Numerical methods for stiff hyperbolic systems

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Outline

Mathematical context

AP/WB schemes for hyperbolic PDE with source terms

Implicit relaxation method for low Mach Euler equations
Mathematical context
Stiff hyperbolic systems

Problem

- We consider the general stiff problem:

\[
\partial_t U + \frac{1}{\epsilon^a} \partial_x F(U) + \frac{1}{\epsilon^b} \partial_x G(U) = \frac{1}{\epsilon^c} R(U) - \frac{\sigma}{\epsilon^d} D(U)
\]

Limit

- First case: \(a = b = c = 1\) and \(\sigma = 0\). long time limit:

\[
\partial_x F(U) + \partial_x G(U) = R(U)
\]

- Second case: \(a = b = 0\), \(c = 1\) and \(\sigma = 0\). relaxation limit:

\[
\partial_t V + \partial_x K_1(V) = 0
\]

- Third case: \(a = b = c = 1\), \(d = 2\) \(\sigma = 1\). diffusion limit:

\[
\partial_t V + \partial_x K_1(V) - \partial_x (K_2(V) \partial_x V) = 0
\]

- 4th: \(a = c = 0\), \(b = 1\) and \(\sigma = 0\). fast wave limit:

\[
\partial_t U + \partial_x \tilde{G}(U) = 0, \quad \partial_x \tilde{F}(U) = 0
\]
Diffusion limit: damped wave equation

Damped wave equation

\[ \begin{align*}
\frac{\partial_t p}{\Delta t} + \frac{1}{\varepsilon} \frac{1}{\Delta x} p &= 0 \\
\frac{\partial_t u}{\Delta t} + \frac{1}{\varepsilon} \frac{1}{\Delta x} u &= -\frac{\sigma}{\varepsilon^2} u
\end{align*} \]

\[ \rightarrow \frac{\partial_t p}{\Delta t} - \frac{\partial}{\partial x} \left( \frac{1}{\sigma} \frac{1}{\Delta x} p \right) = 0 \]

- Ref: Jin-Levermore 96, Gosse-Toscani 01.
- We plug \( u = -\frac{\varepsilon}{\sigma} \frac{1}{\Delta x} p + O(\varepsilon^2) \) in first equation.

Godunov scheme

\[ \begin{align*}
\frac{p_{j+1} - p_j}{\Delta t} + \frac{1}{\varepsilon} \frac{u_{j+1} - u_{j-1}}{\Delta x} - \frac{\Delta x}{2\varepsilon} \frac{p_{j+1} - 2p_j + p_{j-1}}{\Delta x^2} &= 0 \\
\frac{u_{j+1} - u_j}{\Delta t} + \frac{1}{\varepsilon} \frac{p_{j+1} - p_{j-1}}{\Delta x} - \frac{\Delta x}{2\varepsilon} \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} &= -\frac{\sigma}{\varepsilon^2} u_j
\end{align*} \]

- Limit scheme:

\[ \frac{p_{j+1} - p_j}{\Delta t} - \left( \frac{1}{\sigma} + \frac{\Delta x}{2\varepsilon} \right) \frac{p_{j+1} - 2p_j + p_{j-1}}{\Delta x^2} = O(\varepsilon) \]

- CFL condition: \( \Delta t \leq f(\varepsilon) h \)

Diffusion and numerical solutions for \( \varepsilon = 0.001 \).
Long time limit: Euler gravity

Euler gravity

\[
\begin{align*}
\partial_t \rho + \frac{1}{\varepsilon} \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \frac{1}{\varepsilon} \partial_x (\rho u^2) + \frac{1}{\varepsilon} \partial_x p &= -\frac{1}{\varepsilon} \rho \partial_x \phi \\
\partial_t E + \frac{1}{\varepsilon} \partial_x (E u + p u) &= -\frac{1}{\varepsilon} \rho u \partial_x \phi
\end{align*}
\]

- Class of steady solutions: for \( u = 0 \) and \( \partial_x p = -\rho \partial_x \phi \) the system does not move.
- C. Berthon, C. Klingenberg (and al) 15-16-17.

Rusanov scheme

- Example: \( \rho = e^{-x \partial_x \phi}, \ p = e^{-x \partial_x \phi} \) and \( \phi = g x \).

\[
\begin{align*}
\rho^{n+1} &= \rho^n + \frac{\Delta x}{\lambda} \partial_{xx} \rho + O(\Delta x^2) \\
(\rho u)^{n+1} &= (\rho u)^n + \frac{\Delta x}{\lambda} \partial_{xx} (\rho u) + O(\Delta x^2) \\
E^{n+1} &= E^n + \frac{\Delta x}{\lambda} \partial_{xx} E + O(\Delta x^2)
\end{align*}
\]

- with \( \lambda > \max_x (|u| + c) \) with \( c \) the sound speed.
- Conclusion: the equilibriums are not preserved.

Pertubated equilibrium.
Relaxation limit: HRM model

**HRM model**

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t \rho Y + \partial_x (\rho Yu) &= \frac{1}{\varepsilon} (\rho Y_{eq}(\rho) - \rho Y) \\
\partial_t \rho u + \partial_x (\rho u^2 + p) &= 0
\end{align*}
\]

- with $Y$ the mass fraction and $p = p(\rho, Y)$ (Ambrosso 09 etc).
- **Relaxation limit**: the mass fraction is close to given equilibrium.

**Splitting scheme**

- Only write for the mass fraction part

\[
(\rho Y)^* = (\rho Y)^n + \frac{\Delta t}{\varepsilon} (\rho^n Y_{eq}(\rho^n) - \rho^n Y^n)
\]

\[
\frac{(\rho Y)^{n+1} - (\rho Y)^*}{\Delta t} + \frac{(\rho Yu)^*_{j+1} - (\rho Yu)^*_{j-1}}{\Delta x} - \lambda \frac{(\rho Y)^*_{j+1} - 2(\rho Y)^*_{j} + (\rho Y)^*_{j-1}}{\Delta x} = 0
\]

- **Stability** we must take $\Delta t < C\varepsilon \Delta x$. 
Fast wave limit: Low-Mach Euler equation

**Euler low-mach**

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \frac{1}{M} \partial_x p &= 0 \\
\partial_t E + \partial_x (Eu + pu) &= 0
\end{align*}
\]

- **Limit for** $M$ **small**: $u = cts + O(M)$, $p = cts + O(M)$ and $\partial_t \rho + u \partial_x \rho = O(M)$.

**Rusanov scheme**

- At the limit: density advection. Advection scheme:

\[
\partial_t \rho_j + \left( \frac{\rho u_{j+1} - (\rho u)_{j-1}}{\Delta x} \right) - \left| u \right| \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{\Delta x} = 0
\]

- Limit scheme of Rusanov scheme for Euler:

\[
\partial_t \rho_j + \left( \frac{\rho u_{j+1} - (\rho u)_{j-1}}{\Delta x} \right) - \frac{\lambda}{M} \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{\Delta x} = 0
\]

- The scheme for Euler **dissipate too much**.
- **Stability**: $\Delta t \leq C M \Delta x$.
- CFL constrains by ”fast velocity / small amplitude” acoustic waves. **Filter in time/space these waves**.

