The central limit theorem for the geodesics of a hyperbolic surface

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Abstract  In 1960, Sinai [9] stressed the asymptotically chaotic behaviour of geodesics of a cocompact hyperbolic manifold, by proving that they satisfy a central limit theorem. The use of hyperbolic Brownian motion allowed Le Jan [7] to extend this result in 1994, to cofinite hyperbolic manifolds. The same method allowed then to deal with cases of infinite volume too, see [5], and also to handle singular limit theorems (about windings in cusps), see [4].

This short course is devoted to the central limit theorem in the finite volume setting, and restricted to the $d = 2$-dimensional case, which was the case originally addressed by Sinai. It is based on [5], [6] and [7]. The reference book [6] deals with the more involved general case, and contains much more material, in several directions. The easier two-dimensional case allows a lot of simplifications. Among others, a reason is that the structure of $d$-dimensional hyperbolic manifolds is more complicated when $d > 2$. Moreover, the choice made here, of getting rapidly through many details, is intended to allow a faster access into the heart of the purpose.

The hyperbolic Brownian motion is the intrinsic diffusion process associated with the Laplacian on the hyperbolic plane $H^2$. It can be defined by projecting (on $H^2$) some left Brownian motion $(Z_s)$ of the modular group $\text{PSL}(2)$, obtained by solving a linear stochastic differential equation in some affine subgroup of $\text{PSL}(2)$.

A hyperbolic surface is the left quotient $\Gamma \backslash H^2$ of the hyperbolic plane $H^2$ by a Fuchsian group $\Gamma$, and inherits its volume form, Laplace-Beltrami operator and Brownian motion from $H^2$. Its frame bundle can be identified with the left quotient $\mathcal{M} \equiv \Gamma \backslash \text{PSL}(2)$, and its normalized Liouville measure $\mu$ is deduced from the Haar measure of $\text{PSL}(2)$.

The geodesic and the horocyclic flows on $\mathcal{M}$ are merely given by the right action of two one-parameter groups of matrices $(\theta_s)$ and $(\theta^+_s)$ in $\text{PSL}(2)$. The probability measure $\mu$ is invariant for both $(\theta_s)$, $(\theta^+_s)$ and $(Z_s)$, which act on each stable leaf of $\mathcal{M}$. Under $\mu$, $(\theta_s)$ and $(Z_s)$ have the same harmonic measures.

A crucial fact is the ergodicity of the geodesic flow $(\theta_s)$, with respect to $\mu$.

Another crucial fact is that a Poincaré inequality holds for $\mathcal{M}$, which yields a spectral gap for $L^2$ functions on $\mathcal{M}$, and also an exponential decay under the semi-group $(P_t)$ associated to $(Z_s)$, for rotationally Hölderian bounded functions on $\mathcal{M}$. The latter not obviously follows from both the Poincaré inequality and a commutation...
relation in \( \text{PSL}(2) \). The rotational regularity is necessary because \((Z_t)\) evolves only along the stable foliation.

Using the invariance of \( \mu \), the ergodicity of the geodesic flow, time reversal and a change of contour, it can be shown that the central limit theorem for \((\theta_t)\) (under \( \mu \)) reduces to that of the stationary foliated diffusion \((Z_t)\).

Moreover the exponential decay under \((P_t)\) ensures the existence of the potential kernel \( V := \int_0^\infty P_t ds \), roughly yielding an inverse for the foliated Laplacian \( \mathcal{D} \), generator of \((Z_t)\).

Using \( V \) finally allows to reduce the study of the asymptotic law of \( t^{-1/2} \int_0^t f(Z_s) ds \), to that of \( M^f_t / \sqrt{t} \), for some classically handled \( L^2 \) continuous martingale \((M^f_t)\).

Some details and most proofs are not given in this course, and can be found in the reference book [6].

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1 Brownian motion and Itô calculus

This section is a synthetic presentation of the basic stochastic calculus, which is needed then to implement the strategy of comparing geodesics with random paths.

1.1 Brownian Motion

Definition 1.1.1 A standard real Brownian motion is a continuous real process \((B_t \mid t \in \mathbb{R}^+)\) which has:
- independent and stationary increments: for any \(0 = t_0 < t_1 < \ldots < t_N < \infty\), the \(N\) random variables \((B_{t_j} - B_{t_{j-1}})\) are independent, and \((B_{t_j} - B_{t_{j-1}})\) has the same law as \(B_{t_j - t_{j-1}}\);
- Gaussian marginals: for any \(t > 0\), the law of \(B_t\) is centred Gaussian with variance \(t\):
  \[
  \mathbb{E}[f(B_t)] = \int_{-\infty}^{\infty} f(x) e^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}}.
  \]
  Its law is the standard Wiener measure, on the Wiener space \(\mathcal{C}_0([0,1], \mathbb{R})\) of continuous functions from \([0,1]\) into \(\mathbb{R}\) which vanish at 0.

The term standard underlines that \((B_t)\) starts from 0 and has unit variance at time 1. The generic real Brownian motion has the law of \((c_0 + cB_t)\), for constant \(c_0, c\).

Brownian motion \((B_t)\) can be constructed either as the limit of a rescaled random walk \(\varepsilon (X_1 + \cdots + X_{t/\varepsilon^2})\) as \(\varepsilon \downarrow 0\), or by means of a multi-scale series as follows.

Theorem 1.1.2 Let \(\{\phi_k \mid k \in \mathbb{N}^*\}\) be any orthonormal basis of \(L_0^2([0,1], \mathbb{R})\) such that the series \(\sum_{k=1}^{\infty} \phi_k^2\) converges uniformly on \([0,1]\), where \(\phi_k\) is the primitive of \(\phi_k\) which vanishes at 0 and 1. Let \(\{\xi_k \mid k \in \mathbb{N}\}\) be a sequence of independent centred Gaussian random variables having variance 1. Then
\[
B_t := \xi_0 t + \sum_{k \in \mathbb{N}^*} \xi_k \phi_k(t), \quad \text{for all } t \in [0,1],
\]
is a standard real Brownian motion on \([0,1]\).

The following property is straightforward from the definition, since the law of a Gaussian process is prescribed by its mean and its covariance.

Proposition 1.1.3 The standard real Brownian motion \((B_t)\) is the unique real process which is Gaussian centred with covariance function \(\mathbb{R}_+ \ni (s,t) \mapsto \mathbb{E}(B_sB_t) = \min\{s,t\}\).

The processes \(t \mapsto B_{u+t} - B_u\), \(t \mapsto e^{-t}B_{\frac{t}{2}}\), \(t \mapsto tB_1\), and \(t \mapsto (B_T - B_{T-t})\) (for \(0 \leq t \leq T\)) satisfy the same conditions. We therefore deduce the following fundamental properties:
Corollary 1.1.4 The standard real Brownian motion \( (B_t) \) satisfies
- the Markov property: for all \( a \in \mathbb{R}_+ \), \( (B_{a+t} - B_a) \) is also a standard Brownian motion, and is independent from the \( \sigma \)-field \( \mathcal{F}_a \) generated by \( \{B_s | 0 \leq s \leq a\} \);
- the self-similarity: for any \( c > 0 \), \((c^{-1}B_{ct})\) is also a standard real Brownian motion.

An \( \mathbb{R}^d \)-valued process \((B^1_t, \ldots, B^d_t)\) made of \( d \) independent standard Brownian motions \((B^j_t)\) is called a \( d \)-dimensional Brownian motion. Its law, the so-called Wiener measure, is preserved by Euclidean rotations of the vector space \( \mathbb{R}^d \). For any \( v \in \mathbb{R}^d \), \( v + (B^1_t, \ldots, B^d_t) \) is also called a \( d \)-dimensional Brownian motion, starting at \( v \).

1.2 The Itô integral

The Itô integral is the basic continuous time stochastic integral of type
\[
\int_0^t H_s \, dB_s,
\]
where \((B_t)\) is a Brownian motion. As \((B_t)\) is not differentiable and has infinite variation on any interval \([0,t]\), well defining such expression leads to suitably restrict the class \( \Lambda^\infty \) of processes \((H_s)\) to be integrated. A convenient class \( \Lambda^\infty \) is that of right continuous and left limited ("c\`adl\`ag") adapted (i.e., \( H_s \in \mathcal{F}_s \) for all \( s \)) processes such that \( \mathbb{E} \left[ \int_0^\infty H_t^2 \, ds \right] < \infty \).

Adapted step processes \( H^0_t := \sum_{j=0}^{n-1} U_j \mathbf{1}_{[T_j, T_{j+1})}(s) \), with \( T_0 < T_1 < \ldots < T_n \) and bounded \( \mathcal{F}_{T_j} \)-measurable \( U_j \), belong to \( \Lambda^\infty \).

The stochastic (Itô) integral of \((H^0_t)\) is defined as \( \sum_{j=0}^{n-1} U_j (B_{s\wedge T_{j+1}} - B_{s\wedge T_j}) \), and happens to be a continuous martingale. As adapted step processes are dense in \( \Lambda^\infty \) for the \( L^2 \) norm, the Itô integral extends by density to a linear map \( H \mapsto \int_0^t H \, dB \) from \( \Lambda^\infty \) into the space \( \mathcal{M}_c^\infty \) of square integrable continuous martingales bounded in \( L^2 \), and as well from the space \( \Lambda \), of all c\`adl\`ag adapted processes such that \( \mathbb{E} \left[ \int_0^t H_t^2 \, ds \right] < \infty \) for any \( t \geq 0 \), into the space \( \mathcal{M}_c \) of square integrable continuous martingales. The following Itô isometric identity holds.
\[
\mathbb{E} \left[ \left( \int_0^t H dB \right) \times \left( \int_0^t K dB \right) \right] = \mathbb{E} \left[ \int_0^t H_s K_s \, ds \right], \quad \text{for all } H, K \in \Lambda, \ t \in \mathbb{R}_+. \tag{1}
\]

1.3 Itô’s Formula

Itô’s formula is the fundamental result of stochastic calculus.

Define the space \( \mathcal{S}_b \) (respectively \( \mathcal{S} \)) of semimartingales of bounded (respectively \( L^2 \)) type, as the space of square integrable continuous processes which can be
written as a sum
\[ x_0 + \int_0^l H_s \, dB_s + \int_0^l K_s \, ds \]
with \( x_0 \in \mathbb{R} \), and \( H, K \) bounded càdlàg adapted processes (respectively processes in \( \Lambda \)). Note that \( \mathcal{S} \) is made of continuous semimartingales and is included in \( \Lambda \).

**Theorem 1.3.1 (Itô’s Formula)** Consider a semimartingale in \( \mathcal{S}_b \):

\[ X_t := x_0 + \int_0^t H_s \, dB_s + \int_0^t K_s \, ds \]

and a \( C^2 \) function \( \Phi \) having bounded derivatives. Then \( \Phi(X_t) \) is in \( \mathcal{S}_b \), and

\[ \Phi(X_t) - \Phi(x_0) = \int_0^t \Phi'(X_s) H_s \, dB_s + \int_0^t \Phi'(X_s) K_s \, ds + \frac{1}{2} \int_0^t \Phi''(X_s) H_s^2 \, ds. \]  

(2)

The last term is known as the Itô correction, by comparison with the usual calculus (chain rule) formula, which is recovered by taking \( H = 0 \). The formula looks like a second order Taylor formula with \((dB_t)^2 = dt\).

