Preconditioned implicit DG schemes for hyperbolic systems. Application to linear and nonlinear wave problems.

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Summary

Introduction and motivations

Wave equation
- Discrete formulation
- Preconditioning and Schur complement
- Results on preconditioning for Wave equation

Shallow water equation
- Properties of the system
- $\theta$-scheme for Shallow Water and preconditioning
- Wave propagation by $P_{\text{schur}}$
Multi-scale problems

**Time Multi-scale problems:**

- **Models:** hyperbolic systems able to model complex physics through nonlinear conservation laws
- **Properties propagation:** hyperbolic systems have finite propagation speed given by the wave velocities (eigenvalues of the Jacobian).
- **Stability:** the time step is constrained by the fastest waves.
- **Multi-scale problem:** \( V_{\text{max}} \ll V_{\text{min}} \) and \( T_f = O(V_{\text{min}}) \).

- Morphodynamics flows: caused by the movement of a fluid in contact with the topography.
- **Shallow Water + Exner equations:**
  
  \[
  \begin{aligned}
  \partial_t h + \nabla \cdot (hu) &= 0 \\
  \partial_t hu + \nabla \cdot (hu \otimes u) + \nabla p &= -gh\nabla b \\
  \partial_t b + \nabla \cdot Q &= 0
  \end{aligned}
  \]

  where \( h \) is the height, \( u \) the velocity, \( b \) the topography, and \( Q = Q(u) \).

**Time scales:**

- **time step** \( \Delta t \): given by gravity waves’ speeds \( \lambda = \sqrt{hg} \).
- **simulation time** \( T_f \gg \Delta t \): given by the sedimentation behavior.
Implicit scheme:

- To treat this problem, one good option: implicit scheme.
- In the case of implicit schemes we must invert a linear system. Two solutions:
  - exact solvers: too greedy for fine 2D or 3D problems.
  - iterative solvers: the stiff or multi-scale hyperbolic systems are ill-conditioned.
- For iterative solvers, we need to find a robust and efficient preconditioning.

Aim:

To design efficient and robust preconditioned implicit algorithm for hyperbolic systems with DG high-order method on complex geometries.

Objectives:

- Write implicit method (based on GMRES+ Free Jacobian method) for one macro-cell.
- Design and study the physics based preconditioning (based on physical or numerical approximations).
- Full model in Discontinuous Galerkin and preconditioning in Continuous Galerkin.
- Validate the methods on Wave and Shallow Water equations.
Schnaps code: DG method for hyperbolic scheme

DG schemes:
- High order method adapted to the discretization of hyperbolic systems.
- **Principle**: we discretize in each cell the weak form without enforcing continuity between the cells.
- **Reduction CPU**: quadrature using Gauss Lobatto points (diagonal mass matrix).
- **Conditioning**: High-order methods are ill-conditioned.

Complex geometries:
- **Idea**: we decompose the domain between curved macro-cells (GMSH).
- **Macro-cell**: Cartesian in the interior.
General discretization method

We consider systems of equation of the following shape:

\[ \partial_t U + F(U) = S, \]

with \( U \in \mathbb{R}^N \) our unknowns, \( F \) a function acting on \( U \) and \( S \) source terms independent of \( U \).

- **Discretization in space**: Discontinuous Galerkin
  - Polynomial approximation of the solution \( U \) in each cell,
  - Weak formulation: \( \partial_t (U, \varphi)_{L^2} + (f(U), \varphi)_{L^2} = (S, \varphi)_{L^2}, \quad \forall \varphi \) basis function,
  - Discontinuous basis functions,

\[
\varphi_i^+(x_j) = \begin{cases} 
1, & \text{if } j = i^+, \\
0, & \text{if } j \neq i^+,
\end{cases}
\]

where \( x_j \) is a Gauss-Lobatto point.

- **Discretization in time**: \( \theta \)-scheme

\[
U^{n+1} + \Delta t \theta F(U^{n+1}) = U^n - \Delta t (1 - \theta) F(U^n) + \Delta t \theta S^{n+1} + \Delta t (1 - \theta) S^n
\]

Second-order and unconditionally-stable for \( \theta = 0.5 \) (Crank-Nicholson).
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Acoustic wave equations:
\[
\begin{aligned}
\partial_t p + c \nabla \cdot u &= S_p, \\
\partial_t u + c \nabla p &= S_u.
\end{aligned}
\]

with \( u \) the velocity, \( p \) the pressure and \( c \) the speed wave.

