Models and numerical methods for instabilities in the Tokamak

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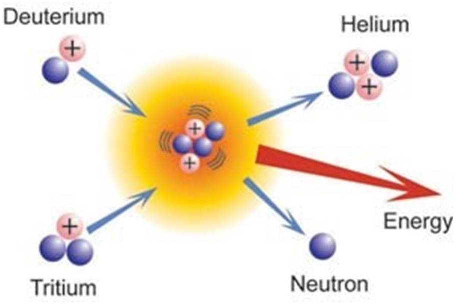
Outline

Mathematical context and JOREK code

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. A neutron and 17.6 MeV of free energy are released. At those energies, the atoms are ionized forming a plasma.
- **Plasma**: For very high temperature, the gas is ionized and gives a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- **ITER**: International project of fusion nuclear plant to validate the nuclear fusion as a power source.



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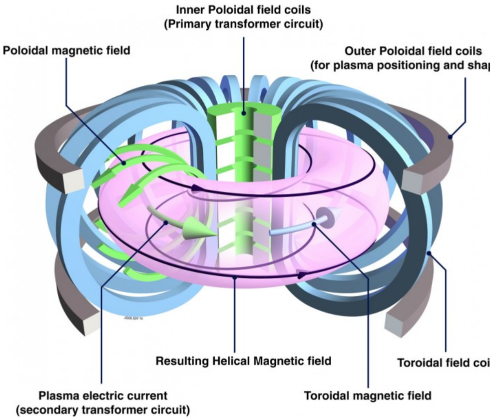
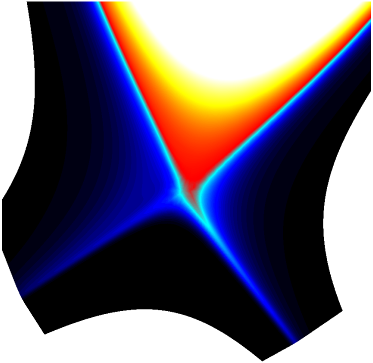


Figure: Tokamak

Physical context : MHD and ELM's

- In the tokamak some instabilities can appear in the plasma or at the edge of the plasma.
- The simulation to these instabilities is an important subject for ITER.
- Exemple of Instabilities in the tokamak :
  - **Disruptions**: Violent instabilities which can damage the tokamak.
  - **Edge Localized Modes (ELM's)**: Periodic edge instabilities which can damage the Tokamak.
- For example the ELM's are linked to the very large gradient of pressure and very large current at the edge.
- These instabilities are described by fluid models (MHD resistive and diamagnetic or extended).

- ELM's simulation



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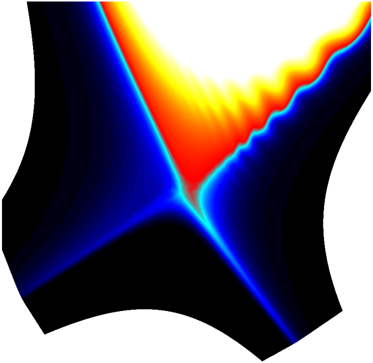

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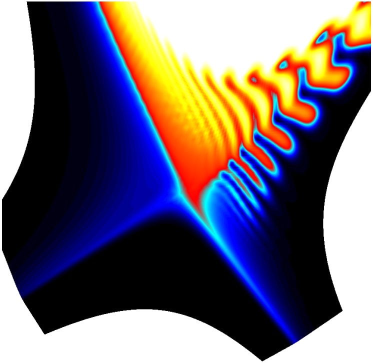

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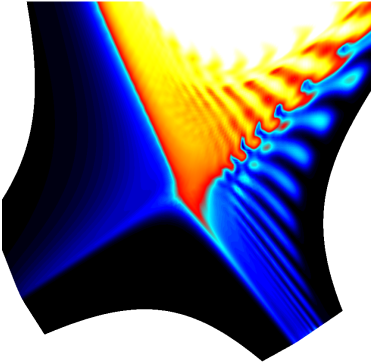