The Itô formula can easily be generalized to functions of \( d \) variables and to the case where we consider \( r \) independent Brownian motions \( B^p \). The space \( \mathcal{S}_b \) (respectively \( \mathcal{S} \)) of semimartingales of bounded (respectively \( L^2 \)) type becomes the space of square integrable continuous processes which can be written as a sum

\[ x_0 + \sum_{p=1}^r \int_0^l H^p_t \, dB^p_t + \int_0^l K_t \, ds, \]

with bounded càdlàg adapted processes (respectively processes in \( \Lambda \)) \( H^p \) and \( K \). Thus we have the following.

**Theorem 1.3.2 (Itô’s Formula)** Consider \( d \) semimartingales in \( \mathcal{S}_b \):

\[ X_t^j := x_0^j + \sum_{p=1}^r \int_0^l H^j_p \, dB^p_t + \int_0^l K^j_t \, ds, \quad 1 \leq j \leq d, \]

and a \( C^2 \) function \( \Phi \) with bounded derivatives on \( \mathbb{R}^d \). Then \( \Phi(X_t) \equiv \Phi(X^1_t, \ldots, X^d_t) \) belongs to \( \mathcal{S}_b \), and we have:

\[ \Phi(X_t) - \Phi(x_0) = \sum_{j=1}^d \sum_{p=1}^r \int_0^l \Phi'_{j,p}(X_s) H^j_s \, dB^p_s + \sum_{j=1}^d \int_0^l \Phi'_{j}(X_s) K^j_s \, ds + \frac{1}{2} \sum_{1 \leq i, j \leq d, p=1} \int_0^l \Phi''_{i,j,p}(X_s) H^i_s H^j_s H^p_s \, ds. \]  

(3)

**Notation** If \( X_t := x_0 + \sum_{p=1}^r \int_0^l H^p_t \, dB^p_t + \int_0^l K_t \, ds \) is in \( \mathcal{S} \), one uses the notation

\[ \langle X, X \rangle_t := \sum_{p=1}^r \int_0^l |H^p_t|^2 \, ds, \]

to denote its quadratic variation. The quadratic covariance \( \langle X, X \rangle_t := \sum_{p=1}^r \int_0^l H^p_t H^p_s \, ds \) is derived by usual polarisation, for any other
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element of $\mathcal{S}$. In this way, the multidimensional Itô formula in $\mathcal{S}_b$ (or in $\mathcal{S}$, when it holds) can be written as:

$$
\Phi(X_t) - \Phi(x_0) = \sum_i \int_0^t \Phi'_i(X_s) dX^i_s + \frac{1}{2} \sum_{i,j} \int_0^t \Phi''_{i,j}(X_s) d\langle X^i, X^j \rangle_s .
$$

(4)

1.4 Linear Stochastic Differential Equations

Solving linear differential equations with respect to Brownian motion happens to be a powerful way to define matrix-valued diffusion processes.

**Theorem 1.4.1** Consider $A_0, A_1, \ldots, A_k \in \mathcal{M}(d)$ (the set of $d \times d$ real matrices), and an $\mathbb{R}^k$-valued standard Brownian motion $W_t = (W^1_t, \ldots, W^k_t)$.

Then there exists a unique continuous $\mathcal{M}(d)$-valued $(\mathcal{F}_t)$-adapted process $(X_t)$, solution of:

$$
X_s = 1 + \sum_{j=1}^k \int_0^s X_t A_j dW^j_t + \int_0^s X_t \left( \frac{1}{2} \sum_{j=1}^k A^2_j + A_0 \right) dt .
$$

(5)

Moreover the right increments of $(X_t)$ are independent and homogeneous: for any $t \geq 0$, the process $s \mapsto X_{t+s}^{-1}X_s$ has the same law as the process $s \mapsto X_s$, and is independent of the $\mathcal{F}_s$ generated by the Brownian motion $(W_t)$ up to time $t$.

The matrix $A_0$ of Eqn (5) is known as the drift component of the process $(X_t)$.

Applying Itô's Formula, we get the following (in which the right Lie derivatives $L^i_A$ are defined on $\mathcal{M}(d)$, as by Eqn (8) below).

**Theorem 1.4.2** Consider the solution $(X_t)$ to the linear S.D.E. (5), and a function $\phi$ of class $C^2$ on $\mathcal{M}(d)$. Then we have

$$
\phi(X_s) = \phi(1) + \sum_{j=1}^k \int_0^s L^i_A \phi(X_t) dW^j_t + \int_0^s \left[ \frac{1}{2} \sum_{j=1}^k (L^i_A)^2 + L^A_0 \right] \phi(X_t) dt .
$$

(6)

The second-order operator $\mathcal{A} := \frac{1}{2} \sum_{j=1}^k (L^i_A)^2 + L^A_0$ is the so-called (infinitesimal) generator of the process $(X_t)$.

**Theorem 1.4.3** Suppose that the coefficients $A_0, A_1, \ldots, A_k$ of the linear S.D.E. (5) belong to some Lie subalgebra $\mathcal{G}$ of $\mathcal{M}(d)$. Then almost surely the solution $(X_t)$ takes its values in the subgroup $G$ of $\text{GL}(d)$ generated by $\exp(\mathcal{G})$.

It is called a left Brownian motion on $G$ (with drift $A_0$).

**Definition 1.4.4** Consider the left Brownian motion $(X_t)$ solving the linear S.D.E. (5), and its generator $\mathcal{A}$ (as in Theorem 1.4.2). The associated semi-group $(P_t)$ is defined by $P_t f(g) = \mathbb{E}[f(gX_t)]$, for any $f \in C_b(G), t \geq 0$ and $g \in G$. 
Proposition 1.4.5 The semi-group \((P_t)\) has the following properties.

(i) \((P_t)\) is a family of non-negative endomorphisms on \(C_b(G)\), such that \(P_0\) is the identity and \(P_1 = 1\) for any \(t \geq 0\).

(ii) \((P_t)\) satisfies the so-called semi-group property: \(P_sP_t = P_{s+t}\) for any \(s, t \geq 0\).

(iii) The semi-group \((P_t)\) is strongly continuous, and Fellerian on \(G\), which means that it maps \(C^b(G)\) into \(C^b(G)\).

(iv) For any \(n \in \mathbb{N}^\ast\), \(g \in G\), \(f_0, \ldots, f_n \in C_b(G)\) and \(0 \leq t_1 \leq \ldots \leq t_n\), we have:

\[ P_t \left[ f_t(gX_{t_1}) \times f_2(gX_{t_2}) \times \cdots \times f_n(gX_{t_n}) \right] = P_t \left[ f_1 P_{t_1} [f_2 P_{t_2} \cdots P_{t_n} f_n] \right](g). \]

(v) \((P_t)\) satisfies the so-called Fokker-Planck (or heat) equation:

\[ \frac{d}{ds} P_s \phi = P_s \mathcal{A} \phi = \mathcal{A} P_s \phi \quad \text{for any } \phi \in C^2_b(G), s \geq 0. \]

2 The modular group \(\text{PSL}(2)\) as a frame bundle

2.1 The modular group \(\text{PSL}(2)\) and its Lie algebra \(\mathfrak{sl}(2)\)

We shall identify the hyperbolic plane \(\mathbb{H}^2\) with the Poincaré upper complex half-plane: \(\mathbb{H}^2 \equiv \{z = x + \sqrt{-1}y \mid x, y \in \mathbb{R}, y > 0\}\). The modular group \(\text{PSL}_2(\mathbb{R}) \equiv \text{PSL}(2)\) is classically parametrized by the Iwasawa coordinates \((z = x + \sqrt{-1}y, \alpha) \in \mathbb{H}^2 \times (\mathbb{R}/2\pi\mathbb{Z})\), in the following way: each \(g \in \text{PSL}(2)\) can be uniquely written \(g = g(z, \alpha) := \theta^+ \alpha(z) \kappa(\alpha)\) (Iwasawa decomposition), where \(n(x), a(y), k(\alpha)\) are the one-parameter subgroups defined by:

\[ \theta^+_x := \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) := \pm \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad k(\alpha) := \pm \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \]

and generated respectively by the following basis elements of the Lie algebra \(\mathfrak{sl}(2)\):

\[ \nu := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \alpha := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \kappa := \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}. \]

Any element \(g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2)\) is usually identified with the homography mapping \(\mathbb{H}^2\) to \(\mathbb{H}^2\), defined by \(gz := \frac{az + b}{cz + d}\).

Note that \(g = g(z, \alpha) \iff [g \sqrt{-1} = z \quad \text{and} \quad g'(\sqrt{-1}) = ye^{\sqrt{-1}\alpha}]\), and that

\[ [\alpha, \nu] = \nu, \quad [\kappa, \nu] = \alpha, \quad [\kappa, \alpha] = \kappa - \nu. \quad (7) \]

The right Lie derivative \(\mathcal{L}_\alpha f\) is defined on differentiable functions \(f\) on \(\text{PSL}(2)\) by:
\[ L_A f(g) := \frac{d}{d\varepsilon} f\left[ g \exp(\varepsilon A) \right], \quad \text{for any } g \in \text{PSL}(2), A \in \mathfrak{sl}(2). \] (8)

This defines a left-invariant vector field \( L_A \) on \( \text{PSL}(2) \), which means that it commutes with left translations on \( \text{PSL}(2) \). The map \( A \mapsto L_A \) from a Lie subalgebra \( \mathcal{G} \) onto the Lie algebra of left-invariant vector fields is an isomorphism of Lie algebras: \([L_A, L_B] = L_{[A,B]} \) for any \( A, B \in \mathcal{G} \). A standard computation shows that

\[ L_\sigma = y \sin \phi \frac{\partial}{\partial y} + y \cos \phi \frac{\partial}{\partial x} - \cos \phi \frac{\partial}{\partial \phi}, \quad L_\alpha = y \cos \phi \frac{\partial}{\partial y} - y \sin \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial \phi} \]

and \( L_\kappa = \frac{\partial}{\partial \phi} \), where \( \sigma := \nu - \kappa = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \).

The measure \( \lambda(dg) := \frac{dx dy d\phi}{2\pi y^2} \) happens to be bi-invariant, under left and right translations on \( \text{PSL}(2) \), hence is the Haar measure of \( \text{PSL}(2) \).

Denote by \( \mathbb{A} \) the affine subgroup of \( \text{PSL}(2) \) generated by all matrices \( \theta^+_x \) and \( a(y) \), which is also the Lie subgroup of \( \text{PSL}(2) \) associated with the Lie subalgebra \( \mathbb{R} \nu + \mathbb{R} \alpha \) of \( \mathfrak{sl}(2) \). Setting \( T_\varepsilon \equiv T_{x,y} := \theta^+_x a(y) = \pm y^{-1/2} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \) for any \( z = x + \sqrt{-1} y \in \mathbb{H}^2 \), we have

\[ \mathbb{A} = \{ T_\varepsilon \equiv T_{x,y} \mid z = x + \sqrt{-1} y \in \mathbb{H}^2 \} \quad \text{and} \quad T_{x,y} T_{x',y'} = T_{x+xy',y'}, \] (9)

which shows that \( \mathbb{A} \) is indeed isomorphic to the usual (direct) affine group of \( \mathbb{R} \). Note that \( \mathbb{A} \) is also the subgroup of the elements of \( \text{PSL}(2) \) which fix \( \infty \in \partial \mathbb{H}^2 \). A particular case of (9) is

\[ a(y) \theta^+_x = \theta^+_x a(y). \] (10)

\( \text{PSL}(2) \) is a model for the unit tangent bundle \( T^1 \mathbb{H}^2 \) to the hyperbolic plane, or equivalently, for the bundle \( O\mathbb{H}^2 \) of (direct) frames over \( \mathbb{H}^2 \), by means of the identification \( g(z,\varphi) \equiv (z,e^{\sqrt{-1} \varphi}) \). It is also identified with \( \mathbb{H}^2 \times \partial \mathbb{H}^2 \), under the assignment \( (z,e^{\sqrt{-1} \varphi}) \leftrightarrow (z,u) \), \( u \) denoting the infinite end of the half-geodesic determined by \( (z,e^{\sqrt{-1} \varphi}) \). Under the former identification, the above Haar measure \( \lambda \) is identified with the Liouville measure on \( T^1 \mathbb{H}^2 \equiv O\mathbb{H}^2 \).

