry of q and that $\displaystyle\int (q_n^z - 1)^+ d\chi' = \int (q_n^z - 1)^- d\chi'$, we see:

$$\begin{aligned}(q_{n+1}^z - 1)(z') &= \int (q_n^z - 1)\, q^{z'}\, d\chi' \le \int (q_n^z - 1)^+\, q^{z'}\, d\chi' - \int (q_n^z - 1)^-\, c\, d\chi' \\ &= \int (q_n^z - 1)^+\, (q^{z'} - c)\, d\chi' \le (1-c)\, \|(q_n^z - 1)^+\|_\infty.\end{aligned}$$

Similarly, $(q_{n+1}^z - 1)(z') \ge \displaystyle\int (q_n^z - 1)^-\, (c - q^{z'})\, d\chi'$, and then
$-(q_{n+1}^z - 1)(z') \le (1-c)\, \|(q_n^z - 1)^-\|_\infty$.

Whence $\|q_{n+1}^z - 1\|_\infty \le (1-c)\, \|q_n^z - 1\|_\infty$, and $\|q_n^z - 1\|_\infty \le c'\,(1-c)^{n-1}$. As a consequence, we get for any $n \ge 2$ and $z \in \partial\mathscr{N}(b+1)$, setting $\zeta_j' := \zeta_j - \zeta_{j-1}$:

$$\Big|\mathbb{E}_z(\zeta_n') - \mathbb{E}(\zeta_n')\Big| = \left|\int (q_{n-1}^z(z') - 1)\mathbb{E}_{z'}(\zeta_1)\, d\chi'(z')\right| \le c'\,(1-c)^{n-2}\, \mathbb{E}_{\chi'}(\zeta_1).$$

Therefore for any $j, n \in \mathbb{N}^*$:

$$\begin{aligned}&\Big|\mathbb{E}\Big((\zeta_j' - \mathbb{E}(\zeta_j'))(\zeta_{j+n}' - \mathbb{E}(\zeta_{j+n}'))\Big)\Big| \\ &= \Big|\mathbb{E}\Big((\zeta_1 - \mathbb{E}(\zeta_1))\mathbb{E}_{Z_{\zeta_1}^{\Phi}}(\zeta_n' - \mathbb{E}(\zeta_n'))\Big)\Big| \\ &\le c'\, \mathbb{E}_{\chi'}(\zeta_1)\,(1-c)^{n-2} \times \mathbb{E}\Big(|\zeta_1 - \mathbb{E}(\zeta_1)|\Big) = \mathscr{O}\Big((1-c)^{n-2}\Big).\end{aligned}$$

3) The step 2) above shows that $Var(\zeta_n) = \mathscr{O}(n)$, and then that for any $q > 1/2$ $n^{-q}\Big(\zeta_n - \mathbb{E}(\zeta_n)\Big)$ goes to 0 in L^2-norm as $n \to \infty$. Thus using the stationarity of

the induced chain $Z^{\Phi}_{\zeta_j}$, we have $\zeta_n = \varrho' n + o(n^q)$ in L^2, where $\varrho' := \mathbb{E}_{\chi'}(\zeta_1) = \mathbb{E}(\zeta_2 - \zeta_1)$. Hence we have in probability: $N_{(\varrho' n + o(n^q))} = n$, and then, by the increasing property of N_t, $N_n = n/\varrho' + o(n^q)$, and $N_t = t/\varrho' + o(t^q)$.
Finally, since we know from lemma 14 that N_t/t converges almost surely to ϱ, we have $1/\varrho' = \varrho$. □

8.6. Windings of Z^{Φ}_t

We use now the preceding sections to prove that we may replace the numbers of excursions achieved at time t by their averages. Of course we set $\psi^i_n := \int_{Z^{\Phi}[\tau^i_n, \zeta^i_n]} \tilde{\omega}$.

