Semi Implicit Relaxation Scheme for Multi-Scale Fluid Problems

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Physical and mathematical context

Relaxation method
Physical and mathematical context
Gas dynamic: Euler equations

- **Context:** Plasma simulation with Euler/MHD equations.

- **Euler equation:**

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{l}_d) &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{u} + p \mathbf{u}) &= 0
\end{align*}
\]

- with \( \rho(t, x) > 0 \) the density, \( \mathbf{u}(t, x) \) the velocity and \( E(t, x) > 0 \) the total energy.

- The pressure \( p \) is defined by \( p = \rho T \) (perfect gas law) with \( T \) the temperature.

- **Hyperbolic system** with nonlinear waves. **Waves speed:** three eigenvalues: \( (\mathbf{u}, \mathbf{n}) \) and \( (\mathbf{u}, \mathbf{n}) \pm c \) with the sound speed \( c^2 = \gamma \frac{p}{\rho} \).
Gas dynamic: Euler equations

- **Context:** Plasma simulation with Euler/MHD equations.

- **Euler equation:**
  \[
  \begin{align*}
  \partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
  \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + pl_d) &= 0 \\
  \partial_t E + \nabla \cdot (E u + pu) &= 0
  \end{align*}
  \]

- with \( \rho(t, x) > 0 \) the density, \( u(t, x) \) the velocity and \( E(t, x) > 0 \) the total energy.

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### Physic interpretation:

- **Two important velocity scales:** \( u \) and \( c \) and the ratio (Mach number) \( M = \frac{|u|}{c} \).

- When \( M \) tends to zero, we obtain incompressible Euler equation:
  \[
  \begin{align*}
  \partial_t \rho + u \cdot \nabla \rho &= 0 \\
  \rho \partial_t u + \rho u \cdot \nabla u + \nabla p &= 0 \\
  \nabla \cdot u &= 0
  \end{align*}
  \]

  In 1D we have just advection of \( \rho \).

- **Aim:** construct an Scheme (AP) valid at the limit with a uniform cost.
Finite Volumes is the natural method to solve hyperbolic systems.

- Default of FV scheme. Consistency:
  \[ \partial_t U + \partial_x F(U) = \Delta x (\partial_x D(U) \partial_x U) + O(\Delta x^2) \]

- We consider \( U_M \) the solution at the low mach limit.
- The scheme can be considered as not adapted/adapted for this regime if
  \[ \lim_{M \to 0} |D(U_M)| \approx M^{-p}, \quad \lim_{M \to 0} |D(U_M)| < C \]

- Example: isolated contact \( p = 1, \nabla \cdot u_0 = 0 \) and \( u_0 \) constant in time.
- Rusanov scheme \( T_f = 2 |u_0| \approx 0.001 \) and 100*100 cells.

Red: exact solution, Blue: numerical solution.
Numerical problem I: time discretization.

- **Explicit scheme**: the CFL condition for low mach flow:
  - The fast phenomena: acoustic waves at velocity $c$
  - The important phenomena: transport at velocity $u$
  - Expected CFL: $\Delta t < \frac{\Delta x}{|u|}$, CFL in practice $\Delta t < \frac{\Delta x}{|c|}$
  - At the end, we use a $\Delta t$ divided by $M$ compared to the expected $\Delta t$

**First solution**

Implicit time scheme. No CFL condition. Taking a larger time step, it allows to "filter" the fast acoustic waves which are not useful in the low-Mach regime.

- Implicit time scheme:
  
  $M_i U^{n+1} = (I_d + \Delta t A(I_d)) U^{n+1} = U^n$

  - We must solve a nonlinear system and after linearization solve some linear systems.

**Problem**

- Direct solver too costly. Approximative conditioning for iterative solver:
  
  $k(M_i) \approx 1 + O \left( \frac{\Delta t}{\Delta x^p M} \right)$

  - We recover the two scales in the conditioning number. The full implicit schemes are difficult to use for this reason.
Numerical problem II: time discretization.

First idea: Semi implicit scheme

- We explicit the slow scale (transport) and implicit the fast scale (acoustic) [CDK12]-[DLVD19]

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p &= 0 \\
\partial_t E + \partial_x (Eu) + \partial_x (pu) &= 0
\end{align*}
\]

Implicit acoustic step:

\[
\begin{align*}
\rho^{n+1} &= \rho^n \\
(\rho u)^{n+1} &= \rho^n u^n - \Delta t \partial_x p^{n+1} + \text{Rhs}_u \\
E^{n+1} &= E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = \text{Rhs}_E
\end{align*}
\]

Plugging this in the second equation, we obtain

\[
E^{n+1} - \Delta t^2 \partial_x \left( \frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = \text{Rhs}(E^n, u^n, \rho)
\]

- Matrix-vector product to compute \(u^{n+1}\).
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p^{n+1}_{\gamma-1} + \frac{1}{2} \rho^n u^n &= E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = Rhs_E
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\frac{p^{n+1}}{\gamma - 1} - \Delta t^2 \partial_x \left( \frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho^n)
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\]

- Matrix-vector product to compute \( u^{n+1} \).

