Proposition of reduced MHD models with diamagnetic effects and energy balance estimate

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Outline

Derivation and study of Fluid models

Hierarchy of Models

- **Microscopic model**: N-Body model. We write the dynamical Newton equation for each particle:

$$\begin{cases} \frac{d\gamma m \boldsymbol{v}_i}{dt} = \sum_j q(\boldsymbol{E}_j + \boldsymbol{v}_i \times \boldsymbol{B}_j) \\ \frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i \end{cases}$$

- Unrealistic approach: we must solve N coupled equations with $N \approx 10^{16} - 10^{20}$.

- **Mesoscopic model**: Kinetic Vlasov model. Taking the limit of the N-Body model we obtain an equation on the distribution of the particles:

$$\partial_t f(t, \boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla f + \boldsymbol{F}_{ext} \cdot \nabla_{\boldsymbol{v}} f = Q(f, f)$$

  with $\boldsymbol{F}_{ext}$ the external force (gravity, Lorentz force, etc).

- **Macroscopic model**: Fluid models (moment models). If we are close to the equilibrium, taking the three first moments of the distribution function we obtain:

$$\partial_t \mathbf{U} + \nabla \cdot F(\mathbf{U}) + \epsilon \nabla \cdot (D(\nabla \mathbf{U})) = 0$$

- Examples : hyperbolic models (Euler, Euler-Lorentz, ideal MHD), parabolic models (Navier-Stokes, Resistive MHD).

Vlasov equations and equilibrium

- First model to describe a plasma : **Two species Vlasov-Maxwell** kinetic equation.
- We define $f_s(t, \mathbf{x}, \mathbf{v})$ the distribution function associated with the species s.

Two-species Vlasov equation

$$\begin{cases} \partial_t f_s + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\boldsymbol{E} + \mathbf{v}\times\boldsymbol{B})\cdot\nabla_{\mathbf{v}} f_s = C_s = \sum_t C_{st}, \\ \frac{1}{c^2}\partial_t \boldsymbol{E} - \nabla\times\boldsymbol{B} = -\mu_0 \boldsymbol{J}, \\ \partial_t \boldsymbol{B} = -\nabla\times\boldsymbol{E}, \\ \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

Energy conservation

$$\frac{d}{dt}\left(\frac{1}{2}\sum_s \int m_s f_s \mid \mathbf{v} \mid^2 d\mathbf{x}d\mathbf{v} + \frac{1}{2\mu_0 c^2}\int \mid \boldsymbol{E} \mid^2 d\mathbf{x} + \frac{1}{2\mu_0}\int \mid \boldsymbol{B} \mid^2 d\mathbf{x}\right) = 0.$$

- Properties of the collision operator
  - For each species: $\int_{R^3} m_s \mathbf{v} C_{ss} d\mathbf{v} = 0$, $\int_{R^3} \frac{1}{2} m_s \mid \mathbf{v} \mid^2 C_{ss} d\mathbf{v} = 0$,
  - No conversion of particles: $\int_{R^3} m_s \mathbf{v} C_{s_1 s_2} d\mathbf{v} = 0$
  - Global momentum and energy conservation: $\int_{R^3} g(\mathbf{v})_s C_{st} d\mathbf{v} + \int_{R^3} g(\mathbf{v})_t C_{ts} d\mathbf{v} = 0$ with $g(\mathbf{v}) = m_s \mathbf{v}$ or $g(\mathbf{v}) = m_s \frac{1}{2} \mid \mathbf{v} \mid^2$

Two-fluid model

- Computing the moments of the Vlasov equation we obtain the following model

Two fluid moments

$$\begin{cases} \partial_t n_s + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s) = 0, \\ \partial_t (m_s n_s \boldsymbol{u}_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla_{\mathbf{x}} p_s + \nabla_{\mathbf{x}} \cdot \overline{\overline{\boldsymbol{\Pi}}}_s = \sigma_s \boldsymbol{E} + \boldsymbol{J}_s \times \boldsymbol{B} + \mathbf{R}_s, \\ \partial_t (m_s n_s \epsilon_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s \epsilon_s + p_s \boldsymbol{u}_s) + \nabla_{\mathbf{x}} \cdot \left(\overline{\overline{\boldsymbol{\Pi}}}_s \cdot \boldsymbol{u}_s + \mathbf{q}_s \right) \\ = \sigma_s \boldsymbol{E} \cdot \boldsymbol{u}_s + Q_s + \mathbf{R}_s \cdot \boldsymbol{u}_s, \\ \\ \frac{1}{c^2} \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J}, \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \\ \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

