Numerical issues for nonlinear MHD Jorek code

Emmanuel Franck
Max Planck Institute For Plasma Physics (Munich)
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Porquerolles

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Physical context and models
**Fusion DT**: Reaction between Deuterium and tritium which product Helium and energy. The deuterium and tritium form a plasma (ionized gas).
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- **Fusion DT**: Reaction between Deuterium and tritium which product Helium and energy. The deuterium and tritium form a plasma (ionized gas).

- **Iter**: International project to prove the efficiency of controlled fusion as a power source. Iter is an experimental power plant using fusion.

- **Magnetic confinement**: The plasma obtained by the reaction is confined in the center of the reactor (tokamak) using a powerful magnetic field.

- **Tokamak**: Toroidal room used for the plasma confinement.

**Figure**: Tokamak
The dynamic of the plasmas in Iter is a very difficult multiscale problem.

**Figure:** Spatial and time scales
Models for Iter

- The dynamic of the plasmas in Iter is a very difficult multiscale problem.

- We have different models for the different time and space scales:
  - Kinetic Vlasov-Maxwell equation not used in practice (CPU cost very important).
  - Gyrokinetic approximation of the Vlasov-Maxwell equation used for the turbulence in the core Tokamak.
  - MagnetoHydrodynamics fluids models (resistive MHD, two fluids MHD) used to simulate the edge instabilities.

![Spatial and time scales](image.png)

**Figure:** Spatial and time scales
An edge-localized mode ("ELM's") is a disruptive instability occurring in the edge region of a tokamak plasma.

The development of edge-localized modes poses an important challenge in magnetic fusion research with tokamaks. Instabilities can damage wall components due to their extremely high energy transfer rate.

**Aim:** simulate the ELM's to estimate the amplitude of these instabilities and understand how control these.

The full - resistive MHD model is given by

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= \nabla \cdot (D_{||} \nabla_{||} \rho + D_{\perp} \nabla_{\perp} \rho) + S_p \\
\rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla (\rho T) &= \mathbf{J} \times \mathbf{B} + \nu \nabla \mathbf{v} \\
\rho \partial_t T + \rho \mathbf{v} \cdot \nabla T + (\gamma - 1) \rho T \nabla \mathbf{v} &= \nabla \cdot (K_{||} \nabla_{||} T + K_{\perp} \nabla_{\perp} T) + S_h \\
\partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \eta \mathbf{J} + S_c \\
\nabla \cdot \mathbf{B} &= 0
\end{align*}
\]  

with \( \rho \) the density, \( \mathbf{v} \) the velocity, \( T \) the temperature, \( \mathbf{B} \) the magnetic field and \( \mathbf{J} = \nabla \times \mathbf{B} \) the current.

- The terms \( D_{||}, D_{\perp}, K_{||}, K_{\perp} \) are anisotropic diffusion tensors.
- We add source terms. \( S_c \) correspond to the current source, \( S_h \) correspond to the heat source, \( S_p \) correspond to the particle source.
We consider the cylindric coordinate \((R, Z, \phi) \in \Omega \times [0, 2\pi]\).

\((R, Z)\) correspond to the poloidal plan and \(\phi\) the toroidal direction.

For the reduced MHD the quantities are \(\rho, T, \Psi, u, v_{||}\) the parallel velocity, \(w\) the vorticity and \(z_j\) the toroidal current.

**Derivation:** Plug \(B\) and \(v\) in the density, magnetic and energy equations. For the equations on \(u\) and \(v_{||}\) we use

\[ e_\phi \cdot \nabla \times (\rho \partial_t v + \rho v \cdot \nabla v + \nabla (\rho T) = J \times B + \nu \Delta v) \]

and

\[ B \cdot (\rho \partial_t v + \rho v \cdot \nabla v + \nabla (\rho T) = J \times B + \nu \Delta v) . \]
Basic Reduced MHD: model 199

With $v_\| = 0$ we obtain the model 199 considered in this talk.