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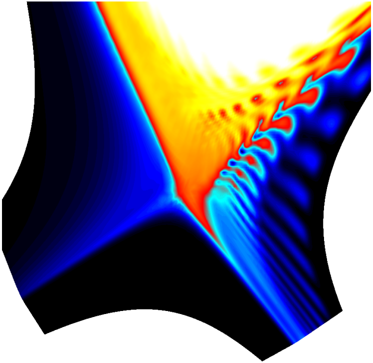

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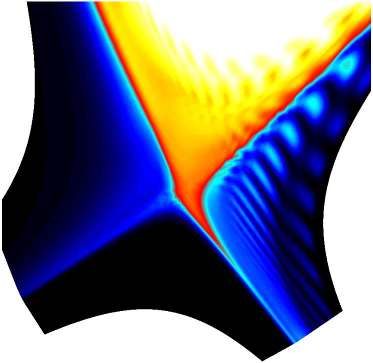

Vlasov equation

- First model to describe a plasma : **Two species Vlasov-Maxwell** kinetic equation.
- We define $f_s(t, \mathbf{x}, \mathbf{v})$ the distribution function associated with the species s. $\mathbf{x} \in D_{\mathbf{x}}$ and $\mathbf{v} \in R^3$.

Two fluids Vlasov equation

$$\begin{cases} \partial_t f_s + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \dfrac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = C_s = \sum_t C_{st}, \\ \frac{1}{c^2}\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \\ \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \cdot \mathbf{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

- Derivation of two-fluid model:
  - We apply this operator $\int_{R^3} g(\mathbf{v})(\cdot)$ on the equation.
  - $g(\mathbf{v})_s = 1, m_s\mathbf{v}, m_s|\mathbf{v}|^2$.
- Using
  - $\int_{D_{\mathbf{v}}} m_s \mathbf{v} C_{ss} d\mathbf{v} = 0, \quad \int_{D_{\mathbf{v}}} m_s |\mathbf{v}|^2 C_{ss} d\mathbf{v} = 0,$
  - $\int_{D_{\mathbf{v}}} g(\mathbf{v})_s C_{st} d\mathbf{v} + \int_{D_{\mathbf{v}}} g(\mathbf{v})_t C_{ts} d\mathbf{v} = 0.$

Two fluid model

- Computing the moment of the Vlasov equations we obtain the two-fluid model

Two fluid moments

$$\begin{cases} \partial_t n_s + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s) = 0, \\ \partial_t (m_s n_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \otimes \mathbf{u}_s) + \nabla_{\mathbf{x}} p_s + \nabla_{\mathbf{x}} \cdot \overline{\overline{\boldsymbol{\Pi}}}_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_s, \\ \partial_t (m_s n_s \epsilon_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \epsilon_s + p_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot \left(\overline{\overline{\boldsymbol{\Pi}}}_s \cdot \mathbf{u}_s + \mathbf{q}_s \right) \\ = \sigma_s \mathbf{E} \cdot \mathbf{u}_s + Q_s + \mathbf{R}_s \cdot \mathbf{u}_s, \\ \\ \frac{1}{c^2} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \\ \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

- $n_s = \int_{D_{\mathbf{v}}} f_s d\mathbf{v}$ the particle number , $n_s \mathbf{u}_s = \int_{D_{\mathbf{v}}} \mathbf{v} f_s d\mathbf{v}$ the momentum, ϵ_s the energy.
- The isotropic pressure are p_s, $\overline{\overline{\boldsymbol{\Pi}}}_s$ the stress tensors and $\mathbf{q}_s$ the heat fluxes.
- $\mathbf{R}_s$ and Q_s associated with the interspecies collision (force and energy transfer).
- The current is given by $\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s \sigma_s \mathbf{u}_s$ with $\sigma_s = q_s n_s$.