### 2.2 Flows and leaves

The right action of the one-parameter subgroups \( a(\mathbb{R}_+^*) \) and \( n(\mathbb{R}) \) defines the two fundamental flows acting on frames, i.e., on \( \text{PSL}(2) \) itself.

**Definition 2.2.1** The geodesic flow is the 1-parameter group defined on \( \text{PSL}(2) \) by:

\[ g \mapsto g \theta_t \equiv g a(e^t), \quad \text{for any } g \in \text{PSL}(2) \text{ and } t \in \mathbb{R}, \text{ setting } \theta_t := a(e^t). \] (11)

The horocyclic flow is the one-parameter group defined on \( \text{PSL}(2) \) by:
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Fig. 1 The correspondence $g(z, \varphi) \equiv (z, e^{\sqrt{-1} \varphi}) \leftrightarrow (z, u)$

$$g \mapsto g \theta_t + x, \quad \text{for any } g \in \text{PSL}(2) \text{ and } x \in \mathbb{R}.$$ (12)

**Proposition 2.2.2** The projection $g \theta_t \sqrt{-1}$ of the orbit of a frame $g \in \text{PSL}(2)$ under the action of the geodesic flow is the geodesic of $\mathbb{H}^2$ determined by the line element associated with $g$, and we have:

$$\text{dist}(g \sqrt{-1}, g \theta_t \sqrt{-1}) = |t|, \quad \text{and} \quad \frac{d}{dt} (g \theta_t) \sqrt{-1} = (g \theta_t)'(\sqrt{-1}).$$

Thus the action of the geodesic flow $(\theta_t)$ on the boundary component of $g \in \mathbb{H}^d \times \partial \mathbb{H}^d$ is trivial, and $\theta_t$ moves the generic line-element along the geodesic it generates, by an algebraic hyperbolic distance ('dist' for short) $t$. In other words, any geodesic $g \theta_{t\mathbb{R}}$ of $\mathbb{H}^2$ is the isometric image under $g$ of the vertical geodesic $e^{t \sqrt{-1}}$ ($t$ being the arc-length).

Proposition 2.2.2 stated in particular that any geodesic is the projection of an orbit of the geodesic flow. Analogously, the following proposition states in particular that any horocycle is the projection of an orbit of the horocyclic flow.

**Proposition 2.2.3** The horocycle through a frame $g \in \text{PSL}(2)$ is the projection of its orbit under the action of the horocyclic flow: $\mathcal{H}(g) = g \theta_{t\mathbb{R}} \sqrt{-1}$. Moreover, we have

$$\mathcal{H}(g) = \mathcal{H}(g') \iff (\exists x \in \mathbb{R}) \ g = g' \theta_{-x}^\perp.$$

Mixing the actions of both flows, we get the notion of stable leaf, as follows.

**Definition 2.2.4** For any boundary point $u \in \partial \mathbb{H}^2$, the stable leaf associated with $u$ is the set $\mathcal{H}_u$ of all frames $g \in \text{PSL}(2)$ pointing at $u$, i.e., such that the base of $\mathcal{H}(g)$ (≡ the future end of $g$) is $u$. 

Note that the flows act on each stable leaf \( \mathcal{H}_u \). Precisely, each \( \mathcal{H}_u \) is an orbit under the right action of the affine subgroup \( \mathbb{A} \): \( \mathcal{H}_u = \{ gT_z \mid T_z \in \mathbb{A} \} \) for any \( g \in \mathcal{H}_u \).

### 2.3 Commutation relations

We establish here the commutation relation between an element \( T_z \equiv T_{z,y} \in \mathbb{A} \) and a rotation \( \rho \in \text{SO}(2) \). According to the Iwasawa decomposition, there exist unique \( T_{z,y} \equiv T_{z,y'} \in \mathbb{A} \) and \( \rho' \in \text{SO}(2) \) such that

\[
T_{z,y} \rho = \rho' T_{z,y'}.
\]

**Lemma 2.3.1** Denote by \( u(\rho) \) the base of \( \mathcal{H}(\rho) \) (\( u(\rho) = \infty \) if and only if \( \rho = \text{Id} \)). Then \( T_z \rho = T_{z,y} \rho = \rho' T_{z,y'} \) implies: \( z = \rho' z' \), and \( u(\rho') = yu(\rho) + x \).

**Proof.** The first claim is merely \( z = T_z \sqrt{-1} = T_{z,y} \rho \sqrt{-1} = \rho' T_{z,y} \sqrt{-1} = \rho' z' \).

Then we easily see on Figure 1 (with \( y = 1, x = 0 \)) that \( u(\rho) = -\cotg \left( \frac{\theta}{2} \right) \) if \( \rho = k(\varphi) \). Finally we have

\[
x + yu(\rho) = T_z u(\rho) = T_{z,y} \rho \theta_0 = \rho' T_{z,y} \theta_0 = \rho' \infty = -\cotg \left( \frac{\theta}{2} \right) = u(\rho').
\]

**Proposition 2.3.2** Denote by \( R_{\alpha} \) the rotation by \(-\alpha \). Then we have the following commutation relations:

(i) for any \( \alpha \in [0, \pi] \) and \( x \in \mathbb{R} \), there exists a unique \( \alpha' \in [0, \pi] \) such that \( \cotg \left( \alpha'/2 \right) = \cotg \left( \alpha/2 \right) - x \) and

\[
\theta^{+}_x R_{\alpha} = R_{\alpha'} \theta^{+}_{x'} \theta_{\log y'}, \quad \text{with} \quad y' := \frac{1 - \cos x}{1 - \cos \alpha}, \quad x' := \frac{\sin \alpha' - \sin x}{1 - \cos \alpha'};
\]

(ii) for any real \( r \) we have a unique \( \alpha_r \in [0, \pi] \) such that:

\[
\theta_r R_{\frac{\pi}{2}} = R_{\alpha_r} \theta^{+}_{sh r} \theta_{\log ch r}, \quad \text{with} \quad \cotg \alpha_r = sh r.
\]

Both a classical and a visual proof are detailed in [6].

### 2.4 Harmonic, Liouville and area measures

The measures of this section play a fundamental role, from the geometrical as well as from the probabilistic or dynamical points of view.

Let us begin with the visual measures of the hyperbolic boundary: the harmonic measure \( \mu_z \) must be understood as the measure of the boundary viewed from the point \( z \). Recall the correspondence \( T^1 \mathbb{H}^2 \leftrightarrow \mathbb{H}^2 \times \partial \mathbb{H}^2 \), under the assignment \( (z, e^{\sqrt{-1} \varphi}) \leftrightarrow (z, u) \), \( u \) being the infinite end of the half-geodesic determined by \( (\bar{z}, e^{\sqrt{-1} \varphi}) \) (as already quoted in Section 2.1 and Figure 1).

**Definition 2.4.1** The harmonic measure \( \mu_z \) on \( \partial \mathbb{H}^2 \) is the image of the uniform law on the unit circle centred at \( z \in \mathbb{H}^2 \) under the map \( e^{\sqrt{-1} \varphi} \mapsto u \).
**Remark 2.4.2** It should be clear from Definition 2.4.1 that for any $g \in \text{PSL}(2)$, $\mu_{gz}$ is the image of $\mu_z$ under $g$: $\mu_z \circ g^{-1} = \mu_{gz}$. This is the so-called geometric property of harmonic measures. Moreover, \{ $\mu_z \mid z \in \mathbb{H}^2$ \} is the unique family of probability measures on $\partial \mathbb{H}^2$ which enjoys this property.

It is well known and straightforward that the density of the harmonic measure $\mu_z \equiv \mu_{(x,y)}$ on $\partial \mathbb{H}^2$ is given by the Poisson kernel of $\mathbb{R} \times \mathbb{R}_+^*$:

$$\mu_z(du) = \pi^{-1} \left( \frac{y}{\sqrt{x^2 + |x-u|^2}} \right) du. \quad (13)$$

Recall from Section 2.1 that the Haar measure $\lambda(dg) = dx dy d\varphi / 2\pi y^2$ is the Liouville measure on $T^1\mathbb{H}^2$ as well, which is therefore invariant under the geodesic and horocycle flows. The area measure of $\mathbb{H}^2$ is the image measure of $\lambda$ under the canonical projection $\pi_0 = [g \mapsto g\sqrt{-1}]$. It is given by $dz = y^{-2} dx dy$, and we clearly have the following disintegration: $\lambda(dg) = \mu_z(du) dz$.

### 3 Left Brownian motion on $\mathbb{A}$ and hyperbolic Brownian motion

Theorem 1.4.2 is used here to introduce two key examples of diffusions.

#### 3.1 Left Brownian motion $(Z_s)$ on the affine subgroup $\mathbb{A}$

The solution $(Z_s)$ of the linear SDE driven by $(W_s) = (W^1_s, W^2_s)$:

$$Z_s = 1 + \int_0^s Z_t (\alpha dW^1_t + \nu dW^2_t) + \frac{1}{2} \int_0^s Z_t (\alpha^2 + \nu^2 - \alpha) dt, \quad (14)$$

is a left Brownian motion on the affine group $\mathbb{A}$, according to Theorem 1.4.2, and the infinitesimal generator of $(Z_s)$ is the (half) Laplace operator

$$\frac{1}{2} \left[ \mathcal{L}_\alpha^2 + \mathcal{L}_\nu^2 - \mathcal{L}_a \right] = \frac{1}{2} \mathcal{D}. \quad (15)$$

**Proposition 3.1.1** Restricted to $\mathbb{A}$, right Lie derivatives and the Laplace operator can be computed as follows: for any $z = (x,y) \in \mathbb{R} \times \mathbb{R}^*_+$,

$$\mathcal{L}_\alpha f(T_z) = y \frac{\partial}{\partial y} f(T_z); \quad \mathcal{L}_\nu f(T_z) = y \frac{\partial}{\partial x} f(T_z); \quad \mathcal{D} = y^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right). \quad (16)$$

**Theorem 3.1.2** Setting $Z_s =: T_{x,y}$, or equivalently $(x_s, y_s) \equiv z_s := Z_s \sqrt{-1}$, for the $\mathbb{A}$-valued left Brownian motion $(Z_s)$ which solves Equation (14), we have:

$$y_s = \exp[W^1_s - s/2], \quad x_s = \int_0^s y_t dW^2_t. \quad (17)$$
The $\mathbb{H}^2$-valued process $(z_s)$ has the half hyperbolic Laplacian

$$\frac{1}{2} \Delta := \frac{1}{2} y^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right)$$

as generator, in the sense that for any $f \in C_b^2(\mathbb{H}^2)$ and $s \geq 0$, we have

$$\mathbb{E}[f(x_s, y_s)] = f(0, 1) + \int_0^s \mathbb{E}[\frac{1}{2} \Delta f(x_t, y_t)] \, dt.$$  

Proof. Multiplying Equation (14) on the right by \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) directly yields the stochastic equations:

$$y_s^{-1/2} = 1 - \frac{1}{2} \int_0^s y_t^{-1/2} \, dW_t^1 + \frac{3}{8} \int_0^s y_t^{-1/2} \, dt,$$

$$y_s^{-1/2} x_s = 1 - \frac{1}{2} \int_0^s y_t^{-1/2} x_t \, dW_t^1 + \int_0^s y_t^{-1/2} x_t dW_t^2 + \frac{3}{8} \int_0^s y_t^{-1/2} x_t dt,$$

from which applying Itô’s Formula we derive:

$$y_s = 1 + \int_0^s y_t \, dW_t^1, \quad y_s^{-1/2} x_s = y_s^{1/2} dW_s^2.$$

The expressions (17) follow directly, as well as (using Itô’s Formula once again) the formula involving $\Delta$. \(\Box\)

The above is actually not enough to conclude that we have really constructed the hyperbolic Brownian motion $(z_s)$ by projecting the left Brownian motion $(Z_s)$. It is indeed necessary to be able to vary the initial point, and for that, to make sure that the projection is symmetrical enough, in order to preserve the Markov property (which generally does not survive to a projection). This is the main meaning of the following two sections.