- $n_s = \int_{R^3} f_s d\mathbf{v}$ the particle number , $m_s n_s \boldsymbol{u}_s = \int_{R^3} m_s \mathbf{v} f_s d\mathbf{v}$ the momentum, ϵ_s the total energy and $\rho_s = m_s n_s$ the density.
- The isotropic pressures are p_s, the stress tensors $\overline{\overline{\boldsymbol{\Pi}}}_s$ and the heat fluxes $\mathbf{q}_s$.
- $\mathbf{R}_s$ and Q_s are associated with the interspecies collision (force and energy transfer).
- The current is given by $\boldsymbol{J} = \sum_s \boldsymbol{J}_s = \sum_s \sigma_s \boldsymbol{u}_s$ with $\sigma_s = q_s n_s$.

Energy conservation

$$\frac{d}{dt}\left(\int_{D_x} (\rho_e \varepsilon_e + \rho_i \varepsilon_i) + \frac{1}{2\mu_0 c^2} \int_{D_x} \mid \boldsymbol{E} \mid^2 d\mathbf{x} + \frac{1}{2\mu_0} \int_{D_x} \mid \boldsymbol{B} \mid^2 d\mathbf{x} \right) = 0$$

MHD: assumptions and generalized Ohm's law

MHD: assumptions

- **quasi neutrality assumption**: $n_i = n_e \Longrightarrow \rho \approx m_i n_i + O(\frac{m_e}{m_i})$, $\mathbf{u} \approx \boldsymbol{u}_i + O(\frac{m_e}{m_i})$
- **Magneto-static assumption** : $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + O(\frac{V_0}{c})$.

- We define $\rho = \rho_i + \rho_e$ and $\mathbf{u} = \frac{\rho_i \boldsymbol{u}_i + \rho_e \boldsymbol{u}_e}{\rho}$.

Velocity relation

- Consequence of the quasi-neutrality:

$$\boldsymbol{u}_e = \boldsymbol{u} - \frac{m_i}{e\rho}\boldsymbol{J} + O\left(\frac{m_e}{m_i}\right)$$

- Summing the mass and moment equation for the two species we obtain:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} + O\left(\frac{m_e}{m_i}\right)$$

- For the pressure equation, we replace the electronic velocity by full velocity using the previous relation.

MHD: derivation

- **Ohm law**: Relation between the electric field and the other variables.
- Taking the electron density and momentum equations we obtain

$$m_e\left(\partial_t(n_e \boldsymbol{u}_e) + \nabla\cdot(n_e \boldsymbol{u}_e \otimes \boldsymbol{u}_e)\right) + \nabla p_e = -en_e \boldsymbol{E} + \boldsymbol{J}_e \times \boldsymbol{B} - \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e + \mathbf{R}_e,$$

- We multiply the previous equation by $-e$ and we define $\boldsymbol{J}_e = -en_e \boldsymbol{u}_e$, we obtain

$$\frac{m_e}{e^2 n_e}\left(\partial_t \boldsymbol{J}_e + \nabla\cdot(\boldsymbol{J}_e \otimes \boldsymbol{u}_e)\right) = \boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B} + \frac{1}{en_e}\nabla p_e + \frac{1}{en_e}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e - \frac{1}{en_e}\mathbf{R}_e,$$

- Using the quasi neutrality $\mathbf{R}_e = \eta \frac{e}{m_i}\rho \boldsymbol{J}$ and $\boldsymbol{u}_e = \boldsymbol{u} - \frac{m_i}{e\rho}\boldsymbol{J}$ we obtain

Generalized Ohm's law

$$\boldsymbol{E} + \underbrace{\boldsymbol{u}\times\boldsymbol{B}}_{\text{drift velocity}} = \underbrace{\eta \boldsymbol{J}}_{\text{resistivity}} \underbrace{+\frac{m_i}{\rho e}\boldsymbol{J}\times\boldsymbol{B}}_{\text{hall term}} \underbrace{-\frac{m_i}{\rho e}\nabla p_e - \frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e}_{\text{pressure term}} + O\left(\frac{m_e}{m_i}\right).$$

