Uniform asymptotic preserving and well-balanced schemes for stiff hyperbolic systems in diffusion regime

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Outline

Mathematic and physical context

Stiff hyperbolic systems

- **Stiff hyperbolic system with source terms**:

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_x F(\mathbf{U}) + \frac{1}{\varepsilon}\partial_y G(\mathbf{U}) = \frac{1}{\varepsilon} S(\mathbf{U}) - \frac{\sigma}{\varepsilon^2} R(\mathbf{U}),\ \mathbf{U} \in \mathbb{R}^n$$

with $\varepsilon \in]0,1]$ et $\sigma > 0$.

- Subset of solutions given by the balance between the source terms and the convective part:
  - **Diffusion solutions** for $\varepsilon \to 0$ and $S(\mathbf{U}) = 0$:

$$\partial_t \mathbf{V} - \operatorname{div}\left(K(\nabla \mathbf{V}, \sigma)\right) = 0, \quad \mathbf{V} \in \operatorname{Ker} R.$$

  - **Steady-state** for $\sigma = 0$ et $\varepsilon \to 0$:

$$\partial_x F(\mathbf{U}) + \partial_y G(\mathbf{U}) = S(\mathbf{U}).$$

- Applications: biology, neutron transport, fluid mechanics, plasma physics, Radiative hydrodynamic (hydrodynamic + linear transport of photon).

Notion of WB and AP schemes

- Acoustic equation with damping and gravity:

$$\begin{cases} \partial_t p + \dfrac{1}{\varepsilon}\partial_x u = 0, \\ \partial_t u + \dfrac{1}{\varepsilon}\partial_x p = -\dfrac{1}{\varepsilon}g - \dfrac{\sigma}{\varepsilon^2}u, \end{cases} \longrightarrow \partial_t p - \partial_x\left(\frac{1}{\sigma}(\partial_x p + g)\right) = 0.$$

- Steady-state: $u = C$, $\partial_x p = -g - \frac{\sigma}{\varepsilon}C$.
- **Godunov-type** schemes give an error homogeneous to $O(\Delta x)$.
- For nearly uniform flows, spurious velocities larger that physical velocity.
- Important deviation of the steady-state.
- **WB scheme**: discretize the steady-state exactly of with high accuracy.
- Ref: S. Jin, A *steady-state capturing method for hyperbolic method with geometrical source terms.*

- Consistency of **Godunov-type** schemes: $O(\dfrac{\Delta x}{\varepsilon} + \Delta t)$.
- CFL condition: $\Delta t(\dfrac{1}{\Delta x \varepsilon} + \dfrac{\sigma}{\varepsilon^2}) \leq 1$.
- Consistency of AP schemes: $O\left(\Delta x + \Delta t\right)$.
- CFL condition: degenerate on parabolic CFL at the limit.
- Ref: S. Jin, D. Levermore *Numerical schemes for hyperbolic conservation laws with stiff relaxation.*

- To construct WB and AP schemes: incorporate the source in the fluxes to capture the balance between source and convective terms.

AP scheme and Uniform convergence in 1D

Jin-Levermore scheme

- **Jin-Levermore scheme**. Plug the balance law $\partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes. We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}})\partial_x p(x_{j+\frac{1}{2}})$$

Coupling the previous relation (and the same for x_{j+1}) with the fluxes

$$\begin{cases} u_j + p_j = u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}\Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\ u_{j+1} - p_{j+1} = u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}\Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}. \end{cases}$$

Jin-Levermore scheme:

$$\begin{cases} \frac{p_j^{n+1} - p_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}} u^n_{j+\frac{1}{2}} - M_{j-\frac{1}{2}} u^n_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{p^n_{j+\frac{1}{2}} - p^n_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{\sigma}{\varepsilon^2} u_j^n = 0, \end{cases}$$

with

$$\begin{cases} u_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2} + \frac{p_j - p_{j+1}}{2} \\ p_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2} + \frac{u_j - u_{j+1}}{2} \end{cases}$$

and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}$.