Assumptions for MHD

- The characteristic velocity $V_0 << c$ which gives $\mu_0 \mathbf{J} = \nabla \times \mathbf{B} + O(\frac{V_0}{c})$.
- Quasi-neutrality $n_i = n_e$ which gives $\rho = m_i n_i + O(\frac{m_e}{m_i})$ and $\mathbf{u} = \mathbf{u}_i + O(\frac{m_e}{m_i})$

Extended MHD: assumptions and generalized Ohm law

- Taking the electronic density and momentum equations we obtain

$$m_e\left(\partial_t(n_e\mathbf{u}_e)+\nabla\cdot(n_e\mathbf{u}_e\otimes\mathbf{u}_e)\right)+\nabla p_e=-en_e\mathbf{E}+\mathbf{J}_e\times\mathbf{B}-\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e+\mathbf{R}_e,$$

- We multiply the previous equation by $-e$ and we define $\mathbf{J}_e=-en_e\mathbf{u}_e$, we obtain

$$\frac{m_e}{e^2n_e}\left(\partial_t\mathbf{J}_e+\nabla\cdot(\mathbf{J}_e\otimes\mathbf{u}_e)\right)=\mathbf{E}+\mathbf{u}_e\times\mathbf{B}+\frac{1}{en_e}\nabla p_e+\frac{1}{en_e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e-\frac{1}{en_e}\mathbf{R}_e,$$

- Using the quasi neutrality, $m_e<<m_i$ and $\mathbf{R}=-\mathbf{R}_e=-\eta\frac{e}{m_i}\rho\mathbf{J}$, we obtain

$$\mathbf{E}+\underbrace{\mathbf{u}\times\mathbf{B}}_{\text{drift velocity}}=\underbrace{\eta\mathbf{J}}_{resistivity}+\underbrace{\frac{m_i}{\rho e}\mathbf{J}\times\mathbf{B}}_{\text{hall term}}-\underbrace{\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e-\frac{m_i}{\rho e}\nabla p_e}_{\text{pressure term}}.$$

- and the the extended MHD:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{J}\times\mathbf{B}-\nabla\cdot\overline{\overline{\mathbf{\Pi}}},\\ \\ \partial_t p+\mathbf{u}\cdot\nabla p+\gamma p\nabla\cdot\mathbf{u}+\nabla\cdot\mathbf{q}=\frac{m_i}{e\rho}\mathbf{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)\\ -\overline{\overline{\mathbf{\Pi}}}:\nabla\mathbf{u}+\overline{\overline{\mathbf{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\mathbf{J}\right)+\eta|\mathbf{J}|^2,\\ \\ \partial_t\mathbf{B}=-\nabla\times\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{J}-\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e-\frac{m_i}{\rho e}\nabla p_e+\frac{m_i}{\rho e}(\mathbf{J}\times\mathbf{B})\right),\\ \nabla\cdot\mathbf{B}=0,\quad\nabla\times\mathbf{B}=\mathbf{J}.\end{cases}$$

Open questions

- **Physical simplification for two fluid plasma**
- The stress tensor: parallel, cross and perpendicular parts. Perp part neglected. Cross part complicate.
- Velocity decomposition
  $\mathbf{u} = (\mathbf{u}, \mathbf{B})\mathbf{B} + \mathbf{E} \times \mathbf{B} + \frac{m_i}{\rho e}\mathbf{B} \times \nabla p_i + O(\frac{\rho_i^*}{L})$
- **GyroViscous-Cancelation** : use this decomposition to kill the cross tensor using the advection terms of the diamagnetic velocity.
- **Problem**: The energy conservation or dissipation is broken.
- **Open-question**: Find a simplified model with energy conservation.

- **Model reduction using potential formulation**
- **Aim**: Reduce the number of variables and eliminate the fast waves.
- Potential formulation:

$$\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$$

$$\mathbf{u} = -R\nabla u \times \mathbf{e}_\phi + v_{||}\mathbf{B} + \tau_{IC}\frac{R}{\rho}(\mathbf{e}_\phi \times \nabla p)$$

- u the electrical potential, ψ the magnetic poloidal flux, $v_{||}$ the parallel velocity.
- **One fluid model**: The reduceds model have the same energy estimate that full MHD (M2AN 2015).
- **Two fluid model**: open question.