Final simplification

$$\left(\frac{m_e}{m_i}\right) << 1 \quad \left(\frac{V_0}{c}\right) << 1$$

Extended MHD: model

Extended MHD

$$
\begin{cases}
\partial_t \rho + \nabla\cdot(\rho\boldsymbol{u}) = 0, \\
\rho\partial_t \boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{J}\times\boldsymbol{B} - \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}, \\
\dfrac{1}{\gamma-1}\partial_t p_i + \dfrac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_i + \dfrac{\gamma}{\gamma-1}p_i\nabla\cdot\boldsymbol{u} + \nabla\cdot\mathbf{q}_i = -\overline{\overline{\boldsymbol{\Pi}}}_i : \nabla\boldsymbol{u}, \\
\dfrac{1}{\gamma-1}\partial_t p_e + \dfrac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_e + \dfrac{\gamma}{\gamma-1}p_e\nabla\cdot\boldsymbol{u} + \nabla\cdot\mathbf{q}_e = \dfrac{1}{\gamma-1}\dfrac{m_i}{e\rho}\boldsymbol{J}\cdot\left(\nabla p_e - \gamma p_e\dfrac{\nabla\rho}{\rho}\right) \\
-\overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right) + \eta|\boldsymbol{J}|^2, \\
\partial_t\boldsymbol{B} = -\nabla\times\left(-\boldsymbol{u}\times\boldsymbol{B} + \eta\boldsymbol{J} - \dfrac{m_i}{\rho e}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e - \dfrac{m_i}{\rho e}\nabla p_e + \dfrac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right), \\
\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{B} = \boldsymbol{J}.
\end{cases}
$$

- **Remark**: We can write easily the equation on the total pressure $p_e + p_i$.

- In Black: ideal MHD. In Black and blue: Viscous-resistive MHD. All the term: Extended MHD.

MHD Invariants

- The mass ρ the momentum $\rho\boldsymbol{u}$ and the total energy $E=\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{B}|^2}{2}+\frac{1}{\gamma-1}p$ with $p=\rho T$ are conserved in time.

Sketch of proof

- Mass and momentum conservation: divergence form + the flux-divergence theorem + null BC.
- Total energy: multiply the first equation by $\frac{|\boldsymbol{u}|^2}{2}$ the second by $\boldsymbol{u}$ and the last one by $\boldsymbol{B}$ we obtain

$$\partial_t E+\nabla\cdot\left[\boldsymbol{u}\left(\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{\gamma}{\gamma-1}p\right)-(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}\right]+\nabla\cdot\mathbf{q}+\nabla\cdot(\overline{\overline{\boldsymbol{\Pi}}}\cdot\boldsymbol{u})+\eta\nabla\cdot(\boldsymbol{J}\times\boldsymbol{B})$$
$$+\nabla\cdot\left[\frac{m_i}{\rho e}\left((\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}-\nabla p_e\times\boldsymbol{B}-\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\times\boldsymbol{B}-\frac{\gamma}{\gamma-1}p_e\boldsymbol{J}-\boldsymbol{J}\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\right)\right]=0$$

  with $\overline{\overline{\boldsymbol{\Pi}}}=\overline{\overline{\boldsymbol{\Pi}}}_i+\overline{\overline{\boldsymbol{\Pi}}}_e$ and $\mathbf{q}=\mathbf{q}_i+\mathbf{q}_e$.
- We conclude with the divergence-flux theorem + BC null.

Closure

Stress tensor and heat flux

- **Closure**: Write the dependency of $\overline{\overline{\Pi}}$ and $\mathbf{q}$ with the variables p, $\mathbf{u}$ and ρ.

$$\overline{\overline{\Pi}} = \overline{\overline{\Pi}}(\mathbf{W}, \mathbf{b}, p) \quad \mathbf{q} = \mathbf{q}(T, \mathbf{b}), \text{ with } \mathbf{W} = \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3}\nabla \cdot \boldsymbol{u}$$