Jin-Levermore scheme

- **Jin-Levermore scheme**. Plug the balance law $\partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes. We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon} u(x_{j+\frac{1}{2}})$$

Coupling the previous relation (and the same for x_{j+1}) with the fluxes

$$\begin{cases} u_j + p_j = u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\ u_{j+1} - p_{j+1} = u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}. \end{cases}$$

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and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}$.

Jin-Levermore scheme II

- **Consistency error** of the Jin-Levermore scheme and classical scheme (uniform mesh):
  - First equation: $O\left(\Delta x + \Delta t\right)$ (ref $\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$ for the classical scheme)
  - Second equation: $O\left(\frac{\Delta x^2}{\varepsilon} + \Delta t\right)$ (ref $\left(\frac{\Delta x^2}{\varepsilon} + \Delta t\right)$ for the classical scheme)

- **Time discretization**:
  - Explicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon + \varepsilon}\right) \leq 1$
  - Semi-implicit CFL : $\Delta t\left(\frac{1}{\Delta x\varepsilon}\right) \leq 1$.

- **Well-balanced property**:
  - Uniform mesh: the scheme is WB,
  - Non-uniform mesh: the scheme is not WB.

Gosse-Toscani scheme

- **Classical strategy**: Localization of the source at the interface and the Riemann problem associated.
- **Other solution**: we take the following source term $\frac{1}{2}(u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}})$ with the JL scheme.

Gosse-Toscani scheme:

$$\begin{cases} \frac{p_j^{n+1}-p_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}}u_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}u_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} \\ \frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}}p_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}p_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} - \frac{M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}}{\Delta x_j \varepsilon} p_j^n + \left(\frac{\sigma_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{2\varepsilon^2 \Delta_j} + \frac{\sigma_{j-\frac{1}{2}}\Delta x_{j-\frac{1}{2}}}{2\varepsilon^2 \Delta_j} \right) u_j^n = 0 \end{cases}$$

with

$$u_{j+\frac{1}{2}} = \frac{u_j^n + u_{j+1}^n}{2} + \frac{p_j^n - p_{j+1}^n}{2}, \qquad p_{j+\frac{1}{2}} = \frac{p_j^n + p_{j+1}^n}{2} + \frac{u_j^n - u_{j+1}^n}{2}$$

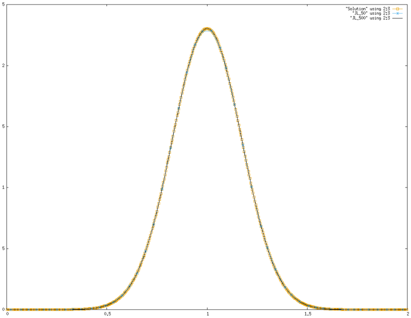
and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}$.

- **Consistency error** of the Gosse-Toscani (uniform mesh): $O(\Delta x + \Delta t)$
- **Time discretization**:
  - Explicit CFL: $\Delta t \left(\frac{1}{\Delta x \varepsilon}\right) \leq 1$, Semi-implicit CFL : $\Delta t \left(\frac{1}{\Delta x \varepsilon + \Delta x^2}\right) \leq 1$.
- **Well-balanced property**: WB scheme on all meshes.

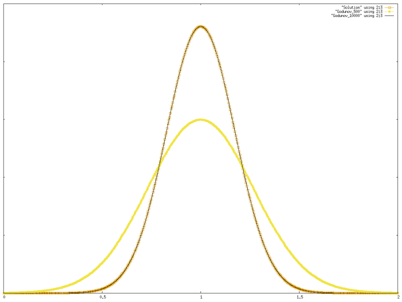
Numerical example

- **Validation test for the AP scheme**: the data are $p(0,x) = G(x)$ with $G(x)$ a Gaussian $u(0,x) = 0$ and $\sigma = 1$, $\varepsilon = 0.001$.

