Project: Preconditioned implicit DG schemes for hyperbolic systems. Application to Shallow water and Exner models.

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Multi-scale problems

Time Multi-scale problems:

- **Models**: hyperbolic systems which can modeled complex physics as nonlinear conservation laws.
- **Properties propagation**: hyperbolic systems have finite propagation speed given by the wave velocities (eigenvalues of the Jacobian).
- **Stability**: the time step is constrained by the fastest waves.
- **Example of Multi-scale problems**:
  - **Stiff problem**: $V_{\text{max}} \ll 1$ and $T_f = O(1)$.
  - **Multi-scale problem**: $V_{\text{max}} \ll V_{\text{min}}$ and $T_f = O(V_{\text{min}})$.
  - **Steady-state problem**: $V_{\text{max}} = O(1)$ and $T_f >> 1$.

Implicit scheme:

- To treat this problem, a good option: *implicit scheme*.
- For implicit schemes we must invert a linear system. Two solutions:
  - **exact solvers**: too greedy for fine 2D or 3D problem.
  - **iterative solvers**: the stiff or multi-scale hyperbolic systems are ill-conditioned.
- For iterative solvers, we need to robust and efficient preconditioning.
Exemple : MHD and radiative transfert

- Ideal MHD (astrophysic, nuclear fusion):

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\partial_t \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \mathbf{J} \times \mathbf{B} \\
\partial_t E + \nabla \cdot \left( \mathbf{u} \left( \frac{\rho |\mathbf{u}|^2}{2} + \frac{\gamma}{\gamma-1} p \right) - (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \right) &= 0 \\
\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \times \mathbf{B} - \mathbf{B} \times \mathbf{u}) &= 0
\end{align*}
\]

with \( \rho \) the density, \( \mathbf{u} \) the velocity, \( \mathbf{B} \) the magnetic field and \( E \) the energy.

time scales:

- simulation time \( T_f : 100\text{--}10000 \ T_a \) (Alfven time).
- time step \( \Delta t (<< T_f) \): gives by magneto-sonic fast wave \( V_f << V_a \) (\( V_a \) Alfvén speed).

- Transport equation (photon, neutron):

\[
\partial_t f + c \Omega \cdot \nabla f = c \sigma \left( \int_{S^2} f d\Omega - f \right), \text{ approximated by} \quad \partial_t \mathbf{U} + c \nabla \cdot \mathbf{F}(\mathbf{U}) = -c \sigma R(\mathbf{U})
\]

with \( \Omega \) the direction, \( c \) the light speed and \( \sigma \) the opacity.

time scales:

- time step \( \Delta t (<< T_f) \): constrains by \( \Delta t < \frac{\hbar}{c} + \frac{1}{c^2 \sigma}, \quad \sigma >> 1 \).
Model of the project: Exner and Shallow-water equations

- Morphodynamics flows: caused by the movement of a fluid in contact with topography (example the sediment layer).
- Many environmental problems and engineering applications.
- Shallow Water + Exner equations:

\[
\begin{align*}
\partial_t h + \nabla \cdot (hu) &= 0 \\
\partial_t hu + \nabla \cdot (hu \otimes u) + \nabla p &= h \nabla b \\
\partial_t b + \zeta \nabla \cdot Q &= 0
\end{align*}
\]

with \( h \) the height, \( u \) the velocity, \( Q = Q(u) \) and \( \zeta \) a constant which depend to the sediment coefficient porosity.

**time scales:**

- **time step** \( dt \): gives by gravity waves \( \lambda = \sqrt{hg} \).
- **simulation time** \( T_f \): gives by the sedimentation behavior.

- \( dt \ll T_f \) consequently we propose to use **implicit scheme**.
- The hyperbolic systems discretized with High-Order methods are ill-conditioned.
DG method for hyperbolic scheme on complex geometries

DG schemes:

- High order method adapted to discretize the hyperbolic systems.
- **Principle**: we discretize in each cell the weak form of the equations without continuity between the cells.
- **Reduction CPU**: quadrature using Gauss Lobatto points (diagonal mass matrix and quick computation of fluxes).