- Total Stress tensor $\overline{\overline{\Pi}} = \overline{\overline{\Pi}}_i + \overline{\overline{\Pi}}_e \approx \overline{\overline{\Pi}}_i$ since $\frac{|\overline{\overline{\Pi}}_e|}{|\overline{\overline{\Pi}}_i|} = O(\frac{m_e}{m_i})$.
- Stress tensor expansion $\overline{\overline{\Pi}} = \overline{\overline{\Pi}}_\parallel + \delta^2 \overline{\overline{\Pi}}_{gv} + \delta^4 \overline{\overline{\Pi}}_\perp \Longrightarrow \overline{\overline{\Pi}} \approx \overline{\overline{\Pi}}_\parallel + \delta^2 \overline{\overline{\Pi}}_{gv}$.
- $\nabla \cdot \overline{\overline{\Pi}}_\parallel$ dissipate energy, not $\nabla \cdot \overline{\overline{\Pi}}_{gv}$ (indeed $\overline{\overline{\Pi}}_{gv} : \nabla \boldsymbol{u} = 0$)
- $\mathbf{q}_{i,e} = -n_{i,e}\left(\chi_\parallel^{i,e}(\mathbf{b}\cdot\nabla T_{i,e})\mathbf{b} + \chi_c^{i,e}\mathbf{b}\times\nabla T_{i,e} + \chi_\perp^{i,e}\mathbf{b}\times(\mathbf{b}\times\nabla T_{i,e})\right)$

Simplification of the velocity

- Velocity expansion of is used in JOREK

$$\| \boldsymbol{B} \|^2 \mathbf{u} = \underbrace{(\mathbf{u}, \boldsymbol{B})\boldsymbol{B}}_{\mathbf{u}_\parallel} + \underbrace{(\boldsymbol{E}\times\boldsymbol{B})}_{\mathbf{u}_E} + \frac{m_i}{\rho e}\underbrace{(\boldsymbol{B}\times\nabla p_i)}_{\mathbf{u}_i} + O(\delta)$$

Reduction in the JOREK model

Full MHD simplification I

- **Parallel viscous heating**: is neglected,
- **Parallel stress tensor**: $\nabla \cdot \overline{\overline{\Pi}}_{\parallel} \approx \Delta \boldsymbol{u}_E + \Delta \boldsymbol{u}_i^*$,
- **Pressure equation**: the following terms are neglected

$$\frac{1}{\gamma - 1} \frac{m_i}{e\rho} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) + \eta |\boldsymbol{J}|^2,$$

- **Momentum equation**: gyro-viscous cancelation

$$\rho \partial_t \boldsymbol{u}_i^* + \rho(\boldsymbol{u}_E + \boldsymbol{u}_i^* + \boldsymbol{u}_{\parallel}) \cdot \nabla \boldsymbol{u}_i^* + \nabla \cdot \overline{\overline{\Pi}}^{gv} \approx \nabla \chi - \rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u}_{\parallel} \text{ with } \nabla \chi << \nabla p.$$

- **Momentum equation**: Perp momentum equation only on $\boldsymbol{u}_E$ with the gv cancelation.

Projection for parallel and poloidal velocity

$$\mathbf{e}_{\phi} \cdot \nabla \times R^2 \left(\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} + \nu \triangle \boldsymbol{u} \right)$$

and

$$\boldsymbol{B} \cdot \left(\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} + \nu \triangle \boldsymbol{u} \right).$$

MHD simplification II

Reduced MHD simplification

- **Magnetic field**:

$$\boldsymbol{B} = \boldsymbol{B}_\phi + \boldsymbol{B}_{pol} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$$

- **Pressure**: $P_e = \frac{P}{1+T_i/T_e}$
- **Parallel velocity**: $\boldsymbol{u}_\parallel \approx \frac{(\mathbf{u},\boldsymbol{B})\boldsymbol{B}}{\|\boldsymbol{B}_\phi\|^2}$
- **Perpendicular velocity**:

$$\boldsymbol{u}_E + \boldsymbol{u}_i^* \approx \frac{(\boldsymbol{E}\times\boldsymbol{B}_\phi)}{\|\boldsymbol{B}_\phi\|^2} + \frac{m_i}{\rho e}\frac{(\boldsymbol{B}_\phi\times\nabla p_i)}{\|\boldsymbol{B}_\phi\|^2} = -R\nabla u \times \mathbf{e}_\phi + \tau_{IC}\frac{R}{\rho}\left(\mathbf{e}_\phi \times \nabla P\right)$$

- **Viscous term**: $\Delta\boldsymbol{u}_E \approx \Delta w$

Full model

- Without the diamagnetic and neoclassical terms, the model of JOREK [1]-[2] dissipate the total energy if we add some small terms [3].
- **Problem**: With diamagnetic terms we don't have an energy estimate signed.