### 3.2 Diffusion processes

It is very natural and useful to extend group-valued left Brownian motions of Section 1.4 to homogeneous spaces, by means of some projection.

**Definition 3.2.1**  
(i) A Fellerian (Markovian) semi-group on a separable metric space $E$ is a family $(P_t)_{t \geq 0}$ of non-negative continuous endomorphisms on $C_b(E)$, such that: $P_0$ is the identity, $P_{s+t} = P_s P_t$, $P_1 = 1$ for any $s, t \geq 0$, and $(P_t)$ is strongly continuous: $\lim_{t \downarrow 0} ||P_t f - f|| = 0$ for any $f \in C_b(E)$.

(ii) Given a separable metric space $E$ endowed with a Borelian probability measure $\mu$ and a Fellerian (Markovian) semi-group $(P_t)$, a continuous $E$-valued process $(X_t)$ such that

$$\mathbb{E}[f_0(X_0) \times f_1(X_{t_1}) \times \ldots \times f_n(X_{t_n})] = \int f_0 P_{t_1} \left[ f_1 P_{t_2-t_1} [f_2 \ldots P_{t_n-t_{n-1}} f_n] \ldots \right] d\mu$$

where $\mu$ is the law of the process $X_t$. \(\Box\)
for any \( n \in \mathbb{N}^* \), \( f_0, \ldots, f_n \in C_b(E) \) and \( 0 \leq t_1 \leq \ldots \leq t_n \), is called a diffusion process on \( E \) with semi-group \((P_t)\) and initial law \( \mu \).

Note that (taking a Dirac mass \( \delta_x \) as \( \mu \), \( n=1 \), and \( f_0 \equiv 1 \)) we have in particular \( P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}[f(X_t) | X_0 = x] \), the index \( x \) in \( \mathbb{E}_x \) specifying the initial value \( X_0 = x \in E \). By Proposition 1.4.5, any left Brownian motion is an example of diffusion process (on a subgroup \( G \) of \( \text{GL}(d) \)). The hyperbolic Brownian motion and other important diffusion processes can be constructed in the following way (up to a possible left quotient by some Fuchsian group, as we shall consider later).

**Proposition 3.2.2** Given a left Brownian motion \((g_t)\) on a group \( G \) and an independent random variable \( g \) on \( G \) with law \( \nu \), consider a continuous map \( p: G \to E \) such that for any \( g \in G \) the law \( Q_t(x, \cdot) \) of \( p(g_t) \) depends only on \( x := p(g) \) and \( t \). Then \( (Q_t) \) is a Fellerian (Markovian) semi-group and \( p(g_t) \) is a diffusion process on \( E \) with semi-group \((Q_t)\) and initial law \( \nu \circ p^{-1} \).

The diffusion process is said started at \( x \in E \) if its initial law is the Dirac mass at \( x \).

**Remark 3.2.2** Note that by Proposition 3.2.2, if \((P_t)\) denotes the semi-group of \((g_t)\) (recall Definition 1.4.4), then for any \( t \geq 0 \) and \( f \in C_b(E) \) we have \((Q_t f) \circ P = P_t (f \circ p)\).

### 3.3 Hyperbolic Brownian motion

We exhibit here a second example of diffusion process, which is essential to our purpose. We consider now the projection \( p = \pi_0: g \mapsto g\sqrt{-1} \) from \( \text{PSL}(2) \) onto \( \mathbb{H}^2 \) and the left Brownian motion \((Z_t)\) on \( \text{PSL}(2) \) (of Section 3.1).

The following shows that they satisfy the assumption of Proposition 3.2.2.

**Lemma 3.3.1** The law of the process \((g Z_t \sqrt{-1})\) depends on \( g \in \text{PSL}(2) \) only through \( g \sqrt{-1} \).

**Proof.** This amounts to show that for any given \( \rho \in \text{SO}(2) \), \((\rho z_t)\) has the same law as \((z_t)\). Now the starting point is \( \sqrt{-1} \) for both, and according to Theorem 3.1.2 (or its proof) we have \( dz_t = y_t dW_t = y_t (dW_t^2 + \sqrt{-1} dW_t^1) \). Note that \( \langle W_t^1, W_t^1 \rangle = 0 \), so that according to Itô’s Formula we have then \( d(\rho z_t) = \rho'(z_t) y_t dW_t = 3(\rho z_t) \cos \phi \frac{\sin \phi}{\cos \phi - \sin \phi} dW_t \equiv 3(\rho z_t) dW'_t \), where \((W'_t)\) is another complex Brownian motion (because the fraction \( \frac{\cos \phi - \sin \phi}{\cos \phi + \sin \phi} \) has unit modulus). This shows that \((\rho z_t)\) satisfies the same linear differential equation as \((z_t)\), hence indeed has the same law. \( \Box \)

This lemma allows to apply Proposition 3.2.2, yielding the hyperbolic heat semi-group, say \((Q_t)\), which satisfies the identity:

\[
Q_t f(z) = \mathbb{E} [f(g Z_t \sqrt{-1})],
\]

for any \( t \geq 0 \), \( f \in C_b(\mathbb{H}^2) \), \( z \in \mathbb{H}^2 \) and \( g \in \text{PSL}(2) \) such that \( g \sqrt{-1} = z \).
The associated diffusion process is called *hyperbolic Brownian motion*. We have the following analogue of Proposition 1.4.5.

**Theorem 3.3.2** The hyperbolic heat semi-group \((Q_t)\) maps \(C^2_b(\mathbb{H}^2)\) into \(C^2_b(\mathbb{H}^2)\), and for any \(f \in C^2_b(\mathbb{H}^2)\) we have:

\[
\frac{d}{dt} Q_t f = \frac{1}{2} \Delta Q_t f = \frac{1}{2} Q_t \Delta f.
\]

Moreover \(Q_t\) (and then the hyperbolic Laplacian \(\Delta\) as well) is self-adjoint with respect to the area measure of \(\mathbb{H}^2\), and covariant with \(PSL(2)\): we have \(Q_t (f \circ g) = (Q_t f) \circ g\) for any \(t \geq 0\) and \(g \in PSL(2)\).

By analogy with Theorem 1.4.2, we say that the half hyperbolic Laplacian \(\frac{1}{2} \Delta\) is the (infinitesimal) generator of the hyperbolic Brownian motion and of its (hyperbolic heat) semi-group \((Q_t)\).

We describe now the main feature of the asymptotic behaviour of the hyperbolic Brownian motion, taking advantage of the integrated formulas (17).

**Proposition 3.3.3** For any starting point \(z \in \mathbb{H}^2\), the hyperbolic Brownian motion \((z_s)\) converges almost surely, as \(s \to \infty\), to a boundary point \(z_\infty \in \partial \mathbb{H}^2\), the law of which is the harmonic measure \(\mu_z\).

**Proof.** First, the covariance with respect to \(g \in PSL(2)\) allows to consider only the case \(z = \sqrt{-1}\). Then it is clear from Formulas (17) that \(y_s\) goes almost surely to 0, and that we almost surely have

\[
x_s \longrightarrow x_\infty := \int_0^\infty e^{W_1 - \frac{1}{2} t} dW_t^2.
\]

This yields a limit point \(x_\infty \in \partial \mathbb{H}^2\). It remains to compute its law. Now for any \(\rho \in SO(2)\), by Lemma 3.3.1 \((\rho z_s)\) has the same law as \((z_s)\), so that the law of \(z_\infty = (x_\infty, 0)\) must be \(SO(2)\)-invariant. As quoted in Remark 2.4.2, this actually yields the definition 2.4.1 of \(\mu_z\). \(\square\)

## 4 Measures and flows on \(M := \Gamma \backslash PSL(2)\)

We shall consider here a fixed Fuchsian group \(\Gamma\) and the associated measures and flows (induced by the measures of Section 2.4 and the flows of Section 2.2) on the quotient space \(M := \Gamma \backslash PSL(2)\), and we investigate their basic properties, when \(\Gamma\) is cofinite, hence geometrically finite too. The main result of this section is a mixing theorem for the action of the geodesic and horocyclic flows on square integrable \(\Gamma\)-invariant functions, namely Theorem 4.4.1. Let us begin with some summary about Fuchsian groups.

### 4.1 Fuchsian groups

The hyperbolic space has infinite volume, which excludes ergodicity properties. To observe them it is necessary to consider periodic functions, i.e., functions on quo-
tient spaces, associated with a cofinite discrete subgroup of isometries. The geodesic flows on such Riemann surfaces are the simplest examples of mixing unstable flows.

Let us briefly recall the geometric theory of Fuchsian groups and their fundamental domains.

**Definition 4.1.1** A Fuchsian group is a discrete subgroup of $\text{PSL}(2)$. Obviously, a Fuchsian group is countable. Classical examples are given by modular groups such as $\Gamma(2)$, $\Gamma(1)$, $D\Gamma(1)$, $\Gamma(3)$.

The main object of study associated to a Fuchsian group $\Gamma$ is its action on $\mathbb{H}^2$, and then its orbit space, that is, the space $\Gamma\backslash\mathbb{H}^2 := \{ \Gamma p \mid p \in \mathbb{H}^2 \}$ of the orbits of $\mathbb{H}^2$ under the (left) action of $\Gamma$. It is natural, in order to visualize this orbit space, to consider subsets of $\mathbb{H}^2$ which essentially contain one and only one representative of each orbit. Precisely, a fundamental domain for a Fuchsian group $\Gamma$ is a connected open subset $D \subset \mathbb{H}^2$ with zero boundary area, such that:

$$\gamma D \cap \gamma' D = \emptyset \text{ for any } \gamma \neq \gamma' \in \Gamma \text{ and } \bigcup \{ \gamma D \mid \gamma \in \Gamma \} = \mathbb{H}^2.$$ 

From the definition of a fundamental domain, the sequence of isometric images of $D$ under $\Gamma$ covers $\mathbb{H}^2$ (up to a negligible subset) without overlapping, drawing a paving, or tesselation of $\mathbb{H}^2$. Of course, there are fundamental domains $D$ which are simpler than others, and more convenient for most purposes. Of particular interest are those which are polygonal, i.e., bounded by a finite set of geodesic lines.

A subset of $\mathbb{H}^2$ is convex if it contains the geodesic segment linking any pair of its points. Let us call convex polygon any convex closed subset $\mathcal{P}$ of $\mathbb{H}^2$, possibly unbounded, having its boundary made out of a finite number of (convex subsets of) geodesics of $\mathbb{H}^2$, namely its sides. A fundamental polygon for $\Gamma$ will be a convex polygon, the interior of which is a fundamental domain for $\Gamma$, and satisfying the following additional property: for each side $S$ of $\mathcal{P}$, there exists some $\gamma \in \Gamma$ such that $S = \mathcal{P} \cap \gamma \mathcal{P}$.