We use the previously described discretization with
\[
U = \begin{pmatrix} p \\ u \end{pmatrix}, \quad F(U) = \begin{pmatrix} c \nabla \cdot u \\ c \nabla p \end{pmatrix}, \quad S = \begin{pmatrix} S_p \\ S_u \end{pmatrix}
\]

Linear hyperbolic system.

Wave propagation:
- Study of solutions given by an equilibrium state and an irrotational perturbation \( p = p_0 + \delta p \) and \( u = u_0 + \delta u \).
- Propagation of the perturbation at the velocity \( \pm c \).
Matrix equation

The weak formulation leads to the matrix equation

\[
\begin{pmatrix}
M & c\theta \Delta t U_1 & c\theta \Delta t U_2 \\
c\theta \Delta t L_1 & M & 0 \\
c\theta \Delta t L_2 & 0 & M
\end{pmatrix}
\begin{pmatrix}
p^{n+1} \\
u^{n+1} \\
v^{n+1}
\end{pmatrix}
= \begin{pmatrix}
M & -c(1 - \theta) \Delta t U_1 & -c(1 - \theta) \Delta t U_2 \\
-c(1 - \theta) \Delta t L_1 & M & 0 \\
-c(1 - \theta) \Delta t L_2 & 0 & M
\end{pmatrix}
\begin{pmatrix}
p^{n} \\
u^{n} \\
v^{n}
\end{pmatrix},
\]

with

\[
M = \left( \int_{\Omega} \varphi_i \varphi_j \, dx \right)_{(i,j) \in [1,N]^2}, \text{ the mass matrix}
\]

\[
U_1 = \left( \int_{\Omega} \partial_x \varphi_j \varphi_i \right)_{(i,j) \in [1,N]^2}, \quad U_2 = \left( \int_{\Omega} \partial_y \varphi_j \varphi_i \right)_{(i,j) \in [1,N]^2},
\]

\[
L_1 = \left( \int_{\Omega} \partial_x \varphi_j \varphi_i \right)_{(i,j) \in [1,N]^2}, \quad L_2 = \left( \int_{\Omega} \partial_y \varphi_j \varphi_i \right)_{(i,j) \in [1,N]^2}.
\]

Aim

Preconditioning of the Jacobian matrix thanks to the Schur theory.
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Schur complement

Property (Schur decomposition)
Let $\mathcal{A}$ be a matrix defined by blocks

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A$ invertible. Assume $A$, $B$, $C$, $D$ are respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices, one has

$$\mathcal{A} = \begin{pmatrix} I_p & 0 \\ CA^{-1} & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_p & A^{-1}B \\ 0 & I_q \end{pmatrix}$$

where $I_p$ is the $p \times p$ identity matrix.

Definition (Schur complement)
If $\mathcal{A}$ is a matrix defined by blocks as in the previous property, the Schur complement of $\mathcal{A}$ is $D - CA^{-1}B$
Preconditioning for Wave equation

Applying this decomposition to our system yields

\[
\begin{pmatrix}
M & c\theta \Delta tU_1 & c\theta \Delta tU_2 \\
c\theta \Delta tL_1 & M & 0 \\
c\theta \Delta tL_2 & 0 & M
\end{pmatrix}
= \\
\begin{pmatrix}
I_N & 0 & 0 \\
c\theta \Delta tLM^{-1} & I_{N^2} & 0 \\
0 & 0 & M - c^2 \theta^2 \Delta t^2 LM^{-1} U
\end{pmatrix}
\begin{pmatrix}
I_N & c\theta \Delta tM^{-1}U \\
0 & I_{N^2}
\end{pmatrix}
\]

with

\[
M = \begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix}, \quad L = \begin{pmatrix}
L_1 \\
L_2
\end{pmatrix}, \quad U = \begin{pmatrix}
U_1 \\
U_2
\end{pmatrix},
\]

and

\[
P_{\text{schur}} = M - c^2 \theta^2 \Delta t^2 LM^{-1} U.
\]
Preconditioning for Wave equation