- Contact with $u = 0.01$. $T_f = 10$.
- Black curve: exact sol.
- Green curve: numerical sol with 100 cells.
Important notion: AP and Well-Balanced schemes

- We consider PDE depending of a small parameter $\varepsilon$ with an asymptotic limit.

**Asymptotic preserving scheme**

- **AP scheme**: a consistent scheme for the initial PDE which gives at the limit a consistent scheme of the limit PDE.
- **Uniform AP scheme**: convergence and stability independent of $\varepsilon$.

**Application**: simulate problem with varying physical parameter and regime. Example: radiative transfer.

**Other application**: use AP scheme to create a new scheme for the limit model. Example: relaxation scheme for Euler equation.

**Well Balanced scheme**

- A scheme which preserve exact (or with high accuracy ?) a steady state of the continuous PDE.
AP/WB schemes for hyperbolic PDE with source terms
Damped wave equation: Godunov scheme

Damped wave equation:

\[
\begin{aligned}
\frac{\partial_t p}{\frac{1}{\varepsilon}} + \frac{1}{\varepsilon} \frac{\partial_x u}{\partial u} &= 0 \\
\frac{\partial_t u}{\frac{1}{\varepsilon}} + \frac{1}{\varepsilon} \frac{\partial_x p}{\partial x} &= -\frac{\sigma}{\varepsilon^2} u
\end{aligned}
\]

- **Riemann Invariant:** \( u + p \) (eigenvalue 1) and \( u - p \) (eigenvalue -1).
- Important relation to obtain the limit: \( \frac{\partial_x p}{\partial x} = -\frac{\sigma}{\varepsilon} u \).

- Upwind scheme for \( \partial_t u + \partial_x (au) = 0 \):

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} = 0
\]

with \( x_j = | x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} | \) and \( u_{j+\frac{1}{2}} = u_j^n \) for \( a > 0 \) and \( u_{j+\frac{1}{2}} = u_{j+1}^n \) for \( a < 0 \).

- Godunov acoustic scheme: **Upwind scheme** on the Riemann invariant. We obtain

\[
\begin{aligned}
\frac{p_j^{n+1} - p_j^n}{\Delta t} + \frac{u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1}}{\varepsilon \Delta x_j} &= 0 \\
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{p_{j+\frac{1}{2}}^{n+1} - p_{j-\frac{1}{2}}^{n+1}}{\varepsilon \Delta x_j} &= 0, \quad \begin{cases}
\frac{u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}}}{2} = u_{j+1}^n + p_{j+1}^n \\
\frac{u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}}}{2} = u_{j+1}^n - p_{j+1}^n.
\end{cases}
\end{aligned}
\]

- **Main drawback:** the fluxes ignore the balance between the pressure gradient and the source.
Damped wave equation: Jin-Levermore AP scheme

Jin-Levermore scheme:

- Plug the balance law \( \partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2) \) in the fluxes (Jin-Levermore 96).
- Scheme write on irregular grids.

- We write

\[
p(x_j) = p(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}}) \partial_x p(x_{j+\frac{1}{2}})
\]

- Coupling the previous relation (and the same for \( x_{j+1} \)) with the fluxes

\[
\begin{align*}
\begin{cases}
  u_j + p_j &= u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{2\varepsilon} u_{j+\frac{1}{2}}, \\
  u_{j+1} - p_{j+1} &= u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{2\varepsilon} u_{j+\frac{1}{2}}.
\end{cases}
\end{align*}
\]

Jin-Levermore scheme:

\[
\begin{align*}
\begin{cases}
  \frac{p_{j+1}^n - p_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n - M_j^n u_{j-\frac{1}{2}}^n}{\varepsilon \Delta x_j} + \frac{\sigma}{\varepsilon^2} u_j^n &= 0, \\
  \frac{u_{j+1}^n - u_j^n}{\Delta t} + \frac{p_{j+\frac{1}{2}}^n - p_{j-\frac{1}{2}}^n}{\varepsilon \Delta x_j} &= \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}.
\end{cases}
\end{align*}
\]

with \( \Delta x_{j+\frac{1}{2}} = |x_{j+1} - x_j| \) and \( M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}. \)
Damped wave equation: Jin-Levermore AP scheme

Jin-Levermore scheme:

- Plug the balance law \( \partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2) \) in the fluxes (Jin-Levermore 96).
- Scheme write on irregular grids.

- We write

\[
p(x_j) = p(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}}) \frac{\sigma}{\varepsilon} u(x_{j+\frac{1}{2}})
\]

- Coupling the previous relation (and the same for \( x_{j+1} \)) with the fluxes

\[
\begin{align*}
\begin{cases}
   u_j + p_j &= u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\
   u_{j+1} - p_{j+1} &= u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}{2\varepsilon} u_{j+\frac{1}{2}}.
\end{cases}
\end{align*}
\]

Jin-Levermore scheme:

\[
\begin{align*}
\begin{cases}
   p_{j+1}^{n+1} - p_j^n &= \frac{\Delta t}{\Delta x_j} \left[ p_j^n - p_j^n \right] + \frac{M_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n - M_j^{-\frac{1}{2}} u_{j-\frac{1}{2}}^n}{\Delta x_j} + \frac{\sigma u_j^n}{\varepsilon^2}, \\
   u_j^n &= \frac{u_j^n + u_{j+1}^n + p_j^n - p_{j+1}^n}{2} + \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}, \\
   p_j^{n+1} &= \frac{p_j^n + p_{j+1}^{n+1}}{2} + \frac{u_j^n - u_{j+1}^n}{2},
\end{cases}
\end{align*}
\]

with \( \Delta x_{j+\frac{1}{2}} = |x_{j+1} - x_j| \) and \( M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}} \).
Damped wave equation: Jin-Levermore AP scheme

Jin-Levermore scheme:

- Plug the balance law $\partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes (Jin-Levermore 96).
- Scheme write on irregular grids.

- We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) - \frac{\Delta x_j}{2} \frac{\sigma}{\varepsilon} u(x_{j+\frac{1}{2}})$$

- Coupling the previous relation (and the same for $x_{j+1}$) with the fluxes

$$\begin{align*}
  u_j + p_j &= u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\
  u_{j+1} - p_{j+1} &= u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}.
\end{align*}$$

Jin-Levermore scheme:

$$\begin{align*}
  \frac{p_{j+1}^n - p_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n - M_{j-\frac{1}{2}}^n u_{j-\frac{1}{2}}^n}{\varepsilon \Delta x_j} &+ \frac{p_{j+\frac{1}{2}}^n - p_{j-\frac{1}{2}}^n}{\Delta t} + \frac{\varepsilon \Delta x_j}{\varepsilon} \frac{\sigma}{\varepsilon^2} u_j^n = 0, \\
  \frac{u_{j+\frac{1}{2}}}{2} = \frac{u_j + u_{j+1}}{2} + \frac{p_j - p_{j+1}}{2}, \\
  \frac{p_{j+\frac{1}{2}}}{2} = \frac{p_j + p_{j+1}}{2} + \frac{u_j - u_{j+1}}{2}
\end{align*}$$

with $\Delta x_{j+\frac{1}{2}} = |x_{j+1} - x_j|$ and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}$. 
Other scheme: Gosse - Toscani scheme.

Derivation of the scheme: Localization of the source on the interface and the Riemann problem associated.

