Cell-Centered Asymptotic preserving schemes for linear transport on unstructured meshes

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Seminary, IPP, Garching
26 june 2012

with : Christophe Buet (CEA), Bruno Després (UPMC).
- Introduction
- AP schemes in 1D and difficulties in 2D
- AP schemes for hyperbolic heat equation in 2D
- AP schemes for angular approximations of the transport equation
- Numerical results
- Conclusion
Physical context and objectives

- **IFC simulations**: interaction between the gas modelized by two-temperatures Euler equations and the radiation modelized by a linear transport equation.

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Physical context and objectives

- **IFC simulations**: interaction between the gas modelized by two-temperatures Euler equations and the radiation modelized by a linear transport equation.

- **Grey linear transport equation**: $f(x, \Omega, t) \geq 0$ the distribution function associated to the particles (photons or neutrons) located in $x$, with a direction $\Omega$. We consider the following equation:

  \[
  \partial_t f(t, x, \Omega) + \Omega \cdot \nabla f(t, x, \Omega) = \sigma \int_{S^2} (f(t, x, \Omega') - f(t, x, \Omega)) d\Omega'.
  \]

- **Diffusion limit**: for $t \gg 1$ and $\sigma \gg 1$, the transport equation, tends toward the following diffusion equation

  \[
  \partial_t E(t, x) - \text{div} \left( \frac{1}{\sigma} \nabla E(t, x) \right) = 0,
  \]

  with $E(t, x) = \int_{\Omega} f(t, x, \Omega) d\Omega$ and $F(t, x) = \int_{\Omega} \Omega f(t, x, \Omega) d\Omega$.

- **Computation cost**: The CPU cost is important, consequently one needs simplified models.
Approximation of transport equation

- Simplified hyperbolic models, depend only on spaces variables.
- **Simplified models:**
  - $P_n$ models: we develop the transport equation on the basis of spherical harmonics.
  - $S_n$ models: we use a quadrature formula to discretize the collision operator.
  - $M_n$ models: non-linear $P_n$ models where the closure is obtained by minimizing the entropy.

\[
\begin{align*}
P_1 \text{ model:} & \\
& \begin{cases}
\partial_t E + \frac{1}{\varepsilon} \text{div} F = 0, \\
\partial_t F + \frac{1}{3\varepsilon} \nabla E = - \frac{\sigma}{\varepsilon^2} F.
\end{cases}
\end{align*}
\]
Approximation of transport equation

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- **Simplified models**:
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  - \( S_n \) models: we use a quadrature formula to discretize the collision operator.
  - \( M_n \) models: non-linear \( P_n \) models where the closure is obtained by minimizing the entropy.

\( P_1 \) model:

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\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \text{div} \mathbf{F} &= 0, \\
\partial_t \mathbf{F} + \frac{1}{3\varepsilon} \nabla E &= -\frac{\sigma}{\varepsilon^2} \mathbf{F}.
\end{align*}
\]

- **Adapted numerical methods**: asymptotic preserving (AP) finite volume schemes capturing the diffusion limit.

**Aims**:

Design of cell-centered finite volume schemes for the simplified models capturing the diffusion limit on unstructured meshes.
Introduction

AP schemes in 1D and difficulties in 2D

AP schemes for hyperbolic heat equation in 2D

AP schemes for angular approximations of the transport equation

Numerical results

Conclusion
Staggered and centered schemes

- Contrary to the Godunov schemes (HLL, Rusanov, upwind) the centered scheme for the hyperbolic heat equation is AP.
- The limit diffusion scheme admit spurious modes.
- The centered scheme is not stable.
Staggered and centered schemes

- Contrary to the Godunov schemes (HLL, Rusanov, upwind) the centered scheme for the hyperbolic heat equation is AP.
- The limit diffusion scheme admit spurious modes.
- The centered scheme is not stable.
- The staggered scheme is also asymptotic preserving:

\[
\begin{align*}
\frac{E_j^{n+1} - E_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^{n+1} - F_{j-\frac{1}{2}}^{n+1} - F_{j+\frac{1}{2}}^n + F_{j-\frac{1}{2}}^n}{2 \varepsilon \Delta x} &= 0, \\
\frac{F_j^{n+1} - F_j^n}{\Delta t} + \frac{E_{j+1} - E_j}{\varepsilon \Delta x} &= -\frac{\sigma}{\varepsilon^2} F_{j+\frac{1}{2}}.
\end{align*}
\]

- But the staggered scheme does not preserve the maximum principle \( E + F > 0, \ E - F > 0 \) in the transport regime.
• Contrary to the Godunov schemes (HLL, Rusanov, upwind) the centered scheme for the hyperbolic heat equation is AP.

• The limit diffusion scheme admit spurious modes.

• The centered scheme is not stable.

• The staggered scheme is also asymptotic preserving:

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\begin{align*}
\frac{E_{j+1}^{n} - E_{j}^{n}}{\Delta t} + \frac{F_{j+\frac{1}{2}}^{n+1} - F_{j-\frac{1}{2}}^{n}}{\varepsilon \Delta x} &= 0, \\
\frac{F_{j+\frac{1}{2}}^{n+1} - F_{j+\frac{1}{2}}^{n}}{\Delta t} + \frac{E_{j+1}^{n+1} - E_{j}^{n+1}}{\Delta x} &= -\frac{\sigma}{\varepsilon^2} F_{j+\frac{1}{2}}^{n+1}.
\end{align*}
\]

• But the staggered scheme does not preserve the maximum principle \( E + F > 0, E - F > 0 \) in the transport regime.

**CEA Constraints**

• Design of schemes which are equal to the upwind scheme when \( \sigma = 0 \) to preserve the transport properties (maximum principle, entropy etc).

• Design of cell-centered schemes. Indeed the hydrodynamic and diffusion codes are coupled with the transport problem use cell-centered methods.
AP schemes: design and examples

Hyperbolic heat equation:

\[
\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E &= -\frac{\sigma}{\varepsilon^2} F,
\end{align*}
\]

\[\Rightarrow \partial_t E - \partial_x \frac{1}{\sigma} \partial_x E = 0.\]

- Consistency error of the **upwind** scheme
  - for the first equation: \( O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right) \),
  - for the second equation: \( O\left(\frac{\Delta x^2}{\varepsilon^2} + \Delta x + \Delta t\right) \).

- CFL condition: \( \Delta t \left(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}\right) \leq 1. \)
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Jin-Levermore scheme

- Principle of design : we introduce the steady state \( \partial_x E = -\frac{\sigma}{\varepsilon} F \) in the fluxes.
- We write the relations

\[
\begin{aligned}
E(x_j) &= E\left(x_{j+\frac{1}{2}}\right) + (x_j - x_{j+\frac{1}{2}}) \partial_x E\left(x_{j+\frac{1}{2}}\right), \\
E(x_{j+1}) &= E\left(x_{j+\frac{1}{2}}\right) + (x_{j+1} - x_{j+\frac{1}{2}}) \partial_x E\left(x_{j+\frac{1}{2}}\right).
\end{aligned}
\]

\[\text{with} \quad M = \frac{2}{\varepsilon^2} + \frac{\sigma}{\varepsilon^2} \Delta x.\]
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\implies \partial_t E - \partial_x \frac{1}{\sigma} \partial_x E = 0.
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\end{align*}
\]

Plugging the discrete equivalent of these relations in the fluxes.

\[
\begin{align*}
F_j + E_j &= F_{j+\frac{1}{2}} + E_{j+\frac{1}{2}}, \\
F_{j+1} - E_{j+1} &= F_{j+\frac{1}{2}} - E_{j+\frac{1}{2}}.
\end{align*}
\]
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Hyperbolic heat equation :

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\begin{align*}
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\[\Longrightarrow \partial_t E - \partial_x \frac{1}{\sigma} \partial_x E = 0.\]

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  - for the first equation : \(O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)\),
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**Jin-Levermore scheme**

