Physic-Based Preconditioning for stiff hyperbolic systems

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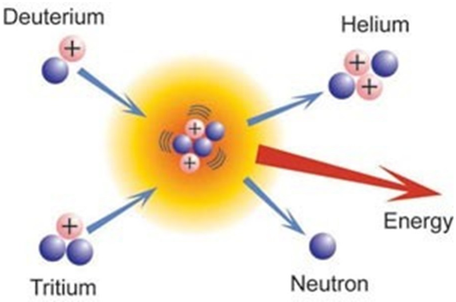
Outline

Mathematical context and JOREK code

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. A neutron and 17.6 MeV of free energy are released. At those energies, the atoms are ionized forming a plasma.
- **Plasma**: For very high temperature, the gas are ionized and gives a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- **ITER**: International project of fusion nuclear plant to validate the nuclear fusion as a power source.



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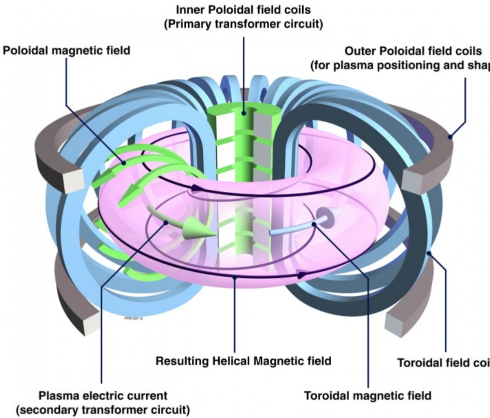
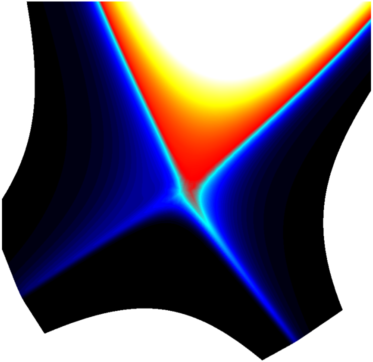


Figure: Tokamak

Physical context : MHD and ELM's

- In the tokamak some instabilities can appear at the edge of the plasmas.
- The simulation to these instabilities is an important subject for ITER.
- Exemple of Edge Instabilities in the tokamak :
  - **Disruptions**: Violent edge instabilities which can damage critically the tokamak.
  - **Edge Localized Modes (ELMs')**: Periodic edge instabilities which can damage the Tokamak.
- These instabilities are linked to the very large gradient of pressure and very large current at the edge.
- These instabilities are described by fluid models (MHD resistive and diamagnetic or extended).

- ELM's simulation



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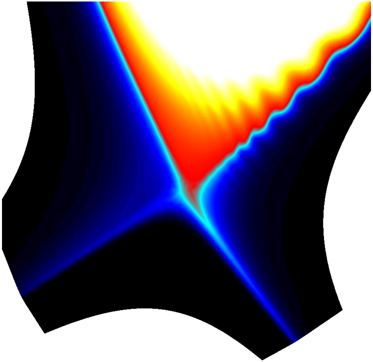

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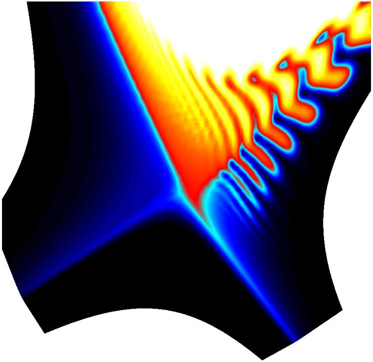

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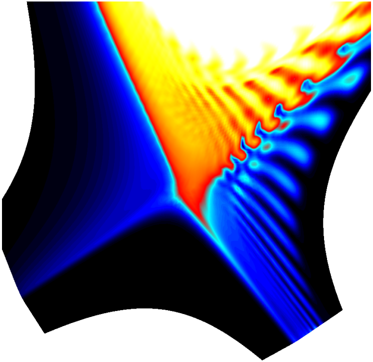

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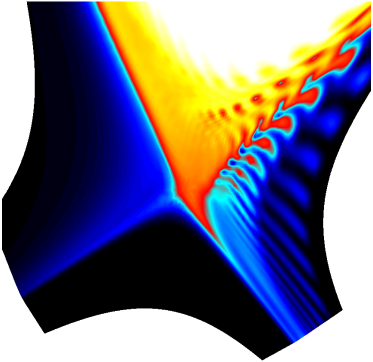

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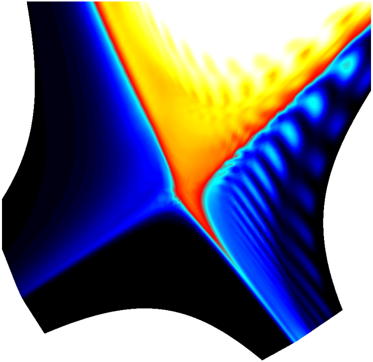

Vlasov equation

- First model to describe a plasma : **Two species Vlasov-Maxwell** kinetic equation.
- We define $f_s(t,\mathbf{x},\mathbf{v})$ the distribution function associated with the species s. $\mathbf{x}\in D_{\mathbf{x}}$ and $\mathbf{v}\in R^3$.