Other solution: we use the following source term $\frac{1}{2}(u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}})$ with the Jin-Levermore scheme.

Gosse-Toscani scheme:

$$\begin{align*}
\frac{p_j^{n+1} - p_j^n}{\Delta t} & + \frac{M_{j+\frac{1}{2}}}{\Delta t} \left( u_{j+\frac{1}{2}} - \frac{1}{2} u_j - \frac{1}{2} u_{j+1} \right) \\
\frac{u_j^{n+1} - u_j^n}{\Delta t} & + \frac{M_{j+\frac{1}{2}}}{\Delta t} \left( p_{j+\frac{1}{2}} - \frac{1}{2} p_j - \frac{1}{2} p_{j+1} \right) - \frac{M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}}{\Delta x_j \varepsilon} p_j^n + \left( \frac{\sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}{2 \varepsilon^2 \Delta x_j} + \frac{\sigma_{j-\frac{1}{2}} \Delta x_{j-\frac{1}{2}}}{2 \varepsilon^2 \Delta x_j} \right) u_j^n = 0
\end{align*}$$

with

$$u_{j+\frac{1}{2}} = \frac{u_j^n + u_{j+1}^n}{2} + \frac{p_j^n - p_{j+1}^n}{2}, \quad p_{j+\frac{1}{2}} = \frac{p_j^n + p_{j+1}^n}{2} + \frac{u_j^n - u_{j+1}^n}{2}$$

and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}$. 
Analysis

Analysis of the Godunov scheme

- **Consistency error:**
  - First equation: \( \left( \frac{\Delta x}{\varepsilon} + \Delta t \right) \). Second equation: \( \left( \frac{\Delta x^2}{\varepsilon} + \Delta t \right) \)

- **Time discretization:**
  - Explicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon + \varepsilon^2} \right) \leq 1 \). Semi-implicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1 \).

Analysis of the Jin–Levermore scheme

- **Consistency error:**
  - First equation: \( (\Delta x + \Delta t) \). Second equation: \( \left( \frac{\Delta x^2}{\varepsilon} + \Delta t \right) \)

- **Time discretization:**
  - Explicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon + \varepsilon^2} \right) \leq 1 \). Semi-implicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1 \).

Analysis of the Gosse–Toscani scheme

- **Consistency error:**
  - First and second equation: \( (\Delta x + \Delta t) \).

- **Time discretization:**
  - Explicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon} \right) \leq 1 \). Semi-implicit CFL: \( \Delta t \left( \frac{1}{\Delta x \varepsilon + \Delta x^2} \right) \leq 1 \).
Numerical example

- **Validation test for the AP scheme**: the data are \( p(0, x) = G(x) \) with \( G(x) \) a Gaussian \( u(0, x) = 0 \) and \( \sigma = 1, \varepsilon = 0.001 \).

### Jin-Levermore scheme

![Jin-Levermore scheme graph]

### Godunov scheme

![Godunov scheme graph]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( L^2 ) error</th>
<th>CPU time</th>
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<tbody>
<tr>
<td>Godunov, 10000 cells</td>
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<td>505 sec</td>
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E. Franck
Numerical example

- **Validation test for the AP scheme**: the data are \( p(0, x) = G(x) \) with \( G(x) \) a Gaussian \( u(0, x) = 0 \) and \( \sigma = 1, \varepsilon = 0.001 \).

### Gosse-Toscani scheme

![Gosse-Toscani scheme](image1)

### Godunov scheme

![Godunov scheme](image2)

### Scheme Summary

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</table>
Test for Well-Balanced property

- We propose to study also the **Well-Balanced property** for the family of steady state:

\[
\begin{align*}
  u(t, x) &= C_1 \\
  p(t, x) &= -\frac{\sigma}{\varepsilon} C_1 x + C_2
\end{align*}
\]

- This steady-state generate also the **affine steady state of the limit equation**.
- For this, we initialize the different schemes with a steady state and simulate with a large final time \((T_f=20)\).
- Results for different scheme and meshes.

<table>
<thead>
<tr>
<th>Scheme/mesh</th>
<th>Uniform Mesh</th>
<th>Random Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>Godunov, 100 cells</td>
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</tr>
<tr>
<td>AP-GT, 1000 cells</td>
<td>3.0E-16</td>
<td>2.8E-15</td>
</tr>
</tbody>
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**Conclusion**

- Only the Gosse-Toscani scheme is WB for all meshes.
Test for uniform convergence in 1D

- We solve the damped wave equation for different values of $\varepsilon$.
- $p(t, x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma} \alpha'(t)) \cos(\pi x), \quad u(t, x) = \left( -\frac{\varepsilon}{\sigma} \alpha(t) \sin(\pi x) \right)$
- Convergence uniform: convergence independent of $\varepsilon$.
- Test: $\varepsilon = h^\gamma$ on uniform and random meshes.

The GT scheme and the JL scheme (only on uniform mesh) are uniform AP with the error homogeneous to $O(h\varepsilon + h^2)$.

On Random mesh the JL scheme is not an uniform AP scheme.
Test for uniform convergence in 1D

- We solve the damped wave equation for different values of $\varepsilon$.

\[ p(t, x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma} \alpha'(t)) \cos(\pi x), \quad u(t, x) = (-\frac{\varepsilon}{\sigma} \alpha(t) \sin(\pi x)) \]

- **Convergence uniform**: convergence independent of $\varepsilon$.
- **Test**: $\varepsilon = h^\gamma$ on uniform and random meshes.

**GT scheme on uniform mesh**

**GT scheme on random mesh**

- The GT scheme and the JL scheme (only on uniform mesh) are uniform AP with the error homogeneous to $O(h\varepsilon + h^2)$.
- On Random mesh the JL scheme is not an uniform AP scheme.
Analysis of AP schemes: modified equations

- The modified equation associated with the Upwind scheme is

\[
\begin{aligned}
\partial_t p + \frac{1}{\varepsilon} \partial_x u - \frac{\Delta x}{2\varepsilon} \partial_{xx} p &= 0, \\
\partial_t u + \frac{1}{\varepsilon} \partial_x p - \frac{\Delta x}{2\varepsilon} \partial_{xx} u &= -\frac{\sigma}{\varepsilon^2} u.
\end{aligned}
\]

- Plugging \(\varepsilon \partial_x p + O(\varepsilon^2) = -\sigma u\) in the first equation, we obtain

\[
\partial_t p - \frac{1}{\sigma} \partial_{xx} p - \frac{\Delta x}{2\varepsilon} \partial_{xx} p = 0.
\]

- **Conclusion**: the regime is captured only on fine grids.

- The modified equation associated to the Gosse-Toscani scheme is

\[
\begin{aligned}
\partial_t p + M \frac{1}{\varepsilon} \partial_x u - M \frac{\Delta x}{2\varepsilon} \partial_{xx} p &= 0, \\
\partial_t u + M \frac{1}{\varepsilon} \partial_x p - M \frac{\Delta x}{2\varepsilon} \partial_{xx} u &= -M \frac{\sigma}{\varepsilon^2} u.
\end{aligned}
\]

- Plugging \(M \varepsilon \partial_x p + O(\varepsilon^2) = -M \sigma u\) in the first equation, we obtain

\[
\partial_t p - \frac{M}{\sigma} \partial_{xx} p - \frac{1 - M}{\sigma} \partial_{xx} p = 0.
\]

- **Conclusion**: the regime is captured on all grids.