Jin-Levermore scheme



Godunov scheme



Scheme	L^2 error	CPU time
Godunov, 10000 cells	0.0376	505 sec
Godunov, 500 cells	0.42	5.31 sec
AP-JL, 500 cells	4.3E-3	5.42 sec
AP-JL, 50 cells	0.012	0.46 sec
AP-GT, 500 cells	1.3E-4	2.38 sec
AP-GT, 50 cells	0.012	0.013 sec

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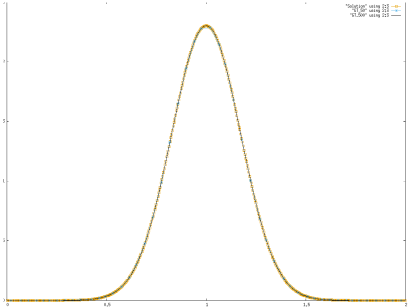
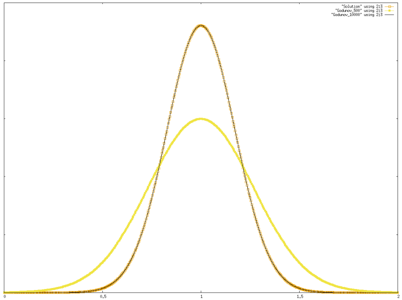
Jin-Levermore scheme



Godunov scheme



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Test for Well-Balanced property

- We propose to validate the Well-Balanced property.
- For this, we initialize the scheme with a steady state and simulate with a large final time (T_f=20).
- Steady state:

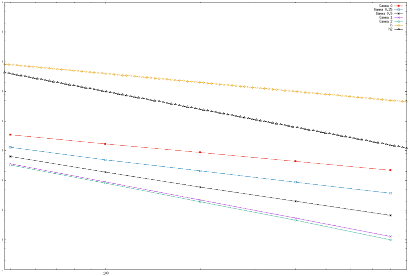
$$\left\{\begin{array}{l} u(t,x)=C_1 \\ p(t,x)=-(g+\frac{\sigma}{\varepsilon}C_1)x+C_2 \end{array}\right.$$

Scheme/mesh	Uniform Mesh	Random Mesh
Godunov, 100 cells	0.0	2.83E-3
Godunov, 1000 cells	5.0E-17	2.7E-4
AP-JL, 100 cells	0.0	3.3E-3
AP-JL, 1000 cells	6.3E-17	3.9E-4
AP-GT, 100 cells	3.1E-16	3.1E-16
AP-GT, 1000 cells	3.0E-16	2.8E-15

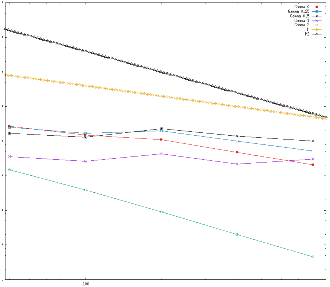
Test for uniform convergence in 1D

- ε dependent periodic solution for the P_1 model.
- $p(t,x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha^{'}(t))\cos(\pi x), \quad u(t,x) = (-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x))$
- Convergence study for $\varepsilon = h^{\gamma}$ on random mesh.

JL scheme on uniform mesh



JL scheme on random mesh



- The GT scheme and the JL scheme (only on uniform mesh) are uniform AP with the error $O(h\varepsilon + h^2)$.
- On Random mesh the JL scheme is not an uniform AP scheme.

Test for uniform convergence in 1D

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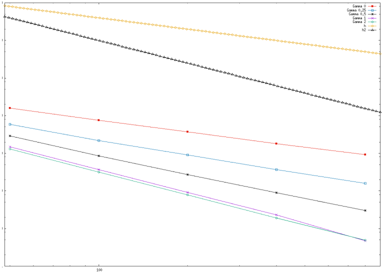
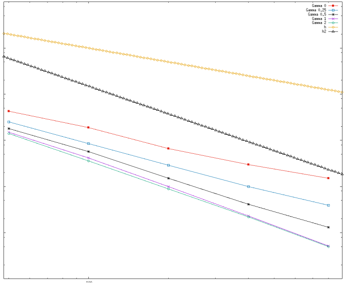
GT scheme on uniform mesh



GT scheme on random mesh



- The GT scheme and the JL scheme (only on uniform mesh) are uniform AP with the error $O(h\varepsilon + h^2)$.
- On Random mesh the JL scheme is not an uniform AP scheme.