New model proposed

Full MHD simplification

Closure

- The closure are detailed in [4].
- No gyro-viscous cancelation and no neglect terms in the pressure equation.
- **Parallel stress tensor**:

$$\nabla \cdot \overline{\overline{\Pi}}_{\parallel} = G\boldsymbol{b} \cdot \nabla \boldsymbol{b} - \frac{1}{3}\nabla G + \mathbf{b} \cdot \nabla G, \quad \overline{\overline{\Pi}}_{\parallel} : \nabla \boldsymbol{u} = -\frac{1}{3\eta_0} G^2$$

  with $G = -\eta_0(2\boldsymbol{b} \cdot \nabla v_{\parallel} - ((\boldsymbol{b} \cdot \nabla \boldsymbol{b}) \cdot \boldsymbol{u}_{\perp}))$.

- **Gyro-viscous stress tensor**:

$$\nabla \cdot \overline{\overline{\Pi}}_{gv} = -\rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u} + p_i \left(\nabla \times \frac{m_i \boldsymbol{b}}{e \parallel \boldsymbol{B} \parallel}\right) \cdot \nabla \boldsymbol{u} + \mathbf{Res}$$

$$\mathbf{Res} = \nabla_{\perp}\left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla \cdot \boldsymbol{b} \times \boldsymbol{u}\right) + \boldsymbol{b} \times \nabla \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla_{\perp} \cdot \boldsymbol{u}\right)$$

- **Open question**: full magnetic field for the reduced perpendicular velocity and projection parallel - perpendicular ?

Reduced velocities and operators

- JOREK terms in Black. Potential new term in red.

B, current and velocity after reduction

- $\mid \boldsymbol{B} \mid^2 = \dfrac{1}{R^2}\left(F_0^2 + \mid \nabla\psi \mid^2\right) \approx \dfrac{F_0^2}{R^2}$, $\boldsymbol{J} = j_\phi \mathbf{e}_\phi + \boldsymbol{J}_{pol} = -\left(\dfrac{1}{R}\Delta^*\psi\mathbf{e}_\phi - \dfrac{1}{R^2}\nabla_{pol}(\partial_\phi\psi)\right)$
- $\boldsymbol{u}_E = R\partial_R u \mathbf{e}_Z - R\partial_Z u \mathbf{e}_R + \left(\dfrac{1}{F_0}\left(R(\nabla_{pol}u, \nabla_{pol}\psi)\mathbf{e}_\phi - \partial_\phi u \nabla_{pol}\psi\right)\right)$
- $\boldsymbol{u}_i^* = \dfrac{m_i}{e\rho}\left(R\partial_R p_i \mathbf{e}_Z - R\partial_Z p_i \mathbf{e}_R\right) + \left(\dfrac{m_i}{F_0 e\rho}\left(R(\nabla_{pol}p_i, \nabla_{pol}\psi)\mathbf{e}_\phi - \partial_\phi p_i \nabla_{pol}\psi\right)\right)$

advection operator

- $\boldsymbol{B}\cdot\nabla f = \dfrac{F_0}{R^2}\partial_\phi f - \dfrac{1}{R}[\psi, f]$ and $\boldsymbol{u}_\parallel \cdot \nabla f = \dfrac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid}\cdot\nabla f$
- $\boldsymbol{u}_E \cdot \nabla f = R[u, f] + \left(\dfrac{1}{F_0}\left((\nabla_{pol}u, \nabla_{pol}\psi)\partial_\phi f - \partial_\phi u(\nabla_{pol}\psi, \nabla_{pol}f)\right)\right)$
- $\boldsymbol{u}_i^* \cdot \nabla f = \dfrac{F_0\tau}{R\rho}[p_i, f] + \left(\dfrac{\tau}{R\rho}\left((\nabla_{pol}p_i, \nabla_{pol}\psi)\partial_\phi f - \partial_\phi p_i(\nabla_{pol}\psi, \nabla_{pol}f)\right)\right)$

- Wa have also new term for the other operator like divergence of he velocities fields.