A Fuchsian group $\Gamma$ admitting a fundamental polygon is said to be geometrically finite. We shall consider only geometrically finite Fuchsian groups.

For any $p \in \mathbb{H}^2$ which is not fixed by any element of a Fuchsian group $\Gamma$, the so-called Dirichlet polygon relative to $\Gamma$ and centred at $p$, i.e.,

$$\mathcal{P}(\Gamma, p) := \{ z \in \mathbb{H}^2 \mid \text{dist}(p, z) \leq \text{dist}(\gamma p, z) \text{ for any } \gamma \in \Gamma \}.$$ 

is a fundamental polygon for $\Gamma$. See [6] for a particular type of Fuchsian group, yielding a parabolic tesselation by an ideal $2n$-gon.

### 4.2 Measures of $\Gamma$-invariant sets

Recall that any $\gamma \in \text{PSL}(2)$ acts (by left translations) as well on the frame bundle $\text{PSL}(2) \equiv T^1\mathbb{H}^2 \equiv O\mathbb{H}^2 \rightarrow \mathbb{H}^2$, and that plainly, if $D \subset \mathbb{H}^2$ is a fundamental domain for $\Gamma$, then the cylindrical domain $\pi_0^{-1}(D) \subset \text{PSL}(2)$ is fundamental for the action (by left translations) of $\Gamma$ on $\text{PSL}(2) \equiv T^1\mathbb{H}^2 \equiv O\mathbb{H}^2$.

From now on, we shall consider mainly $\Gamma$-invariant functions. We first specify the relative (area and Liouville) measures against which we shall integrate them.
We also emphasize the invariance of these relative measures under the flows, which follows from the fact that the Fuchsian group $\Gamma$ acts on the left hand side, while the flows act on the right hand side, so that these two actions commute.

**Proposition 4.2.1** Fix some Fuchsian group $\Gamma$. The following statements hold.

(i) The area of a fundamental domain $D$ depends only on the Fuchsian group $\Gamma$, and will be called the covolume of $\Gamma$, and denoted by $\text{covol}(\Gamma)$.

(ii) The area measure $dz$ of $\mathbb{H}^2$, restricted to a fundamental domain $D$, induces a (relative) area measure $d^r z$ on left $\Gamma$-invariant Borel sets of $\mathbb{H}^2$, which depends only on $\Gamma$.

(iii) The Liouville measure $\lambda$ of $\text{PSL}(2)$, restricted to a cylindrical fundamental domain $\pi_0^{-1}(D)$, induces a (relative) Liouville measure $\lambda^\Gamma$ on left $\Gamma$-invariant Borel sets of $\text{PSL}(2)$, which depends only on $\Gamma$.

(iv) The Liouville measure $\lambda^\Gamma$ is invariant under the (right) action of the geodesic and horocycle flows, and more generally, under the right action of any $g \in \text{PSL}(2)$.

**Notation** We denote henceforth by $\mathcal{M} := \Gamma \setminus \text{PSL}(2)$ the quotient of $T^1 \mathbb{H}^2$ under the left action of the Fuchsian group $\Gamma$, and similarly, by $\mathcal{M} := \Gamma \setminus \mathbb{H}^2$ the quotient of $\mathbb{H}^2$. Then the Borel sets of $\mathcal{M}$ (respectively $\Gamma \setminus \mathbb{H}^2$) are identified with the $\Gamma$-invariant Borel sets of $\text{PSL}(2)$ (respectively of $\mathbb{H}^2$), and the measure $\lambda^\Gamma$ (respectively $d^r z$) is identified with a measure on $\mathcal{M}$ (respectively on $\Gamma \setminus \mathbb{H}^2$). Note that $\Gamma \setminus \mathbb{H}^2 \equiv \mathcal{M} / \text{SO}(2) \equiv \pi_0(\mathcal{M})$.

A Fuchsian group $\Gamma$ is said to be cofinite if it has a finite covolume. It is then necessarily geometrically finite.

### 4.3 Ergodicity

The important mixing and ergodic properties are usually defined as follows.

**Definition 4.3.1** Consider any probability space $(E, \mathcal{E}, \mu)$ and a one-parameter group $(g_t)$ of measure-preserving maps such that for any $f \in L^2(E, \mu)$, $t \mapsto \langle f \circ g_t, f \rangle_{L^2}$ is continuous. The action of $(g_t)$ is said to be mixing if for any $f, \varphi \in L^2$,

$$\lim_{t \to \pm \infty} \langle f \circ g_t, \varphi \rangle_{L^2} = \langle f, 1 \rangle_{L^2} \langle 1, \varphi \rangle_{L^2}.$$

A function $f \in L^2$ is said to be $(g_t)$-invariant if $f \circ g_t = f$ almost surely for any $t \in \mathbb{R}$. The action of $(g_t)$ is said to be ergodic if any $(g_t)$-invariant $f \in L^2(\mu)$ is almost surely constant.

**Proposition 4.3.2** If $(g_t)$ is mixing, then it is ergodic.

Recall the fundamental ergodic theorem, due to Birkhoff. It states that if ergodicity holds, then temporal means converge to spatial mean.

**Theorem 4.3.3** (Ergodic Theorem) If $(g_t)$ is ergodic and if $(t, \beta) \mapsto g_t(\beta)$ is measurable on $\mathbb{R} \times \mathcal{M}$, then for any $f \in L^2$ we have:

$$\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t f \circ g_s ds = \int f \, d\mu, \text{ $\mu$-almost surely and in } L^2.$$
4.4 A mixing theorem

Consider a cofinite Fuchsian group \( \Gamma \), the normalised induced measure \( \mu := \lambda \Gamma / |\lambda \Gamma| = \text{covol}(\Gamma)^{-1} \lambda \Gamma \), the Hilbert space \( L^2 = L^2(M, \mu) \), with inner product \( \langle f, \phi \rangle_{L^2} := \int f \phi \, d\mu \).

**Theorem 4.4.1** The geodesic flow \( (\theta_t) \) and the horocycle flow \( (\theta^+_u) \) are mixing, hence ergodic.

We shall say that a sequence \( (\gamma_n = \rho_n \theta_{r_n} \rho'_n) \) (using the Cartan decomposition) in \( \text{PSL}(2) \) goes to infinity if and only if \( r_n \to \infty \). It is clear that the one-parameter subgroups \( (\theta_t) \) and \( (\theta^+_u) \) go to infinity as \( t \to \pm \infty \), so that the proof of the theorem follows immediately from the following, which happens actually to hold as well.

**Proposition 4.5** If \( (\gamma_n) \subset \text{PSL}(2) \) is a sequence going to infinity, then for any centred \( f, \phi \in L^2_0 \) we have:

\[
\lim_{n \to \infty} \langle f \circ \gamma_n, \phi \rangle_{L^2} = 0.
\]

The proof of this result needs several steps. Again, see [6]. Actually an elegant proof by Y. Guivarc’h (which can be slightly simplified, in the case of \( \text{PSO}(1,d) \), by using Proposition 2.3.2) shows a similar but stronger result: the above statement holds not only for \( \text{PSL}(2) \) and \( \text{PSO}(1,d) \), but even for \( \text{SL}(d) \).

5 Poincaré inequality

The main result of this section is Theorem 5.2.4, which states a Poincaré inequality (i.e., the existence of a spectral gap) for the Laplacian acting on \( \Gamma \)-invariant functions. It is proved by decomposing a fundamental domain into a compact core and cusps neighbourhoods (called solid cusps) which overlap. Actually a Poincaré inequality holds in these parts, and a general proposition states that it is conserved by taking union of overlapping domains. We only need the case of hyperbolic domains, i.e., of connected open subsets of \( \mathbb{H}^2 \), endowed with the restriction of the volume measure \( dz \). However it happens to be easier to begin with the Euclidean case, as long as the Lebesgue measure is provided with weights. Thus we shall first consider the Euclidean framework. All details can be found in the reference book [6].

5.1 The Euclidean case

**Definition 5.1.1** Consider a domain \( D \subset \mathbb{R}^2 \) and two positive measurable functions on \( D \), namely \( \phi \) bounded from above and \( \psi \) bounded away from 0. A Poincaré inequality \( \mathcal{F}(D, \phi, \psi) \) holds when there exists \( C = C(D, \phi, \psi) \) such that for any integrable function \( f \) of class \( C^1 \) on \( D \) satisfying \( \int_D f(x) \phi(x) \, dx = 0 \), we have:

\[
\int_D f(x)^2 \phi(x) \, dx \leq C \int_D |df|^2 \psi(x) \, dx,
\]  

(21)
where \(|df|\) denotes the Euclidean norm of the differential of \(f\) (i.e., of its Euclidean gradient) and \(dx\) the Lebesgue measure of \(\mathbb{R}^2\).

**Lemma 5.1.2** For all functions \(\varphi, \psi\) and \(a > 0\), a Poincaré inequality \(J(D, \varphi, \psi)\) holds on a convex simplex \(D = D_a := \{(x^1, x^2) \in (\mathbb{R}_+^*)^2 \mid x^1 + x^2 < a\}\).

**Proposition 5.1.3** Consider bounded domains \(D, D'\) in \(\mathbb{R}^2\), with closures \(\overline{D}, \overline{D'}\) having some \(C^1\)-diffeomorphic neighbourhoods, and such that Poincaré inequalities \(J(D, \varphi, \psi)\) hold for any \(\varphi, \psi\) as in Definition 5.1.1. Then a Poincaré inequality \(J(D', \varphi, \psi)\) holds on \(D'\) as well, for all \(\varphi\) and \(\psi\).

The following is crucial to allow to consider separately different domains, in order to we obtain a Poincaré inequality on their union.

**Proposition 5.1.4** Let \(D\) and \(D'\) be two intersecting domains in \(\mathbb{R}^2\) and \(\varphi, \psi\) be two positive measurable functions on \(D \cup D'\) such that \(\varphi \in L^1(D \cup D')\). If Poincaré inequalities \(J(D, \varphi, \psi)\) and \(J(D', \varphi, \psi)\) hold, then a Poincaré inequality \(J(D \cup D', \varphi, \psi)\) holds too.

**Proposition 5.1.5** A Poincaré inequality \(J(D, \varphi, \psi)\) holds on a domain \(D \subset \mathbb{R}^2\) whose closure is \(C^1\)-diffeomorphic to a convex polygon, for all \(\varphi\) and \(\psi\).

**Proof.** By Proposition 5.1.3 we just have to consider the case of a convex polygon. Now this is a finite union of adjacent simplexes, \(C^1\)-diffeomorphic to the simplexes considered in Lemma 5.1.2. Moreover we obtain an overlap between adjacent simplexes by means of diffeomorphisms acting non trivially in a small neighbourhood of their adjacent sides. We conclude by applying Propositions 5.1.3 and 5.1.4. \(\Box\)

### 5.2 The case of a fundamental domain in \(\mathbb{H}^2\)

We apply now the proposition 5.1.5 to our main concern: hyperbolic domains, i.e., connected open subsets \(D \subset \mathbb{H}^2\). Any hyperbolic domain \(D\) is endowed with the restriction of the volume measure \(dz\), and with a hyperbolic gradient: namely, to any \(f \in C^2(D)\) we associate its squared hyperbolic gradient:

\[
|\nabla f|^2 := \frac{1}{2} \Delta (f^2) - f \Delta f = y^2 \left(\frac{\partial f}{\partial y}\right)^2 + x^2 \left(\frac{\partial f}{\partial x}\right)^2,
\]

(22)

according to Formula (18). We are henceforth concerned with the following notion of Poincaré inequality.