Analysis of AP schemes: modified equations

- To understand the behavior of the scheme, we use the modified equations method.

- The modified equation associated with the Upwind scheme is

$$\left\{\begin{array}{l}\partial_t p+\frac{1}{\varepsilon}\partial_x u-\frac{\Delta x}{2\varepsilon}\partial_{xx}p=0,\\ \partial_t u+\frac{1}{\varepsilon}\partial_x p-\frac{\Delta x}{2\varepsilon}\partial_{xx}u=-\frac{\sigma}{\varepsilon^2}u.\end{array}\right.$$

- Plugging $\varepsilon\partial_x p+O(\varepsilon^2)=-\sigma u$ in the first equation, we obtain the diffusion limit

$$\partial_t p-\frac{1}{\sigma}\partial_{xx}p-\frac{\Delta x}{2\varepsilon}\partial_{xx}p=0.$$

- **Conclusion**: the regime is captured only on fine grids.

- The modified equation associated to the Gosse-Toscani scheme is

$$\left\{\begin{array}{l}\partial_t p+M\frac{1}{\varepsilon}\partial_x u-M\frac{\Delta x}{2\varepsilon}\partial_{xx}p=0,\\ \partial_t u+M\frac{1}{\varepsilon}\partial_x p-M\frac{\Delta x}{2\varepsilon}\partial_{xx}u=-M\frac{\sigma}{\varepsilon^2}u.\end{array}\right.$$

- Plugging $M\varepsilon\partial_x p+O(\varepsilon^2)=-M\sigma u$ in the first equation, we obtain the diffusion limit

$$\partial_t p-\frac{M}{\sigma}\partial_{xx}p-\frac{1-M}{\sigma}\partial_{xx}p=0$$

- **Conclusion**: the regime is captured on all grids.

Conclusion of the Uniform convergence

AP schemes on uniform grids

- AP schemes modify the numerical diffusion to correct the classical scheme on coarse grid.
- Generally these schemes are uniformly AP on uniform grids.

AP schemes on non-uniform grids

- On non-uniform grids the situation is more complex.
- For example the JL scheme does not converge in the intermediary regimes.
- **Possible Explanation**: since the linear steady states are not preserved the limit diffusion scheme in these regimes does not converge.

Open question

- Link between AP and Well-Balanced schemes for linear steady states. Sufficient condition ? Necessary condition ?

Uniform convergence in space

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$
- **Idea**: use triangular inequalities and AP diagram (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$

$$\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\
P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0
\end{tikzcd}$$

- Intermediary estimations :
  - $||P^\varepsilon - P^0|| \leq C_a\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq C_d h^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C_e\varepsilon^e$,
  - $d \geq c$, $e \geq a$.

- We using $\min(x, y+z) \leq \min(x,y) + \min(x,z)$ and $d \geq c$, $e \geq a$ to obtain

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq C\left(\min(\varepsilon^{-b}h^c, \varepsilon^e) + h^d + \min(\varepsilon^{-b}h^c, \varepsilon^a)\right) \leq 2C\left(h^d + \min(\varepsilon^{-b}h^c, \varepsilon^a)\right)$$

.

- Defining $\varepsilon_{threshold}^{-b}h^c = \varepsilon_{threshold}^a$ we obtain $\min(\varepsilon^{-b}h^c, \varepsilon^a) \leq \varepsilon_{threshold}^a = h^{\frac{ac}{a+b}}$ and

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq h^{\frac{ac}{a+b}}$$

.