Applying this decomposition to our system yields

\[
\begin{pmatrix}
M & c\theta \Delta t U_1 & c\theta \Delta t U_2 \\
c\theta \Delta t L_1 & M & 0 \\
c\theta \Delta t L_2 & 0 & M
\end{pmatrix}
= \begin{pmatrix}
I_N & 0 & 0 \\
c\theta \Delta t L M^{-1} & I_{N^2} & 0 \\
c\theta \Delta t L M^{-1} & I_{N^2} & 0
\end{pmatrix}
\begin{pmatrix}
M & 0 & 0 \\
0 & M - c^2\theta^2 \Delta t^2 L M^{-1} U & 0 \\
0 & 0 & I_{N^2}
\end{pmatrix}
\begin{pmatrix}
I_N & c\theta \Delta t M^{-1} \ U \\
I_N & M - c^2\theta^2 \Delta t^2 L M^{-1} U & I_{N^2}
\end{pmatrix}
\]

with

\[M = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},\]

and

\[P_{schur} = M - c^2\theta^2 \Delta t^2 L M^{-1} U.\]

Hence, the preconditioning for the wave problem unfolds as the following splitting

\[
\begin{cases}
Mp^* = Mp^n - c(1 - \theta) \Delta t MU \begin{pmatrix} u^n \\ v^n \end{pmatrix}, \quad \text{prediction step,} \\
P_{schur} \begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = M \begin{pmatrix} u^n \\ v^n \end{pmatrix} - c(1 - \theta) \Delta t MLp^n - c\theta \Delta t Lp^*, \quad \text{propagation step,} \\
Mp^{n+1} = -c\theta \Delta t U \begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} + Mp^*, \quad \text{correction step.}
\end{cases}
\]
Properties of $P_{\text{schur}}$

- To retrieve from those systems of equations the underlying physics, preconditioning has to follow some properties.
- The splitted system should keep as most physical properties from the original problem as possible.
- $P_{\text{schur}} \equiv l_2 - c^2 \theta^2 \nabla(\nabla \cdot l_2)$

### Properties of $P_{\text{schur}}$

- $P_{\text{schur}}$ should be easy to invert,
- $P_{\text{schur}}$ is self adjoint,
- $P_{\text{schur}}$ propagates an irrotational perturbation with the same speed $\pm c$ as the original problem.

**Proof.** $P_{\text{schur}}$ is the discretization of the motion equation

$$\partial_{tt} \xi - c^2 \theta^2 \nabla (\nabla \cdot \xi) = 0,$$

where $u = \partial_t \xi$.

**Remark**: The preconditioning has the same propagation speed as the full model, which is equivalent at the spectral level.
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Results

Here, we compile some results on different test-cases in Discontinuous Galerkin of fourth order for the acoustic wave equations.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Type of preconditioning</th>
<th>Mesh</th>
<th>Number of iteration / time-step</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>GMRES</td>
<td>20 $\times$ 20</td>
<td>103</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>20 $\times$ 20</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>GMRES</td>
<td>40 $\times$ 40</td>
<td>224</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>40 $\times$ 40</td>
<td>3</td>
</tr>
<tr>
<td>0.05</td>
<td>GMRES</td>
<td>20 $\times$ 20</td>
<td>762</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>20 $\times$ 20</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>GMRES</td>
<td>40 $\times$ 40</td>
<td>1594</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>40 $\times$ 40</td>
<td>20</td>
</tr>
</tbody>
</table>

**Table:** Results for a steady state.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Type of preconditioning</th>
<th>Mesh</th>
<th>Number of iteration / time-step</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>GMRES</td>
<td>30 $\times$ 30</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>30 $\times$ 30</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>GMRES</td>
<td>40 $\times$ 40</td>
<td>220</td>
</tr>
<tr>
<td></td>
<td>GMRES-PC</td>
<td>40 $\times$ 40</td>
<td>12</td>
</tr>
</tbody>
</table>

**Table:** Results for a periodic wave problem.
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Shallow Water equation

Shallow Water equation:

\[
\begin{align*}
\partial_t h + \nabla \cdot (hu) &= 0 \\
\partial_t hu + \nabla \cdot (hu \otimes u) + \nabla p &= -gh\nabla b
\end{align*}
\]

with \( h \) the height, \( u \) the velocity, and the pressure \( p = \frac{gh^2}{2} \).

We can diagonalize the system to obtain the eigenvalues: \((u, n) \pm c\) and \((u, n)\), with \( c = \sqrt{hg} \) the sound speed.

