Relaxation method: a tool to degin time integrators

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Hyperbolic systems et time integration

- We consider a general hyperbolic system with source term:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{R}(\mathbf{U})$$

- **Speeds propagation**: are given by the eigenvalues $\lambda_{1,..,n}$ of $\partial_{\mathbf{U}} \mathbf{F}(\mathbf{U})$.

Hyperbolic system and time integration

- **Classic scheme**: explicit scheme with a CFL $\Delta < \frac{\Delta x}{\lambda_{max}}$.
- **Problem**: it's very penalizing when
  - somes cells are very small,
  - the velocity is locally very high,
  - there is multi-scale problems with slow/fast scales.
- **Solution**: implicit/semi-implicit scheme.

- Implicit time scheme:

$$M_i \boldsymbol{U}^{n+1} = (I_d + \Delta t A(I_d))\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$$

- We must solve a nonlinear system and after linearization solve some linear systems.
- **Conditioning**:

$$k(M_i) \approx 1 + O\left(\frac{\lambda_{max} \Delta t}{\Delta x^p \lambda_{min}}\right)$$

- The implicit schemes for hyperbolic system as Euler/MHD are ill-conditioned.

Relaxation method

- **Relaxation** [XJ95]-[CGS12]-[BCG18]: a way to linearize and decouple the equations. Used to design new schemes.
- **Idea**: Approximate the model

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = 0, \text{ by } \quad \partial_t \mathbf{f} + \mathbf{A}(\mathbf{f}) = \frac{1}{\varepsilon}(Q(\mathbf{f}) - \mathbf{f})$$

- where the structure of the flux $\mathbf{A}(\mathbf{f})$ is more simple.
- At the limit and taking $P\mathbf{f} = \boldsymbol{U}$, $P\mathbf{A}(\mathbf{f}) = \mathbf{F}(\boldsymbol{U})$, we obtain

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = \varepsilon \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + O(\varepsilon^2)$$

- **Time scheme**:
  - we solve

$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \mathbf{A}(\mathbf{f}^{*,n}) = 0$$

  - and after we approximate the stiff source term by

$$\mathbf{f}^{n+1} = \mathbf{f}^* + \omega(Q(\mathbf{f}^*) - \mathbf{f}^*)$$

  with $\omega \in]0, 2]$. The case $\omega = 1$ corresponds to the projection.

Relaxation method II

Application I: Godunov scheme

- The relaxation system is chosen such that it will be easy/possible to write a Godunov scheme for the PDE:
$$\partial_t \mathbf{f} + \partial_x \mathbf{A}(\mathbf{f}) = 0$$
- Applying direclty after, the projection $\omega = 1$ we obtain a scheme for the original system.
- **Ref**: many papers of F. Coquel, F. Bouchut, C. Klingenberg, C. Berthon, C. Chalons, S. Jin etc...

Application II: Implicit/semi-implicit integrator

- The relaxation system allows to decouple/linearize the waves. It is also interesting to design simpler implicit schemes.
- **Principle**: write a semi-implicit/implicit time scheme for
$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \partial_x \mathbf{A}(\mathbf{f}^*) = 0$$
is simpler that for the original system.

Ref: few papers: 2 paper of C. Kligenberg and al, 2 paper of G. Puppo and al and 3 papers of our groups.

Xin-Jin relaxation and implicit

- 1D hyperbolic system:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$$

- Approximation:

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{V} &= 0 \qquad (1) \\ &= \mathbf{F}(\boldsymbol{U}) - \mathbf{V} \qquad (2) \end{aligned}$$

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{V} &= 0 \qquad (3) \\ \partial_t \mathbf{V} + \alpha^2 \partial_x \mathbf{U} &= \frac{1}{\epsilon}(\mathbf{F}(\boldsymbol{U}) - \mathbf{V}) \qquad (4) \end{aligned}$$

- The Xin-Jin relaxation system is stable only if $\alpha > \lambda_{max}$.
- After splitting, we have, as hyperbolic part:

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{V} = 0 \\ \partial_t \mathbf{V} + \alpha^2 \partial_x \mathbf{U} = 0 \end{aligned}$$

- We obtain N linear and independant systems of two variables.
- **Implicit scheme**: we must invert N linear systems of the form

$$\begin{pmatrix} I_d & \Delta t \partial_x \\ \Delta t \alpha^2 \partial_x & I_d \end{pmatrix}$$

- Invert the discreitzation of this matrix is easy using a Schur complement method.