Two fluids Vlasov equation

$$\left\{\begin{array}{l}\partial_t f_s+\mathbf{v}\cdot\nabla_{\mathbf{x}} f_s+\dfrac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s=C_s=\sum_t C_{st},\\ \frac{1}{c^2}\partial_t\mathbf{E}-\nabla\times\mathbf{B}=-\mu_0\mathbf{J},\\ \partial_t\mathbf{B}=-\nabla\times\mathbf{E},\\ \nabla\cdot\mathbf{B}=0\\ \nabla\cdot\mathbf{E}=\frac{\sigma}{\varepsilon_0}.\end{array}\right.$$

- Derivation of two fluid model :
  - We apply this operator $\int_{R^3} g(\mathbf{v})(\cdot)$ on the equation.
  - $g(\mathbf{v})_s=1, m_s\mathbf{v}, m_s|\mathbf{v}|^2$.
- Using
  - $\int_{D_{\mathbf{v}}} m_s\mathbf{v}C_{ss}d\mathbf{v}=0,\quad \int_{D_{\mathbf{v}}} m_s|\mathbf{v}|^2C_{ss}d\mathbf{v}=0,$
  - $\int_{D_{\mathbf{v}}} g(\mathbf{v})_sC_{st}d\mathbf{v}+\int_{D_{\mathbf{v}}} g(\mathbf{v})_tC_{ts}d\mathbf{v}=0.$

Two fluid model

- Computing the moment of the Vlasov equations we obtain the following two fluid model

Two fluid moments

$$
\begin{cases}
\partial_t n_s + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s) = 0, \\
\partial_t (m_s n_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \otimes \mathbf{u}_s) + \nabla_{\mathbf{x}} p_s + \nabla_{\mathbf{x}} \cdot \overline{\overline{\mathbf{\Pi}}}_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_s, \\
\partial_t (m_s n_s \epsilon_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \epsilon_s + p_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot \left(\overline{\overline{\mathbf{\Pi}}}_s \cdot \mathbf{u}_s + \mathbf{q}_s \right) \\
= \sigma_s \mathbf{E} \cdot \mathbf{u}_s + Q_s + \mathbf{R}_s \cdot \mathbf{u}_s, \\
\\
\frac{1}{c^2} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \\
\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \\
\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\sigma}{\varepsilon_0}.
\end{cases}
$$

- $n_s = \int_{D_{\mathbf{v}}} f_s d\mathbf{v}$ the particle number , $m_s n_s \mathbf{u}_s = \int_{D_{\mathbf{v}}} m_s \mathbf{v} f_s d\mathbf{v}$ the momentum, ϵ_s the energy.
- The isotropic pressure are p_s, $\overline{\overline{\mathbf{\Pi}}}_s$ the stress tensors and $\mathbf{q}_s$ the heat fluxes.
- $\mathbf{R}_s$ and Q_s associated with the interspecies collision (force and energy transfer).
- The current is given by $\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s \sigma_s \mathbf{u}_s$ with $\sigma_s = q_s n_s$.

Extended MHD: assumptions and generalized Ohm law

- Taking the electronic density and momentum equations we obtain

$$m_e\left(\partial_t(n_e\mathbf{u}_e)+\nabla\cdot(n_e\mathbf{u}_e\otimes\mathbf{u}_e)\right)+\nabla p_e=-en_e\mathbf{E}+\mathbf{J}_e\times\mathbf{B}-\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e+\mathbf{R}_e,$$

- We multiply the previous equation by $-e$ and we define $\mathbf{J}_e=-en_e\mathbf{u}_e$, we obtain

$$\frac{m_e}{e^2n_e}\left(\partial_t\mathbf{J}_e+\nabla\cdot(\mathbf{J}_e\otimes\mathbf{u}_e)\right)=\mathbf{E}+\mathbf{u}_e\times\mathbf{B}+\frac{1}{en_e}\nabla p_e+\frac{1}{en_e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e-\frac{1}{en_e}\mathbf{R}_e,$$

- Using the quasi neutrality, $m_e<<m_i$ and $\mathbf{R}=-\mathbf{R}_e=-\eta\frac{e}{m_i}\rho\mathbf{J}$, we obtain

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}=\eta\mathbf{J}-\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e+\frac{m_i}{\rho e}\mathbf{J}\times\mathbf{B}-\frac{m_i}{\rho e}\nabla p_e.$$

- and the the extended MHD:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{J}\times\mathbf{B}-\nabla\cdot\overline{\overline{\mathbf{\Pi}}},\\ \\ \frac{1}{\gamma-1}\partial_t p+\frac{1}{\gamma-1}\mathbf{u}\cdot\nabla p+\frac{\gamma}{\gamma-1}p\nabla\cdot\mathbf{u}+\nabla\cdot\mathbf{q}=\frac{1}{\gamma-1}\frac{m_i}{e\rho}\mathbf{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)\\ -\overline{\overline{\mathbf{\Pi}}}:\nabla\mathbf{u}+\overline{\overline{\mathbf{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\mathbf{J}\right)+\eta|\mathbf{J}|^2,\\ \\ \partial_t\mathbf{B}=-\nabla\times\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{J}-\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e-\frac{m_i}{\rho e}\nabla p_e+\frac{m_i}{\rho e}(\mathbf{J}\times\mathbf{B})\right),\\ \nabla\cdot\mathbf{B}=0,\quad\nabla\times\mathbf{B}=\mathbf{J}.\end{cases}$$

Reduced MHD: assumptions and principle of derivation

- **Aim**: Reduce the number of variables and eliminate the fast waves in the reduced MHD model.
- We consider the cylindrical coordinate $(R, Z, \phi) \in \Omega \times [0, 2\pi]$.