Uniform convergence for the Gosse-Toscani scheme

Space result:

We assume that $\|\mathbf{V}^{\varepsilon}(0)-\mathbf{V}_h^{\varepsilon}(0)\|_{L^2(\Omega)} \leq Ch \parallel p(0) \parallel_{H^2}$ and $C_1 h < \Delta x_j < C_2 h \quad \forall j$.
There exist $C(T)>0$ such that:

$$\|\mathbf{V}^{\varepsilon}-\mathbf{V}_h^{\varepsilon}\|_{L^2([0,T]\times\Omega)} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, h+2\varepsilon\right) \parallel p_0 \parallel_{H^3(\Omega)} \leq Ch^{\frac{1}{3}} \parallel p_0 \parallel_{H^3(\Omega)}$$

- **Proof**: we prove all the intermediary estimates. We use the triangular inequality and we conclude.
- **Time discretization**: Using a abstract formulation of implicit scheme (B. Després) we obtain

Final result:

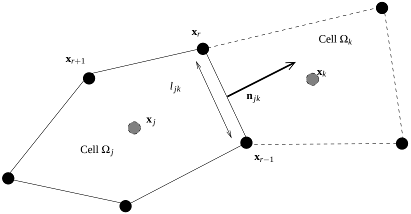
We assume that $\|\mathbf{V}^{\varepsilon}(0)-\mathbf{V}_h^{\varepsilon}(0)\|_{L^2(\Omega)} \leq Ch \parallel p(0) \parallel_{H^2}$ and $C_1 h < \Delta x_j < C_2 h \quad \forall j$.
There exist $C(T)>0$ such that:

$$\|\mathbf{V}^{\varepsilon}(n\Delta t)-\mathbf{V}_h^{\varepsilon}(n\Delta t)\|_{L^2(\Omega)} \leq C(h^{\frac{1}{3}}+\Delta t^{\frac{1}{2}}) \parallel p_0 \parallel_{H^3(\Omega)}$$

2D AP scheme on unstructured meshes

Schémas "Asymptotic preserving" 2D

- **Classical extension in 2D of the Jin-Levermore scheme** : modify the upwind fluxes (1D fluxes write in the normal direction) plugging the steady-state in the fluxes.



- l_{jk} and $\mathbf{n}_{jk}$ the normal and length associated with the edge $\partial\Omega_{jk}$.

Asymptotic limit of the hyperbolic scheme:

$$\mid \Omega_j \mid \partial_t p_j(t) - \frac{1}{\sigma}\sum_k l_{jk}\frac{p_k^n - p_j^n}{d(\mathbf{x}_j, \mathbf{x}_k)} = 0.$$

- $||P_h^0 - P_h|| \to 0$ only on strong geometrical conditions.
- **Additional difficulty in 2D**: The basic extension of AP schemes **do not converge** on 2D general meshes $\forall \varepsilon$.

Nodal scheme : linear case

- Linear case: P_1 model

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\,\mathrm{div}(\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p = -\frac{\sigma}{\varepsilon^2}\mathbf{u}. \end{cases} \longrightarrow \partial_t p - \mathrm{div}\left(\frac{1}{\sigma}\nabla p\right) = 0.$$

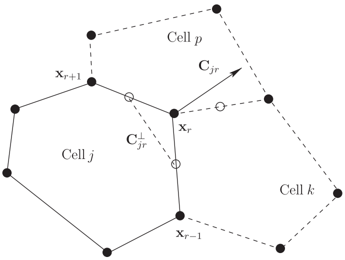
Idea:

Nodal finite volume methods for P_1 model + AP and WB method.

Nodal schemes:

The fluxes are localized at the nodes of the mesh (for the classical scheme this is at the edge).

Notations



- Nodal geometrical quantities $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r}|\Omega_j|$.
- $\sum_j \mathbf{C}_{jr} = \sum_r \mathbf{C}_{jr} = \mathbf{0}$.