- Principle of design : we introduce the steady state \(\partial_x E = -\frac{\sigma}{\varepsilon} F\) in the fluxes.
- We write the relations

\[
\begin{align*}
E(x_j) &= E(x_{j+1/2}) - (x_j - x_{j+1/2}) \frac{\sigma}{\varepsilon} F(x_{j+1/2}), \\
E(x_{j+1}) &= E(x_{j+1/2}) - (x_{j+1} - x_{j+1/2}) \frac{\sigma}{\varepsilon} F(x_{j+1/2}).
\end{align*}
\]

We obtain

\[
\begin{align*}
F_j + E_j &= F_{j+1/2} + E_{j+1/2} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+1/2}, \\
F_{j+1} - E_{j+1} &= F_{j+1/2} - E_{j+1/2} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+1/2}.
\end{align*}
\]
AP schemes : design and examples

Hyperbolic heat equation:

\[ \begin{cases} 
\partial_t E + \frac{1}{\varepsilon} \partial_x F = 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E = -\frac{\sigma}{\varepsilon^2} F, 
\end{cases} \quad \Rightarrow \partial_t E - \partial_x \frac{1}{\sigma} \partial_x E = 0. \]

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  - for the first equation : \( O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right) \),
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\[ \begin{align*}
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E(x_{j+1}) &= E(x_{j+\frac{1}{2}}) - (x_{j+1} - x_{j+\frac{1}{2}}) \frac{\sigma}{\varepsilon} F(x_{j+\frac{1}{2}}).
\end{align*} \]

We obtain

\[ \begin{align*}
E_{j+\frac{1}{2}} &= \left(\frac{E_j + E_{j+1}}{2} + \frac{F_j - F_{j+1}}{2}\right), \\
F_{j+\frac{1}{2}} &= M \left(\frac{F_j + F_{j+1}}{2} + \frac{E_j - E_{j+1}}{2}\right).
\end{align*} \]

with \( M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x} \).
AP schemes in 1D

- The Jin-Levermore scheme is

\[
\begin{align*}
\frac{E_{j}^{n+1} - E_{j}^{n}}{\Delta t} + M \frac{F_{j+1}^{n} - F_{j}^{n}}{2\varepsilon \Delta x} - \frac{E_{j+1}^{n} - 2E_{j}^{n} + E_{j-1}^{n}}{2\varepsilon \Delta x} &= 0, \\
\frac{F_{j}^{n+1} - F_{j}^{n}}{\Delta t} + \frac{E_{j+1}^{n} - E_{j-1}^{n}}{2\varepsilon \Delta x} - \frac{F_{j+1}^{n} - 2F_{j}^{n} + F_{j-1}^{n}}{2\varepsilon \Delta x} + \frac{\sigma}{\varepsilon^2} F_{j}^{n} &= 0.
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(1)

with \( M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x} \).
AP schemes in 1D

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\begin{align*}
\frac{E_j^{n+1} - E_j^n}{\Delta t} + M \left( \frac{F_{j+1}^n - F_j^n}{2\varepsilon \Delta x} \right) + \frac{E_j^{n+1} - 2E_j^n + E_j^{n-1}}{2\varepsilon \Delta x} = 0, \\
\Delta t \left( \frac{F_{j+1}^n - F_j^n}{2\varepsilon \Delta x} \right) + \frac{E_j^{n+1} - 2E_j^n + E_j^{n-1}}{2\varepsilon \Delta x} + \frac{\sigma \varepsilon^2}{2} F_j^n = 0.
\end{align*}
\]  

(1)

with \( M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x} \).

- Consistency error of Jin-Levermore:
  - for the first equation: \( O \left( \Delta x^2 + \varepsilon \Delta x + \Delta t \right) \),
  - for the second equation: \( O \left( \frac{\Delta x^2}{\varepsilon} + \Delta x + \Delta t \right) \).

- CFL condition of explicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2} \right) \leq 1. \)

- CFL condition of semi-implicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1. \)
AP schemes in 1D

Gosse-Toscani scheme

- Principle of design: localization of source terms at the interfaces, which induces a stationary wave in the Riemann problem.

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\begin{align*}
\frac{E_{j}^{n+1} - E_{j}^{n}}{\Delta t} + M \frac{F_{j+1}^{n} - F_{j-1}^{n}}{2\varepsilon \Delta x} - M \frac{E_{j+1}^{n} - 2E_{j}^{n} + E_{j-1}^{n}}{2\varepsilon \Delta x} &= 0, \\
\frac{F_{j}^{n+1} - F_{j}^{n}}{\Delta t} + M \frac{E_{j+1}^{n} - E_{j-1}^{n}}{2\varepsilon \Delta x} - M \frac{F_{j+1}^{n} - 2F_{j}^{n} + F_{j-1}^{n}}{2\varepsilon \Delta x} + M \frac{\sigma}{\varepsilon^2} F_{j}^{n} &= 0.
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\end{align*}
\]

(2)

with \( M = \frac{2\varepsilon}{\varepsilon + \sigma \Delta x} \).

- Consistency error of the Gosse-Toscani scheme:
  - for the first equation: \( O \left( \varepsilon \Delta x + \Delta x^2 + \Delta t \right) \),
  - for the second equation: \( O \left( \Delta x + \Delta t \right) \).

- CFL condition of explicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1 \).

- CFL condition of semi-implicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon + \frac{\Delta x^2}{\sigma}} \right) \leq 1 \).
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\begin{align*}
\frac{E_{j+1}^{n+1} - E_j^n}{\Delta t} + M \frac{F_{j+1}^n - F_j^n}{2\varepsilon \Delta x} - M \frac{E_j^{n+1} - 2E_j^n + E_{j-1}^n}{2\varepsilon \Delta x} &= 0, \\
\frac{F_{j+1}^{n+1} - F_j^n}{\Delta t} + M \frac{E_{j+1}^n - E_{j-1}^n}{2\varepsilon \Delta x} - M \frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{2\varepsilon \Delta x} + M \frac{\sigma}{\varepsilon^2} F_j^n &= 0.
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  - for the first equation: \(O(\varepsilon \Delta x + \Delta x^2 + \Delta t)\),
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- CFL condition of explicit scheme: \(\Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1\).

- CFL condition of semi-implicit scheme: \(\Delta t \left( \frac{1}{\Delta x \varepsilon + \Delta x^2 / \sigma} \right) \leq 1\).

- Remark: The Jin-Levermore scheme (1) with the discretization of the source term \(\frac{1}{2} (F_{j+1/2} + F_{j-1/2})\) is equal to the Gosse-Toscani scheme.
AP schemes in 1D

**Gosse-Toscani scheme**

- Principle of design: localization of source terms at the interfaces, which induces a stationary wave in the Riemann problem.

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\begin{aligned}
\frac{E_{j}^{n+1} - E_{j}^{n}}{\Delta t} + M \frac{F_{j+1}^{n} - F_{j-1}^{n}}{2\varepsilon \Delta x} &= 0, \\
\frac{F_{j}^{n+1} - F_{j}^{n}}{\Delta t} + M \frac{E_{j+1}^{n} - E_{j-1}^{n}}{2\varepsilon \Delta x} &= 0,
\end{aligned}
\]  

\[ M = \frac{2\sigma}{2\varepsilon + \sigma \Delta x}. \]

- Consistency error of the **Gosse-Toscani** scheme:
  - for the first equation: \( O(\varepsilon \Delta x + \Delta x^2 + \Delta t) \),
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- CFL condition of explicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1 \).

- CFL condition of semi-implicit scheme: \( \Delta t \left( \frac{1}{\Delta x \varepsilon + \Delta x^2 / \sigma} \right) \leq 1 \).

- **Remark**: The Jin-Levermore scheme (1) with the discretization of the source term \( \frac{1}{2} (F_{j+1/2} - F_{j-1/2}) \) is equal to the Gosse-Toscani scheme.