Reduced MHD: Assumption

$$\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi \quad \mathbf{u} = -R\nabla u \times \mathbf{e}_\phi + v_{||}\mathbf{B} + \tau_{IC}\frac{R}{\rho}\left(\mathbf{e}_\phi \times \nabla p\right)$$

with u the electrical potential, ψ the magnetic poloidal flux, $v_{||}$ the parallel velocity.

- To avoid high order operators, we introduce the vorticity $w = \Delta_{pol} u$ and the toroidal current $j = \triangle^*\psi = R^2\nabla\cdot(\frac{1}{R^2}\nabla_{pol}\psi)$.
- Derivation: we plug $\mathbf{B}$ and $\mathbf{u}$ in the equations + some computations. For the equations on u and $v_{||}$ we use the following projections

$$\mathbf{e}_\phi \cdot \nabla \times R^2\left(\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B} + \nu\Delta\mathbf{u}\right)$$

and

$$\mathbf{B}\cdot\left(\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B} + \nu\Delta\mathbf{u}\right).$$

Description of the JOREK code

- JOREK: Fortran 90 code, parallel (MPI+OpenMP)
- Main author: Guido Huijmans

- Determine the equilibrium
  - Define the boundary of the computational domain
  - Compute $\psi(R,Z)$ on a first poloidal grid.
- Compute equilibrium solving Grad-Shafranov equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-R^2\frac{\partial p}{\partial\psi}-F\frac{\partial F}{\partial\psi}$$

- Computation of aligned grid
  - Identification of the magnetic flux surfaces
  - Create the aligned grid (with X-point)
  - Interpolate $\psi(R,Z)$ in the new grid and recompute the equilibrium
- **Perturbation of the equilibrium** (small perturbations of non principal harmonics).
- Time-stepping (full implicit)
  - **Poloidal discretization**: 2D Cubic Bezier finite elements.
  - **Toroidal discretization**: Fourier expansion.

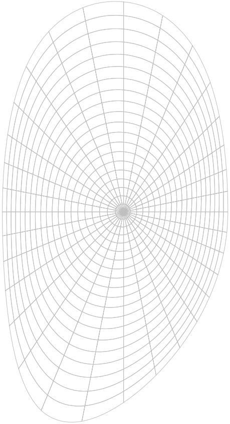

Figure: unaligned grid

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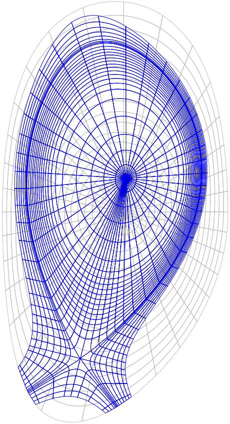


Figure: Aligned grid

Sources of stiffness

Stiffness for explicit scheme

- Fast magneto-sonic waves (for full MHD problems),
- Diffusion operators (anisotropic diffusion and viscous tensor),
- Waves associated to generalized Ohm law,
- low density cases.

- Since the models are stiff we proposed toi use implicit scheme (an alternative is the semi-implicit schemes)

Ill-conditioning for implicit scheme

- Ratio between the waves
- Anisotropic Diffusion operators,
- Nonlinear Hyperbolic structure,
- low density cases.

- The exact solvers are not a good option now because there are so greedy for large cases. Since the problem is ill-conditioned we will need to preconditioning for iterative solvers.

Results for Waves equation

- Comparison between iterative solver for test case in the diffusion limit $\sigma = 1$.

Mesh / solvers		GC	GC-PC	Gmres	Gmres-PC-Jacobi
Mesh 4*4, ε_1	cv iter	✗ -	✗ -	✗ -	✓ 27
Mesh 16*16, ε_1	cv iter	✗ -	✗ -	✗ -	✓ 1.5E+4
Mesh 4*4, ε_2	cv iter	✗ -	✗ -	✗ -	✓ 21000
Mesh 16*16, ε_2	cv iter	✗ -	✗ -	✗ -	✗ -

- $\varepsilon_1 = 10^{-5}$ and $\varepsilon_2 = 10^{-10}$.
- The solver tolerance is 10^{-10} for convergence and iter_max=100000. We compute the average on ten time iterations.
- The GC solver is iunstable and cannot solve this type of problem.

A conclusion

- The results show that it is necessary to use a good preconditioning + robust solver (for general matrix).