Lemma 16. *For each* i, $t^{\frac{-1}{2\delta-1}} \left| \sum_{n=1}^{N^i_t} \psi^i_n - \sum_{n=1}^{[\varrho^i t]} \psi^i_n \right|$ *goes to zero in probability.*

Proof. Fix $\eta \in]0, 1/4[$, take $\gamma \in]0, 1[$, and forget the indices i, useless here.
1) Using lemma 15, we have, for large enough t:

$$\mathbb{P}\left(r_t \left| \sum_{n=1}^{N_t} \psi_n - \sum_{n=1}^{[\varrho t]} \psi_n \right| > \eta \right)$$
$$\leq \mathbb{P}\Big(|N_t - \varrho t| > t^{1/2+\eta} \Big) + \mathbb{P}\left(\sum_{n=[\varrho t - t^{1/2+\eta}]}^{[\varrho t + t^{1/2+\eta}]} |\psi_n| > \eta \Big/ r_t \right)$$
$$\leq \eta + \mathbb{P}\left(\sum_{n=[\varrho t - t^{1/2+\eta}]}^{[\varrho t + t^{1/2+\eta}]} |\psi_n|^\gamma > \eta^\gamma \times t^{\frac{\gamma}{2\delta-1}} \right)$$
$$\leq \eta + \eta^{-\gamma} \times t^{\frac{-\gamma}{2\delta-1}} \sum_{n=[\varrho t - t^{1/2+\eta}]}^{[\varrho t + t^{1/2+\eta}]} \mathbb{E}(|\psi_n|^\gamma).$$

2) Then, using again the modified diffusion $\tilde{Z}_t$ of section 8.1, we have:

$$\begin{aligned}
\mathbb{E}(|\psi_n|^\gamma) &= \mathbb{E}(|\psi'_n|^\gamma \times M_{\zeta_n}) \\
&= \mathbb{E}\Big(M_{\tau_n} \times H(\tilde{Z}_{\tau_n})^{-1} \times \mathbb{E}_{\tilde{Z}_{\tau_n}}(|\psi'|^\gamma \mid \tilde{Z}_{\zeta_n}) \times H(\tilde{Z}_{\zeta_n}) \Big) \\
&\leq \frac{c_2}{c_1} \times \mathbb{E}\Big(M_{\tau_n} \times \mathbb{E}_{\tilde{Z}_{\tau_n}}(|\psi'|^\gamma \mid \tilde{Z}_{\zeta_n}) \Big) \\
&= \frac{c_2}{c_1} \times \mathbb{E}\Big(M_{\tau_n} \times \mathbb{E}_{\tilde{Z}_{\tau_n}}(|\psi'|^\gamma) \Big) \\
&= \frac{c_2}{c_1} \times \mathbb{E}(M_{\tau_n}) \times \mathbb{E}(|\psi'|^\gamma) = \frac{c_2}{c_1} \times \mathbb{E}(|\psi'|^\gamma)
\end{aligned}$$

since the law of ψ' does not depend on $\tilde{Z}_\tau$. Hence we have for all $n \in \mathbb{N}^*$:

$$\mathbb{E}(|\psi_n|^\gamma) \leq c\,\mathbb{E}(|W(Y)|^\gamma) \leq 2c\,\mathbb{E}(Y^{\gamma/2}).$$

Now

$$\mathbb{E}(Y^\gamma) = \mathbb{E}(Y \times Y^{\gamma-1}) = \mathbb{E}\left(\frac{Y}{\Gamma(1-\gamma)}\int_0^\infty e^{-tY} t^{-\gamma}\, dt\right)$$
$$= \frac{1}{\Gamma(1-\gamma)}\int_0^\infty \mathbb{E}(e^{-tY} Y)\, t^{-\gamma}\, dt,$$

and by lemma 11

$$\mathbb{E}(e^{-tY} Y) = -\frac{d}{dt}\mathbb{E}(e^{-tY})$$
$$= -\frac{d}{dt}\left[\tilde{K}_\delta\Big((b+\sqrt{b})\sqrt{2t}\Big)\Big/\tilde{K}_\delta\Big((b+1)\sqrt{2t}\Big)\right],$$

