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Numerical methods for hyperbolic systems

Correction 2 of exercise sheet: advection equation and finite volumes schemes

Exercice 1 We propose to solve the advection equation on the domain $[0, L]$

$$\begin{cases} \dfrac{\partial u}{\partial t} u + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in [0, L], \quad t > 0, \\ u(t = 0, x) = u^0(x), \quad \forall x \in [0, L], \\ u(t, x = 0) = u(t, x = L), \end{cases} \tag{1}$$

with $u^0(x) \in C^1([0, L])$.

We consider the upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a + |a|}{2\Delta x}(u_j^n - u_{j-1}^n) + \frac{a - |a|}{2\Delta x}(u_{j+1}^n - u_j^n) = 0, \tag{2}$$

and the centered scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) = 0, \tag{3}$$

with Δt the time step, Δx the step mesh and u_j^n the approximation to $u(n\Delta t, j\Delta x)$ where $n \in \mathbb{N}$, $j \in \mathbb{N}$.

1. The advection equation satisfies the maximum principle

$$\min_{x\in[0,L]} u(t = 0, x) \le u(t, x) \le \max_{x\in[0,L]} u(t = 0, x).$$

Prove that the upwind scheme satisfies the discrete maximum principle under a CFL condition

$$\min_{j\in[0,N_x]} u_j^n \le u_j^{n+1} \le \max_{j\in[0,N_x]} u_j^n,$$

with $N_x = \frac{L}{\Delta x}$ the number of cells.

To prove the discrete maximum principle we write the scheme as a convex combination

$$u_j^{n+1} = u_j^n - \Delta t \left(\frac{a+|a|}{2\Delta x} - \frac{a-|a|}{2\Delta x} \right) u_j^n + \Delta t \frac{a+|a|}{2\Delta x} u_{j-1}^n - \Delta t \frac{a-|a|}{2\Delta x} u_{j+1}^n.$$

We obtain

$$u_j^{n+1} = \left(1 - \Delta t \left(\frac{|a|}{\Delta x} \right) \right) u_j^n + \Delta t \frac{a+|a|}{2\Delta x} u_{j-1}^n + \Delta t \frac{|a|-a}{2\Delta x} u_{j+1}^n.$$

If $\left(1 - \Delta t \left(\frac{|a|}{\Delta x} \right) \right) \geq 0$ we remark that all coefficients are positive and the sum of the coefficients is equal to 1. Consequently $u_j^{n+1} = C(u_j^n, u_{j-1}^n, u_{j+1}^n)$ with C a convex combination.

Using the properties of convex combination we obtain the discrete maximum principle on the following CFL condition

$$\frac{|a|\Delta t}{\Delta x} \leq 1.$$

2. Prove that the upwind scheme is stable for the L^2 norm using the Neumann analysis.

L^2 stability with continuous Fourier transform

The Neumann analysis is a method to prove the L^2 stability based on Fourier transformation. We begin by introduce the piecewise constant functions $u^n(x)$ defined by $u^n(x) = u_j^n$ for $x \in \left[x_{j-\frac{1}{2}, x+\frac{1}{2}}\right]$. This function belongs to the space $L^2(0,1)$. Consequently we can write this function in the Fourier space. We obtain

$$u^n(x) = \sum_k \hat{u}^n(k) e^{2i\pi k \Delta x}$$

with the Fourier coefficient $\hat{u}^n(k)$. The scheme is stable for L^2 norm if

$$\int_0^1 |u^{n+1}(x)|^2 \leq \int_0^1 |u^n(x)|^2$$

Using the Plancherel equality we show that the previous condition is equivalent to

$$\sum_k |\hat{u}^{n+1}(k)|^2 \leq \sum_k |\hat{u}^n(k)|^2$$

This is this equality that we propose to prove. If we define $v^n(x) = u^n(x + \Delta x)$ therefore $\hat{v}^n(k) = \hat{u}(k) e^{2i\pi k \Delta x}$. The scheme written in the Fourier space is

$$\hat{u}^{n+1}(k) = \hat{u}^n(k) - \Delta t \left(\frac{a + |a|}{2\Delta x} - \frac{a - |a|}{2\Delta x} \right) \hat{u}^n(k) + \Delta t \frac{a + |a|}{2\Delta x} e^{-i\zeta} \hat{u}^n(k) - \Delta t \frac{a - |a|}{2\Delta x} e^{i\zeta} \hat{u}^n(k),$$

with $\zeta = 2\pi k \Delta x$. After some simplifications we obtain

$$\hat{u}^{n+1}(k) = \left(1 - \Delta t \left(\frac{a + |a|}{2\Delta x}(1 - e^{-i\zeta}) + \frac{a - |a|}{2\Delta x}(e^{i\zeta} - 1) \right) \right) \hat{u}^n(k),$$

$$\hat{u}^{n+1}(k) = \left(1 - \Delta t \left(\lambda (e^{i\zeta} - e^{-i\zeta}) + |\lambda|(2 - e^{i\zeta} - e^{-i\zeta}) \right) \right) \hat{u}^n(k),$$

with $\lambda = \frac{a}{2\Delta x}$. Now we use $e^{\pm i\zeta} = \cos(\zeta) \pm i \sin(\zeta)$. We obtain

$$\hat{u}^{n+1}(k) = A(k) \hat{u}^n(k),$$

with $A(k) = 1 - 2\Delta t \left(\lambda i \sin(\zeta) + |\lambda|(1 - \cos)(\zeta) \right)$.