- **Remark**: For the two schemes, the numerical viscosity gives the diffusion limit scheme on coarse grids (\( \frac{\Delta x}{\varepsilon} \gg 1 \)).
Analysis of AP schemes : modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $\|\partial_{t}a_{,x}b E\| \leq C_{a,b}$ and $\|\partial_{t}a_{,x}b F\| \leq \varepsilon C_{a,b}$.
- The modified equation associated to the Upwind scheme is

$$\begin{cases}
\frac{\partial t}{} E + \frac{1}{\varepsilon} \partial x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0, \\
\frac{\partial t}{} F + \frac{1}{\varepsilon} \partial x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F = -\sigma \frac{\varepsilon^2}{2} F.
\end{cases}$$

(3)
Analysis of AP schemes: modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that \( ||\partial_{t,a,b} E|| \leq C_{a,b} \) and \( ||\partial_{t,a,b} F|| \leq \varepsilon C_{a,b} \).
- The modified equation associated to the Upwind scheme is

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\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]

We plug the relation \( \varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F \) in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0.
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To understand the behaviour of the scheme, we use the modified equations method.

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\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F = -\frac{\sigma}{\varepsilon^2} F.
\end{cases}$$

We plug the relation $\varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

On fine grid $\Delta x / \varepsilon << 1$, the diffusion limit is

$$\partial_t E - \frac{1}{\sigma} \partial_{xx} E = 0.$$
To understand the behaviour of the scheme, we use the modified equations method.

We assume that \( ||\partial_{t a_{x b}} E|| \leq C_{a,b} \) and \( ||\partial_{t a_{x b}} F|| \leq \epsilon C_{a,b} \).

The modified equation associated to the Upwind scheme is

\[
\begin{aligned}
\partial_t E + \frac{1}{\epsilon} \partial_x F - \frac{\Delta x}{2\epsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\epsilon} \partial_x E - \frac{\Delta x}{2\epsilon} \partial_{xx} F &= -\frac{\sigma}{\epsilon^2} F.
\end{aligned}
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(3)

We plug the relation \( \epsilon \partial_x E + O(\epsilon^2) = -\sigma F \) in the first equation of (4), we obtain the diffusion limit

On coarse grid \( \frac{\Delta x}{\epsilon} >> 1 \), the diffusion limit is

\[
\partial_t E - \frac{\Delta x}{2\epsilon} \partial_{xx} E = 0.
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Analysis of AP schemes : modified equations

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- We assume that \( \| \partial_{t a, x b} E \| \leq C_{a, b} \) and \( \| \partial_{t a, x b} F \| \leq \varepsilon C_{a, b} \).
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\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]  

(3)

- We plug the relation \( \varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F \) in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0.
\]

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{align*}
\partial_t E + M \frac{1}{\varepsilon} \partial_x F - M \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\varepsilon} \partial_x E - M \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -M \frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]  

(4)
Analysis of AP schemes : modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $||\partial_{t,a,x,b} E|| \leq C_{a,b}$ and $||\partial_{t,a,x,b} F|| \leq \varepsilon C_{a,b}$.
- The modified equation associated to the Upwind scheme is

$$\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon^2} F.
\end{align*} \quad (3)
$$

- We plug the relation $\varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

$$\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2 \varepsilon} \partial_{xx} E = 0.$$

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

$$\begin{align*}
\partial_t E + M \frac{1}{\varepsilon} \partial_x F - M \frac{\Delta x}{2 \varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\varepsilon} \partial_x E - M \frac{\Delta x}{2 \varepsilon} \partial_{xx} F &= -M \frac{\sigma}{\varepsilon^2} F.
\end{align*} \quad (4)
$$

- We plug the relation $M \varepsilon \partial_x E + O(\varepsilon^2) = -M \sigma F$ in the first equation of (4)

$$\partial_t E - \frac{M}{\sigma} \partial_{xx} E - \frac{1-M}{\sigma} \partial_{xx} E = 0.$$
To understand the behaviour of the scheme, we use the modified equations method.

We assume that \( ||\partial_{t,a,x} E|| \leq C_{a,b} \) and \( ||\partial_{t,a,x} F|| \leq \epsilon C_{a,b} \).

The modified equation associated to the Upwind scheme is

\[
\begin{align*}
\partial_t E + \frac{1}{\epsilon} \partial_x F - \frac{\Delta x}{2\epsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\epsilon} \partial_x E - \frac{\Delta x}{2\epsilon} \partial_{xx} F &= -\frac{\sigma}{\epsilon^2} F.
\end{align*}
\]

We plug the relation \( \epsilon \partial_x E + O(\epsilon^2) = -\sigma F \) in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\epsilon} \partial_{xx} E = 0.
\]

The scheme does not capture the diffusion limit.

The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{align*}
\partial_t E + M \frac{1}{\epsilon} \partial_x F - M \frac{\Delta x}{2\epsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\epsilon} \partial_x E - M \frac{\Delta x}{2\epsilon} \partial_{xx} F &= -M \frac{\sigma}{\epsilon^2} F.
\end{align*}
\]

We plug the relation \( M \epsilon \partial_x E + O(\epsilon^2) = -M \sigma F \) in the first equation of (4)

On fine grid \( \frac{\Delta x}{2\epsilon} \ll 1, \ M \to 1 \) and the diffusion coefficient is correct.

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E = 0
\]
Analysis of AP schemes : modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $\| \partial_{t}^{a,x,b} E \| \leq C_{a,b}$ and $\| \partial_{t}^{a,x,b} F \| \leq C_{a,b}$.
- The modified equation associated to the Upwind scheme is

\[
\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon} F.
\end{align*}
\]

- We plug the relation $\varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0.
\]

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{align*}
\partial_t E + M \frac{1}{\varepsilon} \partial_x F - M \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\varepsilon} \partial_x E - M \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -M \frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]

- We plug the relation $M \varepsilon \partial_x E + O(\varepsilon^2) = -M \sigma F$ in the first equation of (4)
- On coarse grid $\frac{\Delta x}{2\varepsilon} \gg 1$, $M \rightarrow 0$ and the diffusion coefficient is correct.

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E = 0
\]
Analysis of AP schemes : modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that \( ||\partial_{t,a,x,b} E|| \leq C_{a,b} \) and \( ||\partial_{t,a,x,b} F|| \leq \varepsilon C_{a,b} \).
- The modified equation associated to the Upwind scheme is

\[
\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon^2} F.
\end{align*}
\] (3)

- We plug the relation \( \varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F \) in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0.
\]

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{align*}
\partial_t E + M \frac{1}{\varepsilon} \partial_x F - M \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\varepsilon} \partial_x E - M \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -M \frac{\sigma}{\varepsilon^2} F.
\end{align*}
\] (4)

- We plug the relation \( M \varepsilon \partial_x E + O(\varepsilon^2) = -M\sigma F \) in the first equation of (4)
- On intermediate grid \( \frac{\Delta x}{2\varepsilon} = 0(1) \), the diffusion coefficient is correct.

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E = 0
\]
Analysis of AP schemes: modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $\|\partial_{t a,x,b} E\| \leq C_{a,b}$ and $\|\partial_{t a,x,b} F\| \leq \varepsilon C_{a,b}$.
- The modified equation associated to the Upwind scheme is

\[
\begin{align*}
\partial_t E + \frac{1}{\varepsilon} \partial_x F - \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + \frac{1}{\varepsilon} \partial_x E - \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -\frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]

(3)

- We plug the relation $\varepsilon \partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E - \frac{\Delta x}{2\varepsilon} \partial_{xx} E = 0.
\]

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{align*}
\partial_t E + M \frac{1}{\varepsilon} \partial_x F - M \frac{\Delta x}{2\varepsilon} \partial_{xx} E &= 0, \\
\partial_t F + M \frac{1}{\varepsilon} \partial_x E - M \frac{\Delta x}{2\varepsilon} \partial_{xx} F &= -M \frac{\sigma}{\varepsilon^2} F.
\end{align*}
\]

(4)

- We plug the relation $M\varepsilon \partial_x E + O(\varepsilon^2) = -M\sigma F$ in the first equation of (4)

\[
\partial_t E - \frac{1}{\sigma} \partial_{xx} E = 0
\]

- The scheme captures the diffusion limit (idem for the Jin-Levermore scheme).
To validate the AP method, we propose the following test case. The parameters are $\sigma = 1$, $\varepsilon = 0.001$. The initial data is given by $E(0, x) = G(x)$ with $G(x)$ a gaussian and $F(0, x) = 0$.

The « asymptotic preserving » scheme is significantly more accurate than the upwind scheme.