**AP schemes**

- AP schemes modify the numerical diffusion to correct the scheme on coarse grid.
- The JL scheme does not converge in the intermediary regimes.
- **Interpretation**: since the linear steady states are not preserved the limit diffusion scheme in these regimes is not consistent.

**Idea**

- The exact preservation of linear steady-state is necessary for uniform AP schemes?
Uniform convergence in space

- Naive convergence estimate: \[ \| P_h^\varepsilon - P^\varepsilon \|_{naive} \leq C_\varepsilon^{-b} h^c \]

\[ \| P_h^\varepsilon - P^\varepsilon \|_{L^2} \leq \min(\| P_h^\varepsilon - P^\varepsilon \|_{naive}, \| P_h^\varepsilon - P_h^0 \| + \| P_h^0 - P^0 \| + \| P^\varepsilon - P^0 \|) \]

- Intermediary estimations:
  - \[ \| P^\varepsilon - P^0 \| \leq C_a \varepsilon^a \]
  - \[ \| P_h^0 - P^0 \| \leq C_d h^d \]
  - \[ \| P_h^\varepsilon - P_h^0 \| \leq C_e \varepsilon^e \]
  - \[ d \geq c, e \geq a. \]

- We using \( \min(x, y + z) \leq \min(x, y) + \min(x, z) \) and \( d \geq c, e \geq a \) to obtain

\[ \| P_h^\varepsilon - P^\varepsilon \|_{L^2} \leq C \left( \min(\varepsilon^{-b} h^c, \varepsilon^e) + h^d + \min(\varepsilon^{-b} h^c, \varepsilon^a) \right) \leq 2C \left( h^d + \min(\varepsilon^{-b} h^c, \varepsilon^a) \right) \]

- Defining \( \varepsilon_{th}^{-b} h^c = \varepsilon_{th}^a \) we obtain \( \min(\varepsilon^{-b} h^c, \varepsilon^a) \leq \varepsilon_{th}^a = h^{ac}_{a+b} \).

**Space result**

We assume that \( \| V^\varepsilon(0) - V_h^\varepsilon(0) \|_{L^2(\Omega)} \leq Ch \| p(0) \|_{H^2} \) and \( C_1 h < \Delta x_j < C_2 h \) \( \forall j \).

\[ \| V^\varepsilon - V_h^\varepsilon \|_{L^2([0, T] \times \Omega)} \leq C \min \left( h^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}, h + 2\varepsilon \right) \| p_0 \|_{H^3(\Omega)} \leq Ch^{\frac{1}{3}} \| p_0 \|_{H^3(\Omega)} \]
Euler equation with external forces

- Euler equation with gravity and friction:

\[
\begin{align*}
\partial_t \rho + \frac{1}{\varepsilon} \partial_x (\rho u) &= 0, \\
\partial_t \rho u + \frac{1}{\varepsilon} \partial_x (\rho u^2) + \frac{1}{\varepsilon} \partial_x p &= -\frac{1}{\varepsilon} (\rho \partial_x \phi + \frac{\sigma}{\varepsilon} \rho u), \\
\partial_t E + \frac{1}{\varepsilon} \partial_x (Eu + pu) &= -\frac{1}{\varepsilon} (\rho u \partial_x \phi + \frac{\sigma}{\varepsilon} \rho u^2).
\end{align*}
\]

- With \( \phi \) the gravity potential, \( \sigma \) the friction coefficient.

Subset of solutions:

- Hydrostatic Steady-state (\( \alpha = 1, \beta = 0 \)):

\[
\begin{align*}
u &= 0, \\
\partial_x p &= -\rho \partial_x \phi.
\end{align*}
\]

- High friction limit (\( \alpha = 0, \beta = 1 \)), no gravity: \( u = 0 \)

- Diffusion limit (\( \alpha = 1, \beta = 1 \)):

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t E + \partial_x (Eu) + p \partial_x u &= 0, \\
u &= -\frac{1}{\sigma} \left( \partial_x \phi + \frac{1}{\rho} \partial_x p \right).
\end{align*}
\]
**Design of AP nodal scheme I**

**Jin Levermore method:**

Plug the relation $\partial_x p + O(\varepsilon) = -\rho \partial_x \phi - \frac{\sigma}{\varepsilon} \rho u$ in the Lagrangian fluxes

- Classical Lagrange+remap scheme (LP scheme):

\[
\begin{align*}
\partial_t \rho_j + \rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} \frac{u^*_{j+\frac{1}{2}} - u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= 0 \\
\partial_t (\rho u)_j + \frac{(\rho u)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - (\rho u)_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{p^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= -\frac{1}{\varepsilon} \left( \rho_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon} \rho_j u_j \right) \\
\partial_t E_j + \frac{E_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - E_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{p^*_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= -\frac{1}{\varepsilon} \left( \rho_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon} \rho_j u_j^2 \right)
\end{align*}
\]

with Lagrangian fluxes

\[
\begin{align*}
p^*_{j+\frac{1}{2}} + (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} &= p_j + (\rho c)_{j+\frac{1}{2}} u_j \\
p^*_{j+\frac{1}{2}} - (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} &= p_{j+1} - (\rho c)_{j+\frac{1}{2}} u_{j+1}
\end{align*}
\]

and the upwind flux

\[
u^*_{j+\frac{1}{2}} f_{j+\frac{1}{2}} = \begin{cases} u^*_{j+\frac{1}{2}} f_j \\ u^*_{j+\frac{1}{2}} f_{j+1} \end{cases}
\]
Design of AP nodal scheme I

Jin Levermore method:

Plug the relation $\partial_x p + O(\varepsilon) = -\rho\partial_x \phi - \frac{\sigma}{\varepsilon} \rho u$ in the Lagrangian fluxes

- Classical Lagrange+remap scheme (LP scheme):

$$\begin{align*}
\partial_t \rho_j + \frac{\rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - \rho_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= 0 \\
\partial_t (\rho u)_j + \frac{(\rho u)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - (\rho u)_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{p^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= -\frac{1}{\varepsilon} \left( \rho_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon} \rho_j u_j \right) \\
\partial_t E_j + \frac{E_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - E_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{p^*_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} &= -\frac{1}{\varepsilon} \left( \rho_j u_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon} \rho_j u_j^2 \right)
\end{align*}$$

with Lagrangian fluxes with the new Lagrangian fluxes

$$\begin{align*}
p^*_{j+\frac{1}{2}} + (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - \frac{\Delta x_{j+\frac{1}{2}}}{2} \left( (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{\varepsilon} \rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} \right) &= p_j + (\rho c)_{j+\frac{1}{2}} u_j \\
p^*_{j+\frac{1}{2}} - (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} + \frac{\Delta x_{j+\frac{1}{2}}}{2} \left( (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{\varepsilon} \rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} \right) &= p_{j+1} - (\rho c)_{j+\frac{1}{2}} u_{j+1}
\end{align*}$$

with $\rho_{j+\frac{1}{2}}$ and $(\rho \partial_x \phi)_{j+\frac{1}{2}}$ averages between the interface and the upwind flux and the upwind flux

$$u^*_{j+\frac{1}{2}} f_{j+\frac{1}{2}} = \begin{cases} u^*_{j+\frac{1}{2}} f_j \\
u^*_{j+\frac{1}{2}} f_{j+1} \end{cases}$$
Jin Levermore method:

