TU München
Zentrum Mathematik
Lehrstuhl M16
E. Sonnendrücker
E. Franck

Numerical methods for hyperbolic systems

Part 1 of correction exercise sheet 2: Galerkin discontinuous for advection equation

Exercice 1 We consider the Lax-Wendroff scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \frac{a^2 \Delta t}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0 \quad (1)$$

with Δt the time step, Δx the step mesh and u_j^n the approximation to $u(n\Delta t, j\Delta x)$ where $n \in \mathbb{N}$, $j \in \mathbb{N}$.

1. Study the L^2 stability.

L^2 stability with continuous Fourier transform

The Neumann analysis is based on Fourier transformation. We introduce the Fourier coefficient $\hat{u}^n(k)$. The scheme written in the Fourier space is

$$\hat{u}^{n+1}(k) = \hat{u}^n(k) - \frac{a\Delta t}{2\Delta x}\left(e^{i\zeta}\hat{u}^n(k) - e^{-i\zeta}\hat{u}^n(k)\right) + \frac{a^2\Delta t}{2\Delta x^2}\left(2\hat{u}^n(k) - e^{i\zeta}\hat{u}^n(k) - e^{-i\zeta}\hat{u}^n(k)\right)$$

with $\zeta = 2\pi k\Delta x$. After simplification we obtain

$$\hat{u}^{n+1}(k) = \left(1 - \left(\frac{\lambda}{2}(e^{i\zeta} - e^{-i\zeta}) + \frac{\lambda^2}{2}(2 - e^{i\zeta} - e^{-i\zeta})\right)\right)\hat{u}^n(k),$$

with $= \frac{a\Delta t}{\Delta x}$. Now we use $e^{\pm i\zeta} = \cos(\zeta) \pm i\sin(\zeta)$. We obtain

$$\hat{u}^{n+1}(k) = A(k)\hat{u}^n(k),$$

with $A(k) = 1 - \left(\lambda i\sin(\zeta) + \lambda^2(1-\cos)(\zeta))\right)$.

To prove the stability we must prove $|\hat{u}^{n+1}(k)|^2 = |A(k)|^2|\hat{u}^n(k)|^2$ with $|A(k)| \leq 1$.
By definition of the complex module we have

$$|A(k)|^2 = (1 - \lambda^2(1-\cos(\zeta)))^2 + \lambda^2\sin^2(\zeta).$$

Expanding the previous expression we obtain

$$|A(k)|^2 = 1 - 2\lambda^2(1-\cos(\zeta)) + \lambda^4(1-\cos(\zeta))^2 + \lambda^2(1-\cos^2(\zeta)).$$

Now we simplify to obtain

$$|A(k)|^2 = 1 - \lambda^2(1 - 2\cos(\zeta) + 2\cos^2(\zeta)) + \lambda^4(1-\cos(\zeta))^2.$$

$$|A(k)|^2 = 1 - \lambda^2(1-\cos(\zeta))^2 + \lambda^4(1-\cos(\zeta))^2 = 1 + (\lambda^4 - \lambda^2)(1-\cos(\zeta))^2.$$

If $\lambda \leq 1$ the term $(\lambda^4 - \lambda^2) \leq 0$ and $|A(k)| \leq 1$. Consequently the scheme is stable

$$\frac{a\Delta t}{\Delta x} \leq 1.$$

L^2 stability with discrete Fourier transform

The Lax wendroff scheme is defined by

$$\mathbf{U}^{n+1} = C\mathbf{U}^n,$$

with C a circulant matrix defined by the coefficients $c_0 = 1-\alpha^2$, $c_{n-1} = \frac{1}{2}(\alpha^2+\alpha)$, $c_1 = \frac{1}{2}(\alpha^2-\alpha)$ and $\alpha = \frac{a\Delta t}{\Delta x}$. Apply the Fourier transform we obtain

$$P^*\mathbf{U}^{n+1} = P^*CPP^*\mathbf{U}^n,$$
$$\hat{\mathbf{U}}^{n+1} = \Lambda_C\hat{\mathbf{U}}^n.$$

Consequently we obtain $||\hat{\mathbf{U}}^{n+1}|| \leq ||\Lambda_C||||\hat{\mathbf{U}}^n||$ and the scheme is stable if $\max_k |\lambda_C^k| \leq 1$. In this case the eigenvalues of Λ_C are defined by

$$\lambda_C^k = 1-\alpha^2 + \frac{1}{2}(\alpha^2-\alpha)e^{\frac{2i\pi k}{n}} + \frac{1}{2}(\alpha^2+\alpha)e^{\frac{2i\pi k(n-1)}{n}},$$
$$\lambda_C^k = 1-\alpha^2 + \frac{1}{2}(\alpha^2-\alpha)e^{\frac{2i\pi k}{n}} + \frac{1}{2}(\alpha^2+\alpha)e^{\frac{-2i\pi k}{n}}.$$

We use the fact that $e^{2i\pi k} = 1$. Using $e^{\frac{2i\pi k}{n}} = \cos(\frac{2\pi k}{n}) \pm i\sin(\frac{2i\pi k}{n})$ we obtain $\lambda_C^k = A(k)$. To finish we use the end of the previous proof.

2. Prove that the consistency error associated to the Lax-Wendroff scheme is $O(\Delta x^2 + \Delta t^2)$ (use the fact that $\partial_{tt}u - a^2\partial_{xx}u = 0$).

Before we study the consistency error, we study the exact solution. u is solution of $\partial_t u + a\partial_x u = 0$. Taking the derivative of the equation we prove u is solution to $\partial_{tt}u + a\partial_{t,x}u = 0$ and $\partial_{t,x}u + a\partial_{xx}u = 0$. Consequently u is solution of

$$\partial_{tt}u - a^2\partial_{xx}u = 0. \tag{2}$$

Now we define $u(x_j, t_n)$ the exact solution. Using the second order Taylor expansion in time and first order expansion in space (as previously), we obtain the following consistency error

$$E = \partial_t u(x_j, t_n) + \frac{\Delta t}{2}\partial_{tt}u(x_j, t_n) + O(\Delta t^2) + a\partial_x u(x_j, t_n)$$
$$-\Delta t\frac{a^2}{2}\partial_{xx}u(x_j, t_n) + O(\Delta x^2 + \Delta x^2\Delta t).$$

We obtain

$$E = \partial_t u(x_j, t_n) + \frac{\Delta t}{2}(\partial_{tt}u(x_j, t_n) - \Delta t\frac{a^2}{2}\partial_{xx}u(x_j, t_n)) + O(\Delta t^2) + a\partial_x u(x_j, t_n)$$
$$-\Delta t\frac{a^2}{2}\partial_{xx}u(x_j, t_n) + O(\Delta x^2 + \Delta x^2\Delta t)$$

Since $u(x_j, t_n)$ is a solution of $\partial_t u + c\partial_x u = 0$ and (2) we obtain $E = O(\Delta x^2 + \Delta t^2)$.