We solve $\partial_t A(U) = B(U, t)$ with

$$B(U) = \left\{ \begin{array}{l}
\left[ \Psi, u \right] - \epsilon \frac{F_0}{R} \partial_\phi u + \frac{\eta(T)}{R} (z_j - S_c(\Psi)) - \eta_n \nabla \cdot (\nabla z_j) \\
\frac{1}{2} [R^2 ||\nabla u||^2, \rho] + [R^2 \hat{\rho}w, u] + [\Psi, z_j] - \epsilon \frac{F_0}{R} \partial_\phi z_j - [R^2, p] \\
+ \nabla \cdot (R\nu(T) \nabla w) - \nu_n \nabla \cdot (\nabla w) \\
\frac{1}{R^2} z_j - \nabla \cdot \left( \frac{1}{R^2} \nabla \Phi \right) \\
w - \nabla \cdot (\nabla u) \\
R^2 [\rho, u] + 2R\rho \partial_Z u + \nabla \cdot (D_{||} \nabla_{||} \rho + D_\perp \nabla_\perp \rho) + S_\rho(\Psi) \\
R^2 [T, u] + 2(\gamma - 1)RT \partial_Z u + \nabla \cdot (K_{||} \nabla_{||} T + K_\perp \nabla_\perp T) + S_h(\Psi) \end{array} \right. $$

with $\hat{\rho} = R^2 \rho$ and $\partial_t A(U) = \left( \frac{1}{R} \partial_t \Psi, \ R \nabla \cdot (\hat{\rho} \nabla (\partial_t u)), \ 0, \ 0, \ R \partial_t \rho, \ R \partial_t T \right)$.

- Physical and numerical resistivity: $\eta$ and $\eta_n$, viscosity coefficients: $\nu$ and $\nu_n$. 

Emmanuel Franck Max Planck Institute For Plasma Physics (M) Numerical issues for nonlinear MHD Jorek code
Jorek Code: description
Description of the jorek code I

- Jorek: code Fortran 90, parallel (MPI+OpenMP) + algebraic libraries (Pastix, MUMPS ...)

- Initialization

- Determinate the equilibrium
  - Define the boundary of the computational domain
  - Create a first grid which is used to compute the aligned grid
  - Compute $\psi(R, Z)$ in the new grid.

- Compute equilibrium
  - Solve the Grad-Shafranov equation

$$R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} = -R^2 \frac{\partial p}{\partial \psi} - F \frac{\partial F}{\partial \psi}$$
Description of the jorek code II

- Computation of aligned grid
  - Identification of the magnetic flux surfaces
  - Create the aligned grid (with x-point)
  - Interpolate $\psi(R, Z)$ in the new grid.

- Recompute equilibrium of the new grid.

- Time-stepping (restart)
  - Construction of the matrix and some profiles (diffusion tensors, sources terms)
  - Solve linear system
  - Update solutions
  - Plot kinetic magnetic energies and restart files.

Figure: Aligned grid
Spatial discretization

- The equation in the **poloidal plane** are discretized using finite element method.
- For the **toroidal direction**: Fourier expansion.
- Basis functions: Cubic Bezier elements
  - Generalization of cubic Hermite elements.
  - The generalization allows the local refinement of each element essential for adaptive mesh refinement.
  - 4 degrees of freedom by node to describe a function (9 for Lagrange cubic finite element).
  - With the isoparametric formulation (discretization of \((R, Z)\) using the Bezier elements) the finite elements can be accurately aligned with the equilibrium flux surfaces.
  - The Cubic Bezier elements assure a \(C^1\) polynomial reconstruction.