which is equivalent to

$$2^{1/2-\delta}\,\Gamma(3/2-\delta)\,(\Gamma(\delta-1/2))^{-1}$$
$$\times\Big((b+\sqrt{b})^{2\delta-1} - (b+1)^{2\delta-1}\Big) \times t^{\delta-3/2} \quad \text{near } 0,$$

and to $\dfrac{(\sqrt{b}-1)\sqrt{b+1}}{\sqrt{2t(b+\sqrt{b})}} \times \exp\Big((1-\sqrt{b})\sqrt{2t}\Big)$ near $+\infty$.

Hence $\mathbb{E}(Y^{\gamma/2})$ is finite for $\delta - 3/2 - \gamma/2 > -1$, that is to say for $\gamma < 2\delta - 1$.

3) By 1) and 2) above, we have for $1/2 + 2\eta \le \dfrac{\gamma}{2\delta-1} < 1$ and large enough t:

$$\mathbb{P}\left(r_t\left|\sum_{n=1}^{N_t}\psi_n - \sum_{n=1}^{[\varrho t]}\psi_n\right| > \eta\right) \le \eta + \eta^{-\gamma} \times t^{\frac{-\gamma}{2\delta-1}} \times 2t^{1/2+\eta} \times c'$$
$$< \eta + \mathcal{O}\Big(t^{(1/2+\eta-\frac{\gamma}{2\delta-1})}\Big) < 2\eta. \qquad \square$$

We verify in the following lemma that only the excursions contribute asymptotically to the windings. N_t is here the number of achieved excursions at time t:

$$N_t := \max\{N_t^i \mid 1 \le i \le N\} = \max\{n \in \mathbb{N} \mid \zeta_n \le t\}. \tag{8.8}$$

Lemma 17. $\limsup_{t\to\infty}\left\| t^{\frac{-1}{2\delta-1}}\left(\int_{Z^\Phi[0,t]}\tilde{\omega} - \sum_{n=1}^{N_t}\psi_n\right) 1_{\{Z_t^\Phi \notin \mathcal{N}(b+1)\}}\right\|_2 = 0.$

Proof. By proposition 2, the drift term of Z_t^Φ is $\nabla \log \Phi$, and thus by using proposition 4 we have on each $[\tau_n, \zeta_n]$ in adapted coordinates: $Z_t^\Phi = (x_t^\Phi, y_t^\Phi)$, with $(y_t^\Phi)^{-1}\, dx_t^\Phi = dw_t + 1_{\{\delta<1\}}\,\mathcal{O}(1/b)\, dt$.
On the other hand, $r_t \le 1/t$ and $r_t = o\,(1/t)$ if $\delta < 1$, and thus using (5.5) we get:

$$\limsup_{t\to\infty}\left\| r_t\left(\int_{Z^\Phi[0,t]}\tilde{\omega} - \sum_{n=1}^{N_t}\psi_n\right) 1_{\{Z_t^\Phi \notin \mathcal{N}(b+1)\}}\right\|_2^2$$

$$\leq c \limsup_{t\to\infty} \left(t^{-2} \left\| \int_0^t 1_{\{b_o \leq y_s^\phi \leq b+\sqrt{b}\}} (y_s^\phi)^2 \, ds \right\|_1 + \left\| o\,(t^{-1}) \int_0^t 1_{\{b_o \leq y_s^\phi \leq b+\sqrt{b}\}} \, ds \right\|_2^2 \right) = 0. \qquad \square$$

We are now able to establish the asymptotic result on the windings of the Φ-diffusion Z_t^Φ. Recall that $r(\mathscr{P}, \omega)$ denotes the residue of ω (and of $\tilde{\omega}$) near the cusp $\mathscr{P}$. Recall also that $\mathscr{P}_1, \ldots, \mathscr{P}_N$ are the cusps of $\mathscr{M}$, and that $\lambda(\mathscr{P}_i)$ was introduced in definition 2.

