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Numerical methods for hyperbolic systems

Exercise sheet 3: Linear hyperbolic systems

Exercice 1

We consider the Maxwell equation

$$\begin{cases} \mu\partial_t B + c\partial_x E = -c\sigma^* B, \\ \varepsilon\partial_t E + c\partial_x B = -\sigma E, \end{cases} \tag{1}$$

with periodic boundary condition on $\Omega = [0, L]$ and $\mu > 0$, $\varepsilon > 0$, $\sigma > 0$, $\sigma^* > 0$.

1. Prove the following energy inequality and the uniqueness of the solutions.

$$\frac{d}{dt}\left(\int_\Omega \varepsilon|E(t,x)|^2 + \mu|B(t,x)|^2 dx\right) = -\int_\Omega \sigma|E(t,x)|^2 + \sigma^*\mu|B(t,x)|^2 dx. \tag{2}$$

We multiply the first equation of (1) by B, the second equation by E and after we integrate the equations.

$$\begin{cases} \mu\displaystyle\int_\Omega B\partial_t B + c\int_\Omega B\partial_x E = -\int_\Omega \sigma^* B^2, \\ \varepsilon\displaystyle\int_\Omega E\partial_t E + \int_\Omega E\partial_x B = -\int_\Omega \sigma E^2, \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega \mu|B|^2 + c\int_\Omega B\partial_x E = -\int_\Omega \sigma^*|B|^2, \\ \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega \varepsilon|E|^2 + c\int_\Omega E\partial_x B = -\int_\Omega \sigma|E|^2. \end{cases} \tag{4}$$

Summing the equations of (4), we obtain

$$E(t) = \frac{1}{2}\frac{d}{dt}\int_\Omega (\mu|B|^2 + \varepsilon|E|^2) + c\int_\Omega \partial_x(EB) = -\int_\Omega (\sigma^*|B|^2 + \sigma|E|^2).$$

We remark that $\int_\Omega \partial_x(EB) = [EB]_\Omega = 0$ because the boundary conditions are periodic. We define two solutions (B_1, E_1) and (B_2, E_2) with $(B_1(t=0,x), E_1(t=0,x)) = (B_2(t=0,x), E_2(t=0,x))$ and the term

$$E_d(t) = \frac{1}{2}\frac{d}{dt}\int_\Omega (\mu|B_1(t,x) - B_2(t,x)|^2 + \varepsilon|E_1(t,x) - E_2(t,x)|^2)$$

Since $E_d(t) \le 0$, $E_d^{'}(t) \le q$ and $E_d(t=0) \ge 0$ we have the uniqueness of the solution.

2. We introduce the plane waves (which are a good approximations of physical waves) defined by $E(t,x) = E_0 e^{i(wt-kx)}$ and $B(t,x) = B_0 e^{i(wt-kx)}$ with $E_0 \in \mathbb{R}$, $B_0 \in \mathbb{R}$, k the wave vector and w the frequency. Give the conditions (called dispersion relation) on w and k such as the planar waves are solutions of (1) for $\sigma = 0$ and $\sigma^* = 0$

We plug the definition of $E(t,x)$ and $B(t,x)$ in (1). We obtain

$$\begin{cases} \mu B_0 i w e^{i(wt-kx)} - ckE_0 i e^{i(wt-kx)} = 0, \\ \varepsilon E_0 i w e^{i(wt-kx)} - ckB_0 i e^{i(wt-kx)} = 0, \end{cases} \tag{5}$$

which are equivalent to

$$\begin{cases} \mu B_0 w - ckE_0 = 0, \\ \varepsilon E_0 w - ckB_0 = 0. \end{cases} \tag{6}$$