*Bezier surfaces and finite elements for MHD simulations*, O. Czarny, G. Huysmans, JCP 2088.
Time scheme in Jorek code

- We recall the model $\partial_t A(U) = B(U, t)$
- For time stepping we use a Crank Nicholson or BDF2 scheme:

$$
(1 + \zeta)A(U^{n+1}) - \zeta A(U^n) + \zeta A(U^{n-1}) = \theta \Delta t B(U^{n+1}) + (1 - \theta) \Delta t B(U^n)
$$

- Defining $G(U) = (1 + \zeta)A(U) - \theta \Delta t B(U)$ and

$$
b(U^n, U^{n-1}) = (1 + 2\zeta)A(U^n) - \zeta A(U^{n-1}) + (1 - \theta) \Delta t B(U^n)
$$

we obtain the non linear problem

$$
G(U^{n+1}) = -G(U^n) + b(U^n, U^{n-1})
$$

- First order linearization

$$
\left(\frac{\partial G(U^n)}{\partial U^n}\right) \delta U^n = -G(U^n) + b(U^n, U^{n-1})
$$

with $\delta U^n = U^{n+1} - U^n$ and $\frac{\partial G(U^n)}{\partial U^n}$ the Jacobian of $G(U^n)$. 
Time scheme in Jorek code

- Linear solver in Jorek:
  - Case 1: Direct solver using Pastix (using when \( n_{tor} = 1 \))
  - Case 2: Iterative solver

- Iterative Solver step 1: Preconditioning
  - Extraction of submatrices associated to each toroidal harmonics.
  - Factorization of each submatrix
  - We solve exactly (with Pastix) each subsystems.
  - We construct the initial vector of GMRES using the solutions of these systems.

- Iterative solver step 2: GMRES solver for the global matrix.
  - The matrix product vector is preconditioned with the solutions of each subsystems.

- Principle: Construction of initial GMRES data + right preconditioning with an approximation of the Jacobian where the coupling between the Fourier harmonics are neglected.

  In practice for some test cases this coupling is strongly nonlinear.
Jorek code: Non convergence

Problem:
- For some test cases the GMRES method does not converge in the nonlinear phase:
  - model 199: non convergence for large time step. With very small time steps we obtain the convergence.
  - other models: in some case the GMRES method does not converge for any time step.

Why?
- The preconditioning is not adapted to obtain a robust GMRES method?
- The spatial poloidal and toroidal discretizations is not adapted?
- The mesh is not adapted?
- The models are not stables or well-posed?
Numerical example

- evolution of energy in time

![Graph showing energy evolution over normalized time](image)

![Density distribution](image)
Numerical example

- **Density**

- evolution of energy in time
Numerical example

- **Density**

- **evolution of energy in time**

![Graph showing energy evolution over time](image-url)
Current works on the time discretization
Inexact Newton scheme

- At the time step $n$, we compute $b(U^n, U^{n-1}), G(U^n)$.
- We choose $U_0 = U^n$ and $\varepsilon_0$.
- Step $k$ of the Newton procedure:
  - We compute $G(U_k)$ and $\left(\frac{\partial G}{\partial U_k}\right)$.
  - We solve the linear system with GMRES:
    
    $$\left(\frac{\partial G(U_k)}{\partial U_k}\right) \delta U_k = \tilde{G}(U_k) = b(U^n, U^{n-1}) - G(U_k)$$

    and the following convergence criterion:

    $$\frac{\| \left(\frac{\partial G}{\partial U_k}\right) \delta U_k + \tilde{G}(U_k) \|}{\| \tilde{G}(U_k) \|} \leq \varepsilon_k, \quad \varepsilon_k = \gamma \left(\frac{\| \tilde{G}(U_k) \|}{\| \tilde{G}(U_{k-1}) \|}\right)^\alpha$$

- We iterate with $U_{k+1} = U_k + \delta U_k$.
- We apply the convergence test (for example $\| \tilde{G}(U_k) \| < \varepsilon_a + \varepsilon_r \| \tilde{G}(U^n) \|$).
- If the newton procedure stops we define $U^{n+1} = U_{k+1}$. 
Preconditioning idea I

- **Aim**: Construct an algorithm which give a good prediction of the solution and which is easy to solve.
  - The algorithm must give a solution of \( A \delta U^n = -G(U^n) + b(U^n, U^{n-1}) \)
    with \( A \approx \frac{\partial G(U^n)}{\partial U^n} \).
  - \( A \) must be well-conditioned. **Idea**: parabolization of the coupled hyperbolic equations.