Theorem 2. *The law of* $t^{\frac{-1}{2\delta-1}} \int_{Z^\Phi[0,t]} \tilde{\omega}$ *converges towards the symmetric stable law with exponent* $(2\delta - 1)$ *and rate* $C(\Gamma, \omega) := \|\Phi\|_2^{-2} \times c(\delta)^2 \times \frac{\Gamma(3/2-\delta)}{\Gamma(\delta-1/2)} \times \sum_{i=1}^N |r(\mathscr{P}_i, \omega)/2|^{(2\delta-1)} \lambda(\mathscr{P}_i)^2$. *This means that*

$$\lim_{t\to\infty} \mathbb{E}\left(\exp\left[\sqrt{-1}\,\alpha\, t^{\frac{-1}{2\delta-1}} \int_{Z^\Phi[0,t]} \tilde{\omega} \right] \right) = \exp\left(-|\alpha|^{2\delta-1} C(\Gamma, \omega) \right), \quad \textit{for any real } \alpha.$$

Proof. Fix some positive ε, and take b large enough so that

$$\mathbb{P}(Z_t^\Phi \in \mathscr{N}(b)) = V^\Phi(\mathscr{N}(b)) < \varepsilon, \quad \text{and} \quad \left| \int a\, d\chi^i \right| < \varepsilon \text{ for } 1 \leq i \leq N.$$

Note that this is possible by lemma 12. Now this implies, with lemmas 16 and 17:

$$\limsup_{t\to\infty} \left| \mathbb{E}\left(\exp\left[\sqrt{-1}\,\alpha\, t^{\frac{-1}{2\delta-1}} \int_{Z^\Phi[0,t]} \tilde{\omega} \right] \right) - \exp\left(-|\alpha|^{2\delta-1} C(\Gamma, \omega) \right) \right|$$

$$\leq \varepsilon + \limsup_{t\to\infty} \left| \mathbb{E}\left(\exp\left[\sqrt{-1}\,\alpha\, t^{\frac{-1}{2\delta-1}} \sum_{n=1}^{N_t} \psi_n \right] \right) - \exp\left(-|\alpha|^{2\delta-1} C(\Gamma, \omega) \right) \right|$$

$$\leq \varepsilon + \limsup_{t\to\infty} \left| \mathbb{E}\left(\exp\left[\sqrt{-1}\,\alpha\, t^{\frac{-1}{2\delta-1}} \sum_{i=1}^{N} \sum_{n=1}^{[\varrho^i t]} \psi_n^i \right] \right) - \exp\left(-|\alpha|^{2\delta-1} C(\Gamma, \omega) \right) \right|$$

$$= \varepsilon + \left| \prod_{i=1}^N \exp\left[-\gamma\left(\alpha\, r(\mathscr{P}_i, \omega) \right) \varrho^i \times \left(1 + \int a\, d\chi^i \right) \right] \right.$$

$$- \exp\left(-|\alpha|^{2\delta-1}\, C(\Gamma,\omega)\right)\Bigg|$$

by section 8.4, and now by definition of the constants $\gamma(\alpha)$ and $C(\Gamma,\omega)$, and by lemma 14, this equals (for $b > \varepsilon^{-2} > 1$):

$$\varepsilon + \Bigg|\exp\Big(-|\alpha|^{2\delta-1}C(\Gamma,\omega)(1+\mathcal{O}(b^{-1/2}))(1+\mathcal{O}(\varepsilon))\Big) - \exp\left(-|\alpha|^{2\delta-1}C(\Gamma,\omega)\right)\Bigg| = \mathcal{O}(\varepsilon). \qquad \square$$

Remark 7. 1) A proof similar to the proof of lemma 4 shows that theorem 2 is valid as well with the form ω instead of the form $\tilde{\omega}$, at least if $|\omega'|$ and $div\,\omega'$ are bounded in $\mathscr{M}\setminus\mathscr{N}(b_o)$.

2) It is possible to show that this theorem is still true with any initial law for the Φ-diffusion, but we do not need this slightly stronger result to deduce below our result on the geodesic flow.

