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Numerical methods for hyperbolic systems

Exercise sheet 4: Nonlinear scalar equations

Exercice 1 We consider the Burgers equation

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2} \right) = 0, \quad \forall x \in \mathbb{R}, \quad t > 0, \\ u(t=0,x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \end{cases} \tag{1}$$

with $u_L \leq u_R$

1. Prove that

$$u(t,x) = \begin{cases} u_L, & \frac{x}{t} < 0.5(u_L + u_R), \\ u_R, & \frac{x}{t} > 0.5(u_L + u_R), \end{cases} \tag{2}$$

and

$$u(t,x) = \begin{cases} u_L, & \frac{x}{t} < u_L, \\ \frac{x}{t}, & u_L < \frac{x}{t} < u_R, \\ u_R, & \frac{x}{t} > u_R, \end{cases} \tag{3}$$

are weak solutions of (1).

A function $u(t,x)$ is a weak solution of (1) if $u(t,x)$ is solution of

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}(u\partial_t\phi + \frac{u^2}{2}\partial_x\phi)dxdt + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx = 0, \quad \phi \in C_0^1, \tag{4}$$

with C_0^1 the space of C^1 functions with compact support ($\phi(t=0,x)=0$ if $|x|>A$ and $t>T$) and if the discontinuity velocity σ of $u(t,x)$ satisfy the Rankine-Hugoniot jumps conditions

$$\sigma = \frac{f(u_R)-f(u_L)}{u_R-u_L},$$

with $f(u)=\frac{u^2}{2}$. Integrating (4) by parts we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}(\partial_t u + \partial_x\frac{u^2}{2})\phi dxdt + \int_{\mathbb{R}}(u_0(x)-u(0,x))\phi(0,x)dx = 0, \quad \phi \in C_0^1. \tag{5}$$

Consequently to prove that (2) and (3) are solutions of (5), we prove that the continuous parts are solutions of (5) and the discontinuity velocity satisfy the Rankine Hugoniot jumps conditions.

First function $u(t,x) = (2)$

On $]-\infty, 0.5(u_L+u_R)[$, $u(t,x)$ is constant, therefore on this interval $u(t,x)$ is solution of (5).
On $]0.5(u_L+u_R), +\infty[$, $u(t,x)$ is constant, therefore on this interval $u(t,x)$ is solution of (5).

Now we apply the Rankine Hugoniot jumps conditions $\sigma = 0.5\frac{u_R^2-u_L^2}{u_R-u_L} = 0.5\frac{(u_R-u_L)(u_L+u_R)}{u_R-u_L} = 0.5(u_L+u_R)$.

Consequently (2) satisfy the Rankine Hugoniot jumps conditions and is a weak solution of (1).

Second function $u(t,x) = (3)$

This function is C^1 by part and globally C^0 thus (3) is a weak solution of (1) if (5) is verified. On $]-\infty, u_L]$ and $]u_R, +\infty[$ it is trivial to prove that (3) satisfy (5) because (3) is constant. To finish we study

$$\int_{u_L}^{u_R}\int_{\mathbb{R}^+}(\partial_t u + \partial_x\frac{u^2}{2})\phi dxdt + \int_{u_L}^{u_R}(u(0,x)-u_0(x))\phi(0,x)dx = 0, \quad \phi \in C_0^1. \tag{6}$$

Firstly $u(0,x) = u_0(x)$.
Since $\partial_t\left(\frac{x}{t}\right) + \partial_x\left(\frac{\frac{x^2}{t^2}}{2}\right) = -\frac{x}{t^2} + \frac{2x}{2t^2} = 0$ then (6) is equal to zero. Consequently (3) is a weak solution of (1).

2. We define the entropy $\eta(u) = \frac{u^{2p}}{2p} + \alpha\frac{u^2}{2}$ ($\alpha > 0$, $p > 2$) associated with (1) and the entropic flux associated with $\xi(u) = \frac{u^{2p+1}}{2p+1} + \alpha\frac{u^3}{3}$. Prove that the function (2) is not a weak entropy solution and the function (3) is a weak entropy solution. Give a condition on u_L and u_R such as (2) is a weak entropy solution.

Corollary useful: for the equation $\partial_t u + \partial_x f(u) = 0$, if f is convexe a shock is entropic if $f^{'}(u_L) > \sigma > f^{'}(u_R)$.

A function $u(t,x)$ is a weak entropic solution of (1) if $u(t,x)$ is solution of

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}(\eta(u)\partial_t\phi+\xi(u)\partial_x\phi)dxdt+\int_{\mathbb{R}}\eta(u_0(x))\phi(0,x)dx\geq 0,\quad \phi\in C_0^1,\phi\geq 0. \tag{7}$$

with C_0^1 the space of C^1 functions with compact support ($\phi(t=0,x)=0$ if $|x|>A$ and $t>T$) and if the discontinuity velocity σ of $u(t,x)$ satisfy

$$\sigma=\frac{f(u_R)-f(u_L)}{u_R-u_L}\ \text{and}\ -\sigma[\eta(u_R)-\eta(u_L)]+\xi(u_R)-\xi(u_L)\leq 0, \tag{8}$$

equivalent to

$$f^{'}(u_L)>\sigma>f^{'}(u_R), \tag{9}$$

when the flux is convex.

Integrating (7) by parts we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}(\partial_t\eta(u)+\partial_x\xi(u))\phi dxdt\leq 0,\quad \phi\in C_0^1,\phi\geq 0. \tag{10}$$

Consequently to prove that (2) and (3) are solutions of (10), we must prove that the continuous parts are solutions of (10) and the discontinuity velocity satisfy the Rankine Hugoniot equality.

First function $u(t,x)=(2)$

We propose to show that (2) satisfy (9) since $f(u)=\frac{u^2}{2}$ is convex. Since $f^{'}(u_L)=u_L$, $f^{'}(u_R)=u_R$ and $u_R>u_L$ the condition (9) is not satisfy. To obtain a entropy solution we must have $u_L>u_R$.

Second function $u(t,x)=(3)$

The function u given by (3), $\eta(u)$ and $\xi(u)$ are C^1 by part and globally C^0 consequently (3) is a weak solution of (1) if (10) is verified. On $]-\infty,u_L]$ and $]u_R,+\infty[$ it is trivial to prove that (3) satisfy (10) because (3) is constant. To finish we study

$$\int_{u_L}^{u_R}\int_{\mathbb{R}^+}(\partial_t\eta(u)+\partial_x\xi(u))\phi dxdt\leq 0,\quad \phi\in C_0^1,\phi\geq 0. \tag{11}$$

Since $\partial_t\eta(\frac{x}{t})+\partial_x\xi(\frac{x}{t})=-\frac{x^{2p}}{t^{2p+1}}-\alpha\frac{x^2}{t^3}+\frac{(2p+1)x^{2p}}{(2p+1)t^{2p+1}}+\alpha\frac{3x^2}{3t^3}=0$ then (11) is equal to zero. Consequently (3) is a weak entropy solution of (1).
If $u_L>u_R$ (3) is a multi-valuate function consequently (3) with $u_L>u_R$ is not a solution.

Remark:
For the elliptic equations the weak solutions are unique. However for the nonlinear hyperbolic equations the weak solutions are not unique consequently we add a physical criterion : the entropy. The weak entropy solutions are unique. For the model (1) the unique solution is (3) for $u_R>u_L$ and (2) for $u_L>u_R$.