Nodal AP schemes

$$\begin{cases} \mid \Omega_j \mid \partial_t p_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \mid \Omega_j \mid \partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r \mathbf{pc}_{jr} = \mathbf{S}_j. \end{cases}$$

- Classical nodal fluxes:

$$\begin{cases} \mathbf{pc}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r), \\ \sum_j \mathbf{pc}_{jr} = \mathbf{0}, \end{cases}$$

with $\widehat{\alpha}_{jr} = \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{\|\mathbf{C}_{jr}\|}$.

- New fluxes obtained plugging steady-state $\nabla p = -\frac{\sigma}{\varepsilon}\mathbf{u}$ in the fluxes:

$$\begin{cases} \mathbf{pc}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \dfrac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{u}_r, \\ \left(\displaystyle\sum_j \widehat{\alpha}_{jr} + \dfrac{\sigma}{\varepsilon}\sum_j \widehat{\beta}_{jr}\right)\mathbf{u}_r = \displaystyle\sum_j p_j \mathbf{C}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j. \end{cases}$$

with $\widehat{\beta}_{jr} = \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$.

- Source term: (1) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \mid \Omega_j \mid \mathbf{u}_j$ ou (2) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2}\sum_r \widehat{\beta}_{jr}\mathbf{u}_r$, $\quad \sum_r \widehat{\beta}_{jr} = \hat{I}_d |\Omega_j|$.
- Using the second source term and rewriting the scheme we obtain an local semi implicit scheme with a CFL independent of ε.

Diffusion scheme

Limit diffusion scheme (P_h^0)

$$\begin{cases} \mid \Omega_j \mid \partial_t p_j(t) - \sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \sum_r \hat{\alpha}_{jr}\mathbf{u}_j = \sum_r \hat{\alpha}_{jr}\mathbf{u}_r, \\ \sigma A_r \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{cases}$$

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\ P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 \end{tikzcd}$$

- **Problem**: estimate $||P_h^\varepsilon - P_h^0||$.
- In practice, we have obtained $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$.

Condition H:

The discrete Hessian of P_h^0 can be bounded or the error estimate $\|P_h^\varepsilon - P_h^0\|$ can be obtained independently of the discrete Hessian.

Diffusion scheme

Limit diffusion scheme (P_h^0)

$$
\begin{cases}
\mid \Omega_j \mid \partial_t p_j(t) - \sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\
\sum_r \hat{\alpha}_{jr} \mathbf{u}_j = \sum_r \hat{\alpha}_{jr} \mathbf{u}_r, \\
\sigma A_r \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j).
\end{cases}
$$

$$
\begin{tikzcd}
P_h^{\varepsilon} \arrow[r, "\partial_t \mathbf{v}_h = 0"] \arrow[d, "h \to 0"'] & {\color{blue} DA_h^{\varepsilon}} \arrow[r, dashed, "\varepsilon \to 0"] \arrow[d, "h \to 0 \text{ and } \varepsilon \to 0"] & P_h^0 \arrow[dl, dashed] \\
P^{\varepsilon} \arrow[r, "\varepsilon \to 0"'] & P^0 &
\end{tikzcd}
$$

- **Problem**: estimate $||P_h^{\varepsilon} - P_h^0||$.
- In practice, we have obtained $||P_h^{\varepsilon} - P_h^0|| \leq C\frac{\varepsilon}{h}$.
- Introduction of an intermediary diffusion scheme DA_h^{ε}.
- DA_h^{ε}: P_h^{ε} scheme with $\partial_t \mathbf{F}_j = \mathbf{0}$.
- In the previous estimation we replace P_h^0 by DA_h^{ε}.

Condition H:

The discrete Hessian of P_h^0 can be bounded or the error estimate $||P_h^{\varepsilon} - P_h^0||$ can be obtained independently of the discrete Hessian.