Kinetic relaxation and implicit I

Kinetic relaxation system

- **Considered model**:
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$
- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $\boldsymbol{U} \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:
$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Consistence condition:
$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{F}(\boldsymbol{U}) \end{cases}$$
- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:
$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$
- Natural extension in 2D/3D.
- **General scheme**: $[D1Q2]^n$, one D1Q2 by macroscopic equation.

Kinetic relaxation and implicit II

- **Property of Kinetic relaxation**: we have n_v independent transport equations to solve the implicit step.
- **Advantages**:
  - the implicit step can be easily parallelized. One MPI process by transport equation for example.
  - We can use method without CFL and matrix invertion: the Semi Lagrangien method.
- **SL Principle**: We use the characteristic method and where the foot is not a mesh point we use an interpolation:

$$f(t^n+\Delta t, x_j) = \Pi_h(f(t^n, x_j - \lambda \Delta t))$$

- SEE NOTEBOOK

Avdantage

- Very simple methof to obtain a CFLless scheme. Possibility to extend to High-Order.

Default

- All the waves are linearized with the same constant velocity : the maximal one. So the coefficient error for all the waves is λ_{max}.
- Not good for multi-scale problem.

Gas dynamic: Euler equations

- **Context**: Plasma simulation with Euler/MHD equations.
- **Euler equation**:

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}+pI_d)=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u}+p\boldsymbol{u})=0\end{array}\right.$$

- with $\rho(t,\mathbf{x})>0$ the density, $\boldsymbol{u}(t,\mathbf{x})$ the velocity and $E(t,\mathbf{x})>0$ the total energy.
- The pressure p is defined by $p=\rho T$ (perfect gas law) with T the temperature.
- **Hyperbolic system** with nonlinear waves. **Waves speed**: three eigenvalues: $(\boldsymbol{u},\mathbf{n})$ and $(\boldsymbol{u},\mathbf{n})\pm c$ with the sound speed $c^2=\gamma\frac{p}{\rho}$.

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Physic interpretation:

- **Two important velocity scales**: $\boldsymbol{u}$ and c and the ratio (Mach number) $M=\frac{|\boldsymbol{u}|}{c}$.
- When M tends to zero, we obtain incompressible Euler equation:

$$\left\{\begin{array}{l} \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho = 0 \\ \rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p_2 = 0 \\ \nabla\cdot\boldsymbol{u} = 0 \end{array}\right.$$

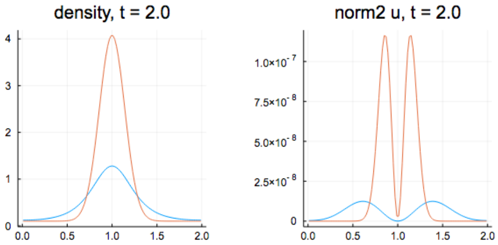
  In 1D we have just advection of ρ.
- **Aim**: contruct an Scheme (Ap) valid at the limit with a uniform cost.

Numerical difficulties in space: Finite volume

- VF method + Rusanov flux. Equivalent equation:

$$\left\{\begin{array}{l}\partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) = \dfrac{S\Delta x}{2}\Delta \rho \\ \rho\partial_t \boldsymbol{u} + \rho \boldsymbol{u}\cdot\nabla \boldsymbol{u} + \dfrac{1}{M^2}\nabla p = \dfrac{S\Delta x}{2}\Delta \boldsymbol{u} \\ \partial_t p + \boldsymbol{u}\cdot\nabla p + \gamma p \nabla\cdot \boldsymbol{u} = \dfrac{S\Delta x}{2}\Delta p\end{array}\right.$$

- **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.
- Example: isolated contact $p = 1$, $\nabla\cdot\boldsymbol{u}_0 = 0$ and $\boldsymbol{u}_0$ constant in time.
- Rusanov scheme $T_f = 2 \mid \boldsymbol{u}_0 \mid \approx 0.001$ and 100*100 cells.



- Red: exact solution, Blue: numerical solution.