In practice a classical scheme for this type of problem is unusable.
We introduce the notations for the edge formulation of finite volume methods.

- \( l_{jk} \) and \( n_{jk} \) are the length and the normal associated to the edge \( \partial \Omega_{jk} \).
- \( \sum_k l_{jk} n_{jk} = 0 \).
- \( (F_{jk}, n_{jk}) \) and \( E_{jk} \) are the fluxes associated to the edge \( \partial \Omega_{jk} \).
Jin-Levermore method: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

\[
\begin{align*}
E(x_j) & \simeq E(x_{jk}) + (x_j - x_{jk}, \nabla E(x_{jk})) \\
E(x_k) & \simeq E(x_{jk}) + (x_k - x_{jk}, \nabla E(x_{jk}))
\end{align*}
\]

Discrete equivalent:

\[
\begin{align*}
E_j & \simeq E_{jk} - \sigma \varepsilon (F_{jk}, x_j - x_{jk}) \\
E_k & \simeq E_{jk} - \sigma \varepsilon (F_{jk}, x_k - x_{jk})
\end{align*}
\]

Plugging the previous relations in the acoustic solver, we obtain:

\[
\begin{align*}
(F_{j}, n_{jk}) + E_j & = (F_{jk}, n_{jk}) + E_{jk} \\
(F_{k}, n_{jk}) - E_k & = (F_{jk}, n_{jk}) - E_{jk}
\end{align*}
\]

To solve this system we need a geometrical assumption.

Assumption: The mesh satisfy the Delaunay condition, therefore:

\[
(x_{jk} - x_j) = d_{jk} n_{jk} \quad \text{et} \quad (x_{jk} - x_k) = -d_{kj} n_{jk}
\]

Asymptotic limit of Jin-Levermore scheme: Two-Points-Flux scheme
2D Extension : difficulties

- **Jin-Levermore method**: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

\[
\begin{align*}
E(x_j) &\approx E(x_{jk}) - \sigma \varepsilon (x_j - x_{jk}, F(x_{jk})), \\
E(x_k) &\approx E(x_{jk}) - \sigma \varepsilon (x_k - x_{jk}, F(x_{jk})).
\end{align*}
\]

Discrete equivalent:

\[
\begin{align*}
E_j &\approx E_{jk} - \sigma \varepsilon (F_{jk}, x_j - x_{jk}), \\
E_k &\approx E_{jk} - \sigma \varepsilon (F_{jk}, x_k - x_{jk}).
\end{align*}
\]

Plugging the previous relations in the acoustic solver, we obtain:

\[
\left.\begin{array}{l}
(F_j, n_{jk}) + E_j = (F_{jk}, n_{jk}) + E_{jk}, \\
(F_k, n_{jk}) - E_k = (F_{jk}, n_{jk}) - E_{jk}.
\end{array}\right\}
\]

To solve this system we need a geometrical assumption.

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\[
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\]

Asymptotic limit of Jin-Levermore scheme: Two-Points-Flux scheme
2D Extension : difficulties

- **Jin-Levermore method**: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

Discrete equivalent

\[
\begin{align*}
E_j &\approx E_{jk} - \frac{\sigma}{\epsilon} (F_{jk}, x_j - x_{jk}), \\
E_k &\approx E_{jk} - \frac{\sigma}{\epsilon} (F_{jk}, x_k - x_{jk}).
\end{align*}
\]

Plugging the previous relations in the acoustic solver, we obtain:

\[
\begin{align*}
(F_j, n_{jk}) + E_j &= (F_{jk}, n_{jk}) + E_{jk}, \\
(F_k, n_{jk}) - E_k &= (F_{jk}, n_{jk}) - E_{jk}.
\end{align*}
\]

To solve this system we need a geometrical assumption.

Assumption: The mesh satisfy the Delaunay condition, therefore:

\[
(x_{jk} - x_j) = d_{jk} n_{jk} \quad \text{et} \quad (x_{jk} - x_k) = -d_{kj} n_{jk}.
\]

Asymptotic limit of Jin-Levermore scheme: Two-Points-Flux scheme

\[
|\Omega_j| E_n + 1_j - E_j \Delta t - \frac{1}{\sigma} \sum_{k,l} d_{jk} E_n k - E_j d (x_j, x_k) = 0.
\]
2D Extension : difficulties

- **Jin-Levermore method**: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

  \[
  \begin{align*}
  E_j & \sim E_{jk} - \sigma \varepsilon (F_{jk}, x_j - x_{jk}), \\
  E_k & \sim E_{jk} - \sigma \varepsilon (F_{jk}, x_k - x_{jk}).
  \end{align*}
  \]

  Plugging the previous relations in the acoustic solver, we obtain:

  \[
  \begin{align*}
  (F_j, n_{jk}) + E_j &= (F_{jk}, n_{jk}) + E_{jk}, \\
  (F_k, n_{jk}) - E_k &= (F_{jk}, n_{jk}) - E_{jk}.
  \end{align*}
  \]
Jin-Levermore method: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

Discrete equivalent

\[
\begin{align*}
E_j &\approx E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, x_j - x_{jk}), \\
E_k &\approx E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, x_k - x_{jk}).
\end{align*}
\]

Plugging the previous relations in the acoustic solver, we obtain:

\[
\begin{align*}
(F_j, n_{jk}) + E_j = (F_{jk}, n_{jk}) + E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, (x_j - x_{jk})), \\
(F_k, n_{jk}) - E_k = (F_{jk}, n_{jk}) - E_{jk} + \frac{\sigma}{\varepsilon} (F_{jk}, (x_k - x_{jk})).
\end{align*}
\]
2D Extension: difficulties

- **Jin-Levermore method**: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

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E_j &\approx E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, x_j - x_{jk}), \\
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\]

Plugging the previous relations in the acoustic solver, we obtain:

\[
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(F_j, n_{jk}) + E_j &= (F_{jk}, n_{jk}) + E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, (x_j - x_{jk})), \\
(F_k, n_{jk}) - E_k &= (F_{jk}, n_{jk}) - E_{jk} + \frac{\sigma}{\varepsilon} (F_{jk}, (x_k - x_{jk})).
\end{align*}
\]

- To solve this system we need a geometrical assumption.

- **Assumption**: The mesh satisfy the Delaunay condition, therefore:

\[
(x_{jk} - x_j) = d_{jk} n_{jk} \text{ et } (x_{jk} - x_k) = -d_{kj} n_{jk}.
\]
**2D Extension : difficulties**

- **Jin-Levermore method**: modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion:

  Discrete equivalent

  \[
  \begin{align*}
  E_j &\approx E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, x_j - x_{jk}), \\
  E_k &\approx E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, x_k - x_{jk}).
  \end{align*}
  \]

  Plugging the previous relations in the acoustic solver, we obtain:

  \[
  \begin{align*}
  (F_j, n_{jk}) + E_j &= (F_{jk}, n_{jk}) + E_{jk} - \frac{\sigma}{\varepsilon} (F_{jk}, (x_j - x_{jk})) , \\
  (F_k, n_{jk}) - E_k &= (F_{jk}, n_{jk}) - E_{jk} + \frac{\sigma}{\varepsilon} (F_{jk}, (x_k - x_{jk})).
  \end{align*}
  \]

- **To solve this system we need a geometrical assumption.**

- **Assumption**: The mesh satisfy the Delaunay condition, therefore:

  \[(x_{jk} - x_j) = d_{jk} n_{jk} \text{ et } (x_{jk} - x_k) = -d_{kj} n_{jk} \].

**Asymptotic limit of Jin-Levermore scheme : Two-Points-Flux scheme**

\[
\left| \Omega_j \right| \frac{E_j^{n+1} - E_j^n}{\Delta t} - \frac{1}{\sigma} \sum_k I_{jk} \frac{E_k^n - E_j^n}{d(x_j, x_k)} = 0. 
\]
Non convergence of Two-Points-Flux diffusion scheme

- Two-Points-Flux scheme does not converge on distorted meshes. Test case: we take as initial condition the fundamental solution of the heat equation at the time $t = 0.001$, final time $t_f = 0.010$.

- Convergence results on Cartesian mesh and Random quadrangular mesh.