- **Example**
  \[
  \begin{align*}
  \partial_t u &= \partial_x v \\
  \partial_t v &= \partial_x u
  \end{align*}
  \rightarrow \begin{align*}
  u^{n+1} &= u^n + \Delta t \partial_x v^{n+1} \\
  v^{n+1} &= v^n + \Delta t \partial_x u^{n+1}
  \end{align*}
  \]

- We obtain \( (1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n \).
- The matrix associated to \( (1 - \Delta t^2 \partial_{xx}) \) is diagonal dominant matrix.

Preconditioning idea II

- To apply easily this method for more complicated equations, we propose a other interpretation.

- We assume that the matrix associated to the previous linear system is

\[
\begin{pmatrix}
D_1 & U \\
L & D_2
\end{pmatrix}
\]

- Using a Schur decomposition we obtain

\[
\begin{pmatrix}
D_1 & U \\
L & D_2
\end{pmatrix} = \begin{pmatrix}
I & UD_2^{-1} \\
0 & I
\end{pmatrix} \begin{pmatrix}
P_{schur} & 0 \\
0 & D_2
\end{pmatrix} \begin{pmatrix}
I & 0 \\
D_2^{-1}L & I
\end{pmatrix}
\]

\[
\begin{pmatrix}
I & -\Delta t \partial_x \\
-\Delta t \partial_x & I
\end{pmatrix} = \begin{pmatrix}
I & -\Delta t \partial_x \\
0 & I
\end{pmatrix} \begin{pmatrix}
P_{schur} & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-\Delta t \partial_x & I
\end{pmatrix}
\]

- All the matrices are triangular or diagonal and easily to invert.

- \( P_{schur} = D_1 - UD_2^{-1}L = (1 - \Delta t^2 \partial_{xx}) \) is diagonal dominant matrix.
Preconditioning with Schur decomposition for MHD

- We apply the Schur decomposition to the model 199. The system solved is

\[
\frac{\partial G(U^n)}{\partial U^n} \delta U^n = \begin{pmatrix}
D_\psi & 0 & D_\psi, T & D_\psi, z_j & 0 & U_{\psi, u} \\
0 & D_\rho & 0 & 0 & 0 & U_{\rho, u} \\
0 & 0 & D_T & 0 & 0 & U_{T,u} \\
D_{z_j, \psi} & 0 & 0 & D_{z_j} & 0 & 0 \\
0 & 0 & 0 & 0 & D_w & D_{w,u} \\
L_{u, \psi} & L_{u, \rho} & L_{u, T} & L_{u, z} & L_{u, w} & D_u
\end{pmatrix} \delta U^n = \tilde{G}(U^n)
\]

with \( \delta U^n = (\delta \psi, \delta \rho, \delta T, \delta z_j, \delta w, \delta u) \) and \( \tilde{G}(U^n) = -G(U^n) + b(U^n, U^{n-1}) \).

- The terms \( D \) contains advection and diffusion operators.
- The terms \( L \) and \( U \) contains non linear coupling hyperbolic operators.
- We reduce the number on variable using the definition of \( w \) and \( z_j \).

\[
\frac{\partial G(U^n)}{\partial U^n} \delta U^* = \begin{pmatrix}
D_\psi^* & 0 & D_\psi^*, T & U_{\psi, u} \\
0 & D_\rho & 0 & U_{\rho, u} \\
0 & 0 & D_T & U_{T,u} \\
L_{u, \psi}^* & L_{u, \rho}^* & L_{u, T}^* & D_u^*
\end{pmatrix} \delta U^*
\]

with \( \delta U = (\delta \psi, \delta \rho, \delta T, \delta u) \)
Preconditioning: Algorithm

The final system with Schur decomposition is given by

\[
\delta U^n = \frac{\partial G(U^n)}{\partial U^n}^{-1} \tilde{G}(U^n) = \left( \begin{array}{cc}
M & U \\
L & D_u
\end{array} \right)^{-1} \tilde{G}(U^n)
\]

\[
= \left( \begin{array}{cc}
I & M^{-1}U \\
0 & I
\end{array} \right) \left( \begin{array}{cc}
M^{-1} & 0 \\
0 & P_{schur}^{-1}
\end{array} \right) \left( \begin{array}{cc}
I & 0 \\
-LM^{-1} & I
\end{array} \right) \tilde{G}(U^n)
\]

with \( P_{schur} = D_u^* - LM^{-1}U \).