Linear Solvers

- We solve a nonlinear problem $G(\mathbf{U}^{n+1}) = b(\mathbf{U}^n, \mathbf{U}^{n-1})$.
- First order linearization

$$\left(\frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}\right)\delta \mathbf{U}^n = -G(\mathbf{U}^n) + b(\mathbf{U}^n, \mathbf{U}^{n-1}) = R(\mathbf{U}^n),$$

  with $\delta \mathbf{U}^n = \mathbf{U}^{n+1} - \mathbf{U}^n$, and $J_n = \frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}$ the Jacobian matrix of $G(\mathbf{U}^n)$.
- Linear solver in JOREK: Left Preconditioning + GMRES iterative solver.
- Principle of the preconditioning step:
  - Replace the problem $J_k \delta \mathbf{U}_k = R(\mathbf{U}^n)$ by $P_k(P_k^{-1} J_k)\delta \mathbf{U}_k = R(\mathbf{U}^n)$.
  - Solve the new system with two steps $P_k \delta \mathbf{U}_k^* = R(\mathbf{U}^n)$ and $(P_k^{-1} J_k)\delta \mathbf{U}_k = \delta \mathbf{U}_k^*$
- If P_k is easier to invert than J_k and $P_k \approx J_k$ the linear solving step is more robust and efficient.

Physic-based Preconditioning of JOREK

- Extraction of the blocks which are associated with each toroidal harmonic.
- Solve exactly with LU decomposition each subsystem associated with a block
- Reconstruction of the solution of $P_k \mathbf{x} = \mathbf{b}$

- **Principle of Physic-based preconditioning**: We neglect in the Jacobian the physical effect associated to the coupling between the Fourier mods (non diagonal block).

JOREK-DJANGO

- JOREK is a large code of physics with complicate geometry, models and test cases. To validate the numeric tools is not a good code.
- **JOREK-DJANGO** : simplified version of JOREK for numerical studies.
- Developers : A. Ratnani (IPP), E. F., C Caldini-Queiros (IPP), L. Mendoza (IPP), B. Nkonga (Uni Nice)
- Future users and developers : E. Sonnendrücker (IPP), H. Guillard (INRIA), V. Grandgirard (CEA), G. Latu (CEA)

Main properties

- Implicit Finite element code in toroidal geometry.
- Generic Splines in quadrangles and triangles (poloidal plane) and Fourier and Splines for Toroidal direction.
- Linear solvers and preconditioning based on PETSC and SPM (interface for spare Matrices).
- Models : 2D and 3d elliptic problems, 2D wave and diffusion equations, 2D and 3D current Hole, 2D Grad Safranov equation and 3D anisotropic diffusion.
- Possible coupling (not finish) with Selalib

Physic based preconditioning for Waves equations

Implicit scheme for Damped waves equations

- Damping wave equation (baby problem used for Inertial fusion confinement)

$$\left\{\begin{array}{l}\partial_t p + c\nabla\cdot\mathbf{u} = 0\\ \\ \partial_t \mathbf{u} + c\nabla p = -c\sigma\mathbf{u}\end{array}\right. \Longleftrightarrow \left\{\begin{array}{l}\partial_t p + \frac{1}{\varepsilon}\nabla\cdot\mathbf{u} = 0\\ \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p = -\frac{\sigma}{\varepsilon^2}\mathbf{u}\end{array}\right.$$

- with σ opacity, c light speed and $\varepsilon \approx \frac{1}{c} \approx \frac{1}{\sigma}$
- When $\varepsilon \longrightarrow 0$ the model can be approximated by $\partial_t p - \nabla\cdot(\frac{1}{\sigma}\nabla p) = 0$.
- This problem is stiff in time. CFL condition is $\Delta t \leq C_1\varepsilon h + C_2\varepsilon^2$.
- Simple way to solve this: implicit scheme but the model is ill-conditioned.
- Two sources of ill-conditioning: the stiff terms (which depend of ε) and the hyperbolic structure.

Philosophy : Divise, reformulate, approximate and rule

- **Divise**: use splitting technic to separate the full coupling system between simple operators (advection, diffusion etc).
- **Reformulate**: rewrite the coupling terms as second order operator simple to invert.
- **Approximate**: use approximations in the previous step to obtain well-posed and ii-conditioning simple operators.
- **Rule**: solve the suitable of sub-systems to obtain an approximation of the solution.

Construction of the preconditioning I

- First we implicit the equation

$$\begin{cases} p^{n+1} + \theta\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1} = p^n - (1-\theta)\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^n \\ \mathbf{u}^{n+1} + \theta\dfrac{\Delta t}{\varepsilon}\nabla p^{n+1} + \theta\dfrac{\Delta t\sigma}{\varepsilon^2}\mathbf{u}^{n+1} = \mathbf{u}^n - (1-\theta)\dfrac{\Delta t}{\varepsilon}\nabla p^n - (1-\theta)\dfrac{\Delta t\sigma}{\varepsilon^2}\mathbf{u}^n \end{cases}$$

- The implicit system is given by

$$\begin{pmatrix} M & U \\ L & D \end{pmatrix}\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1} \end{pmatrix} = \begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $M = I_d$, $D = \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix}$, $U = \begin{pmatrix} \theta\dfrac{\Delta t}{\varepsilon}\partial_x & \dfrac{\Delta t}{\varepsilon}\partial_y \end{pmatrix}$ and $L = \begin{pmatrix} \theta\dfrac{\Delta t}{\varepsilon}\partial_x \\ \theta\dfrac{\Delta t}{\varepsilon}\partial_y \end{pmatrix}$

- The solution of the system is given

$$\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1} \end{pmatrix} = \begin{pmatrix} M & U \\ L & D \end{pmatrix}^{-1}\begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

$$= \begin{pmatrix} I & M^{-1}U \\ 0 & I \end{pmatrix}\begin{pmatrix} M^{-1} & 0 \\ 0 & P_{schur}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -LM^{-1} & I \end{pmatrix}\begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $P_{schur} = D - LM^{-1}U$.