9. Geodesic windings

We gather here the results of the preceding sections: link between geodesic windings and those of $\xi(Z_t^\delta)$ (corollary 4), equality of the diffusions Z_t^Φ and $\xi(Z_t^\delta) = \pi(\xi_t^\delta)$ (Proposition 2), and windings of Z_t^Φ (theorem 2), to deduce below the asymptotic behavior of $J_t(\omega)$ (see (5.1)), thereby completing the proof or our main theorem (stated in section 2), which describes the asymptotic joint law under the Patterson-Sullivan measure m of the geodesic windings.

Before concluding, we need to get rid of the random time change τ_t, inherited from corollary 4. (Whereas τ_n, with an integer subscript, still denotes the n-th start of excursion). This we do in the following lemma.

Lemma 18. $\left| t^{\frac{-1}{2\delta-1}} \int_{\pi\circ\xi^\delta[0,\tau_t]} \tilde{\omega} - t^{\frac{-1}{2\delta-1}} \int_{Z^\Phi[0,t/(\delta-1/2)]} \tilde{\omega} \right|$ *goes to zero in probability as* $t \to \infty$.

Proof. 1) Using lemma $(2, iii)$, we can adapt the proof of lemma 17 as follows:

$$\limsup_{t\to\infty} \left\| r_t \left(\int_{Z^\Phi[0,\tau_t\wedge\tau_{(1+N_{\tau_t})}]} \tilde{\omega} - \sum_{n=1}^{N_{\tau_t}} \psi_n \right) \right\|_2^2$$
$$\leq c \limsup_{t\to\infty} \left(t^{-2} \left\| \int_0^{\tau_t} 1_{\{b_o \leq y_s^\phi \leq b+\sqrt{b}\}} (y_s^\phi)^2\, ds \right\|_1 + \left\| o(t^{-1}) \int_0^{\tau_t} 1_{\{b_o\leq y_s^\phi \leq b+\sqrt{b}\}}\, ds \right\|_2^2 \right)$$
$$\leq c \limsup_{t\to\infty} (b+\sqrt{b})^2\, t^{-2}\, \mathbb{E}(\tau_t) + c \limsup_{t\to\infty} o(t^{-2})\, \mathbb{E}(\tau_t^2) = 0.$$

2) We can extend the validity of lemma 15 to the random time τ_t by using lemma(2, iv) and the increasing property of N_t^i. Indeed, we have with probability converging to 1:

$$(\delta-1/2)^{-1}\varrho^i t - o(t^q) < N^i_{(\delta-1/2)^{-1}t-o(t^q)} \le N_{\tau_t} \le N^i_{(\delta-1/2)^{-1}t+o(t^q)}$$
$$< (\delta-1/2)^{-1}\varrho^i t + o(t^q),$$

with $1/2 < q < 3/4$. Then the proof of lemma 16 remains valid, to prove that $t^{\frac{-1}{2\delta-1}} \left| \sum_{n=1}^{N^i_{\tau_t}} \psi_n^i - \sum_{n=1}^{[(\delta-1/2)^{-1}\varrho^i\, t]} \psi_n^i \right|$ goes to zero in probability.

3) We must now get rid of the possibly incomplete excursion alive at random time τ_t.

Set for each $n \in \mathbb{N}^*$: $\psi_n^* := \sup_{\tau_n<t<\zeta_n} \left| \int_{Z^\Phi[\tau_n,t]} \tilde{\omega} \right|$, and $\psi'^* := \sup_{0<t<\zeta} \left| \int_{\tilde{Z}[0,t]} \tilde{\omega} \right|$.