- \( M, D_u^* \) are associated to the advection and diffusion operators. \( L, U \) are associated to the hyperbolic coupling operators.

Final PC-Algorithm

\[
\begin{cases}
\text{Predictor : } & M \delta v^n_P = (-G_v^n + B_v^n) \\
\text{potential update : } & P_{schur} \delta u^n = (-L \delta v^n_P - G_u^n + B_u^n) \\
\text{Corrector : } & M \delta v^n = M \delta v^n_P - U \delta u^n \\
\text{diffusion, update : } & D_{z_j} \delta z_j^n = D_{z_j,\psi} \delta \psi^n & D_w \delta w^n = D_{w,u} \delta u^n
\end{cases}
\]

with \( \delta v = (\delta \Psi, \delta \rho, \delta T) \), \( G_v \) and \( B_v \) the right hand side for the equations on \( \Psi, \rho \) and \( T \).
The Schur complement $P_{schur} = D_u^* - LM^{-1}U$ necessity to known the matrix $M^{-1}$.

We must approximate $P_{schur}$. Exemple of approximations:

First example of approximation:
- In $P_{schur}$ we assume that $M^{-1} \approx \Delta t$
- **Mathematical problem**: estimate the operator $LU$.

Second example of approximation:
- We introduce a operator $M_*$ (in $u$-space) with $UM_* \approx MU$.
- $P_{Schur} = (D_uM_* - LU)M_*^{-1}$ with $LU$ given by the small flow approximation.
- In this case the Potential udapte step in given by

$$\begin{cases}
\text{potential update I : } (D_uM_* - LU)\delta u^{*,n} = (-L\delta v^p_n - G^n_u + B^n_u) \\
\text{potential update II : } \delta u^n = M_* \delta u^{*,n}
\end{cases}$$

- **Mathematical problem**: estimate the operator $M_*$. 
The PC preconditioning method uses a prediction of the solution based on the approximation of the Schur complement.

It is possible that this prediction of the solution is better than the previous method used in Jorek.

However, it is possible that each step of the PC algorithm also admits a problem of conditioning. But since we have a parabolization of the equations and diagonal dominant matrices, add algebraic preconditioning as multigrid methods can be performing.

For the step where we solve diffusion and advection operators, the previous preconditioning method can be used.
Future way for the time scheme
AP schemes for anisotropic diffusion in Jorek

- Anisotropic diffusion

\[
\partial_t \rho - \nabla \cdot (D_{||} \nabla_{||} \rho + D_\perp \nabla_\perp \rho) = 0 \quad \text{with} \quad D_\perp / D_{||} \ll 1
\]

- It is known that the anisotropic diffusion operators are **ill-conditioned**.

- Big problem of conditioning come from to the hyperbolic coupling terms. But the anisotropic diffusion operators contained in the matrix \(M\) can be generate problems for some test cases.

- The initial Preconditioning algorithm of Jorek is efficient to treat these terms but the CPU time associated is important.

- We propose:
  - Determinate if the conditioning of \(M\) (advection and diffusion terms) is mainly impacted by the anisotropic diffusion.
  - **Use AP scheme for these terms** to avoid to use a preconditioning and decrease the CPU time.
Anisotropic diffusion in jorek

- Application in the jorek code. Example of diffusion operator:

\[-\nabla \cdot \left( (D_{||} - D_{\perp}) \frac{B \otimes B}{||B||^2} \nabla \rho + D_{\perp} \nabla \rho \right) = 0\]

with, for example, the constants $D_{||} = O(1)$, $D^1_{\perp} = O(\varepsilon)$, $D^2_{\perp} = O(1)$, $D^3_{\perp} = O(1)$.