Principle of the preconditioning II

- Using the previous Schur decomposition we can solve the implicit wave equation with the algorithm.

$$\left\{\begin{array}{l} \text{Predictor :} \quad M_h \mathbf{p}^* = R_p \\ \text{Velocity evolution :} \quad P_h \mathbf{u}^{n+1} = (-L_h \mathbf{p}^* + R_{\mathbf{u}}) \\ \text{Corrector :} \quad M_h \mathbf{p}^{n+1} = M_h \mathbf{p}^* - U_h \mathbf{u}_{n+1} \end{array}\right.$$

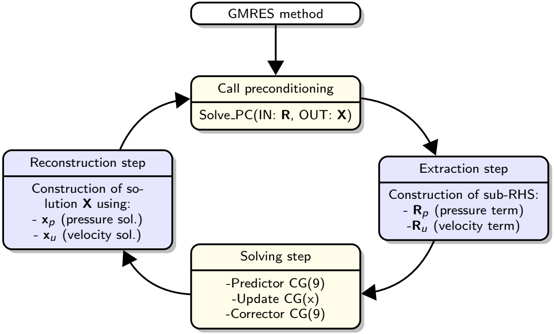
- with the matrices:
  - M_h the mass matrix which discretize the Identity operator
  - U_h discretize the operator U and L_h the discretization of the L operator.
  - P_h discretize the positive and symmetric operator :

$$P_{Schur} = I_d - \nabla(\nabla \cdot I_d) = I_d - \theta^2 \frac{\Delta t^2}{\varepsilon^2} \begin{pmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{pmatrix}$$

- The physic based preconditioning $PB(x)$ solves the previous algorithm with Conjugate-Gradient with $tol = 10^{-x}$ and Jacobi PC

Algorithm of the PhyBas Preconditioning step

- Algorithm and implementation of the $PB(x)$ preconditioning:



- In this case we solve the sub-steps with a GC solver
- We can use also Multi-grid (MG) methods or other methods efficient for symmetric and diagonal dominant matrix.

Results for Waves equation

- Comparison between GMRES method with different preconditioning

Mesh / solvers		Jac	ILU(0)	ILU(4)	MG(2)	SOR	PB
Mesh4*4, ε_1	cv	✓	✓	✓	✓	✓	✓
	iter	27	11		38	8	1
	time	7.2 E-4	1.3E-3	7.7E-3	1.5E-2	1.4E-3	2.1E-3
4*4, ε_2	cv	✓	✓	✓	✗	✓	✓
	iter	2.1E+4	11	1	-	8	1
	time	3.6E-1	1.3E-3	7.7E-3	-	1.5E-3	2.1E-3
16*16, ε_1	cv	✓	✓	✓	✗	✓	✓
	iter	1.5E+4	18	9	140	20	1
	time	5.0E-0	2.3E-2	4.0E-1	5.0E-1	5.0E-2	2.1E-2
16*16, ε_2	cv	✗	✓	✓	✗	✓	✓
	iter	-	18	9	-	20	1
	time	-	2.3E-2	4.0E-1	-	5.0E-2	2.1E-2
64*64, ε_2	cv	✗	✗	✓	✗	✗	x
	iter	-	-	632	-	-	1
	time	-	-	2.0E+1	-	-	4.2E-1

- ILU (Incomplete LU), MG (Multi-grids), SOR, PB (our physic based PC).

A conclusion

- On fine grid our method is the fastest (and the current implementation is not optimal).

Future numerical works for waves

- To obtain the more robust and performing algorithm we must optimize and study some substeps

Preconditioning for sub step

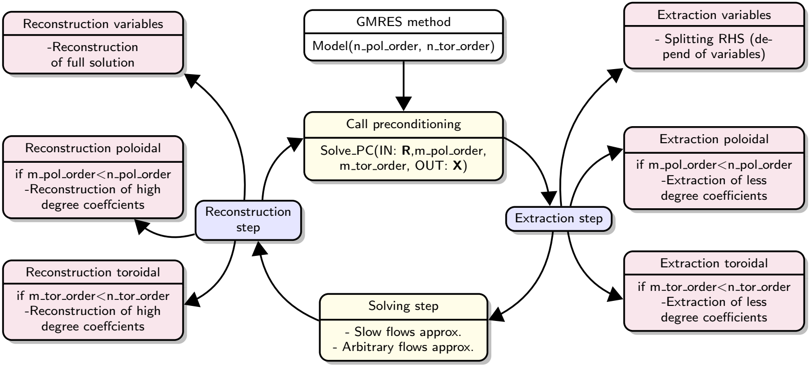
- The Schur is solved with a CG preconditioned. To optimize the resolution of this step we propose to construct a Geometric MG based on the properties of B-Splines (useful also for the nonlinear case).
- The Mass-Matrix for the B-Splines is ill conditioned. We propose an adapted PC based on $M_{2D} \approx \sum_j M_{1D,i} \otimes N_{1D,i}$. The PC will be construct using 1D solving.