Induction equation

- We take the equation

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \Longrightarrow \partial_t \mathbf{A} = -\boldsymbol{E} - F_0 \nabla u$$

- Using $\mathbf{A} = \psi \mathbf{e}_\phi$ we obtain

$$\partial_t \left(\frac{\psi}{R^2} \mathbf{e}_\phi \right) = -\boldsymbol{E} - F_0 \nabla u$$

- To conclude we use the Ohm law and we project

Induction equation (projection in parallel direction)

$$\partial_t \left(\frac{\psi}{R^2} \right) = \frac{\tau}{\rho} \boldsymbol{B} \cdot \nabla p_e + \frac{\eta}{R^2} \Delta^* \psi - \boldsymbol{B} \cdot \nabla u + \mathcal{R}_{\psi,j}$$

with new small term $\mathcal{R}_{\psi,j} = -\frac{\eta}{R^3} \left([\partial_\phi \psi, \psi] \right)$

Induction equation (projection in toroidal direction)

$$\partial_t \left(\frac{\psi}{R^2} \right) = \frac{1}{R} [\psi, u] - \frac{F_0}{R^2} \partial_\phi u - \frac{\tau}{R} [\psi, p_i] + \tau \frac{F_0}{R^2} \partial_\phi (p_i + p_e) + \frac{\eta}{R^2} \Delta^* \psi + \mathcal{R}_{\psi,j}$$

with new small term $\mathcal{R}_{\psi,j} = -O(\tau \boldsymbol{J} \times \boldsymbol{B})$

Density and pressure equations

- The structure of the density equation does not change. We have new term if we keep the new term in the perp velocity field.

Pressure equation

$$\partial_t \frac{1}{\gamma-1} p + \frac{1}{\gamma-1}\left(v_{\parallel}\boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^*\right)\cdot\nabla p + \frac{\gamma}{\gamma-1}\left(\boldsymbol{B}\cdot\nabla v_{\parallel} + \nabla\cdot\boldsymbol{u}_E + \nabla\cdot\boldsymbol{u}_i^*\right) p$$

$$+\nabla\cdot\left((k_{\parallel}-k_{\perp})\frac{\boldsymbol{B}}{\mid \boldsymbol{B}\mid^2}(\boldsymbol{B}\cdot\nabla T) + k_{\perp}\nabla T\right) = R_{p,j}^1 + R_{p,j}^2 + R_{p,u}$$

with

$$\mathcal{R}_{p,j}^1 = \frac{\eta}{R^2}|\Delta^*\psi|^2 + \frac{1}{\gamma-1}\frac{\tau}{R\rho}\Delta^*\psi\left(\partial_\phi p_e - \gamma p_e\frac{\partial_\phi\rho}{\rho}\right)$$

$$\mathcal{R}_{p,j}^2 = \frac{\eta}{R^4}\mid\nabla_{pol}(\partial_\phi\psi)\mid^2 + \frac{1}{\gamma-1}\frac{\tau}{R^2\rho}\nabla_{pol}(\partial_\phi\psi)\cdot\left(\nabla_{pol}p_e - \gamma p_e\frac{\nabla_{pol}\rho}{\rho}\right)$$

$$\mathcal{R}_{p,u} = \frac{\eta_0}{3}(2\boldsymbol{b}\cdot\nabla v_{\parallel} - \frac{R}{F_0^2}(\boldsymbol{B}\cdot\nabla\boldsymbol{B})\cdot(\boldsymbol{u}_E + \boldsymbol{u}_i^*))^2$$

- The new term associated $(\mathcal{R}_{p,u})$ to the heating can be neglected.
- The term $\mathcal{R}_{p,j}^1$ is essential to have energy conservation at the end
- The term $\mathcal{R}_{p,j}^2$ is not essential to have energy conservation at the end.

Momentum equations and future works

Short time Future work

- Write the equations on the perpendicular and parallel momentum equations in the Reduced context.

Open questions

- Classical projection or perp-parallel projection ?
- Simplification of the stress tensor ?
- Keep the new terms which are not necessary for energy dissipation ?
- Keep the Ohmic heating ?

Two reports

- *Hierarchy of fluids models for magnetized and collisional plasmas. Part I: Bi-fluid and Full-MHD models* (finished but need to be verify)
- *Hierarchy of fluids models for magnetized and collisional plasmas. Part II: Reduced MHD models* (not finish)

Long time Future work

- Energy conservation or dissipation for neoclassical terms and neutral terms.

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