Plug the relation $\partial_x p + O(\varepsilon) = -\rho \partial_x \phi - \frac{\sigma}{\varepsilon} \rho u$ in the Lagrangian fluxes

New scheme (LP-AP scheme):

$$
\begin{align*}
\partial_t \rho_j + \frac{\rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* - \rho_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^*}{\varepsilon^\alpha \Delta x_j} &= 0 \\
\partial_t (\rho u)_j + \frac{(\rho u)_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* - (\rho u)_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^*}{\varepsilon^\alpha \Delta x_j} + \frac{p_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^*}{\varepsilon^\alpha \Delta x_j} &= -\frac{1}{\varepsilon^\alpha} \left( (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma}{\varepsilon^\beta} \rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* \right) \\
\partial_t E_j + \frac{E_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* - E_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^*}{\varepsilon^\alpha \Delta x_j} + \frac{p_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^*}{\varepsilon^\alpha \Delta x_j} &= -\frac{1}{\varepsilon^\alpha} \left( (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma}{\varepsilon^\beta} \rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* \right)^2
\end{align*}
$$

with Lagrangian fluxes

$$
\begin{align*}
p_{j+\frac{1}{2}}^* + (\rho c)_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* - \frac{\Delta x_{j+\frac{1}{2}}}{2} (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{\varepsilon^\beta} \rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* &= p_j + (\rho c)_{j+\frac{1}{2}} u_j \\
p_{j+\frac{1}{2}}^* - (\rho c)_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* + \frac{\Delta x_{j+\frac{1}{2}}}{2} (\rho \partial_x \phi)_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}}{\varepsilon^\beta} \rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^* &= p_{j+1} - (\rho c)_{j+\frac{1}{2}} u_{j+1}
\end{align*}
$$

with $\rho_{j+\frac{1}{2}}$ and $(\rho \partial_x \phi)_{j+\frac{1}{2}}$ averages between the interface and the upwind flux

$$
\begin{align*}
u_{j+\frac{1}{2}}^* f_{j+\frac{1}{2}} &= \left\{ \begin{array}{ll}
u_{j+\frac{1}{2}}^* f_j & \\
u_{j+\frac{1}{2}}^* f_{j+1} &
\end{array} \right.
\end{align*}
$$
Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state \( p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}} \) is exactly preserved.

Question: How the scheme preserved the continuous steady state?

First choice:

\[
(\rho \partial_x \phi)_{j+\frac{1}{2}} = \frac{1}{2} (\rho_j + \rho_{j+1}) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}
\]

Only the continuous steady state with \( \rho \partial_x \phi = Cts \) are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.
Properties

**Ap property**
- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

**WB property**
- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}$ is exactly preserved.

- Question: How the scheme preserved the continuous steady state?
- **Second choice:**

$$ (\rho \partial_x \phi)_{j+\frac{1}{2}} = \left( \frac{\rho_{j+1} - \rho_j}{\ln(\rho_{j+1}) - \ln(\rho_j)} \right) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}} $$

- Only the continuous steady state with $\rho = p = e^{-xg}, \phi = gx$ are exactly preserved.

**Idea**
- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

$$ \partial_x p = -\rho \partial_x \phi $$
Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state \( p_{j+1} - p_j = -\Delta x_j \frac{1}{2} (\rho \partial_x \phi)^{j+\frac{1}{2}} \) is exactly preserved.

**Question**: How the scheme preserved the continuous steady state?

**Second choice**:

\[
(\rho \partial_x \phi)^{j+\frac{1}{2}} = \left( \frac{\rho_{j+1} - \rho_j}{\ln(\rho_{j+1}) - \ln(\rho_j)} \right) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}
\]

- Only the continuous steady state with \( \rho = p = e^{-xg}, \phi = gx \) are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

\[
\Delta j+\frac{1}{2} \left( \frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x p \right) = -\Delta j+\frac{1}{2} \left( \frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \rho \partial_x \phi \right)
\]
Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}$ is exactly preserved.

Question: How the scheme preserved the continuous steady state?

Second choice:

$$ (\rho \partial_x \phi)_{j+\frac{1}{2}} = \left( \frac{\rho_{j+1} - \rho_j}{\ln(\rho_{j+1}) - \ln(\rho_j)} \right) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}} $$

Only the continuous steady state with $\rho = p = e^{-xg}$, $\phi = gx$ are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

$$ \Delta_{j+\frac{1}{2}} \left( \frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}} \right) = -\Delta_{j+\frac{1}{2}} \left( \frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}} \partial_x \overline{\phi}_{j+\frac{1}{2}} \right) $$

with $\overline{p}_{j+\frac{1}{2}}$ (same for $\rho$ and $\phi$) average polynomial interpolation.
Properties

Ap property
- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property
- The discrete steady state \( p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}} \) is exactly preserved.

Question: How the scheme preserved the continuous steady state?
Second choice:

\[
(\rho \partial_x \phi)_{j+\frac{1}{2}} = \left( \frac{p_{j+1} - p_j}{\ln(p_{j+1}) - \ln(p_j)} \right) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}
\]

Only the continuous steady state with \( \rho = p = e^{-xg}, \phi = gx \) are exactly preserved.

Idea
- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

the final equilibrium \( p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}^{HO} \)

\[
(\rho \partial_x \phi)_{j+\frac{1}{2}}^{HO} = \Delta x_{j+\frac{1}{2}} \left( \frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \left( \partial_x \tilde{p}_{j+\frac{1}{2}} + \tilde{p}_{j+\frac{1}{2}} \partial_x \tilde{\phi}_{j+\frac{1}{2}} \right) - \frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} \right)
\]
Results

- Comparison between AP and Non AP scheme for Euler equation.

Results

- **Well-Balanced property.**

- **Test case:** $\rho(t, x) = 3 + 2\sin(2\pi x)$ and $\phi(x) = -\sin(2\pi x)$. Random mesh

<table>
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<th>Schemes</th>
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<th>LR-AP (3)</th>
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- **Test case:** $\rho(t, x) = e^{-g x}$, $u(t, x) = 0$, $p(t, x) = e^{-g x}$ et $\phi = g x$. Random mesh

<table>
<thead>
<tr>
<th>Schemes</th>
<th>LR</th>
<th>LR-AP (2)</th>
<th>LR-AP (3)</th>
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</table>

**WB scheme**

Not exact preservation of general steady-state, but **arbitrary high order accuracy around the steady-state**
Implicit relaxation method for low Mach Euler equations
Low Mach and implicit scheme

**Aim: Low Mach Euler equation**

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t \rho u + \nabla \cdot (\rho u \otimes u) + \frac{1}{M} \nabla p &= 0, \\
\partial_t E + \nabla \cdot ((E + p) u) &= 0,
\end{align*}
\]

\[\text{CFL condition } \Delta t \leq hM.\]

- **Aim:** choose a time step adapted to \(u\). Filter the fast waves.
- **Solution:** implicit scheme.

**Implicit scheme**

- **Direct solver:** too expensive in CPU time and memory consumption.
- **Iterative solver:** used in practice. But often ill-conditioning for hyperbolic models.
- **Euler equation:** ill-conditioned mainly in the low-Mach regime.