Now use use $E_0 = \frac{ckB_0}{\varepsilon}$. Plugging this relation in the first equation of (6). We obtain

$$k^2 = \frac{w^2}{c^2}\mu\varepsilon.$$

In this part we assume that $\sigma = 0$ and $\sigma^* = 0$. Now we introduce the DG centered scheme for (1). The mesh Ω_h is defined by $n+1$ points x_i and n cells $K_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. The volume of the cell K_i is $\Delta x_i = |x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}|$. We call a generic cell K. To finish the test function are defined by $v \in V_h = \{v/v|_K \in \mathbb{P}^p(K)\}$. The scheme is given by

$$\begin{cases} \varepsilon \sum_{l=0}^{k} \int_{K_i} \phi_l^i \phi_m^i \left(\frac{E_{l,i}^{n+1} - E_{l,i}^n}{\Delta t}\right) - c\sum_{l=0}^{k} B_{l,i}^n \int_{K_i} \phi_l^i \partial_x \phi_m^i + c\sum_{l=0}^{k} \left[B\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad \forall 0 \le m \le k, \\ \mu \sum_{l=0}^{k} \int_{K_i} \phi_l^i \phi_m^i \left(\frac{B_{l,i}^{n+1} - B_{l,i}^n}{\Delta t}\right) - c\sum_{l=0}^{k} E_{l,i}^n \int_{K_i} \phi_l^i \partial_x \phi_m^i + c\sum_{l=0}^{k} \left[E\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad \forall 0 \le m \le k, \end{cases} \tag{7}$$

with $\left[B\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}\left(B_{l,i+1}^n \phi_l^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}) + B_{l,i}^n \phi_l^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}})\right)$
$-\frac{1}{2}\left(B_{l,i-1}^n \phi_l^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}) + B_{l,i}^n \phi_l^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})\right)$.

3. We consider $V_h = P^1(K)$. We propose to use the Lagrange polynomial associated with the point $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$. Prove that the family is a basis of V_h. Write the scheme in a cell K_i.

We we study the family $\left(\frac{x-x_{j-\frac{1}{2}}}{\Delta x}, \frac{x_{j+\frac{1}{2}}-x}{\Delta x}\right)$.

$$\lambda_1 \frac{x-x_{j-\frac{1}{2}}}{\Delta x} + \lambda_2 \frac{x_{j+\frac{1}{2}}-x}{\Delta x} = 0,$$

is equivalent to

$$(\lambda_1-\lambda_2)\frac{x}{\Delta x} + \frac{x_{j+\frac{1}{2}}\lambda_2 - x_{j-\frac{1}{2}}\lambda_1}{\Delta x} = 0.$$

This relation is true for all x if $(\lambda_1-\lambda_2) =$ and $\frac{x_{j+\frac{1}{2}}\lambda_2 - x_{j-\frac{1}{2}}\lambda_1}{\Delta x} = 0$. Consequently $\lambda_1 = \lambda_2$ and

$$\frac{x_{j+\frac{1}{2}}\lambda_1 - x_{j-\frac{1}{2}}\lambda_1}{\Delta x} = \lambda_1 \frac{x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}}{\Delta x} = \lambda_1 = 0$$

Consequently the family of vectors is free. Since $\dim P^1(K_i) = 2$ the family is a basis.

To write the scheme we begin by a remark$\phi_0^i = \frac{x_{j+\frac{1}{2}}-x}{\Delta x} = \hat{\phi}_0(\frac{x-x_{j-\frac{1}{2}}}{\Delta x})$ and $\phi_1^i = \frac{x-x_{j-\frac{1}{2}}}{\Delta x} = \hat{\phi}_1(\frac{x-x_{j-\frac{1}{2}}}{\Delta x})$ with $\hat{\phi}_1 = a$ and $\hat{\phi}_0 = 1-a$.