To our knowledge, there were no AP schemes on unstructured meshes before this study.
AP schemes in 1D and difficulties in 2D

AP schemes for hyperbolic heat equation in 2D

AP schemes for angular approximations of the transport equation

Numerical results

Conclusion

AP schemes for hyperbolic heat equation in 2D
Notations for nodal finite volume schemes

**Idea:**

use **nodal** formulation of finite volume methods introduced in Lagrangian hydrodynamics to discretize the wave equation and couple this scheme with the Jin-Levermore method.
Notations for nodal finite volume schemes

Idea:

use nodal formulation of finite volume methods introduced in Lagrangian hydrodynamics to discretize the wave equation and couple this scheme with the Jin-Levermore method.

- We introduce the nodal formulation
- $l_{jr} \mathbf{n}_{jr} = \left( \begin{array}{c} -y_{r-1} + y_{r+1} \\ x_{r-1} - x_{x+1} \end{array} \right)$.
- $\sum_j l_{jr} \mathbf{n}_{jr} = \sum_r l_{jr} \mathbf{n}_{jr} = 0$.
- $F_r$ and $E_{n_{jr}}$ are the fluxes associated to the node $X_r$.
- $V_r$ is the control volume around the node $X_r$. 
Construction of the nodal scheme

GLACE AP scheme

\[
\begin{aligned}
\Omega_j & \quad \partial_t E_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr}(F_r, n_{jr}) = 0, \\
\Omega_j & \quad \partial_t F_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr} E_{njr} = S_j.
\end{aligned}
\]

- Classical nodal solver:

\[
\begin{aligned}
E_{njr} - E_j n_{jr} &= \hat{\alpha}_{jr}(F_j - F_r), \\
\sum_j l_{jr} E_{njr} &= 0,
\end{aligned}
\]

with \( \hat{\alpha}_{jr} = n_{jr} \otimes n_{jr} \).
Introduction

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Construction of the nodal scheme

GLACE AP scheme

\[
\begin{align*}
|\Omega_j| \partial_t E_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr}(F_r, n_{jr}) &= 0, \\
|\Omega_j| \partial_t F_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr} E_{jr} &= S_j.
\end{align*}
\]

- Classical nodal solver:

  \[
  \begin{cases}
    E_{jr} - E_j n_{jr} = \hat{\alpha}_{jr}(F_j - F_r), \\
    \sum_j l_{jr} E_{jr} = 0,
  \end{cases}
  \]

  with \( \hat{\alpha}_{jr} = n_{jr} \otimes n_{jr} \).

- Modified nodal solver: plugging \( \nabla E = -\frac{\sigma}{\varepsilon} F \) in the fluxes

  \[
  \begin{cases}
    E_{jr} - E_j n_{jr} = \hat{\alpha}_{jr}(F_j - F_r), \\
    \left( \sum_j l_{jr} \hat{\alpha}_{jr} \right) F_r = \sum_j l_{jr} E_j n_{jr} + \sum_j l_{jr} \hat{\alpha}_{jr} F_j.
  \end{cases}
  \]
Construction of the nodal scheme

GLACE AP scheme

| \Omega_j | \partial_t E_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr}(F_r, n_{jr}) = 0, \\
| \Omega_j | \partial_t F_j(t) + \frac{1}{\varepsilon} \sum_r l_{jr} E_{njr} = S_j.

- Classical nodal solver:

\[
\begin{align*}
E_{njr} - E_j n_{jr} &= \hat{\alpha}_{jr} (F_j - F_r), \\
\sum_j l_{jr} E_{njr} &= 0,
\end{align*}
\]

with \( \hat{\alpha}_{jr} = n_{jr} \otimes n_{jr} \).

- Modified nodal solver: plugging \( \nabla E = -\frac{\sigma}{\varepsilon} F \) in the fluxes

\[
\begin{align*}
E_{njr} - E_j n_{jr} &= \hat{\alpha}_{jr} (F_j - F_r) - \frac{\sigma}{\varepsilon} \hat{\beta}_{jr} F_r, \\
(\sum_j l_{jr} \hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j l_{jr} \hat{\beta}_{jr}) F_r &= \sum_j l_{jr} E_j n_{jr} + \sum_j l_{jr} \hat{\alpha}_{jr} F_j,
\end{align*}
\]

with \( \hat{\beta}_{jr} = n_{jr} \otimes (x_r - x_j) \).
## GLACE AP scheme

\[
\begin{align*}
|\Omega_j| \left| \partial_t E_j(t) + \frac{1}{\epsilon} \sum_r l_{jr} (F_r, n_{jr}) \right| &= 0, \\
|\Omega_j| \left| \partial_t F_j(t) + \frac{1}{\epsilon} \sum_r l_{jr} E_{njr} \right| &= S_j.
\end{align*}
\]

- **Classical nodal solver:**
  \[
  \begin{align*}
  E_{njr} - E_j n_{jr} &= \hat{\alpha}_{jr} (F_j - F_r), \\
  \sum_j l_{jr} E_{njr} &= 0,
  \end{align*}
  \]
  with \( \hat{\alpha}_{jr} = n_{jr} \otimes n_{jr} \).

- **Modified nodal solver:** plugging \( \nabla E = -\frac{\sigma}{\epsilon} F \) in the fluxes
  \[
  \begin{align*}
  E_{njr} - E_j n_{jr} &= \hat{\alpha}_{jr} (F_j - F_r) - \frac{\sigma}{\epsilon} \hat{\beta}_{jr} F_r, \\
  \left( \sum_j l_{jr} \hat{\alpha}_{jr} + \frac{\sigma}{\epsilon} \sum_j l_{jr} \hat{\beta}_{jr} \right) F_r &= \sum_j l_{jr} E_j n_{jr} + \sum_j l_{jr} \hat{\alpha}_{jr} F_j,
  \end{align*}
  \]
  with \( \hat{\beta}_{jr} = n_{jr} \otimes (x_r - x_j) \).

- **Source term discretization:** (1) \( S_j = -\frac{\sigma}{\epsilon^2} |\Omega_j| F_j \), (2) \( S_j = -\frac{\sigma}{\epsilon^2} \sum_r \hat{\beta}_{jr} F_r \).
Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
- The scheme with the discretization (2) of the source term is equal to

\[
\begin{align*}
|\Omega_j| \frac{E_j^{n+1} - E_j}{\Delta t} + \frac{1}{\varepsilon} \sum_r l_{jr} (M_r F_r, n_{jr}) &= 0, \\
|\Omega_j| \frac{F_j^{n+1} - F_j}{\Delta t} + \frac{1}{\varepsilon} \sum_r l_{jr} E_{jr} &= -\frac{1}{\varepsilon} \left( \sum_r l_{jr} \hat{\alpha}_{jr} (\hat{l}_{jr} - M_r) \right) F_j^{n+1}.
\end{align*}
\]

with

\[
\begin{align*}
E_{jr} - E_{j} n_{jr} &= \hat{\alpha}_{jr} M_r (F_j - F_r), \\
(\sum_j l_{jr} \hat{\alpha}_{jr}) F_r &= \sum_j l_{jr} E_{jr} n_{jr} + \sum_j l_{jr} \hat{\alpha}_{jr} F_j.
\end{align*}
\]

\[
M_r = \left( \sum_j l_{jr} \hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j l_{jr} \hat{\beta}_{jr} \right)^{-1} \left( \sum_j l_{jr} \hat{\alpha}_{jr} \right).
\]
Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
- The scheme with the discretization (2) of the source term is equal to

\[
\begin{align*}
\left\{ \begin{array}{l}
\Omega_j \left| \frac{E_{j}^{n+1} - E_{j}^{n}}{\Delta t} \right| + \frac{1}{\varepsilon} \sum_{r} l_{jr} (M_r F_r, n_{jr}) = 0, \\
\Omega_j \left| \frac{F_{j}^{n+1} - F_{j}^{n}}{\Delta t} \right| + \frac{1}{\varepsilon} \sum_{r} l_{jr} E_{n_{jr}} = -\frac{1}{\varepsilon} \left( \sum_{r} l_{jr} \hat{\alpha}_{jr} (I_d - M_r) \right) F_{j}^{n+1}.
\end{array} \right. \\
\end{align*}
\]

with

\[
M_r = \left( \sum_{j} l_{jr} \hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_{j} l_{jr} \hat{\beta}_{jr} \right)^{-1} \left( \sum_{j} l_{jr} \hat{\alpha}_{jr} \right).
\]