**Idea**

- Using relaxation model and AP schemes to obtain implicit scheme **without matrices**.
Relaxation scheme

- We consider the relaxation model (Jin-Xin 95) for a scalar system \( \partial_t u + \partial_x F(u) = 0 \):

\[
\begin{align*}
\partial_t u + \partial_x v &= 0 \\
\partial_t v + \alpha^2 \partial_x u &= \frac{1}{\varepsilon}(F(u) - v)
\end{align*}
\]

Limit

- The limit scheme of the relaxation system is

\[
\partial_t u + \partial_x F(u) = \varepsilon \partial_x (\lambda^2 - |\partial F(u)|^2) \partial_x u + O(\varepsilon^2)
\]

- **Stability**: the limit system is dissipative if \( (\lambda^2 - |\partial F(u)|^2) > 0 \).

- We diagonalize the hyperbolic matrix

\[
\begin{pmatrix}
0 & 1 \\
\lambda^2 & 0
\end{pmatrix}
\]


\[
\begin{align*}
\partial_t f_- - \lambda \partial_x f_- &= \frac{1}{\varepsilon}(f_{eq}^- - f_-) \\
\partial_t f_+ + \lambda \partial_x f_+ &= \frac{1}{\varepsilon}(f_{eq}^+ - f_+)
\end{align*}
\]

with \( u = f_- + f_+ \) and \( f_{eq}^\pm = \frac{u}{2} \pm \frac{F(u)}{2\lambda} \).

Remark

- **Main property**: the transport is diagonal (D1Q2 model) which can be easily solved.
**Generic kinetic relaxation scheme**

**Kinetic relaxation system**

- **Considered model:**
  \[
  \partial_t U + \partial_x F(U) = 0
  \]

- **Lattice:** \( W = \{ \lambda_1, \ldots, \lambda_{n_v} \} \) a set of velocities.

- **Mapping matrix:** \( P \) a matrix \( n_c \times n_v \) (\( n_c < n_v \)) such that \( U = Pf \), with \( U \in \mathbb{R}^{n_c} \).

- **Kinetic relaxation system:**
  \[
  \partial_t f + \Lambda \partial_x f = \frac{1}{\varepsilon} (f_{eq}(U) - f)
  \]

- We define the macroscopic variable by \( Pf = U \).

- Consistence condition (R. Natalini, D. Aregba-Driollet, F. Bouchut):
  \[
  C \left\{ \begin{array}{l}
  Pf_{eq}(U) = U \\
  Pf_{eq}(U) = F(U)
  \end{array} \right.
  \]

- **In 1D:** same property of stability that the classical relaxation method.

- **Limit of the system:**
  \[
  \partial_t U + \partial_x F(U) = \varepsilon \partial_x \left( (P\Lambda^2 \partial f_{eq} - |\partial F(U)|^2) \partial_x U \right) + O(\varepsilon^2)
  \]

**First Generalization**

- **Generalization** \([D1Q2]^n\): one Xin-Jin or D1Q2 model by macroscopic variable.
Time scheme

- **Property**: the nonlinearity is local and non-locality is linear.
- **Main idea**: time splitting scheme between transport and source.

Consistency in time

- We define the two operators for each step:
  \[ T_{\Delta t} : e^{\Delta t \partial_x} f^{n+1} = f^n \]
  \[ R_{\Delta t} : f^{n+1} + \theta \frac{\Delta t}{\varepsilon} (f^{eq}(U) - f^{n+1}) = f^n - (1 - \theta) \frac{\Delta t}{\varepsilon} (f^{eq}(U) - f^n) \]
- **Final scheme**: \( \Psi(\Delta t) = T_{\Delta t} \circ R_{\Delta t} \) is consistent with
  \[ \partial_t U + \partial_x F(U) = \left( \frac{(2 - \omega)\Delta t}{2\omega} \right) \partial_x (D(U)\partial_x U) + O(\Delta t^2) \]
  with \( \omega = \frac{\Delta t}{\varepsilon + \theta \Delta t} \) and \( D(U) = (P \Lambda^2 \partial_U f^{eq} - A(U)^2) \).

Drawback

- For \([D1Q2]^2\) scheme we have a large error: \( D(U) = (\lambda^2 I_d - A(U)^2) \)
High-order extension

**High order scheme**
- Second order splitting
  \[ \Psi(\Delta t) = T \left( \frac{1}{2} \Delta t \right) \circ R(\Delta t) \circ T \left( \frac{1}{2} \Delta t \right) \]
- Higher order scheme using composition:
  \[ M_p(\Delta t) = \Psi(\gamma_1 \Delta t) \circ \Psi(\gamma_2 \Delta t) \circ \ldots \circ \Psi(\gamma_s \Delta t) \]
- With \( \gamma_i \in [-1, 1] \), we obtain a \( p \)-order schemes.
- Susuki scheme: \( s = 5, p = 4 \). Kahan-Li scheme: \( s = 9, p = 6 \).
- High-order convergence only for macroscopic variables.

**Space solver**
- **Exact transport**: the choice of the velocities link time and space discretization.
- **Semi-Lagrangian**: Interpolation \( 2q + 1 \) gives a consistency error \( O(\frac{h^{2d+2}}{\Delta t}) \).
- **Implicit DG**: DG (\( k \) polynomial and Gauss-Lobatto) point gives a consistency error \( O(h^k) + O(\Delta t^2) \).
Burgers: convergence results

- **Model:** Burgers equation

\[ \partial_t \rho + \partial_x \left( \frac{\rho^2}{2} \right) = 0 \]

- Spatial discretization: SL-scheme, 2000 cells, degree 11.

- **Test:** \( \rho(t = 0, x) = \sin(2\pi x) \). \( T_f = 0.14 \) (before the shock) and no viscosity.

- Scheme: splitting schemes and Suzuki composition + splitting.

<table>
<thead>
<tr>
<th>Δt</th>
<th>SPL 1, ( \theta = 1 )</th>
<th>SPL 1, ( \theta = 0.5 )</th>
<th>SPL 2, ( \theta = 0.5 )</th>
<th>Suzuki</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error</td>
<td>order</td>
<td>Error</td>
<td>order</td>
</tr>
<tr>
<td>0.005</td>
<td>( 2.6E^{-2} )</td>
<td>-</td>
<td>( 1.3E^{-3} )</td>
<td>-</td>
</tr>
<tr>
<td>0.0025</td>
<td>( 1.4E^{-2} )</td>
<td>0.91</td>
<td>( 3.4E^{-4} )</td>
<td>1.90</td>
</tr>
<tr>
<td>0.00125</td>
<td>( 7.1E^{-3} )</td>
<td>0.93</td>
<td>( 8.7E^{-5} )</td>
<td>1.96</td>
</tr>
<tr>
<td>0.000625</td>
<td>( 3.7E^{-3} )</td>
<td>0.95</td>
<td>( 2.2E^{-5} )</td>
<td>1.99</td>
</tr>
</tbody>
</table>

- Scheme: second order splitting scheme.

- Same test after the shock:
Numerical results: 2D-3D fluid models

- **Model**: liquid-gas Euler model with gravity.
- **Kinetic model**: $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme**: 2 order Implicit DG scheme. 3rd order in space. CFL around 6.
- **Test case**: Rayleigh-Taylor instability.