Conclusion

- **Semi implicit**: only one scale in the implicit symmetric positive operator.
- Strong gradient of \( \rho \) generates ill-conditioning. Assembly at each time (costly).
- Nonlinear solver can have bad convergence for if \( \Delta t \gg 1 \) and \( \partial_x p \) not so small.
Relaxation method
Relaxation method I

- **Relaxation** [XJ95]-[CGS12]-[BCG18]: a way to linearize and decouple the equations. Used to design new schemes.

- **Idea:** Approximate the model

  \[ \partial_t U + \partial_x F(U) = 0, \text{ by } \partial_t f + A(f) = \frac{1}{\varepsilon} (Q(f) - f) \]

  At the limit and taking \( Pf = U \) we obtain

  \[ \partial_t U + \partial_x F(U) = \varepsilon \partial_x (D(U) \partial_x U) + O(\varepsilon^2) \]

- **Time scheme:**
  - we solve

    \[ \frac{f^* - f^n}{\Delta t} + A(f^{*,n}) = 0 \]

  - and after we approximate the stiff source term by

    \[ f^{n+1} = f^* + \omega (Q(f^*) - f^*) \]

    with \( \omega \in ]0, 2] \).

**Why ?**

- In general, we construct \( A \) with a simpler structure than \( F \) to design numerical flux in FV.

- Here, we construct \( A \) with a simpler structure to design simple implicit scheme.
Relaxation method II

- **Problem:** the nonlinearity of the implicit acoustic step generates difficulties.
- Non conservative form and acoustic term:

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= 0 \\
\frac{\partial}{\partial t} p + u \frac{\partial}{\partial x} p + \rho c^2 \frac{\partial}{\partial x} u &= 0 \\
\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + \frac{1}{\rho} \frac{\partial}{\partial x} p &= 0
\end{align*}
\]

- **Idea:** Relax only the acoustic part ([BCG18]) to linearize the implicit part.

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v) &= 0 \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u v + \Pi) &= 0 \\
\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} (E v + \Pi v) &= 0 \\
\frac{\partial}{\partial t} \Pi + v \frac{\partial}{\partial x} \Pi + \phi \lambda^2 \frac{\partial}{\partial x} v &= \frac{1}{\varepsilon} (p - \Pi) \\
\frac{\partial}{\partial t} v + v \frac{\partial}{\partial x} v + \frac{1}{\phi} \frac{\partial}{\partial x} \Pi &= \frac{1}{\varepsilon} (u - v)
\end{align*}
\]

- **Limit:**

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= \varepsilon \frac{\partial}{\partial x} [A \frac{\partial}{\partial x} p] \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) &= \varepsilon \frac{\partial}{\partial x} [(A u \frac{\partial}{\partial x} p) + B \frac{\partial}{\partial x} u] \\
\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} (E u + p u) &= \varepsilon \frac{\partial}{\partial x} [AE \frac{\partial}{\partial x} p + A \frac{\partial}{\partial x} \frac{p^2}{2} + B \frac{\partial}{\partial x} \frac{u^2}{2}]
\end{align*}
\]

with \( A = \frac{1}{\rho} \left( \frac{\rho}{\phi} - 1 \right) \) and \( B = (\rho \phi \lambda^2 - \rho^2 c^2) \).

- **Stability:** \( \phi \lambda > \rho c^2 \) and \( \rho > \phi \).

**Advantage**

- We keep the conservative form for the original variables and obtain a **fully linear acoustic**.
Splitting

Dynamical splitting

- **Splitting**: we solve sub-part of the system one by one. **Dynamic case**: Splitting time depending for low-mach [IDGH2018]

- For large acoustic waves (Mach number not small) we want capture all the phenomena. Consequently use an explicit scheme.

- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

**Splitting**: Explicit convective part/Implicit acoustic part.

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho uv + M^2(t) \Pi) &= 0 \\
\partial_t E + \partial_x (E v + M^2(t) \Pi v) &= 0 \\
\partial_t \Pi + v \partial_x \Pi + \phi \lambda_c^2 \partial_x v &= 0 \\
\partial_t v + v \partial_x v + \frac{M^2(t)}{\phi} \partial_x \Pi &= 0
\end{align*}
\]

\[
\begin{align*}
\partial_t \rho &= 0 \\
\partial_t (\rho u) + (1 - M^2(t)) \partial_x \Pi &= 0 \\
\partial_t E + (1 - M^2(t)) \partial_x (\Pi v) &= 0 \\
\partial_t \Pi + \phi (1 - M^2(t)) \lambda_a^2 \partial_x v &= 0 \\
\partial_t v + (1 - M^2(t)) \frac{1}{\phi} \partial_x \Pi &= 0
\end{align*}
\]

with \( M(t) \approx \max \left( M_{\text{min}}, \min \left( \max_x \frac{|u|}{c}, 1 \right) \right) \)

- Eigenvalues of Explicit part: \( v, v \pm M(t) \lambda_c \). Implicit part 0, \( \pm (1 - M^2(t)) \lambda_a \approx c \)

- **At the end**: we make the projection \( \Pi = p \) and \( v = u \) (can be viewed as a discretization of the stiff source term).
Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

\[
\begin{align*}
\rho^{n+1} &= \rho^n \\
(\rho u)^{n+1} + \Delta t(1 - M^2(t_n)) \partial_x \Pi^{n+1} &= (\rho u)^n \\
E^{n+1} + \Delta t(1 - M^2(t_n)) \partial_x (\Pi v)^{n+1} &= E^n \\
\Pi^{n+1} + \Delta t(1 - M^2(t_n)) \phi \lambda^2 \phi \partial_x v^{n+1} &= \Pi^n \\
v^{n+1} + \Delta t(1 - M^2(t_n)) \frac{1}{\phi} \partial_x \Pi^{n+1} &= v^n
\end{align*}
\]

- We plug the equation on \( v \) in the equation on \( \Pi \). We obtain the following algorithm:
  - Step 1: we solve
    \[
    (l_d - (1 - M^2(t_n))^2 \Delta t^2 \lambda^2 \partial_{xx}) \Pi^{n+1} = \Pi^n - \Delta t(1 - M^2(t_n)) \phi \lambda^2 \partial_x v^n
    \]
  - Step 2: we compute
    \[
    v^{n+1} = v^n - \Delta t(1 - M^2(t_n)) \frac{1}{\phi} \partial_x \Pi^{n+1}
    \]
  - Step 3: we compute
    \[
    (\rho u)^{n+1} = (\rho u)^n - \Delta t(1 - M^2(t_n)) \partial_x \Pi^{n+1}
    \]
  - Step 4: we compute
    \[
    E^{n+1} = E^n - \Delta t(1 - M^2(t_n)) \partial_x (\Pi^{n+1} v^{n+1})
    \]

Advantage

- We solve only a constant Laplacian. We can assembly matrix one time.
- No problem of conditioning, which comes from to the strong gradient of \( \rho \)
Spatial scheme in 1D

- **Idea**: FV Godunov fluxes for the explicit part + Central fluxes for the implicit part.
- Main problem of the explicit part: design numerical flux.
- **First possibility**: since the maximal eigenvalue is $O(Mach)$ a Rusanov scheme.
- Other solution: construct a Godunov scheme for the relaxation system. Principle:
  - eigenvalues: $\nu - \mathcal{E}(t)\lambda_c$, $\nu(x3)$, $\nu + \mathcal{E}(t)\lambda_c$
  - Strong invariants of external waves:
    \[
    \partial_t(\nu \pm \phi\lambda_c\pi) + (\nu \pm \mathcal{E}(t)\lambda_c)\partial_x(\nu \pm \phi\lambda_c\pi) = 0
    \]
  - Strong invariants of central wave:
    \[
    \partial_t\left(\frac{1}{\rho} + \frac{\pi}{\rho\phi\lambda^2}\right) + \nu\partial_x\left(\frac{1}{\rho} + \frac{\pi}{\rho\phi\lambda^2}\right) = 0
    \]
    \[
    \partial_t\left(u - \frac{\phi}{\rho}\nu\right) + \nu\partial_x\left(u - \frac{\phi}{\rho}\nu\right) = 0
    \]
    \[
    \partial_t\left(\rho e + \frac{\pi^2}{2\rho\phi\lambda^2} + \frac{(v - u)^2}{2(\frac{\rho}{\phi} - 1)}\right) + \nu\partial_x\left(\rho e + \frac{\pi^2}{2\rho\phi\lambda^2} + \frac{(v - u)^2}{2(\frac{\rho}{\phi} - 1)}\right) = 0
    \]
- **Important**: strong invariant are weak invariant (conserved) on other wave.
  - **Exemple**: $(\pi, \nu)$ preserved on central wave.
- We obtain all the intermediary states using these previous result.
Results 1D I: contact