To prove the stability we must prove $|\hat{u}^{n+1}(k)|^2 = |A(k)|^2 |\hat{u}^n(k)|^2$ with $|A(k)| \leq 1$.
By definition of the complex module we have

$$|A(k)|^2 = (1 - 2\Delta t|\lambda| + 2\Delta t|\lambda| \cos(\zeta))^2 + 4\Delta t^2 \lambda^2 \sin^2(\zeta).$$

Expanding the previous expression we obtain

$$|A(k)|^2 = (1 - 2\Delta t|\lambda|)^2 + (1 - 2\Delta t|\lambda|)(2\Delta t|\lambda| \cos(\zeta)) + 4\Delta t^2 \lambda^2.$$

Expanding the first term we can show that $|A(k)|^2 \leq 1$ if

$$(1 - 2\Delta t|\lambda|)(4\Delta t|\lambda| - 2|\lambda|\Delta t \cos(\zeta)) \geq 0.$$

To finish we obtain that this expression is satisfied if $(1 - 2\Delta t|\lambda|)$, therefore $\frac{|a|\Delta t}{\Delta x} \leq 1$.

L^2 stability with discrete Fourier transform

This method of proof the discrete Fourier transform and the properties of circulant matrix.

To begin we define the symmetric matrix P with $P_{jk} = \frac{1}{\sqrt{n}} e^{\frac{2i\pi jk}{n}}$.
The discrete Fourier transform is given by $\hat{\mathbf{U}} = P^*\mathbf{U}$ for the vector $\mathbf{U}$ with $P^* = \overline{P}^t = P^{-1}$. The inverse Fourier transform is given by $\mathbf{U} = P\hat{\mathbf{U}}$.

The circulant matrix are defined by

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & & c_{n-3} \\ \vdots & & & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

If a matrix C is a circulant matrix we can diagonalize the matrix with the decomposition $C = P\Lambda_C P^*$ and the diagonal matrix Λ_C defined by

$$\lambda_C^k = \sum_{j=0}^{n-1} c_j e^{\frac{2i\pi jk}{n}}.$$

This result shows that when we write the scheme on the Fourier space we obtain the scheme under the diagonal form, more easily to study. The upwind scheme is defined by

$$\mathbf{U}^{n+1} = C\mathbf{U}^n,$$

with $c_0 = 1 - |\alpha|$, $c_{n-1} = \frac{1}{2}(\alpha + |\alpha|)$, $c_1 = \frac{1}{2}(|\alpha| - \alpha)$ and $\alpha = \frac{a\Delta t}{\Delta x}$. Apply the Fourier transform we obtain

$$P^*\mathbf{U}^{n+1} = P^*CPP^*\mathbf{U}^n,$$
$$\hat{\mathbf{U}}^{n+1} = \Lambda_C\hat{\mathbf{U}}^n.$$

Consequently we obtain $||\hat{\mathbf{U}}^{n+1}|| \leq ||\Lambda_C||||\hat{\mathbf{U}}^n||$ and the scheme is stable if $\max_k |\lambda_C^k| \leq 1$.
In this case the eigenvalues of Λ_C are defined by

$$\lambda_C^k = 1 - |\alpha| + \frac{1}{2}(|\alpha| - \alpha)e^{\frac{2i\pi k}{n}} + \frac{1}{2}(|\alpha| + \alpha)e^{\frac{2i\pi k(n-1)}{n}},$$
$$\lambda_C^k = 1 - |\alpha| + \frac{1}{2}(|\alpha| - \alpha)e^{\frac{2i\pi k}{n}} + \frac{1}{2}(|\alpha| + \alpha)e^{\frac{-2i\pi k}{n}}.$$

We use the fact that $e^{2i\pi k} = 1$. Using $e^{\frac{2i\pi k}{n}} = \cos(\frac{2\pi k}{n}) \pm i\sin(\frac{2i\pi k}{n})$ we obtain $\lambda_C^k = A(k)$. To finish we use the end of the previous proof.