Final results

Space result:

We assume that the meshes and the data are regular and the initial datas well-prepaped. There exist $C(T) > 0$ such that:

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq Cf(h,\varepsilon) \parallel p_0 \parallel_{H^4(\Omega)} \leq Ch^{\frac{1}{4}} \parallel p_0 \parallel_{H^4(\Omega)}$$

with

$$f(h,\varepsilon) = \min\left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + h + (h+\varepsilon) + \varepsilon\right)$$

- Introducing $\varepsilon_{thresh} = h^{\frac{1}{2}}$ we prove that the worst case is $\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\| \leq C_2 h^{\frac{1}{4}}$.

Space-time result:

Wa assume that the assumptions are verified. There exist $C > 0$ such that:

$$\|\mathbf{V}^\varepsilon(t_n) - \mathbf{V}_h^\varepsilon(t_n)\|_{L^2(\Omega)} \leq C\left(f(h,\varepsilon) + \Delta t^{\frac{1}{2}}\right) \parallel p_0 \parallel_{H^4(\Omega)}$$

Remark: The condition H is not satisfied. The diffusion scheme used is DA_ε.

Intermediary results I

Estimation of $||\mathbf{V}^{\varepsilon} - \mathbf{V}^{\varepsilon}_h||$:

We assume that assumptions are verified. There exist $C > 0$ such that:

$$\|\mathbf{V}^{\varepsilon}_h - \mathbf{V}^{\varepsilon}\|_{L^{\infty}((0,T):L^2(\Omega))} \leq C\sqrt{\frac{h}{\varepsilon}}.$$

- Principle of proof:
  - Control the stability of the discrete quantities $\mathbf{u}_r$ and $\mathbf{u}_j$ by ε
  - We define the error $E(t) = ||\mathbf{V}^{\varepsilon} - \mathbf{V}^{\varepsilon}_h||_{L^2}$ and we estimate $E^{'}(t)$ using Young and Cauchy-Schwartz inequalities, stability estimates and integration in time.

Estimation of $||DA^{\varepsilon}_h - P^0||$:

Wa assume that the assumptions are verified. There exist $C_1 > 0$ such that:

$$||\mathbf{V}^0_h - \mathbf{V}^0||_{L^2(\Omega)} \leq C_1(T)(h + \varepsilon), \qquad 0 < t \leq T.$$

- Principle of proof:
  - Control the stability of the discrete quantities $\nabla_r p$ and p_j.
  - Consistance study of Div and Grad discrete operators.
  - L^2 estimate using consistency error and Gronwall lemma.

Intermediary results II

Estimate $||P_h^\varepsilon - DA_h^\varepsilon||$:

We assume that the assumptions are verified. There exist $C_2(T) > 0$ such that:

$$||\mathbf{V}_h^\varepsilon - \mathbf{V}_h||_{L^2(\Omega)} \leq C_2(T)\varepsilon \max\left(1, \sqrt{\varepsilon h^{-1}}\right) + Ch, \qquad 0 < t \leq T.$$

Estimate $||P^\varepsilon - P^0||$:

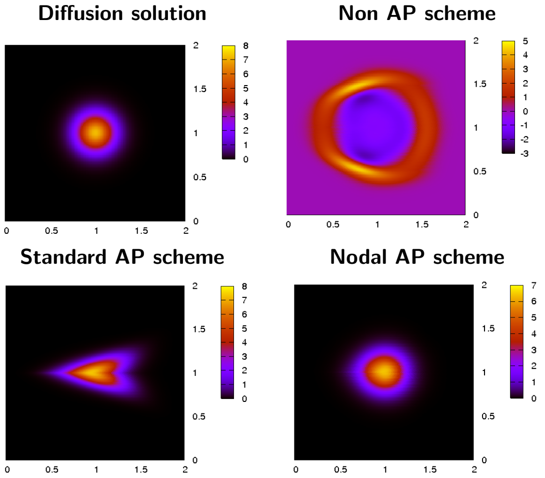
We assume that the assumptions are verified. There exist $C_3(T) > 0$ such that:

$$||\mathbf{V}^\varepsilon - \mathbf{V}^0||_{L^2(\Omega)} \leq C_3(T)\varepsilon, \qquad 0 < t \leq T.$$

- Principe of proof:
  - Write $P^0 = P^\varepsilon + R$ (resp $DA_h^\varepsilon = P_h^\varepsilon + R$) with R a residue.
  - Find a bound with ε of the residue.
  - L^2 estimate of the difference between the two models and between the two schemes.

