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Numerical methods for hyperbolic systems

Exercise sheet 2: Galerkin discontinuous for advection equation

Exercice 1 We consider the Lax-Wendroff scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{a}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)-\frac{a^2\Delta t}{2\Delta x^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n)=0 \tag{1}$$

with Δt the time step, Δx the step mesh and u_j^n the approximation to $u(n\Delta t, j\Delta x)$ where $n\in\mathbb{N}$, $j\in\mathbb{N}$.

1. Study the L^2 stability.

2. Prove that the consistency error associated to the Lax-Wendroff scheme is $O(\Delta x^2+\Delta t^2)$ (use the fact that $\partial_{tt}u-a^2\partial_{xx}u=0$).

Exercice 2 In this exercise we propose to study the high order DG approximation for the advection equation with $a>0$. Ω is the domain. The mesh Ω_h is defined by $N+1$ points x_i and n cells $K_i=\left[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right]$. We call a generic cell K. Finally the test functions are defined by $v\in V_h=\{v/v|_K\in\mathbb{P}^p(K)\}$ with $\mathbb{P}^p(K)$ a space of p-order polynomials defined on K.

1. Write the weak formulation of the equation for an element K.

We define $\left\{\phi_l^i\right\}_{l=0}^p$ a basis of V_h. The numerical solution in the element K_i is noted $u_h^i=u_{h_{|K_i}}=\sum_{l=0}^p u_l^i\phi_l^i$.

The DG-centered scheme is given by

$$\sum_{l=0}^p \partial_t\int_{K_i}u_h^i\phi_m^i - a\sum_{l=0}^p\int_{K_i}u_h^i\partial_x\phi_m^i + a\sum_{l=0}^p\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=0,\quad 0\le m\le k, \tag{2}$$

with

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=\frac{1}{2}(u_h^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}})+u_h^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}))$$
$$-\frac{1}{2}(u_h^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})+u_h^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})),$$

and in matrix form by

$$\sum_{l=0}^{p} \partial_t u_l^i \int_{K_i} \phi_l^i \phi_m^i - a \sum_{l=0}^{p} u_l^i \int_{K_i} \phi_l^i \partial_x \phi_m^i + a \sum_{l=0}^{p} \left[u\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad 0 \le m \le k, \tag{3}$$

with

$$\left[u\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}(u_l^i \phi_l^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}) + u_l^{i+1}\phi_l^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}))$$
$$-\frac{1}{2}(u_l^{i-1}\phi_l^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}) + u_l^i \phi_l^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}))$$

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2. Assuming that boundary conditions are periodic, prove that $\frac{1}{2}\frac{d}{dt}\int_K u_h^2 dx = 0$.

3. Derive the classical finite volumes centered scheme starting the DG scheme (**??**).

4. Design a flux $\left[u\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$ which is the DG extension of the upwind finite volume scheme.