Relaxation method

- **Problem**: the nonlinearity of the implicit acoustic step generates difficulties.
- Non conservative form and acoustic term:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t p + u\partial_x p + \rho c^2 \partial_x u = 0 \\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x p = 0 \end{array}\right.$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{array}\right.$$

- **Limit**:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = \varepsilon\partial_x\left[A\partial_x p\right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon\partial_x\left[(Au\partial_x p) + B\partial_x u\right] \\ \partial_t E + \partial_x(E u + p u) = \varepsilon\partial_x\left[AE\partial_x p + A\partial_x \frac{p^2}{2} + B\partial_x \frac{u^2}{2}\right] \end{array}\right.$$

- with $A = \frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)$ and $B = \left(\rho\phi\lambda^2 - \rho^2 c^2\right)$.
- **Stability**: $\phi\lambda > \rho c^2$ and $\rho > \phi$.

Avdantage

- We keep the conservative form for the original variables and obtain a fully linear acoustic.

Dynamical splitting

- **Splitting**: we solve sub-part of the system one by one. **Dynamic case**: Splitting time depending for low-mach [IDGH2018]
- For large acoustic waves (Mach number not small) we want capture all the phenomena. Consequently use an explicit scheme.
- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

Splitting: Explicit convective part/Implicit acoustic part.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho uv+\mathcal{M}^2(t)\Pi)=0\\ \partial_t E+\partial_x(Ev+\mathcal{M}^2(t)\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda_c^2\partial_x v=0\\ \partial_t v+v\partial_x v+\frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi=0\end{array}\right., \quad \left\{\begin{array}{l}\partial_t\rho=0\\ \partial_t(\rho u)+(1-\mathcal{M}^2(t))\partial_x\Pi=0\\ \partial_t E+(1-\mathcal{M}^2(t))\partial_x(\Pi v)=0\\ \partial_t\Pi+\phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v=0\\ \partial_t v+(1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi=0\end{array}\right.$$

with $\mathcal{M}(t)\approx max\left(\mathcal{M}_{min}, min\left(max_x\frac{|u|}{c},1\right)\right)$

- Eigenvalues of Explicit part: v, $v\pm\mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$. Implicit part 0, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$
- **At the end**: we make the projection $\Pi=p$ and $v=u$ (can be viewed as a discretization of the stiff source term).

Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

$$\left\{\begin{array}{l} \rho^{n+1}=\rho^n \\ (\rho u)^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\partial_x \Pi^{n+1}=(\rho u)^n \\ E^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\partial_x(\Pi v)^{n+1}=E^n \\ \Pi^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2\partial_x v^{n+1}=\Pi^n \\ v^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x\Pi^{n+1}=v^n \end{array}\right.$$

- We plug the equation on v in the equation on Π. We obtain the following algorithm:
  - Step 1: we solve

$$(I_d-(1-\mathcal{M}^2(t_n))^2\Delta t^2\lambda_a^2\partial_{xx})\Pi^{n+1}=\Pi^n-\Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2\partial_x v^n$$

  - Step 2: we compute

$$v^{n+1}=v^n-\Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x\Pi^{n+1}$$

  - Step 3: we compute

$$(\rho u)^{n+1}=(\rho u)^n-\Delta t(1-\mathcal{M}^2(t_n))\partial_x\Pi^{n+1}$$

  - Step 4: we compute

$$E^{n+1}=E^n-\Delta t(1-\mathcal{M}^2(t_n))\partial_x(\Pi^{n+1}v^{n+1})$$

Advantage

- We solve only a constant Laplacian. We can assembly matrix one time.
- No problem of conditioning, which comes from to the strong gradient of ρ

Results 1D I: contact

- Smooth contact :

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	Old relax Rusanov	Relax Rus	Relax PC-FVS
250	0.042	$3.6E^{-4}$	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

- **Old relax**: other relaxation scheme where the implicit Laplacian is not constant and depend of ρ^n.
- Comparison time scheme:

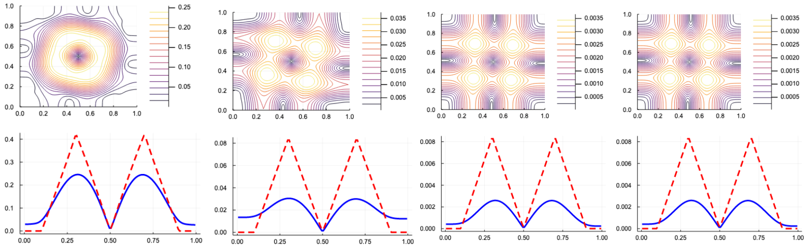
Scheme	λ	Δt
Explicit	$\max(\mid u-c \mid, \mid u+c \mid)$	$2.2E^{-4}$
SI Old relax	$\max(\mid u-\mathcal{M}(t_n))\frac{\lambda}{\rho} \mid, \mid u+\mathcal{M}(t_n))\frac{\lambda}{\rho} \mid)$	0.0075
SI new relaxation	$\max(\mid v-\mathcal{M}(t_n))\lambda \mid, \mid v+\mathcal{M}(t_n))\lambda \mid)$	0.04