2D case in annulus

3D case in cylinder

**Figure**: Plot of the mass fraction of gas
Numerical results: 2D-3D fluid models

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- **Test case**: Rayleigh-Taylor instability.

2D case in annulus

![2D case in annulus](image1.png)

2D cut of the 3D case

![2D cut of the 3D case](image2.png)

**Figure**: Plot of the mass fraction of gas
Classical kinetic representation

Limitation

- High-order extension allows to correct the main default of relaxation: large error.
- In two situations the **High-order extension is not sufficient**:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: Sod problem.
- **Second order time scheme + SL scheme**:

  ![Graphs](image_url)

  - Left: density $\Delta t = 1.0^{-4}$. Right: density $\Delta t = 4.0^{-4}$

- **Conclusion**: shock and high order time scheme needs **limiting methods**.
Classical kinetic representation

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- High-order extension allows to correct the main default of relaxation: large error.
- In two situations the High-order extension is not sufficient:
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- **Euler equation**: smooth contact \((u = \text{cts}, p = \text{cts})\).  
- First/Second order time scheme + SL scheme. \(T_f = \frac{2}{M}\) and 100 time step.

- Order 1 Left: \(M = 0.1\). Right: \(M = 0.01\)
- **Conclusion**: First order method too much dissipative for low Mach flow (dissipation with acoustic coefficient).
Classical kinetic representation

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Generic vectorial D1Q3

Idea

- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- Consistency condition:

\[
\begin{align*}
\begin{cases}
  f_+^k + f_0^k + f_-^k &= U_k, & \forall k \in \{1..N_c\} \\
  \lambda_- f_-^k + \lambda_0 f_0^k + \lambda_+ f_+^k &= F_k(U), & \forall k \in \{1..N_c\}
\end{cases}
\end{align*}
\]

- We assume a decomposition of the flux (Bouchut 03, Natalini -Aregba 00)

\[
F_k(U) = F_0^{k,-}(U) + F_0^{k,+}(U) + \lambda_0 I_d
\]

- We obtain the following equation for the equilibrium

\[
\begin{align*}
\begin{cases}
  f_+^k + f_0^k + f_-^k &= U_k, & \forall k \in \{1..N_c\} \\
  (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k &= F_0^{k,-}(U) + F_0^{k,+}(U), & \forall k \in \{1..N_c\}
\end{cases}
\end{align*}
\]

- By analogy of the kinetic theory and kinetic flux splitting scheme we propose the following decomposition \( \sum_{v>0} v f^k = F_0^{k,+}(U) \) and \( \sum_{v<0} v f^k = F_0^{k,-}(U) \).
Generic vectorial D1Q3

Idea

- Add a **central velocity** (equal or close to zero) to capture the slow dynamics.

- The lattice $[D1Q3]^N$ is defined by the velocity set $V = [\lambda_-, \lambda_0, \lambda_+]$ and

\[
\begin{align*}
    f_{-}^{eq}(U) &= -\frac{1}{(\lambda_0 - \lambda_-)} F_0^-(U) \\
    f_{0}^{eq}(U) &= \left(U - \left(\frac{F_0^+(U)}{(\lambda_+ - \lambda_0)} - \frac{F_0^-(U)}{(\lambda_0 - \lambda_-)}\right)\right) \\
    f_{+}^{eq}(U) &= \frac{1}{(\lambda_+ - \lambda_0)} F_0^+(U)
\end{align*}
\]

Stability

- Condition only on the **macroscopic flux splitting**.

- Condition for **entropy stability**:
  - $F_0^+$ and $F_0^-$ is an entropy decomposition of the flux
  - $\partial F_0^+, -\partial F_0^-$ and $1 - \frac{\partial F_0^+ - \partial F_0^-}{\lambda}$ are positive.
D1Q3 for scalar case

- First choice: **D1Q3 Rusanov** ($\lambda_0 = 0$)

\[
F^-(\rho) = -\lambda_- \frac{(F(\rho) - \lambda_+ \rho)}{\lambda_+ - \lambda_-}, \quad F^+(\rho) = \lambda_+ \frac{(F(\rho) - \lambda_- \rho)}{\lambda_+ - \lambda_-}
\]

- Consistency (for $\lambda_- = -\lambda_+$):

\[
\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left( \lambda^2 - |\partial F(\rho)|^2 \right) \partial_x \rho + O(\Delta t^2)
\]

- Second choice: **D1Q3 Upwind**

\[
F^-(\rho) = \chi_{\{\partial F(\rho) < \lambda_0\}} (F(\rho) - \lambda_0 \rho) \quad F^+(\rho) = \chi_{\{\partial F(\rho) > \lambda_0\}} (F(\rho) - \lambda_0 \rho)
\]

with $\chi$ the indicatrice function.

- Consistency:

\[
\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left( \lambda |\partial F(\rho)| - |\partial F(\rho)|^2 \right) \partial_x \rho + O(\Delta t^2)
\]

- Third choice: **D1Q3 Lax-Wendroff** ($\lambda_0 = 0$)

\[
F^-(\rho) = \frac{1}{2} \left( F(\rho) + \frac{\alpha}{\lambda} \int^\rho (\partial F(u))^2 \right) \quad F^+(\rho) = \frac{1}{2} \left( F(\rho) + \frac{\alpha}{\lambda} \int^\rho (\partial F(u))^2 \right)
\]

with $\lambda_0 = 0$ and $\lambda_- = -\lambda_+$ and $\alpha \geq 1$.

- Consistency:

\[
\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left( (\alpha - 1) |\partial F(\rho)|^2 \right) \partial_x \rho + O(\Delta t^2).
\]

- The last one is not entropy stable and $L^2$ stability in some case.
D1Q3 for Euler equation II

- **Low Mach case:**

\[
\begin{cases}
\partial_t \rho + \partial_x (\rho u) = 0 \\
\partial_t \rho u + \partial_x \left( \rho u^2 + \frac{p}{M} \right) = 0 \\
\partial_t E + \partial_x (Eu + pu) = 0
\end{cases}
\]

- We want to preserve as possible the limit:

\[
p = cts, \quad u = cts, \quad \partial_t \rho + u \partial_x \rho = 0
\]

- **Idea:** Splitting of the flux (E. Toro 12):

\[
F(U) = \begin{pmatrix}
(\rho)u \\
(\rho u)u + p \\
(E)u + pu
\end{pmatrix}
\]

- **Idea:** Lax-Wendroff Flux splitting for convection and AUSM-type (M. Liou 93) for the pressure term.
- Use only \( u, p \) and \( \lambda (\approx c) \) to reconstruct pressure. Important to preserve the low mach limit.
- We obtain

\[
F^\pm(U) = \frac{1}{2} \begin{pmatrix}
(\rho u \pm \frac{u^2}{\lambda} \rho) + p \\
(\rho u^2 \pm \frac{u^2}{\lambda} q) + p(1 \pm \gamma \frac{u}{\lambda}) \\
(E u \pm \frac{u^2}{\lambda} E) + (pu \pm \frac{1}{\lambda} \gamma (u^2 + \lambda^2) p)
\end{pmatrix}
\]