Fix $\varepsilon > 0$, and $1/2 < q < 3/4$, and consider, using lemma (2, iv), t large enough so that $\mathbb{P}\Big(|\tau_t - (\delta-1/2)^{-1}t| > o(t^q)\Big) < \varepsilon$, and, using lemma 15, so that $\mathbb{P}\Big(|N_{((\delta-1/2)^{-1}t\,\pm o(t^q))} - \varrho\,((\delta-1/2)^{-1}t\ \pm o(t^q))| > o(t^q)\Big) < \varepsilon$. Then we have for such t:

$$\mathbb{P}\Big(r_t \times \psi^*_{(1+N_{\tau_t})} > \varepsilon\Big) - 2\varepsilon \le \mathbb{P}\Big(r_t \times \max\{\psi_n^* \,|\, \varrho\,(\delta-1/2)^{-1}t - o(t^q) < \tau_n$$
$$< \varrho\,(\delta-1/2)^{-1}t + o(t^q)\} > \varepsilon\Big)$$
$$\le \mathbb{P}\Big(r_t \times \max\{\psi_n^* \,|\, N_{(\varrho\,(\delta-1/2)^{-1}t-o(t^q))} \le n$$
$$\le N_{(\varrho\,(\delta-1/2)^{-1}t+o(t^q))}\} > \varepsilon\Big)$$
$$\le 2\varepsilon + \mathbb{P}\Big(r_t \times \max\{\psi_n^* \,|\, \varrho^2\,(\delta-1/2)^{-1}t - o(t^q)$$
$$\le n \le \varrho^2\,(\delta-1/2)^{-1}t + o(t^q)\} > \varepsilon\Big)$$
$$\le 2\varepsilon + \sum_{n=ct-o(t^q)}^{n=ct+o(t^q)} \mathbb{P}\Big(\psi_n^* > \varepsilon/r_t\Big)$$
$$\le 2\varepsilon + \sum_{n=ct-o(t^q)}^{n=ct+o(t^q)} (\varepsilon/r_t)^{-\gamma} \times \mathbb{E}\Big((\psi_n^*)^\gamma\Big)$$
$$\le 2\varepsilon + (c_2/c_1) \times o(t^q) \times (\varepsilon/r_t)^{-\gamma} \times \mathbb{E}\Big((\psi'^*)^\gamma\Big)$$
$$\le 2\varepsilon + t^{q-(\gamma/(2\delta-1))} \times \mathbb{E}\Big((\psi'^*)^\gamma\Big)$$

as in the proof of lemma 16.

Finally we just have to verify that $(\psi'^*)^\gamma$ is integrable for $\gamma < 2\delta - 1$. Now

$$\psi'^* = r \sup_{0<t<\zeta} \left| \int_0^t \tilde{y}_s\, dw_s \right| \overset{(law)}{=} r \sup_{0<t<1} |W_t| \times \left(\int_0^\zeta (\tilde{y}_s)^2\, ds \right)^{1/2}$$

$$= r \sup_{0<t<1} |W_t| \times \sqrt{Y},$$

where w and W are two standard real Brownian motions, independent of Y. Then

$$\mathbb{E}\Big((\psi'^*)^\gamma\Big) = r^\gamma \times \mathbb{E}\left(\sup_{0<t<1} |W_t|^\gamma\right) \times \mathbb{E}\Big(Y^{\gamma/2}\Big) < \infty$$

for $\gamma < 2\delta - 1$, as in the proof of lemma 16.

4) So far, thanks to proposition 2, we just proved that in probability $\limsup_{t\to\infty} t^{\frac{-1}{2\delta-1}} \left| \int_{\pi\circ\xi^\delta[0,\tau_t]} \tilde{\omega} - \sum_{i=1}^{N} \sum_{n=1}^{[(\delta-1/2)^{-1}\varrho^i t]} \psi_n^i \right| = 0$. But since we can deal with the incomplete excursion possibly alive at time t as in the proof of theorem 2, or as in 3) above (in an easier way however), lemmas 16 and 17 allow us to conclude, observing that the expression in the statement does not depend on b. □

Finally we deduce our main theorem (stated in section 2) from (5.1), (5.3), corollary 4, lemma 18 and theorem 2, with merely $C'(\Gamma, \omega) = C(\Gamma, \omega)/(\delta - 1/2)$.

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