- The matrix $M_r$ generalize the coefficient $M$ introduced in the 1D schemes.
Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
- The scheme with the discretization (2) of the source term is equal to

\[
\begin{aligned}
| \Omega_j | \left\{ \begin{array}{l}
\frac{E_j^{n+1} - E_j}{\Delta t} + \frac{1}{\varepsilon} \sum_l l_{jr} (M_r F_r, n_{jr}) = 0, \\
\frac{F_j^{n+1} - F_j}{\Delta t} + \frac{1}{\varepsilon} \sum_l l_{jr} E_{n_{jr}} = -\frac{1}{\varepsilon} \left( \sum_l l_{jr} \hat{\alpha}_{jr} (\hat{l}_d - M_r) \right) F_{j}^{n+1}.
\end{array} \right.
\end{aligned}
\]

with

\[
\begin{aligned}
\{ & E_{n_{jr}} - E_j n_{jr} = \hat{\alpha}_{jr} M_r (F_j - F_r), \\
& (\sum_l l_{jr} \hat{\alpha}_{jr}) F_r = \sum_l l_{jr} E_j n_{jr} + \sum_j l_{jr} \hat{\alpha}_{jr} F_j.
\end{aligned}
\]

\[
M_r = \left( \sum_j l_{jr} \hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j l_{jr} \hat{\beta}_{jr} \right)^{-1} \left( \sum_j l_{jr} \hat{\alpha}_{jr} \right).
\]

- The matrix $M_r$ generalize the coefficient $M$ introduced in the 1D schemes
- We implicit the source term to obtain a scheme with a CFL condition independent of $\varepsilon$. 

Diffusion limit

**Proposition**

When $\varepsilon$ tends to zero the limit scheme is:

$$\begin{aligned}
|\Omega_j| \frac{\partial_t E_j(t)}{\partial t} - \sum_r l_{jr}(F_r, n_{jr}) &= 0, \\
\sigma A_r F_r &= \sum_j l_{jr} E_j n_{jr}, \\
A_r &= -\sum_j l_{jr} n_{jr} \otimes (x_r - x_j).
\end{aligned}$$
Diffusion limit

**Proposition**

When $\varepsilon$ tends to zero the limit scheme is:

\[
\begin{align*}
\Omega_j \mid \partial_t E_j(t) - \sum_r l_{jr}(F_r, n_{jr}) &= 0, \\
\sigma A_r F_r &= \sum_j l_{jr} E_j n_{jr}, \quad A_r = -\sum_j l_{jr} n_{jr} \otimes (x_r - x_j).
\end{align*}
\]

\[
\| e(t) \|_{L^2(\Omega)}^2 = \sum_j | \Omega_j | (E_j(t) - E(x_j, t))^2.
\]

\[
\| f(t) \|_{L^2([0,t] \times \Omega)}^2 = \int_0^t \sum_r | V_r | (F_r(t) - \nabla E(x_r, t))^2.
\]

**Theorem**

If the matrix $A_r^S$ satisfies $A_r^S \geq \alpha V_r$ with $\alpha$ a constant then the semi-discrete diffusion scheme is convergent for all time $T > 0$ with the estimation:

\[
\| E(t) \|_{L^2(\Omega)} + \| f(t) \|_{L^2([0,t] \times \Omega)} \leq C(T) h.
\]
Diffusion limit

Proposition

When $\varepsilon$ tends to zero the limit scheme is:

\[
\begin{cases}
|\Omega_j| \partial_t E_j(t) - \sum_r l_{jr} (F_r, n_{jr}) = 0, \\
\sigma A_r F_r = \sum_j l_{jr} E_j n_{jr}, \quad A_r = -\sum_j l_{jr} n_{jr} \otimes (x_r - x_j).
\end{cases}
\]

\[
\| e(t) \|_{L^2(\Omega)}^2 = \sum_j |\Omega_j| (E_j(t) - E(x_j, t))^2.
\]

\[
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If the matrix $A^S_r$ satisfies $A^S_r \geq \alpha V_r$ with $\alpha$ a constant then the semi-discrete diffusion scheme is convergent for all time $T > 0$ with the estimation:

\[
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\]

- The nodal AP scheme with the discretization (2) of the source term is stable in norm $L^2$. 

Presentation 20 / 36
AP schemes for angular approximations of the transport equation
We introduce the Friedrichs system with stiff source term

\[ \partial_t u + \frac{1}{\varepsilon} A \partial_x u + \frac{1}{\varepsilon} B \partial_y u = -\frac{\sigma}{\varepsilon^2} R u, \quad u \in \mathbb{R}^n \]

- \( A, B, R \) are symmetric matrices and \( R \) is positive.
Friedrichs systems with stiff source terms

- We introduce the Friedrichs system with stiff source term
  \[ \partial_t u + \frac{1}{\varepsilon} A \partial_x u + \frac{1}{\varepsilon} B \partial_y u = -\frac{\sigma}{\varepsilon^2} Ru, \quad u \in \mathbb{R}^n \]

- \( A, B, R \) are symmetric matrices and \( R \) is positive.

**Lemma**

We note \( E_i \) the eigenvectors of \( R \) with \( \text{Ker} R = \text{vect}(E_1...E_p) \). There are two particular vectors associated to the eigenvalues \( \lambda_{p+1}, \lambda_{p+2} \). We assume that

\[
\begin{align*}
AE_i &= \gamma_i E_{p+1}, \quad \forall i \in \{1..p\}, \\
BE_i &= \delta_i E_{p+2}, \quad \forall i \in \{1..p\},
\end{align*}
\]

therefore \( ((u, E_1), ..., (u, E_p)) \) tends to \( v \in \mathbb{R}^p \) when \( \varepsilon \) tends to zero with

\[
\partial_t v - \frac{1}{\lambda_{i_1} \sigma} K_1 \partial_{xx} v - \frac{1}{\lambda_{i_2} \sigma} K_2 \partial_{yy} v = 0,
\]

and \( K_1, K_2 \) symmetric positive matrices.
\textbf{\(P_n\) and \(S_n\) models}

- \textbf{\(S_n\) models}: (discrete ordinate methods) which approximate the scattering operator with a quadrature formula.
- \textbf{Properties of \(S_n\) models}: \(A, B\) diagonal matrices, \(\dim \ker R = 1\), \(R\) symmetric positive for the variable \(u_i = \sqrt{w_i} f(\Omega_i)\).
- \(w_i\) the quadrature weight, \(\Omega_i\) the quadrature speed and \(f\) the solution of the transport equation.
**$P_n$ and $S_n$ models**

- **$S_n$ models**: (discrete ordinate methods) which approximate the scattering operator with a quadrature formula.

- **Properties of $S_n$ models**: $A, B$ diagonal matrices, $\dim \text{Ker } R = 1$, $R$ symmetric positive for the variable $u_i = \sqrt{w_i} f(\Omega_i)$.

- $w_i$ the quadrature weight, $\Omega_i$ the quadrature speed and $f$ the solution of the transport equation.

- **$P_n$ models**: projection of the transport equation on the spherical harmonics basis.

- **Properties of $P_n$ models**: symmetrizable system, $R$ is defined by $R_{11} = 0$ and $R_{ii} = 1$ ($i \neq 0$).
**$P_n$ and $S_n$ models**

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**Proposition**

The $P_n$ and $S_n$ models satisfy the previous assumption of structure.
Proposition

Then we write a $P_n$ or $S_n$ model in the eigenvectors basis of $R$, we obtain

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon} \mathbf{A}' \partial_x \mathbf{v} + \frac{1}{\varepsilon} \mathbf{B}' \partial_y \mathbf{v} = -\frac{\sigma}{\varepsilon^2} \mathbf{D} \mathbf{v} \tag{5}$$

with $\mathbf{D}$ a diagonal matrix defined by $D_{11} = 0$ et $D_{ii} = 1$ ($i \neq 0$). If the assumption of structure is satisfied, then

$$\mathbf{A}' = P_{1,x} + \mathbf{A}'', \quad \mathbf{B}' = P_{1,y} + \mathbf{B}'',$$

with $\mathbf{A}_{0,j}'' = 0$, $\mathbf{A}_{i,0}'' = 0$, $\mathbf{B}_{0,j}'' = 0$, $\mathbf{B}_{i,0}'' = 0$.