- We define $\varepsilon = D^1_{\perp}$, $\nabla_{||} = \frac{B}{||B||} \cdot (\frac{B}{||B||} \cdot \nabla \rho)$ to obtain

\[-\nabla \cdot \left( \frac{1}{\varepsilon} A_{||} \nabla_{||} \rho + A_{\perp} \nabla_{\perp} \rho \right) = 0\]

with $A_{||} = \varepsilon D_{||}$ and $A_{\perp} = D_{\perp} = O(\varepsilon)$.

- In this formulation we can apply the AP scheme.

*Asymptotic-Preserving schemes. Modeling, simulation and mathematical analysis of magnetically confined plasmas, C. Negulescu.*
Extension for others reduced MHD and full MHD

- It will be important to extend the PC-algorithm for the models with parallel velocity and full MHD.
- For the models with parallel velocity the operators $U$ and $P_{Schur}$ are applied on $u$ and $v_{||}$.
- For the full-MHD the operators $U$ and $P_{Schur}$ are applied on the complete velocity field.
- For the full-MHD we have

\[ \mathbf{v} = R\nabla u \times \mathbf{e}_\phi + Rw\mathbf{e}_\phi + \frac{1}{R^2} \nabla \perp \chi \]

with $u$, $w$, $\chi$ scalar fluxes.

- $u$ is associated mainly with the Alfven wave.
- $w$ is associated mainly with the slow wave.
- $\chi$ is associated mainly with the fast wave.
Extension for others reduced MHD and full MHD

- In the model 199, the choice of the velocity field show the Alfven wave dominate.

- In the reduced MHD with parallel velocity and the full-MHD, the different types of waves are present.

- The ratio between the different waves is very important. Consequently the conditioning is impacted by the ratio.

- If this problem impact the efficiency of the PC-algorithm we can use a method proposed by S. Jardin coupled with the previous algorithm.

**Jardin method for Schur matrix**

- Use projection operator to isolate the physics associated with the different wave types in different blocks weakly coupled.

- Each submatrix are corrected conditioned.
Current works on the models
Recently B. Després and R. Sart have proposed a more rigorous method to deduce the reduced MHD models (moment method).

To obtain an energy balanced estimate we must had a term on the poloidal magnetic flux equation.

For all the models the equation $\psi$ come from

$$\frac{\partial}{\partial t} \frac{\psi}{R} = [\psi, u] - \epsilon \frac{F_0}{R} \partial \phi u + \eta(\frac{T}{R}) (z_j - S_c(\psi)) - \eta_n \nabla \cdot (\nabla z_j) + Q$$

For the model 199 the term $Q$ satisfy $\triangle^* Q = 0$.

For the models with parallel velocity the term $Q$ satisfy $\triangle^* Q = b(F_0, \nu_{||}, \psi)$.

**Work:** add this term in Jorek and study the stability of time schemes

*Derivation of hierarchies of reduced MHD models in Tokamak geometry, B. Després, Rémy Sart, 2013.*
Conclusion
Ongoing and future works

**Time discretization**

- Finish the PC-algorithm (model 199) and analyze the different approximations of $P_{Schur}$.

- Analyze the conditioning of the matrix $P_{schur}$.

- If this matrix is ill-conditioned, use classical method as ILU method or multigrid method in the "update velocity" step.

- Analyze the conditioning of the matrix $M$.

- If the conditioning is impacted by the anisotropic diffusion, try to reduced the computational cost using AP schemes.

- Adapt the PC-algorithm for the reduced MHD models with parallel velocity.
Stability of the models
- Add the stabilization terms in the reduced models with parallel velocity.
- Analyze the impact of these terms on the numerical methods.

Spatial discretization and mesh generation
- Spatial discretization using general Splines with different types of refinement (A. Rathani).
- Mesh generation using generic Splines and isoGeometric analysis (A. Rathani).
- Stabilization of Galerkin method (B. Nkonga).
- Use Splines to discretize the toroidal direction (B. Nkonga).