Using this remark we propose to compute the different integral and terms

$$\int_{K_i} \phi_1^i \phi_1^i = \Delta x \int_0^1 \hat{\phi}_1^i \hat{\phi}_1^i = \int_0^1 a = \frac{\Delta x}{3}.$$

The same principle of computation give

$$\int_{K_i} \phi_0^i \phi_1^i = \int_{K_i} \phi_1^i \phi_0^i = \frac{\Delta x}{6}, \quad \int_{K_i} \phi_1^i \phi_1^i = \frac{\Delta x}{3}.$$

$$\int_{K_i} \phi_0^i \partial_x \phi_0^i = \int_{K_i} \phi_1^i \partial_x \phi_0^i = -\frac{\Delta x}{2}, \quad \int_{K_i} \phi_0^i \partial_x \phi_1^i = \int_{K_i} \phi_0^i \partial_x \phi_0^i = \frac{\Delta x}{2}.$$

We have also

$$\phi_0^i\phi_0^i(x_{j+\frac{1}{2}}) = 0, \quad \phi_0^i\phi_1^i(x_{j+\frac{1}{2}}) = \phi_1^i\phi_0^i(x_{j+\frac{1}{2}}) = \phi_1^i\phi_1^i(x_{j+\frac{1}{2}}) = 1,$$
$$\phi_0^i\phi_0^i(x_{j-\frac{1}{2}}) = 1, \quad \phi_0^i\phi_1^i(x_{j-\frac{1}{2}}) = \phi_1^i\phi_0^i(x_{j-\frac{1}{2}}) = \phi_1^i\phi_1^i(x_{j-\frac{1}{2}}) = 0,$$
$$\phi_0^{i+1}\phi_1^i(x_{j+\frac{1}{2}}) = 1, \quad \phi_0^{i+1}\phi_0^i(x_{j+\frac{1}{2}}) = \phi_1^{i+1}\phi_0^i(x_{j+\frac{1}{2}}) = \phi_1^{i+1}\phi_1^i(x_{j+\frac{1}{2}}) = 0,$$
$$\phi_1^{i-1}\phi_0^i(x_{j-\frac{1}{2}}) = 1, \quad \phi_0^{i-1}\phi_0^i(x_{j-\frac{1}{2}}) = \phi_0^{i-1}\phi_1^i(x_{j-\frac{1}{2}}) = \phi_1^{i-1}\phi_1^i(x_{j-\frac{1}{2}}) = 0.$$

Now we can compute the scheme. We define $\mathbf{E}_i^n = (E_{0,i}^n, E_{1,i}^n)$ and $\mathbf{B}_i^n = (B_{0,i}^n, B_{1,i}^n)$. The scheme is given by

$$\left\{\begin{array}{l} \varepsilon M_i\left(\frac{\mathbf{E}_i^{n+1}-\mathbf{E}_i^n}{\Delta t}\right) - cD_i\mathbf{B}_i^n + cK_{i,+}\mathbf{B}_{i+1}^n + cK_i\mathbf{B}_i + cK_{i,-}\mathbf{B}_{i-1} = 0, \quad \forall 0\leq m\leq k, \\ \mu M_i\left(\frac{\mathbf{B}_i^{n+1}-\mathbf{B}_i^n}{\Delta t}\right) - cD_i\mathbf{E}_i^n + cK_{i,+}\mathbf{E}_{i+1}^n + cK_i\mathbf{E}_i + cK_{i,-}\mathbf{E}_{i-1} = 0, \quad \forall 0\leq m\leq k, \end{array}\right. \tag{8}$$

with

$$M_i = \frac{\Delta x}{3}\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}\right), \quad K_i = \frac{\Delta x}{2}\left(\begin{array}{cc} -1 & -1 \\ 1 & 1 \end{array}\right)$$

and

$$K_{i,+} = \frac{1}{2}\left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \quad K_i = \frac{1}{2}\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \quad K_{i,-} = \frac{1}{2}\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$$

4. In this exercise we propose to study the numerical dispersive relation which define the numerical wave vector $\tilde{k}$ for $V_h = P^0(K) = \text{Span}(1)$. Write the scheme for this basis. Now we define $B_i^n = B_0 e^{j(wn\Delta t - \tilde{k}i\Delta x)}$ and $E_i^n = E_0 e^{j(wn\Delta t - \tilde{k}i\Delta x)}$ with j the complex number. Gives the relation between w and $\tilde{k}$ such as the discrete plane waves are solutions of (7). Show that the numerical dispersive relation is $\tilde{k}^2 = \frac{w^2}{c^2} + O(\Delta x^p + \Delta t^q)$ with $p > 2$ and $q > 2$.