- The matrices $P_{1,x}$, $P_{1,y}$ are the matrices associated to the $P_1$ system.
Proposition

Then we write a $P_n$ or $S_n$ model in the eigenvectors basis of $R$, we obtain

$$\partial_t v + \frac{1}{\varepsilon} A' \partial_x v + \frac{1}{\varepsilon} B' \partial_y v = -\frac{\sigma}{\varepsilon^2} Dv$$

with $D$ a diagonal matrix defined by $D_{11} = 0$ et $D_{ii} = 1$ ($i \neq 0$). If the assumption of structure is satisfied, then

$$A' = P_{1,x} + A''$$
$$B' = P_{1,y} + B''$$

with $A'_{0,j} = 0$, $A''_{i,0} = 0$, $B'_{0,j} = 0$, $B''_{i,0} = 0$.

- The matrices $P_{1,x}$, $P_{1,y}$ are the matrices associated to the $P_1$ system.
- **Conclusion**: The $P_n$ and $S_n$ models can be split between a $P_1$ system and a system which does not play a role in the diffusion regime.

- **Numerical method (micro-macro decomposition?)**: Split the system, discretize the $P_1$ system with an AP scheme and the other system with a classical scheme.
Final algorithm

- Decomposition algorithm for the system

\[ \partial_t u + \frac{1}{\varepsilon} A_1 \partial_x u + \frac{1}{\varepsilon} A_2 \partial_y u = - \frac{\sigma}{\varepsilon^2} Ru. \]  

(6)
Final algorithm

- Decomposition algorithm for the system

\[
\partial_t u + \frac{1}{\varepsilon} A_1 \partial_x u + \frac{1}{\varepsilon} A_2 \partial_y u = -\frac{\sigma}{\varepsilon^2} Ru. \tag{6}
\]

- **First step**: We write the system (6) in the eigenvectors basis of \( R \) to obtain

\[
\partial_t v + \frac{1}{\varepsilon} A'_1 \partial_x v + \frac{1}{\varepsilon} A'_2 \partial_y v = -\frac{\sigma}{\varepsilon^2} Du, \tag{7}
\]

with \( v = Q'u, \ A'_1 = Q'A_1 Q \) et \( A'_2 = Q'A_2 Q \).
Final algorithm

- Decomposition algorithm for the system

\[
\partial_t u + \frac{1}{\varepsilon} A_1 \partial_x u + \frac{1}{\varepsilon} A_2 \partial_y u = - \frac{\sigma}{\varepsilon^2} Ru. \tag{6}
\]

- First step: We write the system (6) in the eigenvectors basis of R to obtain

\[
\partial_t v + \frac{1}{\varepsilon} A'_1 \partial_x v + \frac{1}{\varepsilon} A'_2 \partial_y v = - \frac{\sigma}{\varepsilon^2} D u, \tag{7}
\]

with \( v = Q'u \), \( A'_1 = Q'A_1 Q \) et \( A'_2 = Q'A_2 Q \).

- Second step: We split the diagonalized system (7). We obtain

\[
\partial_t v + \frac{1}{\varepsilon} \left( P_{1,x} \partial_x v + P_{1,y} \partial_y v \right) + \frac{1}{\varepsilon} \left( A''_1 \partial_x v + A''_2 \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D v. \tag{8}
\]
Final algorithm

- Decomposition algorithm for the system

\[
\partial_t u + \frac{1}{\varepsilon} A_1 \partial_x u + \frac{1}{\varepsilon} A_2 \partial_y u = - \frac{\sigma}{\varepsilon^2} R u. \tag{6}
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\[
\partial_t v + \frac{1}{\varepsilon} A'_1 \partial_x v + \frac{1}{\varepsilon} A'_2 \partial_y v = - \frac{\sigma}{\varepsilon^2} D u, \tag{7}
\]

with \( v = Q^t u, \ A'_1 = Q^t A_1 Q \) et \( A'_2 = Q^t A_2 Q \).

- **Second step**: We split the diagonalized system (7). We obtain

\[
\partial_t v + \frac{1}{\varepsilon} \left( P_{1,x} \partial_x v + P_{1,y} \partial_y v \right) + \frac{1}{\varepsilon} \left( A''_1 \partial_x v + A''_2 \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D' v, \tag{8}
\]

with \( D' \) defined by \( D'_{22} = D'_{33} = 1 \) et \( D'_{ii} = 1 \) si \( ii \neq 22, 33 \).

- **Third step**: The system (9) is discretized with an AP scheme

\[
\partial_t v + \frac{1}{\varepsilon} \left( P_{1,x} \partial_x v + P_{1,y} \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D' v, \tag{9}
\]
Final algorithm

- Decomposition algorithm for the system

\[ \partial_t u + \frac{1}{\varepsilon} A_1 \partial_x u + \frac{1}{\varepsilon} A_2 \partial_y u = - \frac{\sigma}{\varepsilon^2} R u. \]  

(6)

- **First step**: We write the system (6) in the eigenvectors basis of \( R \) to obtain

\[ \partial_t v + \frac{1}{\varepsilon} A_1' \partial_x v + \frac{1}{\varepsilon} A_2' \partial_y v = - \frac{\sigma}{\varepsilon^2} D u, \]  

(7)

with \( v = Q'u, A_1' = Q'A_1 Q \) et \( A_2' = Q'A_2 Q \).

- **Second step**: We split the diagonalized system (7). We obtain

\[ \partial_t v + \frac{1}{\varepsilon} \left( P_{1,x} \partial_x v + P_{1,y} \partial_y v \right) + \frac{1}{\varepsilon} \left( A_1'' \partial_x v + A_2'' \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D' v. \]  

(8)

- **Third step**: The system (9) is discretized with an AP scheme

\[ \partial_t v + \frac{1}{\varepsilon} \left( P_{1,x} \partial_x v + P_{1,y} \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D' v, \]  

(9)

with \( D' \) defined by \( D'_{22} = D'_{33} = 1 \) et \( D'_{ii \neq 22,ii \neq 33} \).

- **Fourth step**: The system (10) is discretized with a classical scheme (Upwind, Rusanov)

\[ \partial_t v + \frac{1}{\varepsilon} \left( A_1'' \partial_x v + A_2'' \partial_y v \right) = - \frac{\sigma}{\varepsilon^2} D'' v, \]  

(10)

with \( D'' \) defined by \( D''_{11} = D''_{22} = D''_{33} = 0 \) et \( D''_{ii} = 1 \) \( i \geq 4 \).
Numerical results
Examples of unstructured meshes

Two classical examples of unstructured meshes.

Random mesh

Kershaw mesh
• In transport regime ($\varepsilon = O(1)$ and $\sigma = O(1)$), at first order the scheme converges.

• **Diffusion regime**: The initial data is given by the fundamental solution of the heat equation at the time $t = 0.001$. Final time $t_f = 0.010$.

<table>
<thead>
<tr>
<th>Mesh/ $\varepsilon$</th>
<th>$\varepsilon = 10^{-3}$</th>
<th>$\varepsilon = 10^{-4}$</th>
<th>$\varepsilon = 10^{-6}$</th>
<th>$\varepsilon = 10^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian 60-120 cells</td>
<td>1.8</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Cartesian 80-160 cells</td>
<td>1.75</td>
<td>1.97</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Cartesian 120-240 cells</td>
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<td>1.95</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Random quad. 60-120 cells</td>
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<td>2</td>
<td>2</td>
</tr>
<tr>
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<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
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<td>1.92</td>
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<td>2</td>
</tr>
<tr>
<td>Kershaw 60-120 cells</td>
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<td>2.1</td>
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<tr>
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- The scheme converges on triangular meshes with an order between 1 and 2.
- The error between the diffusion solution and the solution of the $P_1$ model is homogeneous to $O(\varepsilon)$.
- For $\Delta x = O(1)$ the order decreases. Indeed we compare the numerical solution of $P_1$ system and the exact diffusion solution.
In transport regime ($\varepsilon = O(1)$ and $\sigma = O(1)$), at first order the scheme converges.

