TU München
Zentrum Mathematik
Lehrstuhl M16
E. Sonnendrücker
E. Franck

Numerical methods for hyperbolic systems

Exercise sheet 3: Linear hyperbolic systems

Exercice 1

We consider the wave equation

$$\left\{\begin{array}{l}\partial_t p + \partial_x u = 0,\\ \partial_t u + \partial_x p = 0,\end{array}\right. \tag{1}$$

1. Diagonalize the system and show that the upwind scheme is given by

$$\left\{\begin{array}{l}\dfrac{v_j^{n+1}-v_j^n}{\Delta t} + \dfrac{v_j^n - v_{j-1}^n}{\Delta x} = 0,\\[2ex] \dfrac{w_j^{n+1}-w_j^n}{\Delta t} - \dfrac{w_{j+1}^n - w_j^n}{\Delta x} = 0,\end{array}\right. \tag{2}$$

with $v = p + u$ and $w = p - u$.

We sum and subtract the two equations of (1) to obtain

$$\left\{\begin{array}{l}\partial_t (p+u) + \partial_x (p+u) = 0,\\ \partial_t (p-u) - \partial_x (p-u) = 0,\end{array}\right. \tag{3}$$

This computation shows that the eigenvalues of (1) are 1 and -1. The eigenvectors are $(1, 1)$ and $(1, -1)$. Now we define $v = p + u$, $w = p - u$.

When we apply the upwind scheme to advection equation $\partial_t u + a\partial_x u = 0$ the flux is defined by

$$u_{j+\frac{1}{2}} = \left\{\begin{array}{ll} u_j & \text{if } a > 0,\\ u_{j+1} & \text{if } a < 0.\end{array}\right.$$

Consequently the upwind scheme for the system (3) is

$$\left\{\begin{array}{l}\dfrac{v_j^{n+1}-v_j^n}{\Delta t} + \dfrac{v_j^n - v_{j-1}^n}{\Delta x} = 0,\\[2ex] \dfrac{w_j^{n+1}-w_j^n}{\Delta t} - \dfrac{w_{j+1}^n - w_j^n}{\Delta x} = 0,\end{array}\right. \tag{4}$$

with $v = p + u$ and $w = p - u$.

2. Prove that the upwind scheme (2) for the initial system (1) can be write on the following form

$$\begin{cases} \dfrac{p_j^{n+1}-p_j^n}{\Delta t}+\dfrac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}-\dfrac{p_{j+1}^n-2p_j^n+p_{j-1}^n}{2\Delta x}=0, \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+\dfrac{p_{j+1}^n-p_{j-1}^n}{2\Delta x}-\dfrac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{2\Delta x}=0, \end{cases} \tag{5}$$

and use the scheme (5) to compute the consistency error.

Firstly we sum the equations and multiply by 0.5, secondly we subtract the equations and multiply by 0.5. We obtain

$$\begin{cases} \dfrac{p_j^{n+1}-p_j^n}{\Delta t}+\dfrac{-w_{j+1}^n+(w_j^n+v_j^n)-v_{j-1}^n}{2\Delta x}=0, \\ \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+\dfrac{w_{j+1}^n+(v_j^n-w_j^n)-v_{j-1}^n}{2\Delta x}=0, \end{cases} \tag{6}$$

Using the definition of v and w we obtain the result.

Now we propose to prove the result of consistency error. We define $u(x_j,t_n)$ the exact solution.

$$\begin{cases} \dfrac{p(x_j,t^{n+1})-p(x_j,t^n)}{\Delta t}+\dfrac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2\Delta x}-\dfrac{p(x_{j+1},t^n)-2p(x_j,t^n)+p(x_{j-1},t^n)}{2\Delta x}=0 \\ \\ \dfrac{u(x_j,t^{n+1})-u(x_j,t^n)}{\Delta t}+\dfrac{p(x_{j+1},t^n)-p(x_{j-1},t^n)}{2\Delta x}-\dfrac{u(x_{j+1},t^n)-2u(x_j,t^n)+u(x_{j-1},t^n)}{2\Delta x}=0 \end{cases} \tag{7}$$

Using the Taylor expansion as the previous exercise sheet we obtain

$$\begin{cases} \partial_t p(x_j,t^n)+O(\Delta t)+\partial_x u(x_j,t^n)+O(\Delta x^2)-\dfrac{\Delta x}{2}\partial_{xx}p(x_j,t^n), \\ \partial_t u(x_j,t^n)+O(\Delta t)+\partial_x p(x_j,t^n)+O(\Delta x^2)-\dfrac{\Delta x}{2}\partial_{xx}u(x_j,t^n), \end{cases} \tag{8}$$

Since $p(x_j,t^n)$ and $u(x_j,t^n)$ are solution the consistency error is $O(\Delta x+\Delta t)$ for the two equations.