The DG scheme for $V_h = P^0(K) = \mathrm{Span}(1)$ is

$$\begin{cases} \varepsilon \dfrac{E_i^{n+1} - E_i^n}{\Delta t} - \dfrac{B_{i+1}^n - B_{i-1}^n}{2\Delta x} = 0, \\ \\ \mu \dfrac{B_i^{n+1} - B_j^n}{\Delta t} + \dfrac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} = 0, \end{cases} \tag{9}$$

Plugging the definition of E_i^n and B_i^n in (9) we obtain

$$\begin{cases} \varepsilon E_0 \left(e^{jw\Delta t} - 1\right) = -\dfrac{c\Delta t}{2\Delta x} B_0 \left(e^{j\tilde{k}\Delta x} - e^{j\tilde{k}\Delta x}\right), \\ \\ \mu B_0 \left(e^{jw\Delta t} - 1\right) = -\dfrac{c\Delta t}{2\Delta x} E_0 \left(e^{j\tilde{k}\Delta x} - e^{j\tilde{k}\Delta x}\right), \end{cases} \tag{10}$$

$$\begin{cases} \varepsilon E_0 \left(e^{jw\Delta t} - 1\right) = -\dfrac{jc\Delta t}{2\Delta x} B_0 \left(\sin(\tilde{k}\Delta x)\right), \\ \\ \mu B_0 \left(e^{jw\Delta t} - 1\right) = -\dfrac{jc\Delta t}{2\Delta x} E_0 \left(\sin(\tilde{k}\Delta x)\right), \end{cases} \tag{11}$$

$$\begin{cases} \varepsilon E_0 \left(e^{j\frac{w\Delta t}{2}} - e^{-j\frac{w\Delta t}{2}}\right) = -\dfrac{jc\Delta t}{2\Delta x} B_0 \left(\sin(\tilde{k}\Delta x)\right), \\ \\ \mu B_0 \left(e^{j\frac{w\Delta t}{2}} - e^{-j\frac{w\Delta t}{2}}\right) = -\dfrac{jc\Delta t}{2\Delta x} E_0 \left(\sin(\tilde{k}\Delta x)\right), \end{cases} \tag{12}$$

$$\begin{cases} \varepsilon E_0 \left(2j \sin\left(\dfrac{w\Delta t}{2}\right)\right) e^{j\frac{w\Delta t}{2}} = -\dfrac{jc\Delta t}{2\Delta x} B_0 \left(\sin(\tilde{k}\Delta x)\right), \\ \\ \mu B_0 \left(2j \sin\left(\dfrac{w\Delta t}{2}\right)\right) e^{j\frac{w\Delta t}{2}} = -\dfrac{jc\Delta t}{2\Delta x} E_0 \left(\sin(\tilde{k}\Delta x)\right), \end{cases} \tag{13}$$

Plugging the second equation of (11) in the first equation to obtain

$$\left(\sin^2\left(\frac{w\Delta t}{2}\right)\right) e^{j\frac{w\Delta t}{2}} = \frac{c^2 \Delta t^2}{4\Delta x^2} \sin^2(\tilde{k}\Delta x)$$

This relation is the numerical dispersive relation. Using limited expansion we obtain

$$\left(\frac{w\Delta t}{2} + O(\Delta t^3)\right)^2 (1 + O(\Delta t)) = \frac{c^2 \Delta t^2}{4\Delta x^2} \left(\tilde{k}\Delta x + O(\Delta x^2)\right)^2$$

which is equivalent to

$$\left(\frac{w\Delta t}{2}\right)^2 = \frac{c^2 \Delta t^2}{\Delta x^2} \left(\frac{\tilde{k}\Delta x}{2}\right)^2 + O(\Delta t^3) + O(\Delta x^3).$$

To finis we observe that the previous equality is equal to $\tilde{k}^2 = \left(\frac{w}{c}\right)^2 + O(\Delta t^3) + O(\Delta x^3)$.