- Conditioning:

Schemes	Δt	conditioning
Si old relax	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

Results in 2D: Gresho vortex

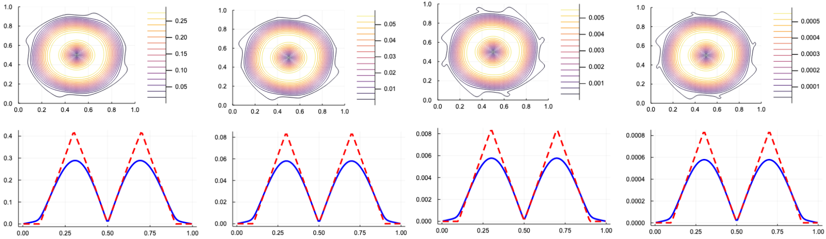
- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$



- Explicit Lagrange+remap scheme Norm of the velocity (2D plot). 1D initial (red) and final (blue) time .From left to right: $M_0 = 0.5$ ($\Delta t = 1.4E^{-3}$), $M_0 = 0.1$ ($\Delta t = 3.5E^{-4}$), $M_0 = 0.01$ ($\Delta t = 3.5E^{-5}$), $M_0 = 0.001$ ($\Delta t = 3.5E^{-6}$).

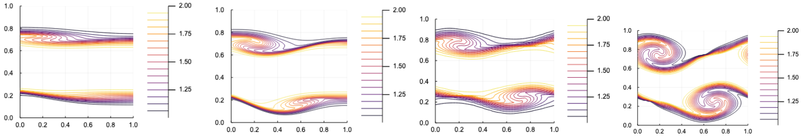
Results in 2D: Gresho vortex

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$



- Relaxation scheme. Norm of the velocity (2D plot). 1D initial (red) and final (blue) times. From left to right: $M = 0.5$, $\Delta t = 2.5E^{-3}$, $M = 0.1$, $\Delta t = 2.5E^{-3}$, $M = 0.01$, $\Delta t = 2.5E^{-3}$, $M = 0.001$, $\Delta t = 2.5E^{-3}$.

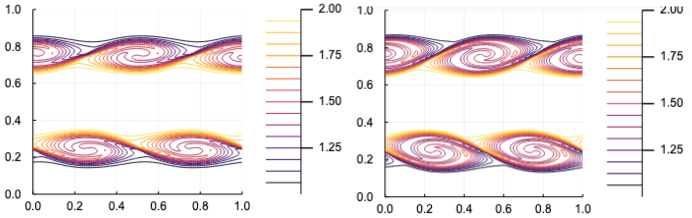
- kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 1$, $M_0 = 0.1$. Explicit Lagrange-Remap scheme with 120×120 (left) and 360×360 cells (middle left), SI two-speed relaxation scheme ($\lambda_c = 18$, $\lambda_a = 15$, $\phi = 0.98$) with 42×42 (middle right) and 120×120 cells (right).

Results in 2D: Kelvin helmholtz

- kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 2$, $M_0 = 0.01$ with SI two-speed relaxation scheme ($\lambda_c = 180$, $\lambda_a = 150$, $\phi = 0.98$). Left: 120×120 cells. Right: 240×240 cells.

Conclusion

Resume

- Using the Xin-Jin relaxation we obtain a implicit scheme with only N linear elliptic problems to solve and N matrix vector products.
- Using the kinetic relaxation we obtain a implicit scheme with $n_v > 2N$ transport equations solved without matrix using a SL scheme.
- Using the modified Suliciu relaxation we obtain a semi-implicit scheme with a linear/constant epllitpic problem to invert and 3 matrix-vector products. The scheme is AP in the low-Mach limit.

Conclusion

- The relaxation is a good tools to construct simpler implicit solvers: smaller, with good conditoning, without nonlinear iterator and sometimes with matrix invertion.
- **Other application**: Ripa model and WB semi-implicit scheme.