TU München
Zentrum Mathematik
Lehrstuhl M16
E. Sonnendrücker
E. Franck

Numerical methods for hyperbolic systems

Numerical exercises: Discontinuous Galerkin Explicit solvers for linear hyperbolic systems

The aim of the numerical exercises is to write a Discontinuous Galerkin-Finite volume solver with explicit higher order time integration (Runge Kutta Methods) for linear and nonlinear hyperbolic systems. We advise you to write the most modular code as possible (small generic subroutines for different computations).

Exercice 1

Aim: Write the centered and upwind Discontinuous Galerkin schemes for the advection equation in a Matlab program. The higher order integrator (TVD Runge Kutta Methods) is always present in the initial code.
The initialization of the data, the time stepping are write in the file "MainAdvection.m"

1. Write a function which computes the Lagrange polynomials and the degrees of freedom associated: "Degrees_freedom.m", "Basis_functions.m".

2. A Discontinuous Galerkin scheme with first order scheme in time cab ne write of the following form

$$M\mathbf{U}_j^{n+1} = M\mathbf{U}_j^n - \Delta t \times (D\mathbf{U}_j^n + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}).$$

Write a function which computes the local matrices M_j and D_j (using a Gauss quadrature formula) : "Matrix_local_integral.m", "quadrature.m".

3. Write functions which compute the upwind and centered fluxes for the advection equation. Write a function "Solvers_advection.m" which compute the spatial discretization

$$-M^{-1} \times (D\mathbf{U}_j^n + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}).$$

4. Write these boundary conditions (using the ghost cells) in the code: "BC_advection.m"

- $u(t, L_G) = u(t, L_D)$ (periodic condition on the domain $[L_G, L_D]$)
- $F(u) = 0$ $F(u)$ the numerical flux

5. Write a function which compute the discrete L^p norm and the error between the numerical and exact solutions : "norm_LP.m"

6. Find a adapted boundary condition for the following transport test cases. Test the schemes for positive or negative velocity a and for two initial data $u^0(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_c)^2}{2\sigma^2}}$ with x_c the center of the domain and

$$u^0(x) = \begin{cases} 1, & x < \frac{1}{2}L, \\ 0, & x > \frac{1}{2}L. \end{cases}$$

Initial datas and exact solutions can be write the functions: "init_datas_advection.m", "Solution_advection.m".

Find the exact solutions associated with these initial datas.

Which convergence rates do you observe for these tests case, respectively ?

7. Explicit the behavior of the schemes relative to the maximum principle and the L^p stability.

8. Explicit the behavior of the schemes when the CFL condition is not respected.

Exercice 2
Aim: Write the centered and upwind Discontinuous Galerkin schemes for the Maxwell equations in a Matlab program.

1. Write a file MainMaxwell.m containing the initialization, the time stepping and the plot functions for the Maxwell equations (we can adapt the files "MainAdvection.m", "RK_solving_advection", "Solvers_advection.m").

2. Write functions which computes the upwind and centered fluxes for the Maxwell equations.

3. Test the schemes for two initial data. The first is $E^0(x) = E_0 \cos(2\pi x)$, $B^0(x) = B_0 \cos(2\pi x)$. The second is

$$E^0(x) = \begin{cases} 1, & 0.4 \times L < x < 0.6 \times L, \\ 0, & \text{else} \end{cases}$$

and $B^0(x) = 0$

Find the exact solutions associated with these initial datas (use diagonalization of the system).

Which convergence rates do you observe for these test cases, respectively ?

4. Compare the mass conversation, the discrete L^2 energy associated with the Maxwell equations for different schemes and different polynomial order.