AP scheme vs classical scheme

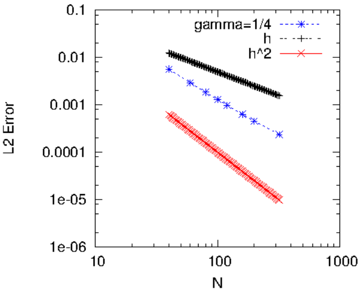
- Test case: heat fundamental solution. Results for different hyperbolic scheme with $\varepsilon = 0.001$ on Kershaw mesh.



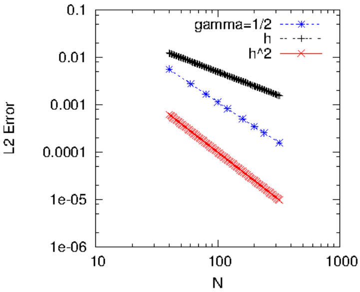
Uniform convergence

- ε dependent periodic solution for the P_1 model.
- $p(t,\mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{u}(t,\mathbf{x}) = \left(-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y),\quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x)\right)$
- Convergence study for $\varepsilon = h^\gamma$ on random mesh.

$\gamma = \frac{1}{4}$



$\gamma = \frac{1}{2}$

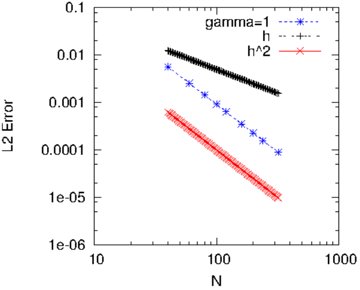


- Numerical results show that the error is homogenous to $O(h\varepsilon + h^2)$.
- Theoretical estimate that we can hope: $O((h\varepsilon)^{\frac{1}{2}} + h)$.
- Non optimal estimation in the intermediary regime.

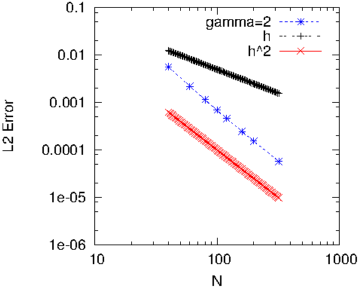
Uniform convergence

- ε dependent periodic solution for the P_1 model.
- $p(t, \mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t)) \cos(\pi x) \cos(\pi y)$
- $\mathbf{u}(t, \mathbf{x}) = (-\frac{\varepsilon}{\sigma}\alpha(t) \sin(\pi x) \cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t) \sin(\pi y) \cos(\pi x))$
- Convergence study for $\varepsilon = h^{\gamma}$ on random mesh.

$\gamma = 1$



$\gamma = 2$



- Numerical results show that the error is homogenous to $O(h\varepsilon + h^2)$.
- Theoretical estimate that we can hope: $O((h\varepsilon)^{\frac{1}{2}} + h)$.
- Non optimal estimation in the intermediary regime.

Conclusion of the 2D case

AP schemes on unstructured grids

- Contrary the classical scheme the nodal scheme allows to obtain the uniform AP property
- However there are spurious mods for non smooth datas (possible stabilization).
- **Other scheme**: MPFA-AP scheme without spurious mods but the uniform convergence is an open question.

Extension

- We propose AP schemes for Friedrich's systems using a particular splitting between the P_1 model and a rest (close to micro-macro decomposition).
- The nodal scheme is also use to construct an AP scheme for Euler with friction and the M_1 model.

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