Linearized Shallow Water Homogeneous equation: we consider that the solutions are given by an equilibrium and a perturbation \( h = h_0 + \delta h \) and \( u = u_0 + \delta u \).

The linearized system propagates these perturbations at the velocity \((u_0, n) \pm \sqrt{h_0}g\) and \((u_0, n)\).

Aim of the Physic-Based Preconditioner:

To obtain a simpler operator (well-conditioned) which propagates the perturbations with velocities close to the original problem.
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\(\theta\)-scheme and free Jacobian method (I)

\(\theta\)-scheme applied on Shallow Water equation yields

\[
\begin{aligned}
\begin{cases}
    h^{n+1} + \theta \Delta t \nabla \cdot (h^{n+1} u^{n+1}) = h^n - \Delta t (1 - \theta) \nabla \cdot (h^n u^n), \\
    h^{n+1} u^{n+1} + \theta \Delta t h^{n+1} (u^{n+1} \cdot \nabla) u^{n+1} + \theta \Delta t h^{n+1} g \nabla h^{n+1} + \theta \Delta t g h^{n+1} \nabla b \\
    = h^n u^n - \Delta t (1 - \theta) h^n (u^n \cdot \nabla) u^n - \Delta t (1 - \theta) g h^n \nabla h^n - (1 - \theta) \Delta t g h^n \nabla b.
\end{cases}
\end{aligned}
\]

This system can be rewritten in the form

\[
\begin{aligned}
G\left(\begin{bmatrix} h^{n+1} \\ u^{n+1} \end{bmatrix}\right) = B\left(\begin{bmatrix} h^n \\ u^n \end{bmatrix}\right),
\end{aligned}
\]

with

\[
G: \begin{bmatrix} h \\ u \end{bmatrix} \mapsto \begin{bmatrix} h + \theta \Delta t \nabla \cdot (h u) \\ h u + \theta \Delta t h (u \cdot \nabla) u + \theta \Delta t g h \nabla h + \theta \Delta t g h \nabla b \end{bmatrix},
\]

and

\[
B: \begin{bmatrix} h \\ u \end{bmatrix} \mapsto \begin{bmatrix} h - \Delta t (1 - \theta) \nabla \cdot (h u) \\ h u - \Delta t (1 - \theta) h (u \cdot \nabla) u - \Delta t (1 - \theta) g h \nabla h - (1 - \theta) \Delta t g h \nabla b \end{bmatrix}.
\]
A linearization of $G$ gives

$$Jac^n_G \left( \begin{array}{c} \delta h^n \\ \delta u^n \end{array} \right) = B \left( \begin{array}{c} h^n \\ u^n \end{array} \right) - G \left( \begin{array}{c} h^n \\ u^n \end{array} \right),$$

with $\delta h^n = h^{n+1} - h^n$, $\delta u^n = u^{n+1} - u^n$ and $Jac^n_G$ the Jacobian matrix of $G$ at $\left( \begin{array}{c} h^n \\ u^n \end{array} \right)$,

$$Jac^n_G = \left( \begin{array}{cc} D_1 & U \\ L & D_2 \end{array} \right)$$

with

$$D_1 = I_1 + \theta \Delta t \nabla \cdot (u^n I_1), \quad U = \theta t \nabla \cdot (h^n I_2),$$

$$L = u^n I_1 + \theta \Delta t g \nabla (h^n I_1) + \theta \Delta t l_1 (u^n \cdot \nabla) u^n + \theta g I_1 \Delta t \nabla b,$$

$$D_2 = h^n I_2 + \theta \Delta t h^n (u^n \cdot \nabla) I_2 + \theta \Delta t h^n (l_2 \cdot \nabla) u^n.$$
Preconditioning Algorithm based on $P_{\text{schur}}$

The Schur decomposition gives the following algorithm

$$
\begin{align*}
D_1 \delta h^* &= -\Delta t \nabla \cdot (h^n u^n), \\
(D_2 - LD_1^{-1} U) \delta u^{n+1} &= -L \delta h^* - \Delta t h^n (u^n \cdot \nabla) u^n - \Delta t gh^n \nabla h^n - \Delta t \theta gh^n \nabla b, \\
D_1 \delta h^{n+1} &= D_1 \delta h^* - U \delta u^{n+1},
\end{align*}
$$

while its complement for the Shallow Water system is

$$
P_{\text{schur}} = D_2 - LD_1^{-1} U.
$$

Different flows

We want to study different Schur approximations introduced by L. Chacón for MHD flows:

- slow flow,
- arbitrary flow,

in order to compute the linear wave propagation of $P_{\text{schur}}$.