Problem : low density limit

- In the low density limit the bi-fluid terms generated ill-conditioned Jacobian.
- **Open question**: find another formulation well-conditioned.

Numerical works

JOREK-DJANGO

- JOREK is a large code of physics with complicate geometry, models and test cases. To validate the numeric tools is not a good code.

- **JOREK-DJANGO** : simplified version of JOREK for numerical studies.
- Developers : A. Ratnani (IPP), E. F., C Caldini-Queiros (IPP), L. Mendoza (IPP), B. Nkonga (Uni Nice)
- Future users and developers : E. Sonnendrücker (IPP), H. Guillard (INRIA), V. Grandgirard (CEA), G. Latu (CEA), M. Holzl (IPP)

Main properties

- Implicit Finite element code in toroidal geometry.
- Generic Splines in quadrangles and triangles (poloidal plane) and Fourier and Splines for Toroidal direction.
- Linear solvers and preconditioning based on PETSC and SPM (interface for spare Matrices).
- Models : 2D and 3d elliptic problems, 2D wave and diffusion equations, 2D and 3D current Hole, 2D Grad Safranov equation and 3D anisotropic diffusion.
- Possible coupling (not finish) with Selalib.

Compatible discretization

- For resistive and extended MHD the discretization chosen is
  - High-order isogeometric finite element method for poloidal plane: B-Splines
  - Fourier expansion or high-order finite element for toroidal direction.
- Advantages of the B-Splines
  - **Isogeometry**: Allows to describe with high accuracy the geometry and the function evaluate by the FE method.
  - Basis function with h and p-refinement (mesh and degree refinement).
  - k-refinement allows to adapt the regularity (matrices smaller, better-conditioned) and treat high-order operator.

	Nb dof		error		time solving	
	C^0	C^{p-1}	C^0	C^{p-1}	C^0	C^{p-1}
p=3	36481	4225	3.5E-8	1.1E-7	1.1E-2	9.6E-3
p=4	65205	4356	1.6E-10	1.9E-9	4.8E-2	2.3E-2

- Comparison of different regularity for a Poisson problem solved with CG-Jacobi.

Future works

- Study the Compatible spaces ($H(div)$ and $H(curl)$) and DeRham sequence for the B-Splines and apply this to
  - Maxwell equations
  - Stokes-Maxwell equations
  - Full - MHD

Implicit scheme for wave equation

- Damping wave equation (baby problem used for Inertial fusion confinement)

$$\begin{cases} \partial_t p + c\nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + c\nabla p = \varepsilon \Delta \mathbf{u} \end{cases}$$

- This problem is stiff in time for fast waves. CFL condition closed to $\Delta t \leq C_1 \frac{h}{c}$.
- Simple way to solve this: implicit scheme but the model is ill-conditioned.
- Two sources of ill-conditioning: the stiff terms (which depend of ε) and the hyperbolic structure.

Philosophy : Divise, reformulate, approximate and rule

- **Divise**: use splitting technic to separate the full coupling system between simple operators (advection, diffusion etc).
- **Reformulate**: rewrite the coupling terms as second order operator simple to invert.
- **Approximate**: use approximations in the previous step to obtain well-posed and well-conditioning simple operators.
- **Rule**: solve the suitability of sub-systems to obtain an approximation of the solution.

Principle of the preconditioning

- The implicit system is given by

$$\begin{pmatrix} M & U \\ L & D \end{pmatrix}\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1} \end{pmatrix} = \begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $M = I_d$, $D = \begin{pmatrix} I_d - c\theta\varepsilon\Delta & 0 \\ 0 & I_d - c\theta\varepsilon\Delta \end{pmatrix}$, $L = \begin{pmatrix} \theta c\Delta t\partial_x \\ \theta c\Delta t\partial_y \end{pmatrix}$ and $U = L^t$.