Adaptivity

- It is not necessary to solve the subsystems of the PC with the same accuracy to the full problem.
- Consequently to reduce the size of the sub-matrices we can use B-Splines with less order or different regularity between the model and the PC.
- **Regularity of the B-Splines** : if the regularity is high we have less accuracy but smaller matrices with better conditioning.

Algorithm of the adaptive PhyBas Preconditioning step

- Algorithm and implementation of the $APB(x)$ preconditioning:



- In the future it is important to perform the extraction and reconstruction parts.

Future theoretical works for waves

- Now we propose to study the discretization of the problem $P_{schur}u = R$.
- The weak form associated is given by $a_p(\mathbf{u},\mathbf{v}) = l(\mathbf{v})$ with $\mathbf{u},\mathbf{v} \in H_0(div,\Omega) \cap H_0(curl,\Omega)$, $l(\mathbf{v}) = \int_\Omega Rv$ and

$$a_p(\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{u},\mathbf{v}) + \theta^2 \frac{a^2}{\varepsilon^2}\Delta t^2 \int_\Omega (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})$$

- A classical estimation is $\parallel \nabla\cdot\mathbf{u} \parallel_{L^2}^2 + \parallel \nabla\times\mathbf{u} \parallel_{L^2} 2 \geq C \parallel \mathbf{u} \parallel_{L^2}^2$. Using this estimation we remark that we obtain

$$a_p(\mathbf{u},\mathbf{v}) \geq (1 + C\theta^2 \frac{a^2}{\varepsilon^2}\Delta t^2) \parallel \mathbf{u} \parallel_{L^2}^2 - \theta^2 \frac{a^2}{\varepsilon^2}\Delta t^2 \parallel \nabla\times\mathbf{u} \parallel_{L^2}^2$$

- In the limit regime $\mathbf{u} = -\varepsilon\nabla p$, consequently $\nabla\times\mathbf{u} = 0$, the problem is coercive.
- When ε is close to one the coercivity is not sure.

Future works

- Study the existence and uniqueness of the solutions using mixed-formulation and inf-sup condition.
- Study the discrete problem using same framework and discrete H(div) and H(curl) spaces for B-splines

Physic based preconditioning for MHD equations

Current Hole and preconditioning associated

- Current Hole : reduced problem in cartesian coordinates.
- The model

$$\left\{\begin{array}{l}\partial_t\psi = [\psi, u] + \eta\Delta\psi \\ \\ \partial_t\Delta u = [\Delta u, u] + [\psi, \Delta\psi] + \nu\Delta^2 u\end{array}\right.$$

with $w = \Delta u$ and $j = \Delta\psi$.

- In this formulation we split evolution and elliptic equations.
- For the time discretization we use a Cranck-Nicholson scheme and linearized the nonlinear system to obtain

$$\begin{pmatrix} M & U \\ L & D \end{pmatrix}\begin{pmatrix} \Delta\psi^n \\ \Delta u^n \end{pmatrix} = \begin{pmatrix} R_\psi \\ R_u \end{pmatrix}$$

or

$$\begin{pmatrix} I_d - \Delta t\theta[\cdot, u^n] - \Delta t\theta\Delta & -\Delta\theta[\psi^n, \cdot] \\ \\ -\Delta t\theta[\psi^n, \Delta\cdot] - \Delta t\theta[\cdot, \Delta\psi^n] & \Delta - \Delta t\theta([\Delta\cdot, u^n] + [\cdot, \Delta u^n] + \Delta^2) \end{pmatrix}\begin{pmatrix} \delta\psi^n \\ \delta u^n \end{pmatrix} = \begin{pmatrix} R_\psi \\ R_u \end{pmatrix}$$

Design of the preconditioning for reduced MHD

PB-PC for Current Hole

$$\begin{cases} \text{Predictor}: & M\delta\psi_p^n = R_\psi \\ \text{potential update}: & P_{schur}\delta u^n = \left(-L\delta\psi_p^n + R_u\right) \\ \text{Corrector}: & M\delta\psi^n = M\delta\psi_p^n - U\delta u^n \\ \text{Current update}: & \delta z_j^n = \Delta\delta\psi^n \\ \text{Vorticity update}: & \delta w^n = \Delta\delta u^n \end{cases}$$

- The schur complement is given by $P_{schur} = D - LM^{-1}U$
- Two approximations for M^{-1}:
  - **Slow flow**: $M^{-1} = \Delta t$
  - **Arbitrary flow**: find M^* such that $UM^* \approx MU$. Consequently

$$P^{-1} = (D - LM^{-1}U)^{-1} \approx M^*(DM^* - LU)^{-1},$$

  we obtain

$$\begin{cases} \text{potential update I}: & (DM^* - LU)\delta u^{**} = \left(-L\delta\psi_p^n + R_u\right) \\ \text{potential update II}: & \delta u^n = M^*\delta u^{**} \end{cases}$$

- **Last question** : Computation of the operator LU (second order form of the coupling hyperbolic operators).