Complex geometries:

- **Idea**: we decompose the domain between curved macro-cells (GMSH).
- **Macro-cell**: inside the mesh is Cartesian.
Project

- Participant: E. Franck (Inria Nancy, Tonus team), P. Helluy (Inria Nancy, Tonus team IRMA) and H. Guillard (Inria Sophia, Castor team).
- Founding: IPL Fusion FRATRES

Aim:
Design efficient and robust preconditioned implicit algorithm for hyperbolic systems with DG high-order method on complex geometries

Objectives Cemracs:
- Write implicit method (based on GMRES) for one macro-cell (Cartesian mesh).
- Study two ways to construct physic based preconditioning.
- Validate the methods on Wave and Shallow Water equations (SH+Exner also if possible).

Post Cemracs:
- Extension to the multi macro-cells (complex geometry) case.
- Study the preconditioning (optimization, well-balanced property etc).
- Positivity and slope limiters for nonlinear problems.
First preconditioning: Directional splitting

- In each macro-cell: Cartesian mesh.
- We propose to use a directional splitting to design the preconditioning.
- Example: advection equation

\[ \partial_t u + a_x \partial_x u + a_y \partial_y u = 0 \]

- Implicit scheme:

\[ (I_d + \Delta t a_x \partial_x + \Delta t a_y \partial_y) u^{n+1} = u^n \]

Idea:

- Time Splitting: \((I + \Delta t a_x \partial_x + \Delta t a_y \partial_y) = (I + \Delta t a_x \partial_x)(I + \Delta t a_y \partial_y) + O(\Delta t^2)\)

- Algorithm to solve \(Px = b\) (P preconditioning): \(x^* = (I + \Delta t a_x \partial_x)^{-1}b\) and \(x = (I + \Delta t a_y \partial_y)^{-1}x^*\)

- Exact solver for each 1D problem in the PC (tridiagonal matrices for transport and small profile matrices for hyperbolic systems).
Second preconditioning: operator splitting

**Idea:**
- Coupling hyperbolic problem are ill-conditioned contrary to simple diffusion and advection operators.
- **Idea:** Use operator splitting and a reformulation to approximate the Jacobian by a suitability of simple problems (advection or diffusion).
- For each subproblem we use an adapted solver as multi-grid solver.

- Implicit scheme for wave: we solve
  \[
  \begin{align*}
  \partial_t u &= \partial_x v \\
  \partial_t v &= \partial_x u
  \end{align*}
  \rightarrow
  \begin{align*}
  u^{n+1} &= u^n + \Delta t \partial_x v^{n+1} \\
  v^{n+1} &= v^n + \Delta t \partial_x u^{n+1}
  \end{align*}
  
  which is strictly equivalent to solve one parabolic problem
  \[
  (1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n
  \]
  
  and applies one matrix vector product: \( v^{n+1} = v^n - \Delta t \partial_x u^{n+1} \).

**Preconditioning:**
- The solution of \( Px = b \) given by: \( u = (1 - \Delta t^2 \partial_{xx})^{-1} (u^n + \Delta t \partial_x v^n) \) and \( v = v^n - \Delta t \partial_x u^* \) with \( x = (u, v) \) and \( b = (u^n, v^n) \).
- The implicit step can be solved with multi-grid solver (small accuracy).
Second preconditioning: optimization and open questions

Models

- The operator splitting is written for the wave problem.
- **Shallow water**: we need to write the second order operator which is a reformulation of the coupling term (approximation).
- **Exner water**: how extend to the method for the Exner problem (additional equation on the topography)?

Implementation and discretization

- Discretization of full problem: DG scheme with Gauss-Lobatto points.
- Discretization of the PC operators: **EF method** with the same degrees of freedom (matrices smaller and Jacobian matrices simples).
- Gmres method + **Free jacobian method** for the full model and **multi-grid method** for the PC.

Optimization

- **Adaptivity of order**: we can use a less order approximation in the preconditioning that for the full problem.
Thanks