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Slow flow approximation of the Schur complement

**Slow flow hypothesis**

We assume that the flow is small, consequently $\Delta t |u^n| \ll 1$. Consequently we obtain that $D_1 \approx I_1$ in this regime.

For a constant velocity $u^n$, $P_{\text{schur}}$ becomes

$$P_{\text{schur}} = D_2 - L I_1 U = h^n I_2 + \theta t^n (u^n \cdot \nabla) I_2 + \theta \Delta t h^n (I_2 \cdot \nabla) u^n - LU,$$

and

$$LU = \theta \Delta t (u^n + \theta \Delta t (u^n \cdot \nabla) u^n) \nabla \cdot (h^n I_2) + \theta^2 \Delta t^2 \nabla [l_2 \cdot \nabla p^n + 2p^n \nabla \cdot l_2].$$
Slow flow hypothesis

We assume that the flow is small, consequently \( \Delta t |\mathbf{u}^n| \ll 1 \). Consequently we obtain that \( D_1 \approx I_1 \) in this regime.

For a constant velocity \( \mathbf{u}^n \), \( P_{\text{schur}} \) becomes

\[
P_{\text{schur}} = D_2 - L_1 U = h^n l_2 + \theta th^n (\mathbf{u}^n \cdot \nabla) l_2 + \theta \Delta th^n (l_2 \cdot \nabla) \mathbf{u}^n - LU,
\]

and

\[
LU = \theta \Delta t (\mathbf{u}^n + \theta \Delta t (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n) \nabla \cdot (h^n l_2) + \theta^2 \Delta t^2 \nabla [l_2 \cdot \nabla p^n + 2p^n \nabla \cdot l_2].
\]

**Hypothesis 1.** We neglect the advection term in \( LU \), to obtain the dispersion relation

\[
\omega = \left( \theta \frac{\mathbf{u}^n \cdot n}{2} \pm \theta \sqrt{h^n g - \frac{(\mathbf{u}^n \cdot n)^2}{4}} \right) \| \mathbf{k} \|.
\]

**Hypothesis 2.** We consider now the full \( LU \) operator, and we obtain

\[
\omega = \pm \theta \sqrt{gh^n} \| \mathbf{k} \|.
\]

**Proof.** To prove those two results, we write the motion equation on \( \xi \) with \( \partial_t \xi = \mathbf{u} \) and inject an irrotational linear plane wave.
The approximation $D_1 \approx I_1$ is not valid anymore, we have to consider $D_1^{-1}$ in $P_{\text{schur}}$.

We introduce the construction of an operator $M$ such that $UM \approx D_1 U$ consequently we obtain that

$$P_{\text{schur}} = (D_2 M - LU)M^{-1}.$$

The solution of the equation $P_{\text{schur}} \delta u = 0$ is given by

$$\begin{align*}
(D_2 M - LU) \delta u^* &= 0, \\
\delta u &= M \delta u^*.
\end{align*}$$

We choose

$$M = I_2 + \theta \Delta t u^n (\nabla \cdot I_2).$$

For a constant velocity $u^n$,

**Hypothesis 1.** We neglect advection terms in $LU$ to obtain the dispersion relation

$$\omega = \pm \theta \sqrt{gh^n \|k\|}.$$

**Hypothesis 2.** We consider each term of the $LU$ operator to obtain the following dispersion relation

$$\omega = \left(-\theta \frac{u^n \cdot n}{2} \pm \theta \sqrt{h^n g - \frac{3}{4}(u^n \cdot n)^2}\right) \|k\|.$$
Theoretical perspectives

- Propose two new approximations of the Schur that are
  - non-negative,
  - partly or fully symmetric,
  - spectrally close in the fast-flow regime:
    \[ u \cdot n \pm c \] for the first approximation, and \( \pm (u \cdot n + c) \) for the second one.

Numerical perspectives

- Optimize preconditioning for the wave system,
- Validation of the Shallow Water preconditioning with Jacobian free method,
- Variation of the approximation degrees between the preconditioned and the full model.
We are grateful to our supervisors for their continuous support!

Thank you for your attention