Semi Implicit Relaxation Scheme for Multi-Scale Fluid Problems

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Outline

Physical and mathematical context

Gas dynamic: Euler equations

- **Context**: Plasma simulation with Euler/MHD equations.
- **Euler equation**:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + pI_d) = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{u} + p \boldsymbol{u}) = 0 \end{cases}$$

- with $\rho(t, \mathbf{x}) > 0$ the density, $\boldsymbol{u}(t, \mathbf{x})$ the velocity and $E(t, \mathbf{x}) > 0$ the total energy.
- The pressure p is defined by $p = \rho T$ (perfect gas law) with T the temperature.

- **Hyperbolic system** with nonlinear waves. **Waves speed**: three eigenvalues: $(\boldsymbol{u}, \mathbf{n})$ and $(\boldsymbol{u}, \mathbf{n}) \pm c$ with the sound speed $c^2 = \gamma \frac{p}{\rho}$.

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- The pressure p is defined by $p=\rho T$ (perfect gas law) with T the temperature.
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Physic interpretation:

- **Two important velocity scales**: $\boldsymbol{u}$ and c and the ratio (Mach number) $M=\frac{|\boldsymbol{u}|}{c}$.
- When M tends to zero, we obtain incompressible Euler equation:

$$\left\{\begin{array}{l}\partial_t\rho+\boldsymbol{u}\cdot\nabla\rho=0\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p_2=0\\ \nabla\cdot\boldsymbol{u}=0\end{array}\right.$$

  In 1D we have just advection of ρ.
- **Aim**: contruct an Scheme (AP) valid at the limit with a uniform cost.

Finite Volumes

Finite Volumes is the natural method to solve hyperbolic systems.

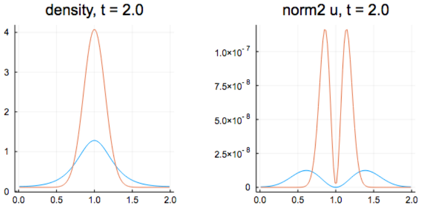
- Default of FV scheme. Consistency :

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta x(\partial_x D(\boldsymbol{U})\partial_x \boldsymbol{U}) + O(\Delta x^2)$$

- We consider $\boldsymbol{U}_M$ the solution at the low mach limit.
- The scheme can be considered as not adapted/adapted for this regime if

$$lim_{M\rightarrow 0} \mid D(\boldsymbol{U}_M) \mid \approx M^{-p}, \quad lim_{M\rightarrow 0} \mid D(\boldsymbol{U}_M) \mid < C$$

- Example: isolated contact $p = 1$, $\nabla \cdot \boldsymbol{u}_0 = 0$ and $\boldsymbol{u}_0$ constant in time.
- Rusanov scheme $T_f = 2 \mid \boldsymbol{u}_0 \mid \approx 0.001$ and 100*100 cells.



- Red: exact solution, Blue: numerical solution.

Numerical problem I: time discretization.

- **Explicit scheme**: the CFL condition for low mach flow:
  - The fast phenomena: acoustic waves at velocity c
  - The important phenomena: transport at velocity u
  - Expected CFL: $\Delta t < \frac{\Delta x}{|u|}$, CFL in practice $\Delta t < \frac{\Delta x}{|c|}$
  - At the end, we use a Δt divised by M compare to the expected Δt

First solution

Implicit time scheme. No CFL condition. Taking a larger time step, it allows to "filter" the fast acoustic waves which are not useful in the low-Mach regime.

- Implicit time scheme:

$$M_i \boldsymbol{U}^{n+1} = (I_d + \Delta t A(I_d))\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$$

- We must solve a nonlinear system and after linearization solve some linear systems.

Problem

- Direct solver too costly. Approximative conditioning for iterative solver:

$$k(M_i) \approx 1 + O\left(\frac{\Delta t}{\Delta x^p M}\right)$$

- We recover the two scales in the conditioning number. The full implicit schemes are difficult to use for this reason.

Numerical problem II: time discretization.

First idea: Semi implicit scheme

- We explicit the slow scale (transport) and implicit the fast scale (acoustic) [CDK12]-[DLVD19]

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x(\rho u) = 0 \\
\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\
\partial_t E + \partial_x(E u) + \partial_x(p u) = 0
\end{array}
\right.
$$

Implicit acoustic step:

$$
\left\{
\begin{array}{l}
\rho^{n+1} = \rho^n \\
(\rho u)^{n+1} = \rho^n u^n - \Delta t \partial_x p^{n+1} + Rhs_u \\
E^{n+1} = E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = Rhs_E
\end{array}
\right.
$$

Plugging this in the second equation, we obtain

$$
E^{n+1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho)
$$

- Matrix-vector product to compute u^{n+1}.