Approximation of the Schur complement I

- Computation of Schur complement for (slow flow approximation $M^{-1} \approx \Delta t$)

$$P_{schur} = \frac{\Delta\delta u}{\Delta t} + \mathbf{u}^n \cdot \nabla(\Delta\delta u) + \delta\mathbf{u} \cdot \nabla(\Delta u^n) - \theta\nu\Delta^2\delta u - \theta^2\Delta t LU$$

- Operator $LU = \mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla\delta u)) + \frac{\partial j^n}{\partial\psi^n}\mathbf{B}^n_{pol} \cdot \nabla(\mathbf{B}^n \cdot \nabla\delta u)$.
- $\mathbf{B}^n \cdot \nabla\delta u = -[\psi^n, \delta u]$ and $\mathbf{u}^n \cdot \nabla\delta u = -[\delta u, u^n]$ et $\delta\mathbf{u} \cdot \nabla u^n = -[u^n, \delta u]$.

- **Remark**: the LU operator is the parabolization of coupling hyperbolic terms which contains only the Alfvén waves (rigorous proof missing).

Properties of LU operator

- We consider the L^2 space. The operator LU is not self adjoint and not positive for all δu

$$< LU\delta u, \delta u >_{L^2} = \int |\nabla(\mathbf{B}^n \cdot \nabla\delta u)|^2 - \int \frac{\partial j^n}{\partial\psi^n}(\mathbf{B}^n_{pol} \cdot \nabla\delta u)(\mathbf{B}^n \cdot \nabla\delta u)$$

- We propose the following approximation $LU^{approx} = \mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla\delta u))$.
- The operator LU^{approx} is positive and self-adjoint.

- There are different methods to solve the Schur complement using splitting to solve smaller and more simple operators.

Solving the different steps of the PC

- **Question** How solve each step ?
- The first simple and efficient solver is to use the Multi-Grids methods (MG) efficient for second order and advection operators.
- But perhaps it can be more efficient to split some terms in the sub-systems to use the most adapted solver for each operator.
- Example for the Schur complement (L. Chacon paper) using a splitting and an approximation:

$$\begin{cases} \text{Schur solver I} : \Delta\delta u^{*} = RHS \\ \text{Schur solver II} : \left(\frac{I_d}{\Delta t} + \mathbf{u}^n \cdot \nabla I_d - \theta\nu\Delta\right)\delta u^{**} = \delta u^{*} \\ \text{Schur solver III} : \left(\frac{\Delta I_d}{\Delta t} - \mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla I_d))\right)\delta u^{n+1} = \delta u^{**} \end{cases}$$

- MG methods are adapted for advection diffusion problems.
- GC is more adapted for symmetric and positive anisotropic operator (smoother for MG are more complicated for anisotropic problem).
- **L. Chacon remark**: to replace $\mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla I_d))$ by $\Delta(\mathbf{B}^n \cdot \nabla(\mathbf{B}^n \cdot \nabla I_d))$ generate noise.

Results for Current Hole Model

- Comparison between GMRES method with different preconditioning
- 50 time step in the linear phase (kink instability ?). $tol = 10^{-8}$, $iter\_max = 10000$.

Mesh / solvers		Jac	ILU(0)	ILU(4)	MG	SOR	PB(6)	PB(4)
16*16 dt=0.5	cv	✗	✓	✓	✗	✓	✓	✓
	iter	-	14	6	-	12	1	1
	time	-	1.2E-1	1.4E+0	-	1.8E-1	2.6E+0	2.3E+0
32*32 dt=1	cv	✗	✓	✓	✗	✗	✓	✓
	iter	-	26	9	-	-	1	1
	time	-	6.8E-1	7.2E+0	-	-	9.8E+0	8.9E+0
64*64 dt=4	cv	✗	✓	✓	✗	✗	✓	✓
	iter	-	404	84	-	-	1	1
	time	-	2.4E+1	3.9E+1	-	-	3.9E+1	3.8E+1

- On fine grid our method is the more robust and competitive
- This is not optimal because :
  - The matrices (7 in this case) are assembled one by one and not at the same time.
  - The extraction and reconstruction are made one by one.
  - The assembly of the matrices in Django are not optimal (PETSC configuration).
  - We solve each sub-system with a GMRES-MG(2) and not just a MG solver.
- 75% of the solving time comes from to the construction of the sub-matrices. In the future we will assume that it is possible to decrease this part by 5-6.

Futur work on the PC for MHD problem

PC Full MHD

- The matrix M contains advection and diffusion operators for ρ, T and $\mathbf{B}$
- To treat anisotropic operators splitting technics or adapted MG methods can be used.
- The LU operator (called ideal MHD force operator in the book of Schnack) is given by

$$(LU)\delta\mathbf{v} = \left[\mathbf{B} \times \nabla \times \nabla \times (I_d \times \mathbf{B}) - \mathbf{J} \times \nabla \times (I_d \times \mathbf{B}) - \nabla(I_d \cdot \nabla p + \gamma p \nabla \cdot I_d)\right] \delta\mathbf{v}$$

- Extension to bi-fluid (or extended) MHD is possible.

Discretization for Full MHD

- Compatible discretization for full MHD (DeRham Diagram for Splines)
- Study of the problem associated with the LU operator: existence, uniqueness, discretization compatible and stable.
- Discretization adapted to treat low density problem.