- Preserve contact. Diffusion error for \( \rho \) in \( O(u^2) \).
**Burgers**

- **Model:** Viscous Burgers equations

\[ \partial_t \rho + \partial_x \left( \frac{\rho^2}{2} \right) = 0 \]

- **Test case 1:** \( \rho(t = 0, x) = \sin(2\pi x) \). 10000 cells. Order 17. First order time scheme.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>Rusanov Error</th>
<th>Rusanov Order</th>
<th>Upwind Error</th>
<th>Upwind Order</th>
<th>Lax Wendroff ( \alpha = 1 ) Error</th>
<th>Lax Wendroff ( \alpha = 1 ) Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3.9E-2</td>
<td>-</td>
<td>1.1E-2</td>
<td>-</td>
<td>2.3E-3</td>
<td>-</td>
</tr>
<tr>
<td>0.005</td>
<td>2.1E-2</td>
<td>0.89</td>
<td>6.4E-3</td>
<td>0.78</td>
<td>6.0E-4</td>
<td>1.94</td>
</tr>
<tr>
<td>0.0025</td>
<td>1.1E-2</td>
<td>0.93</td>
<td>3.5E-3</td>
<td>0.87</td>
<td>1.5E-4</td>
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</tr>
<tr>
<td>0.00125</td>
<td>5.4E-3</td>
<td>1.03</td>
<td>1.8E-3</td>
<td>0.96</td>
<td>3.9E-5</td>
<td>1.95</td>
</tr>
</tbody>
</table>

- Shock wave. First order scheme in time.

- Left \( \Delta t = 0.002 \). Right \( \Delta t = 0.01 \). Reference (black), Rusanov (yellow), Upwind (violet), Lax-Wendroff (green), Lax-Wendroff \( \alpha = 1.5 \) (blue).
**Model:** Viscous Burgers equations

\[ \partial_t \rho + \partial_x \left( \frac{\rho^2}{2} \right) = 0 \]

**Test case 1:** \( \rho(t = 0, x) = \sin(2\pi x) \). 10000 cells. Order 17. First order time scheme.

<table>
<thead>
<tr>
<th>Error Order</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Rusanov</td>
<td>Upwind</td>
<td>Lax Wendroff ( \alpha = 1 )</td>
</tr>
<tr>
<td>( \Delta t = 0.01 )</td>
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<td>-</td>
</tr>
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</tr>
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<td>( \Delta t = 0.00125 )</td>
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<td>1.03</td>
</tr>
</tbody>
</table>

**Rarefaction wave. First order scheme in time.**

Left \( \Delta t = 0.002 \). Right \( \Delta t = 0.01 \). Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff \( \alpha = 1 \) (blue), Lax-Wendroff \( \alpha = 2 \) (Yellow).

E. Franck 36/39
1D Euler equations II

- **Test case**: Smooth contact. We take $p = 1$ and $u$ is also constant.
- **Final aim**: take $\Delta t = O\left(\frac{1}{u}\right)$ when $u$ decrease to have the same error.

We choose $\Delta t = 0.02$ and $T_f = 2$. 4000 cells. First order time scheme. We compare different D1Q3 schemes.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Rusanov</th>
<th>VL</th>
<th>Osher</th>
<th>Low Mach</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 10^{-2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(t, x)$</td>
<td>0.26</td>
<td>$1.0E^{-1}$</td>
<td>$8.4E^{-2}$</td>
<td>$1.0E^{-3}$</td>
</tr>
<tr>
<td>$u(t, x)$</td>
<td>0</td>
<td>$3.4E^{-3}$</td>
<td>$6.0E^{-7}$</td>
<td>0</td>
</tr>
<tr>
<td>$p(t, x)$</td>
<td>0</td>
<td>$5.0E^{-4}$</td>
<td>$4.3E^{-8}$</td>
<td>0</td>
</tr>
<tr>
<td>$u = 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(t, x)$</td>
<td>0.26</td>
<td>$1.0E^{-1}$</td>
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</tr>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>$u = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(t, x)$</td>
<td>0.26</td>
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<td>$5.0E^{-4}$</td>
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<td>0</td>
</tr>
</tbody>
</table>

- **Drawback**: When the time step is too large we have dispersive effect.
- **Possible explanation**: the error would be homogeneous to

\[
| \rho^n(x) - \rho(t, x) | \approx O(\Delta tu^2) + O(\Delta t^2 u\lambda^q).
\]

with $\lambda$ closed to the sound speed.

- **Problem**: At the second order we recover partially the problem since $\lambda$ is closed to the sound speed.
1D Euler equations III

- **Possible solution**: decrease $\lambda$ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:
  \[
  V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]
  \]
  slow scale
- The convective part at the slow scale. The acoustic part at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t = \frac{0.001}{u}$:

<table>
<thead>
<tr>
<th>Error</th>
<th>$u = 10^{-1}$</th>
<th>$u = 10^{-2}$</th>
<th>$u = 10^{-3}$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 1$</td>
<td>$2.5E^{-3}$</td>
<td>$2.5E^{-3}$</td>
<td>$2.5E^{-3}$</td>
<td>$2.5E^{-3}$</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td>2</td>
<td>0.2</td>
<td>0.02</td>
<td>0.002</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>20</td>
<td>200</td>
<td>2000</td>
<td>2000</td>
</tr>
</tbody>
</table>

**Conclusion**

- **Conclusion**: the error would be homogeneous to
  \[
  | \rho^n(x) - \rho(t, x) | \approx [O(\Delta t u^2) + O(\Delta t^2 u \lambda_s^g)]
  \]
  with $\lambda_s$ which can be take small.
- **Drawback**: For the stability it seems necessary to have
  \[
  \lambda_s \lambda_f \geq C \max_x (u + c)
  \]
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**Conclusion**

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$$| \rho^n(x) - \rho(t, x) | \approx [O(\Delta t u^2) + O(\Delta t^2 u \lambda_s^q)]$$

  - with $\lambda_s$ which can be take small.

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$$\lambda_s \lambda_f \geq C \max_x (u + c)$$
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- We propose **two-scale kinetic model**.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]$$

**coupling**

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**Conclusion**

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- with $\lambda_s$ which can be taken small.
- **Drawback:** For the stability it seems necessary to have

$$\lambda_s \lambda_f \geq C \max_x (u + c)$$
## Conclusion

### Ap schemes for diffusion limit

- **AP scheme**: plug the term source effect in the fluxes.
- **Uniform AP**: scheme: previous construction not sufficient. WB also?
- **Other Works:**
  - 2D extension on unstructured meshes for damped wave equations [BDF12], [FHNG11], [BDFL16].
  - Extension on 2D unstructured meshes for Friedrich’s systems [BDF14].
  - Extension on 2D unstructured meshes for nonlinear radiative problem [BDF11], [BDF12] and Euler equations [F14], [FM16].

### Kinetic relaxation schemes

- **Implicit schemes**: without matrices based on kinetic relaxation schemes.
- **High order time extension** [CFHMN17], [CFHMN18] and parallel algorithm [Cemracs18].
- **Future Works:**
  - D1Q3 schemes for hyperbolic problem in 1D (in redaction). Extension in 2D/3D application to low-Mach Euler equation.
  - Implicit Kinetic schemes for anisotropic diffusion (in redaction).
  - Boundary conditions (Post doc of F. Drui).
  - Incompressibility, divergence constrains.