3. Give the consistency error associated to the upwind scheme.

To study the consistency error associated to a scheme we plug the exact solution in the scheme. We define $u(x_j, t_n)$ the exact solution. Firstly we can prove that the upwind (2) scheme can be rewritten on the following form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \frac{|a|}{2\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0.$$

Now we plug the exact solution in the previous form of the scheme

$$\begin{aligned} &\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} + \frac{a}{2\Delta x}(u(x_{j+1}, t^n) - u(x_{j-1}, t^n)) \\ &+ \frac{|a|}{2\Delta x}\left(u(x_{j+1}, t^n) - 2u(x_j, t^n) + u(x_{j-1}, t^n)\right) = 0. \end{aligned} \tag{4}$$

Using a Taylor expansion we prove that

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t \partial_t u(x_j, t^n) + \frac{\Delta t^2}{2}\partial_{tt} u(x_j, t^n) + O(\Delta t^3), \tag{5}$$

$$u(x_{j+1}, t^n) = u(x_j, t^n) + \Delta x \partial_x u(x_j, t^n) + \frac{\Delta x^2}{2}\partial_{xx} u(x_j, t^n) + O(\Delta x^3), \tag{6}$$

and

$$u(x_{j-1}, t^n) = u(x_j, t^n) - \Delta x \partial_x u(x_j, t^n) + \frac{\Delta x^2}{2}\partial_{xx} u(x_j, t^n) + O(\Delta x^3). \tag{7}$$

Subtracting (6) and (7) we obtain

$$\frac{a}{2\Delta x}(u(x_{j+1}, t^n) - u(x_{j-1}, t^n)) = a\partial_x u(x_j, t^n) + O(\Delta x^2).$$

Adding (6) and (7) we obtain

$$\frac{|a|}{2\Delta x}(u(x_{j+1}, t^n) - 2u(x_j, t^n) + u(x_{j-1}, t^n)) = |a|\Delta x \partial_{xx} u(x_j, t^n) + O(\Delta x^2).$$

Finally (5) yields

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = \partial_t u(x_j, t^n) + O(\Delta t).$$

Plugging these relations in (4) we obtain

$$\partial_t u(x_j, t^n) + O(\Delta t) + a\partial_x u(x_j, t^n) + O(\Delta x^2) - \frac{|a|}{2}\Delta x \partial_{xx} u(x_j, t^n). \tag{8}$$

Since $u(x_j, t^n)$ is a solution then

$$(8) = O(\Delta t) + O(\Delta x^2) - \frac{|a|}{2}\Delta x \partial_{xx} u(x_j, t^n) = O(\Delta t) + O(\Delta x).$$

4. Discuss the discrete maximum principle for the centered scheme.

To prove the discrete maximum principle we write the scheme as a convex combination

$$u_j^{n+1} = u_j^n - \Delta t \frac{a}{2\Delta x} u_{j+1}^n + \Delta t \frac{a}{2\Delta x} u_{j-1}^n.$$

The coefficient associated to u_{j+1}^n is negative thus $u_j^{n+1} = C(u_j^n, u_{j-1}^n, u_{j+1}^n)$ is not a convex combination. Therefore the maximum principle is not preserved.

For example we take as initial data $u_j^0 = 1$ if $j \leq j_0$ and $u_j^0 = 2$ if $j > j_0$. For $n = 1$ and $j = j_0$ $u_{j_0}^1 = 1 - \frac{a\Delta t}{2\Delta x} \leq 1$. The maximum principle is not preserved.

5. Study the L^2 stability and the consistency error associated to the centered scheme.

L^2 stability with continuous Fourier transform

We begin by the L^2 stability. We introduce the Fourier coefficient $\hat{u}^n(k)$. The scheme written in the Fourier space is

$$\hat{u}^{n+1}(k) = \left(1 - \Delta t \left(\lambda(e^{i\zeta} - e^{-i\zeta})\right)\right) \hat{u}^n(k),$$

with $\zeta = 2\pi k \Delta x$ and $\lambda = \frac{a}{2\Delta x}$. Now we use $e^{\pm i\zeta} = \cos(\zeta) \pm i \sin(\zeta)$. We obtain

$$\hat{u}^{n+1}(k) = A(k)\hat{u}^n(k).$$

with $A(k) = 1 - 2\Delta t\left(2\lambda i \sin(\zeta)\right))$.

To prove the stability we must prove that $|\hat{u}^{n+1}(k)|^2 = |A(k)|^2 |\hat{u}^n(k)|^2$ with $|A(k)| \leq 1$.
By definition of the complex module we have

$$|A(k)|^2 = 1 + 4\Delta t^2 \lambda^2 \sin^2(\zeta).$$

Consequently $|A(k)|^2 \geq 1$. The scheme is not stable.

The proof using the discrete transform is detailed for this scheme in the note of the lecture.

Now we study the consistency of the scheme
We define $u(x_j, t_n)$ the exact solution. We plug the exact solution in the previous form of the scheme

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} + \frac{a}{2\Delta x}\left(u(x_{j+1}, t^n) - u(x_{j-1}, t^n)\right) \tag{9}$$

Subtracting (6) and (7) we obtain

$$\frac{a}{2\Delta x}(u(x_{j+1}, t^n) - u(x_{j-1}, t^n)) = a\partial_x u(x_j, t^n) + O(\Delta x^2).$$

Using (5) we obtain

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = \partial_t u(x_j, t^n) + O(\Delta t).$$

Plugging these relations in (9) we obtain

$$\partial_t u(x_j, t^n) + O(\Delta t) + a\partial_x u(x_j, t^n) + O(\Delta x^2). \tag{10}$$

Since $u(x_j, t^n)$ is solution then

$$(10) = O(\Delta t) + O(\Delta x^2).$$