**Definition 5.2.1** A Poincaré inequality holds on a hyperbolic domain \(D\) when there exists \(C_D > 0\) such that, for any integrable function \(f\) of class \(C^1\) on \(D\), we have

\[
\int_D f(x,y)y^{-2} \, dx \, dy = 0 \implies \int_D f^2(x,y)y^{-2} \, dx \, dy \leq C_D \int_D |\nabla f|^2(x,y) \, dx \, dy.
\]

(23)
Note that we now particularize the weights $\varphi, \psi$ of the preceding section 5.1 to $\varphi \equiv y^{-2}$ and $\psi \equiv 1$. Applying Proposition 5.1.5 to a bounded hyperbolic domain $D$, we obtain the following.

**Corollary 5.2.2** A Poincaré inequality (23) holds on any bounded domain $D \subset \mathbb{H}^2$, whose closure is $C^1$-diffeomorphic to a convex polygon.

To handle unbounded fundamental domains, we need in particular to address their unbounded ends. We call **solid cusp** the image in the orbit space $\Gamma \backslash \mathbb{H}^2$ of a horoball $\mathcal{H}^+$ based at some parabolic point $u$, and small enough in order that $\Gamma \backslash \mathcal{H}^+$ can be identified with $\Gamma_u \backslash \mathcal{H}^+$, where $\Gamma_u$ denotes the maximal parabolic subgroup of $\Gamma$ fixing $u$, isomorphic to $\mathbb{Z}$.

**Proposition 5.2.3** A Poincaré inequality (23) holds on any solid cusp.

We can now conclude with the aim of the present section 5, which yields the spectral gap result we need.

**Theorem 5.2.4** A Poincaré inequality (23) holds on a fundamental domain $D$ of a cofinite (hence geometrically finite) Fuchsian group $\Gamma$.

**Proof.** We use Corollary 5.2.2, and Propositions 5.2.3 and 5.1.4. For this, we note that we can cover a fundamental polygon by a finite union of solid cusps and a so-called core, which is compact and diffeomorphic to a convex polygon. Indeed, the solid cusps are obtained by cutting each of the unbounded ends of the fundamental polygon by a sufficiently small horodisc (based at the corresponding infinite end, i.e., cusp). The remaining surface is the so-called core. This core can be slightly enlarged, by cutting each unbounded end by a smaller horodisc, in order to intersect the interior of every solid cusp (and to be still diffeomorphic to a convex polygon).

\[\square\]

### 6 Central Limit Theorem for geodesics

In this final section we sketch a proof of the Sinai Central Limit Theorem, generalized to the case of a cofinite (and then geometrically finite) Fuchsian group. This theorem, which completes the mixing property of Section 4.4, shows that asymptotically geodesics behave chaotically, and yields a quantitative expression of this phenomenon.

The strategy proceeds by establishing such a result first for Brownian trajectories, which is easier because of their strong independence properties. Then geodesics are compared to Brownian trajectories, by means of a change of contour and time reversal. This requires in particular considering diffusion paths on the stable foliation and deriving the existence of a key potential kernel, using the spectral gap exhibited in Section 5 and the commutation relation of Section 2.3 (which is related to the instability of the geodesic flow).

This strategy was initiated in [7] and developed in particular in [4] and [5]. The whole detailed proof, in any dimension, can be found in the reference book [6].

Fix a cofinite Fuchsian group $\Gamma$ admitting a spectral gap, for example a cofinite one (according to Theorem 5.2.4). Recall that $\Gamma \backslash \mathbb{H}^2 \equiv \mathcal{M} / \text{SO}(2) \equiv \pi_0(\mathcal{M})$. 
6.1 Dual $\mathbb{A}$-valued left Brownian motions

We return now to the $\mathbb{A}$-valued left Brownian motion $(Z_t)$ solving Equation (14) in Section 3.1. Recall from 3.1.1 that it has generator
\[
\frac{1}{2} D = \frac{1}{2} \left[ \mathcal{L}_v^2 + \mathcal{L}_\alpha^2 - \mathcal{L}_\alpha \right],
\]
and that according to Section 3.3, it projects on $\mathbb{H}^2$ under $\pi_0 = \{g \mapsto g\sqrt{-1}\}$ to the hyperbolic Brownian motion; precisely, for any $g \in \text{PSL}(2)$, $\pi_0(g Z_t)$ is a hyperbolic Brownian motion on $\mathbb{H}^2$, started at $\pi_0(g) = g\sqrt{-1}$.

Recall also that according to Formulas (17) for any $t \geq 0$ and for some $\mathbb{R}^2$-valued Brownian motion $(w, W)$
\[
Z_t = T_{y_t}, \quad \text{with} \quad y_t = e^{-t/2} \quad \text{and} \quad x_t = \int_0^t y_s \, dW_s.
\]
We shall also use as a key ingredient the dual diffusion process $(Z_t^*)$, which is the $\mathbb{A}$-valued left Brownian motion solving the equation obtained from (14) by changing merely $\alpha$ into $-\alpha$. Its generator is the adjoint of $\frac{1}{2} D$ with respect to $\mu$:
\[
\frac{1}{2} D^* = \frac{1}{2} \left[ \mathcal{L}_v^2 + \mathcal{L}_\alpha^2 + \mathcal{L}_\alpha \right],
\]
Denote by $P_s^* = \exp[(s/2)D^*]$ its semi-group, adjoint to the semi-group $(P_s)$ of $(Z_t)$ with respect to $\mu$.

**Lemma 6.1.1** For any $s$, $Z_s^*$ and $Z_s^{-1}$ have the same law.

As for $(Z_t)$ in Formula (24), there exists a unique $\hat{Z}_t = (\hat{x}_t, \hat{y}_t) \in \mathbb{R} \times \mathbb{A}^*$ such that for some independent Brownian motions $w, W$ and for any $t \geq 0$:
\[
Z_t^* = T_{\hat{y}_t}, \quad \text{with} \quad \hat{y}_t = e^{-t/2} \quad \text{and} \quad \hat{x}_t = \int_0^t \hat{y}_s \, dW_s.
\]
We shall need at a crucial step the following technical lemma, the main ingredient of which is the calculation of the Lie derivatives of the convolution of a smooth function by the laws of $(Z_t), (Z_t^*)$. We still denote the semi-groups corresponding to the right action of $(Z_t), (Z_t^*)$ on $\mathcal{M}$ by $(P_s), (P_s^*)$.

**Lemma 6.1.2**

(i) For any $n \in \mathbb{N}$, both semi-groups $(P_s)$ and $(P_s^*)$ act on $\mathcal{C}^n$ bounded functions on $\text{PSL}(2)$ which have bounded Lie derivatives of order $\leq n$. Moreover $(P_s)$ and $(P_s^*)$ act on functions on $\mathbb{A}$ having bounded $\mathcal{L}_\alpha, \mathcal{L}_v$-derivatives.

(ii) The hyperbolic heat semi-group $(Q_s)$ acts on $\Gamma$-left invariant functions on $\mathbb{H}^2$.

(iii) The relative area measure $d^F z$ (recall Proposition 4.2.1(ii)) is invariant under the hyperbolic Brownian semi-group.

Note that (ii) above follows at once from Theorem 3.3.2, and for (iii), that we have:
\[
\int Q_s f \, d^F z = \int \pi_0^{-1}(D) (Q_s f) \circ \pi_0 \, d\lambda = \int \pi_0^{-1}(D) P_s (f \circ \pi_0) \, d\lambda
\]
\[
= \int \pi_0^{-1}(D) f \circ \pi_0 \, d\lambda = \int f \, d^F z.
\]
6.2 Two dual diffusions

We now use the right action of $\text{PSL}(2)$ on $\mathcal{M}$, and Proposition 4.2.1.

**Definition 6.2.1** Set $\mu := \lambda^\Gamma / \text{cvol}(\Gamma)$, and fix a $\mathcal{M}$-valued random variable $\xi_0$, independent from $(Z_s), (Z_s^*)$ and having law $\mu$. Set

$$\xi_s := \xi_0 Z_s \quad \text{and} \quad \xi_s^* := \xi_0 Z_s^*, \quad \text{for any } s \geq 0.$$  

**Proposition 6.2.2** Both diffusions $(\xi_s)$ and $(\xi_s^*)$ are stationary (with law $\mu$) on $\mathcal{M}$, and dual of each other: for any test-functions $\varphi, \psi$ on $\mathcal{M}$ and any $s \geq 0$, we have

$$\mathbb{E}[\varphi(\xi_s) \psi(\xi_0)] = \mathbb{E}[\varphi(\xi_s^*) \psi(\xi_0^*)].$$

**Proof.** The stationarity is clear from the $\text{PSL}(2)$-right-invariance of the Liouville measure (recall Proposition 4.2.1(iv)). Then using Lemma 6.1.1, Proposition 4.2.1(iv) and Definition 6.2.1, we have:

$$\mathbb{E}[\varphi(\xi_s) \psi(\xi_s^*)] = \mathbb{E}[\varphi(\xi_0) \psi(\xi_s^*)] = \mathbb{E}[\varphi(\xi_0) \psi(\xi_0^*)].$$

Thus the diffusion $(\xi_s)$ projects under $\pi_0$ to the stationary hyperbolic Brownian motion on $\Gamma \setminus \mathbb{H}^2$.

**Remark 6.2.3** Formulas (20) of Theorem 3.3.2 remain valid on the quotient $\Gamma \setminus \mathbb{H}^2$: for any bounded measurable $h$ on $\Gamma \setminus \mathbb{H}^2$ we have $P_t(h \circ \pi_0) = (Q_t h) \circ \pi_0$, and then $\frac{d}{dt} Q_t h = \frac{1}{2} \Delta Q_t h = \frac{1}{2} Q_t \Delta h$. Moreover by Formula (19), for any $p \in \Gamma \setminus \mathbb{H}^2$, such that $\pi_0(\beta) = p$, $z_0 := \pi_0(\beta Z_t)$ defines a diffusion on $\Gamma \setminus \mathbb{H}^2$, such that $Q_t f(z) = \mathbb{E}[f(z_t)]$ for any test-function $f$ on $\Gamma \setminus \mathbb{H}^2$. Such $(z_t)$ is a hyperbolic Brownian motion (started at $p$) on $\Gamma \setminus \mathbb{H}^2 \equiv \pi_0(\mathcal{M})$.

**Theorem 6.2.4** The hyperbolic Brownian semi-group $(Q_t)$ is self-adjoint in $L^2(dz)$. By Theorem 5.2.4, the hyperbolic Brownian semigroup $(Q_t)$ satisfies a Poincaré inequality: for any $h \in L^2(d^2 z)$ having zero mean and any $s > 0$, we have:

$$\frac{d}{ds} \int_D |Q_t h|^2(z) d^2 z = \int_D Q_t h dQ_t h = - \int_D \gamma(Q_t h) \leq -C_D^{-1} \int_D |Q_t h|^2(z) d^2 z,$$

since by Proposition 4.2.1 $\int Q_t h(z) d^2 z = 0$. This entails the following.