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$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = 0 \end{cases}$$

Implicit acoustic step:

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} = \rho^n u^n - \Delta t \partial_x p^{n+1} + Rhs_u \\ \frac{p^{n+1}}{\gamma-1} + \frac{1}{2}\rho^n u^n = E^n - \Delta t \partial_x(p^{n+1}u^{n+1}) = Rhs_E \end{cases}$$

Plugging this in the second equation, we obtain

$$\frac{p^{n+1}}{\gamma - 1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho^n)$$

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Plugging this in the second equation, we obtain

$$\frac{p^{n+1}}{\gamma - 1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho^n)$$

- Matrix-vector product to compute u^{n+1}.

Conclusion

- **Semi implicit**: only one scale in the implicit symmetric positive operator.
- Strong gradient of ρ generates ill-conditioning. Assembly at each time (costly).
- Nonlinear solver can have bad convergence for if $\Delta t >> 1$ and $\partial_x p$ not so small.

Relaxation method

Relaxation method I

- **Relaxation** [XJ95]-[CGS12]-[BCG18]: a way to linearize and decouple the equations. Used to design new schemes.
- **Idea**: Approximate the model

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = 0, \text{ by } \quad \partial_t \mathbf{f} + \mathbf{A}(\mathbf{f}) = \frac{1}{\varepsilon}(Q(\mathbf{f}) - \mathbf{f})$$

- At the limit and taking $P\mathbf{f} = \boldsymbol{U}$ we obtain

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = \varepsilon \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + O(\varepsilon^2)$$

- **Time scheme**:
  - we solve

$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \mathbf{A}(\mathbf{f}^{*,n}) = 0$$

  - and after we approximate the stiff source term by

$$\mathbf{f}^{n+1} = \mathbf{f}^* + \omega (Q(\mathbf{f}^*) - \mathbf{f}^*)$$

  with $\omega \in]0, 2]$.

Why ?

- In general, we construct $\mathbf{A}$ with a simpler structure than $\mathbf{F}$ to design numerical flux in FV.
- Here, we construct $\mathbf{A}$ with a simpler structure to design simple implicit scheme.

Relaxation method II

- **Problem**: the nonlinearity of the implicit acoustic step generates difficulties.
- Non conservative form and acoustic term:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t p+u\partial_x p+\rho c^2\partial_x u=0\\ \partial_t u+u\partial_x u+\frac{1}{\rho}\partial_x p=0\end{array}\right.$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho u v+\Pi)=0\\ \partial_t E+\partial_x(E v+\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda^2\partial_x v=\frac{1}{\varepsilon}(p-\Pi)\\ \partial_t v+v\partial_x v+\frac{1}{\phi}\partial_x\Pi=\frac{1}{\varepsilon}(u-v)\end{array}\right.$$

- **Limit**:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=\varepsilon\partial_x\left[A\partial_x p\right]\\ \partial_t(\rho u)+\partial_x(\rho u^2+p)=\varepsilon\partial_x\left[(Au\partial_x p)+B\partial_x u\right]\\ \partial_t E+\partial_x(Eu+pu)=\varepsilon\partial_x\left[AE\partial_x p+A\partial_x\frac{p^2}{2}+B\partial_x\frac{u^2}{2}\right]\end{array}\right.$$

- with $A=\frac{1}{\rho}\left(\frac{\rho}{\phi}-1\right)$ and $B=\left(\rho\phi\lambda^2-\rho^2c^2\right)$.
- **Stability**: $\phi\lambda>\rho c^2$ and $\rho>\phi$.

Avdantage

- We keep the conservative form for the original variables and obtain a fully linear acoustic.

Dynamical splitting

- **Splitting**: we solve sub-part of the system one by one. **Dynamic case**: Splitting time depending for low-mach [IDGH2018]
- For large acoustic waves (Mach number not small) we want capture all the phenomena. Consequently use an explicit scheme.
- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

Splitting: Explicit convective part/Implicit acoustic part.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho uv+\mathcal{M}^2(t)\Pi)=0\\ \partial_t E+\partial_x(Ev+\mathcal{M}^2(t)\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda_c^2\partial_x v=0\\ \partial_t v+v\partial_x v+\frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi=0\end{array}\right. ,\quad \left\{\begin{array}{l}\partial_t\rho=0\\ \partial_t(\rho u)+(1-\mathcal{M}^2(t))\partial_x\Pi=0\\ \partial_t E+(1-\mathcal{M}^2(t))\partial_x(\Pi v)=0\\ \partial_t\Pi+\phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v=0\\ \partial_t v+(1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi=0\end{array}\right.$$

with $\mathcal{M}(t)\approx max\left(\mathcal{M}_{min}, min\left(max_x\frac{|u|}{c},1\right)\right)$

- Eigenvalues of Explicit part: v, $v\pm\mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$. Implicit part 0, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$
- **At the end**: we make the projection $\Pi=p$ and $v=u$ (can be viewed as a discretization of the stiff source term).

Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

$$\left\{\begin{array}{l} \rho^{n+1}=\rho^n \\ (\rho u)^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\partial_x\Pi^{n+1}=(\rho u)^n \\ E^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\partial_x(\Pi v)^{n+1}=E^n \\ \Pi^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2\partial_x v^{n+1}=\Pi^n \\ v^{n+1}+\Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x\Pi^{n+1}=v^n \end{array}\right.$$

- We plug the equation on v in the equation on Π. We obtain the following algorithm:
  - Step 1: we solve

$$(I_d-(1-\mathcal{M}^2(t_n))^2\Delta t^2\lambda_a^2\partial_{xx})\Pi^{n+1}=\Pi^n-\Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2\partial_x v^n$$

  - Step 2: we compute

$$v^{n+1}=v^n-\Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x\Pi^{n+1}$$

  - Step 3: we compute

$$(\rho u)^{n+1}=(\rho u)^n-\Delta t(1-\mathcal{M}^2(t_n))\partial_x\Pi^{n+1}$$

  - Step 4: we compute

$$E^{n+1}=E^n-\Delta t(1-\mathcal{M}^2(t_n))\partial_x(\Pi^{n+1}v^{n+1})$$

Advantage

- We solve only a constant Laplacian. We can assembly matrix one time.
- No problem of conditioning, which comes from to the strong gradient of ρ

Spatial scheme in 1D

- **Idea**: FV Godunov fluxes for the explicit part + Central fluxes for the implicit part.
- Main problem of the explicit part: design numerical flux.
- **First possibility**: since the maximal eigenvalue is $O(Mach)$ a Rusanov scheme.
- Other solution: construct a Godunov scheme for the relaxation system. Principle:
  - eigenvalues: $v-\mathcal{E}(t)\lambda_c, v(x3), v+\mathcal{E}(t)\lambda_c$
  - Strong invariants of external waves:

$$\partial_t(v\pm\phi\lambda_c\pi)+(v\pm\mathcal{E}(t)\lambda_c)\partial_x(v\pm\phi\lambda_c\pi)=0$$

  - Strong invariants of central wave:

$$\partial_t\left(\frac{1}{\rho}+\frac{\pi}{\rho\phi\lambda^2}\right)+v\partial_x\left(\frac{1}{\rho}+\frac{\pi}{\rho\phi\lambda^2}\right)=0$$

$$\partial_t\left(u-\frac{\phi}{\rho}v\right)+v\partial_x\left(u-\frac{\phi}{\rho}v\right)=0$$

$$\partial_t\left(\rho e+\frac{\pi^2}{2\rho\phi\lambda_c^2}+\frac{(v-u)^2}{2(\frac{\rho}{\phi}-1)}\right)+v\partial_x\left(\rho e+\frac{\pi^2}{2\rho\phi\lambda_c^2}+\frac{(v-u)^2}{2(\frac{\rho}{\phi}-1)}\right)=0$$

  - **Important**: strong invariant are weak invariant (conserved) on other wave. **Exemple**: (π,v) preserved on central wave.
  - We obtain all the intermdiary states using these previous result.

Results 1D I: contact

- Smooth contact :

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	Old relax Rusanov	Relax Rus	Relax PC-FVS
250	0.042	$3.6E^{-4}$	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

- **Old relax**: other relaxation scheme where the implicit Laplacian is not constant and depend of ρ^n.
- Comparison time scheme:

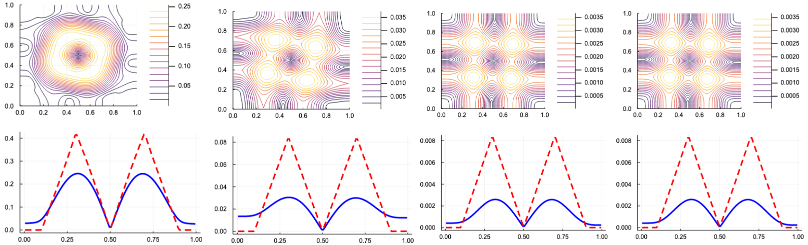
Scheme	λ	Δt
Explicit	$\max(\mid u-c \mid, \mid u+c \mid)$	$2.2E^{-4}$
SI Old relax	$\max(\mid u-\mathcal{M}(t_n))\frac{\lambda}{\rho} \mid, \mid u+\mathcal{M}(t_n))\frac{\lambda}{\rho} \mid)$	0.0075
SI new relaxation	$\max(\mid v-\mathcal{M}(t_n))\lambda \mid, \mid v+\mathcal{M}(t_n))\lambda \mid)$	0.04

- Conditioning:

Schemes	Δt	conditioning
Si old relax	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

Results in 2D: Gresho vortex

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$



- Explicit Lagrange+remap scheme Norm of the velocity (2D plot). 1D initial (red) and final (blue) time .From left to right: $M_0 = 0.5$ ($\Delta t = 1.4E^{-3}$), $M_0 = 0.1$ ($\Delta t = 3.5E^{-4}$), $M_0 = 0.01$ ($\Delta t = 3.5E^{-5}$), $M_0 = 0.001$ ($\Delta t = 3.5E^{-6}$).

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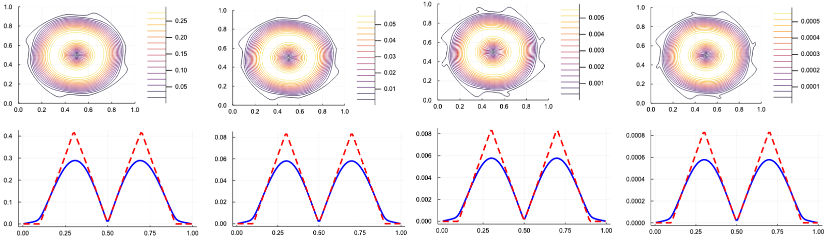
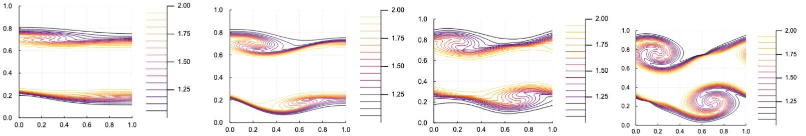


- Relaxation scheme. Norm of the velocity (2D plot). 1D initial (red) and final (blue) times. From left to right: $M = 0.5$, $\Delta t = 2.5E^{-3}$, $M = 0.1$, $\Delta t = 2.5E^{-3}$, $M = 0.01$, $\Delta t = 2.5E^{-3}$, $M = 0.001$, $\Delta t = 2.5E^{-3}$.

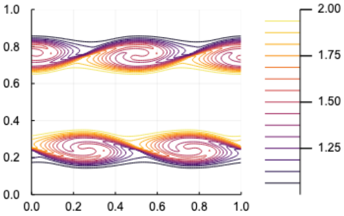
Results in 2D: Kelvin helmholtz

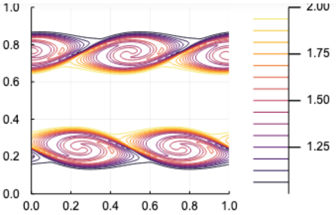
- kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 1$, $M_0 = 0.1$. Explicit Lagrange-Remap scheme with 120×120 (left) and 360×360 cells (middle left), SI two-speed relaxation scheme ($\lambda_c = 18$, $\lambda_a = 15$, $\phi = 0.98$) with 42×42 (middle right) and 120×120 cells (right).

- kelvin-Helmholtz instability. Density:





- Density at time $T_f = 3$, $k = 2$, $M_0 = 0.01$ with SI two-speed relaxation scheme ($\lambda_c = 180$, $\lambda_a = 150$, $\phi = 0.98$). Left: 120×120 cells. Right: 240×240 cells.

Conclusion

Resume

- Introducing Dynamic splitting scheme we separate the scales.
- Introducing implicit scheme for the acoustic wave we can filter these waves.
- Introducing relaxation we simplify at the maximum the implicit scheme.
- A well-adapted spatial scheme is also very important.
- **At the end**: we capture the incompressible limit.

Perspectives:

- **To avoid some spurious mods**: Use compatible discretization for the linear wave part (mimetic/staggered DF, compatible finite element).
- Extension to High Order, MUSCL firstly and after DG and HDG schemes.
- Extension to Shallow-Water/Ripa models and MHD (main goal). For MHD the relaxation it is ok but the splitting is less clear.