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- The error between the diffusion solution and the solution of the $P_1$ model is homogeneous to $O(\varepsilon)$.
- For $\frac{\Delta x}{\varepsilon} = O(1)$ the order decreases. Indeed we compare the numerical solution of $P_1$ system and the exact diffusion solution.
We solve the $P_1$ model with previous test case and $\varepsilon = 0.001$. The results for the hyperbolic schemes are computed on Kershaw mesh.
Numerical results for Friedrichs systems

- Diffusion regime: previous test case.
- The order of convergence is computed with two meshes (14400 and 57600 cells):

**Table:** Order for the $P_3$ numerical solution

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<tr>
<td>Trig. reg.</td>
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<tr>
<td>Random. trig.</td>
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</tr>
<tr>
<td>Kershaw</td>
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**Table:** Order for the $S_2$ numerical solution

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<tbody>
<tr>
<td>Cartesian</td>
<td>1.80</td>
<td>1.95</td>
</tr>
<tr>
<td>Random. quad.</td>
<td>1.85</td>
<td>2</td>
</tr>
<tr>
<td>Trig. reg.</td>
<td>1.9</td>
<td>2</td>
</tr>
<tr>
<td>Random. trig.</td>
<td>1.35</td>
<td>1.35</td>
</tr>
<tr>
<td>Kershaw</td>
<td>1.85</td>
<td>1.95</td>
</tr>
</tbody>
</table>

**TAB.:** Order for the $P_3$ numerical solution

**TAB.:** Order for the $S_2$ numerical solution

- Transport test case: fundamental solution

**Fundamental solution of $P_3$**

**Fundamental solution of $S_2$**
\textit{M}_1 \text{ model}

The non-linear two moments $M_1$ model, obtained by maximizing the photon entropy, is:

\begin{align*}
\frac{\partial_t E}{\varepsilon} + \nabla \cdot F &= 0 \\
\frac{\partial_t F}{\varepsilon} + \nabla \left( \hat{P} \right) &= -\frac{\sigma}{\varepsilon^2} F,
\end{align*}

(11)

E is the energy, $F$ the radiative flux and

$$\hat{P} = \frac{1}{2} \left( (1 - \chi(f)) I_d + (3\chi(f) - 1) \frac{f \otimes f}{\|f\|} \right) E \in \mathbb{R}^{2 \times 2}$$

the radiative pressure.
The non-linear two moments $M_1$ model, obtained by maximizing the photon entropy, is:

\[
\begin{aligned}
\partial_t E + \frac{1}{\varepsilon} \nabla \cdot \mathbf{F} &= 0 \\
\partial_t \mathbf{F} + \frac{1}{\varepsilon} \nabla (\widehat{P}) &= -\frac{\sigma}{\varepsilon^2} \mathbf{F},
\end{aligned}
\]  

(11)

$E$ is the energy, $\mathbf{F}$ the radiative flux and

\[
\widehat{P} = \frac{1}{2} ((1 - \chi(f)) \text{Id} + (3\chi(f) - 1) \frac{\mathbf{f} \otimes \mathbf{f}}{\| \mathbf{f} \|^2}) E \in \mathbb{R}^{2\times2}
\]

the radiative pressure. We define $f = |\mathbf{F}| / E$ and $\chi(f) = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}}$.

The $M_1$ model satisfies

- the diffusion limit, $\varepsilon \to 0 : \partial_t E - \text{div} \left( \frac{1}{3\sigma} \nabla E \right) = 0$, First Tools : AP scheme
- the entropy property : $\partial_t S + \frac{1}{\varepsilon} \text{div} (\mathbf{Q}) \geq 0$, Second Tools : Reformulation
- the maximum principle : $E > 0$, $|f| < 1$, like a dynamic gas system

with

\[
S = \frac{E^{3/4} (1 - |u|^2)}{(3 + |u|^2)^2}, \quad u = \frac{(3\chi - 1)f}{2|f|^2}, \quad \mathbf{Q} = uS.
\]
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### The $M_1$ model satisfies

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$$

Idea : we formulate the $M_1$ model like a dynamic gas system :

- to use Lagrange+remap nodal scheme and obtain a consistent limit diffusion scheme,
- to use the entropy to preserve the maximum principle.
**Numerical method**

**New formulation**

\[
\begin{align*}
\partial_t \rho + \frac{1}{\varepsilon} \text{div}(\rho \mathbf{u}) &= 0 & \text{mass conservation} \\
\partial_t \rho \mathbf{v} + \frac{1}{\varepsilon} \text{div}(\rho \mathbf{u} \otimes \mathbf{v}) + \frac{1}{\varepsilon} \nabla q &= - \frac{\sigma}{\varepsilon^2} \rho \mathbf{v} & \text{momentum conservation} \\
\partial_t \rho e + \frac{1}{\varepsilon} \text{div}(\rho \mathbf{u} e + q \mathbf{u}) &= 0 & \text{total conservation energy} \\
\partial_t \rho s + \frac{1}{\varepsilon} \text{div}(\rho \mathbf{u} s) &\geq 0 & \text{Entropy inequality}
\end{align*}
\]

- **\( q = \frac{1 - \chi}{2} E \)** the radiative flux \( E = \rho e \) the radiative energy \( S = \rho s \).
- The \( M_1 \) is independent of the density.
- **\( F = \rho \mathbf{v} \)** the radiative flux \( E = \rho e \) the radiative energy \( S = \rho s \).
- **\( \hat{P} = \mathbf{u} \otimes F + q I_d \)**
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\]

- \( q = \frac{1 - \chi}{2} E \), \( \mathbf{u} = \frac{3\chi - 1}{2} \frac{\mathbf{f}}{||\mathbf{f}||^2} \)
- The \( M_1 \) is independent of the density.
- \( \mathbf{F} = uE + q \mathbf{u} \), \( \mathbf{P} = u \otimes \mathbf{F} + qI \)

Numerical discretization

- We use the Lagrange+remap nodal GLACE scheme coupled with the Jin-Levermore method.
- We use a second order advection scheme for the remap part (when \( \varepsilon \) is small).
Numerical method

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Properties

- The scheme is entropic and preserve the maximum principle.
- The scheme is AP with a second order positive limit scheme.
- We can define a semi-implicit variant with a CFL condition independant of \( \varepsilon \).
Conclusion

- We have designed and studied AP schemes for hyperbolic heat equation valid on unstructured meshes ([1] – [2]).
- Using the previous decomposition we have obtained AP schemes for $S_n$ and $P_n$ models ([3]).
- Using the proximity between the $M_1$ model and the Euler equations, we have proposed an AP, entropic and positive scheme for $M_1$ model (non-linear radiative transfer model) based on a Lagrange+remap method ([4]).

Publications

1. C. Buet, B. Després, E. Franck *Design of asymptotic preserving schemes for hyperbolic heat equation on unstructured meshes*. Numerisch Mathematik, Online.


3. C. Buet, B. Després, E. Franck *AP schemes for Friedrichs systems with stiff relaxation on unstructured meshes. Applications to the angular discretization in transport*. In redaction.

4. C. Buet, B. Després, E. Franck *An asymptotic preserving scheme with the maximum principle for the $M_1$ model on distorted meshes*. Submitted.
Ongoing works and future works

- **Ongoing works**
  - Design of asymptotic preserving and positive scheme for $S_n$ models based on edge finite volume scheme. We propose to couple a "even-odd" formulation with a nonlinear diffusion scheme (LMP scheme).
  - Extension of the nodal scheme for Euler and Shallow water equations with gravity and friction using a Lagrange+remap approach.

- **Future works**
  - Theoretical study of the AP schemes for the $P_1$ model and the Euler equations (with C. Buet, B. Després).
  - Extension of the nodal scheme on unstructured conical meshes.
  - Design of asymptotic preserving schemes for multi-groups models.
Thank you for your attention