Other projects

Cemracs project I

- Project: "Adaptive physic based preconditioning for a linearized Discontinuous Galerkin Shallow water scheme", E.F, Philippe Helluy and Hervé Guillard.
- Exner equations for sedimentation

$$\left\{\begin{array}{l} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0 \\ \\ \partial_t h\mathbf{u} + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \nabla p = h\nabla b \\ \\ \partial_t b + \zeta\nabla\cdot\mathbf{Q} = 0 \end{array}\right.$$

with h the height, $\mathbf{u}$ the velocity, $\mathbf{Q}=\mathbf{Q}(\mathbf{u})$ and ζ a constant which depend to the sediment coefficient porosity.

time scales:

- **time step** dt: gives by gravity waves $=\sqrt{hg}$.
- **simulation time** T_f : gives by the sedimentation behavior.

- $dt << T_f$ consequently we propose to use implicit scheme.
- The hyperbolic system discretized with High-Order methods are ill-conditioned.
- **Aim** : Design efficient and robust less order preconditioning for Linearized Shallow water and Exner equations With DG schemes.

Code

- SCHNAPS a 2D and 3D DG code using macro-cells method and Gauss-Lobatto point.
- PARALUTION a library of iterative linear solvers (GRMES, MG, CG etc).

Aim

- Write an implicit version of Linearized Shallow water and Exner equations in SCHNAPS
- Write and study the physic preconditioning for these equations (question for the best extension to Exner).
- Use a continuous Galerkin method with the same degree of freedom for second order operators.
- Study and validate the less-order preconditioning.

Other possible works

- Find a way to assure positivity and stability (hyper-viscosity or flux-limiter).
- Implicit scheme and PC "well-balanced" for steady states.
- Newton method and problem and positivity.

Radiative transfert I

- Project: "implicit scheme with lower order PC for P_n models" with Xavier Blanc, Emmanuel Labourasse + Master student ?

Transport equation (photonics neuronic):

$$\partial_t f + c\mathbf{\Omega} \cdot \nabla f = c\sigma \left(\int_{S^2} f d\mathbf{\Omega} - f \right)$$

with $\mathbf{\Omega}$ the direction, c the light speed and σ the opacity.

- Important regimes:
  - Free transport regime $(\sigma \rightarrow 0)$: exact transport of the solution
  - Diffusion regime $(\sigma \rightarrow \infty)$: the solution can be approximated by

$$\partial_t E - \nabla \cdot \left(\frac{1}{3\sigma} \nabla E \right) = 0, \quad \text{with } E = \int_{S^2} f d\mathbf{\Omega}.$$

P_n models:

- The kinetic equations are approximated by linear hyperbolic P_n systems (expend the distribution on the spherical harmonics basis)

$$\partial_t \mathbf{U} + cA_x \partial_x \mathbf{U} + cA_y \partial_y \mathbf{U} + cA_z \partial_Z \mathbf{U} = -c\sigma R \mathbf{U}$$

Radiative transfert II

- **Typical simulation for IFC**: we use P_{15} model. The regime is close to free transport regime at the beginning of the simulation and in the diffusion regime after.
- **Problems for explicit scheme** : Very large and stiff hyperbolic system. Stiff hyperbolic CFL for Explicit schemes, Stiff hyperbolic + parabolic CFL condition for AP schemes.
- **Problems for implicit scheme** : large hyperbolic system (bad structure) and large ratio between wave velocities ($\{\lambda_0 c, ..., \lambda_p c\}$ with $\lambda_0 \approx 0$ and $\lambda_1 \approx 1$).
- We must add a preconditioning.
- For the previous model the velocity equation couple all the others equations consequently the Schur (parabolization of coupling terms) is write on the velocity.
- For the P_n model the coupling is more complicate.

Physic Based PC for P_n model

- Find how decompose the matrix to write the Schur decomposition.
- Write the parabolic form associated (SP_n models).
- Study the limits (diffusion and free transport regimes) of the Preconditioning operator.
- Use less spherical harmonics in the PC.

Conclusion

Conclusion

Conclusion:

- The idea to design a PC is to write the solving step as a suitability of simple operators (easy to invert) using splitting and reformulation (second order formulation) methods.
- The possible approximations gives the PC algorithm.
- **Problem**: the proposed method is dependent of the problem and use a lot of methods (CG, MG, GMRES etc) $\Longrightarrow$ lot of work to treat all the models.

Possible approximations:

- **Solving approximation**: each sub step can be solved with a small accuracy.
- **Physical approximation**: each subsystem can be simplified to obtain well-conditioned operators (necessary in the MHD case).
- **Discretization approximation**: the systems associated with the PC can be solved with less order numerical methods or coarser grids.
- **Multi-discretization approximation**: the PC models and the model can be discretized with different methods (finite element for PC and DG for the full system).

Perspectives

Middle term:

- Study of weak form for wave problem and Schur for wave equations.
- Geometric multi-grids and PC for mass Matrix with tensor product property.
- Extension to 3D current Hole in toroidal geometry and study Schur Splitting.
- Directional splitting (toroidal and poloidal) for the PC in the 3D case.
- Extension to the reduced MHD model with pressure and density.
- CEMRACS

Long term:

- Extension the full MHD and extended MHD with compatible discretization.
- Study of weak form and compatible discretization for the Schur decomposition.
- Adaptivity of the discretization (order, regularity) between PC and model.
- Extension to radiative P_n model.