3. Prove that the scheme (4) satisfy the maximum principle for the quantities under a CFL condition.

We consider the scheme (4)

$$\begin{cases} \dfrac{v_j^{n+1} - v_j^n}{\Delta t} + \dfrac{v_j^n - v_{j-1}^n}{\Delta x} = 0, \\ \\ \dfrac{w_j^{n+1} - w_j^n}{\Delta t} - \dfrac{w_{j+1}^n - w_j^n}{\Delta x} = 0, \end{cases} \tag{9}$$

As previously we write the scheme on a convex combination form.

$$\begin{cases} v_j^{n+1} = \left(1 - \dfrac{\Delta t}{\Delta x}\right) v_j^n + \dfrac{\Delta t}{\Delta x} v_{j-1}^n = 0, \\ \\ w_j^{n+1} = \left(1 - \dfrac{\Delta t}{\Delta x}\right) w_j^n + \dfrac{\Delta t}{\Delta x} w_{j+1}^n = 0, \end{cases} \tag{10}$$

For each equation we obtain convex combinations on the CFL condition $\frac{\Delta t}{\Delta x} < 1$.

4. Prove that the scheme (2) is stable for all l^q norms ($1 \leq q \leq \infty$) using the previous result and convex functions. The l^q norm is defined by

$$||(v, w)||_{l^q} = \left(\Delta x \sum_j ||(v_j, w_j)||_q^q \right)^{\frac{1}{q}},$$

with $||(v_j, w_j)||_q^q = |v_j|^q + |w_j|^q$.

We define $\alpha = \frac{\Delta t}{\Delta x} < 1$, consequently

$$\begin{cases} v_j^{n+1} = (1-\alpha)\, v_j^n + \alpha v_{j-1}^n = 0, \\ \\ w_j^{n+1} = (1-\alpha)\, w_j^n + \alpha w_{j+1}^n = 0, \end{cases} \tag{11}$$

We define the convex function $f(x) = |x|^p$, since this function is convex we have

$$\begin{cases} f(v_j^{n+1}) \leq (1-\alpha)\, f(v_j^n) + \alpha f(v_{j-1}^n), \\ \\ f(w_j^{n+1}) \leq (1-\alpha)\, f(w_j^n) + \alpha f(w_{j+1}^n), \end{cases} \tag{12}$$

Now we introduce the l^q norm associated with the (9)

$$||(v^{n+1}, w^{n+1})||_{l^q}^q = \left(\Delta x \sum_j ||(v_j^{n+1}, w_j^{n+1})||_q^q \right) = \Delta x \sum_j |v_j^{n+1}|^q + |w_j^{n+1}|^q.$$

Using (12) we obtain

$$\Delta x \sum_j |v_j^{n+1}|^q + |w_j^{n+1}|^q \leq \Delta x \sum_j (1-\alpha)|v_j^n|^q + \alpha |v_{j-1}^n|^q + (1-\alpha)|w_j^n|^q + \alpha |w_{j+1}^n|^q.$$

Since the boundary are periodic $\sum_j v_j^n = \sum_j v_{j-1}^n$ and $\sum_j |v_{j-1}|^q = \sum_j |v_j|^q$ thus we have

$$||(v^{n+1}, w^{n+1})||_{l^q}^q = \Delta x \sum_j |v_j^{n+1}|^q + |w_j^{n+1}|^q \leq \Delta x \sum_j |v_j^n|^q + |w_j^n|^q = ||(v^n, w^n)||_{l^q}^q.$$

For the L^∞ norm it is simple. The norm is defined by $||(v,w)||_\infty = \{\max(C_1, c_2), \max_j |v_j| \leq C_1, \max_j |w_j| \leq C_2\}$. Since the the maximum principle is preserved the scheme is stable for L^∞ norm with $C_1 = \max_j v_j^0$ and $C_2 = \max_j w_j^0$.

Additional question

We introduce the damped wave equation

$$\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = -\sigma u, \end{cases} \tag{13}$$

and the upwind scheme associated

$$\begin{cases} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} + \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \dfrac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} - \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x} = -\sigma u_j^n, \end{cases} \tag{14}$$

5. We call "steady states" the solutions of the systems defined by $\partial_x u = 0$ and $\partial_x p = -\sigma u$. Prove that (14) preserve exactly the steady states.

We take $u_j^n = a$ and $p_j^n = -a\sigma x_j + b$. This choice correspond to the discretization of the steady state.
Plugging these definitions in the fluxes we remark that $\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$ and $\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x} = 0$. We obtain

$$\begin{cases} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} - \dfrac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} = -\sigma a, \end{cases} \tag{15}$$

Now we remark that

$$\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = \frac{a\sigma}{2\Delta x}(x_{j+1} - 2x_j + x_{j-1}),$$
$$\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = \frac{a\sigma}{2\Delta x}(\Delta x - \Delta x) = 0,$$

and

$$\frac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} = -\frac{a\sigma}{2\Delta x}(x_{j+1} - x_{j-1}),$$
$$\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = -\frac{a\sigma}{2\Delta x}(2\Delta x) = -a\sigma.$$

Consequently

$$\begin{cases} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \sigma a = -\sigma a, \end{cases} \tag{16}$$

and $p_j^{n+1} = p_j^n$, $u_j^{n+1} = u_j^n$.