**Corollary 6.2.5** The hyperbolic Brownian semigroup $(Q_t)$ admits a spectral gap (hence is mixing) in $L^2(d^2 z)$: for any $h \in L^2(d^2 z)$ having zero mean and any $s \geq 0$ we have:

$$\int |Q_t h|^2(z) d^2 z \leq e^{-C_D s} \int h^2(z) d^2 z.$$
6.3 Spectral gap along the foliation

Till now we have a Poincaré inequality, and then a spectral gap, at the level of a cofinite fundamental domain of the hyperbolic space. We need to extend this crucial spectral gap property to the foliation defined by the geodesic and horocyclic flows. This will be done using appropriate norms. The exponential decay does not hold for all $L^2$ functions, but for the rotationally Hölderian ones, due to the fact that the diffusion evolves along a foliation. This will immediately imply the existence of a potential kernel for the foliated Brownian semi-group $(P_t)$.

We shall use the coordinate $u(\rho)$ considered in Lemma 2.3.1 (and Figure 1).

We need to consider the following projection of generic functions on $\mathbb{H}^2 \times \partial \mathbb{H}^2$ onto functions on $\mathbb{H}^2$, got by averaging over $\text{SO}(2)$.

**Definition 6.3.1** ([5], [6]) For any non-negative Borelian function $F$ on $\mathbb{H}^2 \times \partial \mathbb{H}^2$, and any $z \in \mathbb{H}^2$, set

$$F(z) := \int_{\partial \mathbb{H}^2} F(z, u) \mu_c(du),$$

where $\mu_c$ is the harmonic measure at $z$ (recall Definition 2.4.1).

Note that according to Section 2.4, we have

$$\int_{\mathbb{H}^2} F(z) \, dz = \int_{\mathbb{H}^2 \times \partial \mathbb{H}^2} F \, d\lambda.$$

Note also that, thanks to the geometric property of harmonic measures (recall Remark 2.4.2), if $F$ is $\Gamma$-invariant, then $F$ is $\Gamma$-invariant too. And then, for any $\Gamma$-invariant function $F$ recalling Proposition 4.2.1 we have

$$\int F(z) \, d\Gamma_z = \int F \, d\lambda_\Gamma.$$

**Definition 6.3.2** ([5], [6]) Given $r > 0$, a Borelian function $F$ on $\mathbb{H}^2 \times \partial \mathbb{H}^2$ such that

$$\|F\|_r := \sup_{g \in \text{PSL}(2), \rho \in \text{SO}(2) \setminus \{\text{Id}\}} |F(g\rho) - F(g)| \times \ell(\rho)^{-r} < \infty$$

is said to be rotationally $r$-Hölderian. Here $\ell(\rho)$ denotes the distance from $\rho$ to $\text{Id}$ (which can be replaced by $|\sin(\frac{\phi}{2})|$ or $\frac{1}{|\sin(\frac{\phi}{2})| + 1}$).

**Remark 6.3.3** The semi-group $(P_t)$ of $(\xi_t)$ is contracting in $L^2(\mu)$: if $t \geq 0$ and $F \in L^2(\mu)$, then $\|P_tF\|_2 \leq \|F\|_2$. \[ \square \]

**Proof.** Indeed, by Schwarz inequality and right-invariance of $\mu$:

$$\|P_tF\|^2_{L^2(\mu)} = \int_{\mathcal{M}} |P_tF|^2 d\mu \leq \int_{\mathcal{M}} P_tF(P^2)d\mu = \|F\|^2_{L^2(\mu)}, \square$$

The following crucial result quantifies the necessary control in the transversal (rotational) direction, in order to handle the foliated diffusion $(Z_t)$. Its delicate proof (see [5] or [6]) crucially uses Lemma 2.3.1, Formulas (24), and Corollary 6.2.5.

**Theorem 6.3.4** ([5], [6]) For any centred $F \in L^\infty(\mu)$ which is rotationally $r$-Hölderian, there exist $C$ and $\delta = \delta(r, \Gamma) > 0$ such that

$$\|P_tF\|_{L^2(\mu)} \leq C (\|F\|_\infty + \|F\|_r) e^{-\delta t} \text{ for all } t \geq 0.$$
6.4 Resolvent kernel and conjugate function

In this section we introduce a resolvent (potential) kernel, which allows to exhibit conjugate functions to a given function \( f \) on \( \mathbb{H}^2 \times \partial \mathbb{H}^2 \), provided it has some regularity. This construction will be crucial below, to compare geodesics to Brownian paths by means of a contour deformation.

The following simplified theory of differential 1-forms on \( \text{PSL}(2) \) contains just what is necessary to implement the contour deformation.

**Definition 6.4.1** Call 1-form any \( C^1 \) map from \( \text{PSL}(2) \) into the dual Lie algebra \( \mathfrak{sl}(2)^* \), and longitudinal 1-form any \( C^1 \) map from \( \text{PSL}(2) \) into the dual subalgebra \((\mathbb{R} \alpha + \mathbb{R} \nu)^*\). The longitudinal 1-form \( \omega = \omega_\alpha \alpha^* + \omega_\nu \nu^* \) is closed if and only if \( (\mathcal{L}_\alpha - 1) \omega_\nu = \mathcal{L}_\nu \omega_\alpha \).

**Proposition 6.4.2** If a longitudinal 1-form \( \omega \) is closed, then for any \( g \in \text{PSL}(2) \), \( T_{s,y} \in \mathbb{L} \), and any \( C^1 \) path \( \gamma \equiv (T_{s,y}, q) \), from 1 to \( T_{s,y} \), the line integral

\[
\int_{g\gamma[0,1]} \omega := \int_0^1 \omega_\alpha(gT_{s,y,s}) y_s^{-1} dy_s + \int_0^1 \omega_\nu(gT_{s,y,s}) y_s^{-1} dx_s
\]

does not depend on the choice of the path \( \gamma \). Moreover

\[
\int_{g\gamma[0,1]} \omega = \int_0^1 \omega(gT_s) (T_s^{-1} T_{s+ds}).
\]

**Definition 6.4.3** 1) Let us introduce the resolvent kernel, for any \( q \in \mathbb{N}^* \) and any bounded measurable function \( \phi \) on \( \text{PSL}(2) \):

\[
\mathcal{U}^q \phi (g) := \int_0^\infty e^{-qt} \phi(g \theta_t) dt.
\]

2) For any \( \Gamma \)-invariant bounded Borelian function \( f \) on \( \text{PSL}(2) \) such that

\[
\int f d\lambda \Gamma = 0 \quad \text{and the derivatives } \mathcal{L}_\nu^2 f, \mathcal{L}_\nu^4 f \text{ exist and are bounded, let us set:}
\]

\[
\tilde{f} := -\mathcal{U}^1 \mathcal{L}_\nu f, \quad \text{and} \quad \omega^f := f \alpha^* + \tilde{f} \nu^*.
\]

**Lemma 6.4.4** ([7]) For \( f \) as in Definition 6.4.3 above, the lifted 1-form \( \omega^f \) is closed, with bounded coefficients \( f, \tilde{f} \). Moreover, the derivatives \( \mathcal{L}_\nu \tilde{f} = -\mathcal{U}^2 \mathcal{L}_\nu^4 f \) and \( \mathcal{L}_\alpha \tilde{f} = \mathcal{L}_\nu f + \tilde{f} \nu^* \) exist and are bounded on \( \text{PSL}(2) \).

**Proof.** The commutation relation (10) implies:

\[
\mathcal{L}_\nu \mathcal{U}^q \phi (g) = \frac{d}{ds} \int_0^\infty e^{-qt} \phi(g \theta_t^+ \theta_s^-) dt = \frac{d}{ds} \int_0^\infty e^{-qt} \phi(g \theta_t \theta_s^-) dt = \mathcal{U}^{q+1} \mathcal{L}_\nu \phi (g),
\]

provided that \( \phi \) has a bounded \( \mathcal{L}_\nu \)-derivative. On the other hand for bounded \( \phi \) we have:

\[
\mathcal{L}_\alpha \mathcal{U}^q \phi (g) = \int_0^\infty e^{-qt} \frac{d}{dt} \phi(g \theta_t) dt = q \mathcal{U}^q \phi (g) - \phi (g).
\]

Applying this to \( \phi = -\mathcal{L}_\nu f \), we obtain the existence and boundedness of \( \mathcal{L}_\nu \tilde{f} \). This yields the result, according to Definition 6.4.1. \( \square \)
6.5 Contour deformation

In this section we use the closed form $\omega^f$ exhibited in the preceding section, to change the integration path in $\frac{1}{\sqrt{t}} \int_0^t f(\beta \theta_s) \, ds$: we substitute the diffusion path $\xi[0,2t] := \{ \xi_s \mid 0 \leq s \leq 2t \}$ for the geodesic path $g[0,t] := \{ g_{\theta_s} \mid 0 \leq s \leq t \}$. In this contour deformation two residual terms appear, which will be asymptotically negligible. Recall that the diffusion $(\xi_s)$ was introduced in Definition 6.2.1, and has stationary law $\mu = \text{covol}(\Gamma)^{-1} \lambda$ (recall Proposition 6.2.2).

Precisely, the aim of this section is to establish the following.

Theorem 6.5.1 ([5],[6]) Let $f$ be any $\Gamma$-invariant bounded Borelian function on $\text{PSL}(2)$, such that $\int f \, d\lambda = 0$ and its derivatives $\mathcal{L}_\nu f, \mathcal{L}_\nu^2 f$ are bounded. Consider the associated 1-form $\omega^f$ as in Definition 6.4.3. Then for any real $a$ we have:

$$\lim_{t \to \infty} \left\{ \int \exp \left( \frac{a \cdot \sqrt{t}}{\sqrt{t}} \int_0^t f(\theta_s) \, ds \right) \mu(dg) - \mathbb{E} \left[ \exp \left( \frac{a \cdot \sqrt{t}}{\sqrt{t}} \int_{\xi[0,t]} \omega^f \right) \right] \right\} = 0.$$

Indications for the proof. Using (26), we have to consider the law of $\int_g^{g_{\theta_s/2}} \omega^f$. Using Proposition 6.4.2, Lemma 6.4.4 and $T_{\tilde{\xi}, \tilde{\gamma}_t} = Z_t^*$ allows the change of contour:

$$\int_g^{g_{\theta_s/2}} \omega^f = \int_g^{g_{T_{0, \tilde{\gamma}_t}}} \omega^f - \int_g^{g_{0, \tilde{\gamma}_t}} \omega^f - \int_g^{g_{T_{0, \tilde{\gamma}_t}}} \omega^f,$$

in which $(\xi_t, \gamma_t)$ is given by Formula (25). The term $\int_g^{g_{T_{0, \tilde{\gamma}_t}}} \omega^f$ can be neglected since it is controlled by

$$\frac{|\tilde{\gamma}_t|}{\tilde{\gamma}_t} = \int_0^t e^{w_i - w_i(1-s)/2} \, dW_s \quad \text{l律} \int_0^t e^{-w_i - s/2} \, dW_s \to x_\infty := \int_0^\infty e^{-w_i - s/2} \, dW_s \in \mathbb{R},$$

by Formula (25) and Corollary 1.1.4(4). Then, under the law $\mu \otimes \mathbb{P}$ we have:

$$\frac{1}{\sqrt{t}} \int_g^{g_{T_{0, \tilde{\gamma}_t}}} \omega^f = \frac{1}{\sqrt{t}} \int_0^{w_i} f(\theta_s) \, ds \quad \text{l律} \quad \frac{1}{\sqrt{t}} \int_0^{w_i \sqrt{t}} f(\theta_s) \, ds \to \int f \, d\mu = 0$$

by ergodicity (recall Theorems 4.4.1 and 4.3.3). We are thus left with $\int_g^{g_{T_{0, \tilde{\gamma}_t}}} \omega^f$.