- The solution of the system is given

$$\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1} \end{pmatrix} = \begin{pmatrix} I & M^{-1}U \\ 0 & I \end{pmatrix}\begin{pmatrix} M^{-1} & 0 \\ 0 & P_{schur}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -LM^{-1} & I \end{pmatrix}\begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $P_{schur} = D - LM^{-1}U$.

- Using the previous Schur decomposition we can solve the implicit wave equation with the algorithm.

$$\begin{cases} \text{Predictor :} \quad Mp^{*} = R_p \\ \text{Velocity evolution :} \quad P\mathbf{u}^{n+1} = (-Lp^{*} + R_{\mathbf{u}}) \\ \text{Corrector :} \quad Mp^{n+1} = M_h p^{*} - U\mathbf{u}_{n+1} \end{cases}$$

- with the matrices:
  - P discretize the positive and symmetric operator :

$$P_{Schur} = I_d - c\varepsilon\theta\Delta I_d - \nabla(\nabla\cdot I_d) = I_d - c\theta\varepsilon\Delta I_d - c2\theta^2\Delta t^2\begin{pmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{pmatrix}$$

Results for the PC with pressure Schur

- Results for classical Preconditioning (no diffusion).

	Cells	Jacobi		ILU(2)		ILU(4)		ILU(8)	
		iter	Err	iter	Err	iter	Err	iter	Err
$c\Delta t$=1	16*16	-	-	140	2.8E-1	55	4.8E-1	90	1.4E+0
	32*32	-	-	-	-	-	-	180	5.E+0
	64*64	-	-	-	-	-	-	-	-
$c\Delta t$=100	16*16	-	-	88	2.4E-1	58	4.9E-1	88	1.4E+0
	32*32	-	-	-	-	-	-	110	5.6E+0
	64*64	-	-	-	-	-	-	2000	8.8E+1

- Results for the new preconditioning.

	Cells	PB_p		PB_u	
		iter	Err	iter	Err
$a\Delta t$=1	16*16	4	4.9E-2	3	6.8E-2
	32*32	2	9.2E-2	1	1.2E-1
	64*64	2	4.2E-1	1	24
$a\Delta t$=100	16*16	7	1.1E-1	8	4.5E-1
	32*32	6	5.3E-1	6	2.8E+0
	64*64	6	1.E+0	-	-

- For each sub-system we use a CG+Jacobi solver.
- Velocity Schur operator (coupled diffusion operator) not easy to invert and generate a large additional cost.
- On fine grid we use CG+MG 2-cycle for velocity Schur operator.

Some remarks

- **Schur complement on the velocity** since In fluid mechanics and plasma physics the velocity couple all the other equations.
- **Problem** : Schur complement on the velocity not so well-conditioned.

- Wave problem of the hyperbolic problem :
  - Pressure and $(\mathbf{u}, \mathbf{n})$ propagate at the speed $\pm c$,
  - $(\mathbf{u} \times \mathbf{n})$ propagate at the speed 0.
- **Idea**: split the propagation (static and non static waves) in the Schur complement using the vorticity equation:

$$\partial_t \mathbf{u} + c\nabla p = \mathbf{f}_u \Longrightarrow \partial_t(\nabla \times \mathbf{u}) = \nabla \times \mathbf{f}_u$$

$$\begin{cases} \text{Predictor : } & M p^* = R_p \\ \text{Vorticity evolution : } & \mathbf{w}^{n+1} = R(R_{\mathbf{u}}) \\ \text{Velocity evolution : } & P\mathbf{u}^{n+1} = \left(\alpha R(\mathbf{w}^{n+1}) - L p^* + R_{\mathbf{u}}\right) \\ \text{Corrector : } & M p^{n+1} = M_h p^* - U\mathbf{u}_{n+1} \end{cases}$$

- with R the matrix of the curl operator, $\alpha = c^2\theta^2\Delta t^2$ and $P_{Schur} = I_d - (\varepsilon c\theta + \alpha)\Delta$.

Remarks

- The method, the propagation properties and the vorticity prediction can be generalized for compressible fluid mechanics.

- **Adaptivity** : In the PC, we want use different discretization that for the full model (less order, other toroidal discretization etc).