Then Proposition 6.2.2 and Lemma 6.1.1 allow the time reversal $Z_t^\ast := Z_t^{-1}$, so that using the right invariance of $\mu$ under $Z_t$, under the law $\mu \otimes \mathbb{P}$ we have:

$${\int_g}^{g_{Z_t^\ast}} \omega^f = {\int_{g_{\xi_t}}}^{g_{\xi_0}} \omega^f = {\int_{g_{\xi_t}}}^{g_{\xi_t}} \omega^f = -{\int_{g_{\xi[0,t]}}}^{g_{\xi_t}} \omega^f \, .$$
6.6 Divergence of the lifted 1-form $\omega^f$

Fix a bounded function $f$ on $\text{PSL}(2)$, with bounded Hölderian derivatives of order 1 and 2, and such that $\int f \, d\lambda^\Gamma = 0$. The following technical result can be derived from the commutation relations in $\text{PSL}(2)$, as presented in Section 2.3.

**Lemma 6.6.1** The potentials $\mathcal{U}^1 f$ and $\mathcal{U}^2 f$ are bounded and Hölderian on $\text{PSL}(2)$.

The following general statement is easily derived from Itô’s Formula.

**Proposition 6.6.2** Consider a closed longitudinal 1-form $\Omega$ having continuous $\mathcal{L}_\alpha, \mathcal{L}_\nu$-derivatives. Then for any $t \geq 0$ we almost surely have:

$$\int_{[0,t]} \Omega = M^\Omega_t + \frac{1}{2} \int_0^t \text{div} \Omega (\xi_s) \, ds,$$

with $\text{div} \Omega := \mathcal{L}_\alpha(\Omega(\alpha)) + \mathcal{L}_\nu(\Omega(\nu)) - \Omega(\alpha)$

and a continuous martingale $M^\Omega_t$ having quadratic variation:

$$\langle M^\Omega \rangle_t = \int_0^t (\Omega(\alpha)^2 + \Omega(\nu)^2)(\xi_s) \, ds.$$  

Lemmas 6.4.4 and 6.6.1 allow to apply Proposition 6.6.2 to $\omega^f$ of Formula (28). Since $\mathcal{L}_\nu f = -\mathcal{U}^2 \mathcal{L}_\nu f$ by Formula (28) and Lemma 6.4.4, we easily obtain

$$\int_{[0,t]} \omega^f = M^f_t + \frac{1}{2} \int_0^t K f(\xi_s) \, ds,$$  \hspace{1cm} (29)

where

$$K f := \mathcal{L}_\alpha f - \mathcal{U}^2 \mathcal{L}_\nu f - f$$  \hspace{1cm} (30)

and $M^f_t$ is a continuous martingale having quadratic variation given by:

$$\langle M^f \rangle_t = \int_0^t (f^2 + \tilde{f}^2)(\xi_s) \, ds.$$  \hspace{1cm} (31)

**Lemma 6.6.3** The function $K f$ of Formula (30) is $\Gamma$-invariant and bounded: $K f \in L^\infty(\mathcal{M}, \mu)$. Moreover is is Hölderian.

**Proof.** We already observed in Lemma 6.4.4 that $\tilde{f}$ and $\mathcal{L}_\nu \tilde{f}$ are bounded. It remains to show that $K f$ is Hölderian. By Formula (29), this amounts to verify that $\mathcal{U}^2 \mathcal{L}_\nu f$ is Hölderian, which is provided by Lemma 6.6.1 (which can be applied directly to the Hölderian function $\mathcal{L}_\nu f$). \quad $\square$

**Proposition 6.6.4** We have $\int K f \, d\lambda^\Gamma = 0$ ($K$ was defined in Formula (30)).
6.7 Sinai’s Central Limit Theorem

The aim of this section is to complete the description of the proof of the following theorem, essentially due to Y. Sinai [9], who treated the cocompact case. See also [8] in the cocompact case too, but in non-constant curvature.

**Remark 6.7.1** The cofinite case appeared in [7], and the infinite case in [5]. By a totally different (martingale) method, [3] establishes the Sinai CLT below without the $C^2$ constraint on the function $f$. See also [1] and [2] for a related problem and the use of a coding method.

**Remark 6.7.2** The stochastic calculus method presented in these notes allowed as well to obtain singular limit theorems on the geodesic flow, namely the asymptotic (stable) law (under the Liouville measure, or the Patterson-Sullivan one if the volume is infinite) of line integrals $t^{-\rho} \int_{0}^{t} \omega$, where $\omega$ is a closed 1-form which does not need to be a cusp form as in the Sinai theorem below: it can have non-vanishing residues in the cusps; so that the exponent $\rho$ can be larger than $1/2$. See in particular [4] and [5].

**Theorem 6.7.3** ([5],[6],[7]) Let $f$ be a $\Gamma$-invariant, bounded real function on $\text{PSL}(2)$, of class $C^2$ with bounded and Hölderian derivatives, such that $\int f \, d\lambda^\Gamma = 0$. Then for all $a \in \mathbb{R}$ we have:

$$\lim_{t \to \infty} \int \exp \left( \frac{a\sqrt{-1}}{\sqrt{t}} \int_{0}^{t} f(g \theta_s) \, ds \right) \lambda^\Gamma(dg) = \text{covol}(\Gamma) \times \exp \left( - \frac{a^2}{2} V(f) \right),$$

(32)

where ($Kf$ being given by Formula (30))

$$V(f) := \frac{2}{\text{covol}(\Gamma)} \int \left[ (f + \frac{1}{2} \mathscr{L}_a V K f)^2 + (\tilde{f} + \frac{1}{2} \mathscr{L}_\nu V K f)^2 \right] d\lambda^\Gamma$$

(33)

vanishes if and only if $f$ equals $\mathscr{L}_a h$, for some $h \in L^2(\text{PSL}(2), \lambda^\Gamma)$.

The kernel $V$ which appears in Theorem 6.7.3, in the definition of the variance $\mathcal{V}$, is the potential kernel of the semi-group $(P_t)$:

$$V := \int_{0}^{\infty} P_t \, dt,$$

(34)

which, as we saw in Theorem 6.3.4, makes sense on centred $\Gamma$-invariant functions which are regular enough, such as $Kf$ of Section 6.6 (by Lemma 6.6.3 and Proposition 6.6.4).

To give sense to the expression (33) in Theorem 6.7.3, on the one hand we need to justify the existence of $L^2$-derivatives $\mathscr{L}_a V K f$. On the other hand, to prove Theorem 6.7.3 we want to use Proposition 6.6.2 and its consequence Formula (29) (in Section 6.6), but with $[\omega^f + \frac{1}{2} d(V K f)]$ instead of $\omega^f$, in order to get rid of the problematic part $\int_{0}^{t} Kf(\xi_s) \, ds$ in Formula (29). Hence we must express this term in the form of:
\[
\int_0^t Kf(\xi_s) \, ds = VKf(\xi_t) - VKf(\xi_0) + \text{a martingale term}.
\]

To do this, the idea is to show the following technical result, which will apply to \( F = Kf \), thanks to Lemma 6.6.3 and Proposition 6.6.4.

**Proposition 6.7.4** ([5],[6]) Any centred and bounded rotationally \( r \)-Hölderian function \( F \) on \( \mathcal{M} \) admits a bounded potential \( VF \) with \( L^2 \)-derivatives \( \mathcal{L}VF \) such that, setting \( V_b := \int_0^b p_t \, dt \):

1. \( \|VF - V_b F \, dt\|_{L^2(\mathcal{M})} \leq (\|F\|_{\infty} + \|F\|_{r}) \times O(e^{-\delta b}) \);
2. \( \|\mathcal{L}VF - \mathcal{L}VbF \, dt\|_{L^2(\mathcal{M})} \leq (\|F\|_{\infty} + \|F\|_{r}) \times O(e^{-\delta b/2}). \)

Lemma 6.6.3 and Proposition 6.6.4 allow to apply Proposition 6.7.4 to the function \( Kf = \text{div} \omega^f \) of Formula (29), which gives full sense to the expression (33) in Theorem 6.7.3. This allows also to apply Itô’s Formula to \( VKf \), or alternatively, to apply Formula (29) of Section 6.6 with \( \omega^f + \frac{1}{2} d(VKf) \) instead of \( \omega^f \). By means of another approximation procedure (of \( Kf \) by a regularized sequence \( (F_n) \)), and using that \( \frac{1}{2} \text{div}(dV_b) = \frac{1}{2} \partial V_b = P_b - P_0 \) by the Fokker-Planck equation of Proposition 1.4.5 and Theorem 3.3.2, this yields the following.

**Proposition 6.7.5** For any \( t \geq 0 \) we have:

\[
\int_{\xi [0,t]} \omega^f = \frac{1}{2} VKf(\xi_0) - \frac{1}{2} VKf(\xi_t) + \mathcal{M}_t^f,
\]

where \( (\mathcal{M}_t^f) \) is a continuous martingale having quadratic variation:

\[
\langle \mathcal{M}_t^f \rangle = \int_0^t \left[ (f + \frac{1}{2} \mathcal{L}_aVKf)^2 + (\tilde{f} + \frac{1}{2} \mathcal{L}_bVKf)^2 \right] (\xi_s) \, ds.
\]

So far, it remains to prove a Central Limit Theorem for the martingale \( (\mathcal{M}_t^f) \). Recall from Definition 6.2.1 and Proposition 6.2.2 that the diffusion \( (\tilde{\xi}_s) \) is stationary with semi-group \( (P_t) \) and law \( \mu \) (proportional to \( \lambda^T \)) on \( \mathcal{M} \).

**Lemma 6.7.6** Let \( M_t = \sum_{j=1}^2 \int_0^t \psi_j(\xi_s) \, dW^j_s \) be a continuous real martingale, for \( \psi_1, \psi_2 \in L^2(\mathcal{M}, \mu) \). Then the law of \( M_t/\sqrt{t} \) converges, as \( t \to \infty \), to the centred Gaussian law with variance \( \int_{\mathcal{M}} (\psi_1^2 + \psi_2^2) \, d\mu \).

**Lemma 6.7.7** As \( t \to \infty \), the law of \( t^{-1/2} \int_{\xi [0,t]} \omega^f \) converges towards the centred Gaussian law with variance:

\[
\mathcal{V}_0(f) := \int \left[ (f + \frac{1}{2} \mathcal{L}_aVKf)^2 + (\tilde{f} + \frac{1}{2} \mathcal{L}_bVKf)^2 \right] \, d\mu.
\]

Finally, Theorem 6.5.1 and the previous lemma 6.7.7 imply at once that:

\[
\lim_{t \to \infty} \exp \left( a \frac{\sqrt{t} - 1}{\sqrt{t}} \int_0^{t/2} f(g \theta_s) \, ds \right) d\mu(g) = \exp \left( -a^2 \mathcal{V}_0(f)/2 \right).
\]
Recalling that (in Definition 6.2.1) \( \mu = \text{covol}(\Gamma)^{-1}\lambda^\Gamma \), we deduce immediately the first claim of Sinai’s Central Limit Theorem 6.7.3.

The last assertion of Theorem 6.7.3 is easy: by Formula (33) \( \mathcal{V}(f) = 0 \) implies \( f = \mathcal{L}_\alpha(-\frac{1}{2}VKf) \), and \( VKf \in L^2(\lambda^\Gamma) \) by Proposition 6.7.4. And reciprocally, if \( f = \mathcal{L}_\alpha h \), then by the invariance of Proposition (4.2.1, iv) we have

\[
\int_0^t f(g_\theta s) \, ds = t^{-1/2} \left[ h(g_\theta t) - h(g) \right] \to 0
\]

in \( \mu \)-probability, which forces \( \mathcal{V}(f) = 0 \).

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