- **Smooth contact:**
  \[
  \begin{align*}
  \rho(t, x) &= \chi_{x<x_0} + 0.1\chi_{x>x_0} \\
  u(t, x) &= 0.01 \\
  p(t, x) &= 1
  \end{align*}
  \]

- **Error**

<table>
<thead>
<tr>
<th>cells</th>
<th>Ex Rusanov</th>
<th>Ex LR</th>
<th>Old relax Rusanov</th>
<th>Relax Rus</th>
<th>Relax PC-FVS</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>0.042</td>
<td>3.6E^{-4}</td>
<td>1.4E^{-3}</td>
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<td>4.1E^{-4}</td>
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<tr>
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<td>0.024</td>
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<td>6.9E^{-4}</td>
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<td>2.0E^{-4}</td>
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<tr>
<td>1000</td>
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<td>2000</td>
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<td>1.7E^{-4}</td>
<td>9.8E^{-5}</td>
<td>4.9E^{-5}</td>
</tr>
</tbody>
</table>

- **Old relax:** other relaxation scheme where the implicit Laplacian is not constant and depend of \( \rho^n \).

- **Comparison time scheme:**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( \lambda )</th>
<th>( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit</td>
<td>( \max(</td>
<td>u-c</td>
</tr>
<tr>
<td>SI Old relax</td>
<td>( \max(</td>
<td>u - \mathcal{M}(t_n)) \frac{\lambda}{\rho}</td>
</tr>
<tr>
<td>SI new relaxation</td>
<td>( \max(</td>
<td>v - \mathcal{M}(t_n)) \lambda</td>
</tr>
</tbody>
</table>

- **Conditioning:**

<table>
<thead>
<tr>
<th>Schemes</th>
<th>( \Delta t )</th>
<th>conditioning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si old relax</td>
<td>0.00757</td>
<td>3000</td>
</tr>
<tr>
<td>Si new relax</td>
<td>0.041</td>
<td>9800</td>
</tr>
<tr>
<td>Si new relax</td>
<td>0.0208</td>
<td>2400</td>
</tr>
<tr>
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<td>0.0075</td>
<td>320</td>
</tr>
</tbody>
</table>
Results in 2D: Gresho vortex

- Gresho vortex: $\nabla \cdot \mathbf{u} = 0$ and $p = \frac{1}{M^2} + p_2(x)$

- Explicit Lagrange+remap scheme
  Norm of the velocity (2D plot). 1D initial (red) and final (blue) time.
  - From left to right: $M_0 = 0.5$ ($\Delta t = 1.4E^{-3}$), $M_0 = 0.1$ ($\Delta t = 3.5E^{-4}$), $M_0 = 0.01$ ($\Delta t = 3.5E^{-5}$), $M_0 = 0.001$ ($\Delta t = 3.5E^{-6}$).
Results in 2D: Gresho vortex

- Gresho vortex: \( \nabla \cdot \mathbf{u} = 0 \) and \( p = \frac{1}{M^2} + p_2(x) \)

- Relaxation scheme. Norm of the velocity (2D plot). 1D initial (red) and final (blue) times. From left to right: \( M = 0.5, \Delta t = 2.5 \times 10^{-3} \), \( M = 0.1, \Delta t = 2.5 \times 10^{-3} \), \( M = 0.01, \Delta t = 2.5 \times 10^{-3} \), \( M = 0.001, \Delta t = 2.5 \times 10^{-3} \).
Results in 2D: Kelvin helmholtz

- kelvin-Helmholtz instability. Density:

- Density at time $T_f = 3$, $k = 1$, $M_0 = 0.1$. Explicit Lagrange-Remap scheme with $120 \times 120$ (left) and $360 \times 360$ cells (middle left), SI two-speed relaxation scheme ($\lambda_c = 18$, $\lambda_a = 15$, $\phi = 0.98$) with $42 \times 42$ (middle right) and $120 \times 120$ cells (right).
Results in 2D: Kelvin helmholtz

- kelvin-Helmholtz instability. Density:

- Density at time $T_f = 3$, $k = 2$, $M_0 = 0.01$ with SI two-speed relaxation scheme ($\lambda_c = 180$, $\lambda_a = 150$, $\phi = 0.98$). Left: $120 \times 120$ cells. Right: $240 \times 240$ cells.
Conclusion

Resume

- Introducing **Dynamic splitting scheme** we separate the scales.
- Introducing **implicit scheme** for the acoustic wave we can filter these waves.
- Introducing **relaxation** we simplify at the maximum the implicit scheme.
- A well-adapted spatial scheme is also very important.
- **At the end**: we capture the incompressible limit.

Perspectives:

- **To avoid some spurious mods**: Use compatible discretization for the linear wave part (mimetic/staggered DF, compatible finite element).
- Extension to **High Order**, MUSCL firstly and after DG and HDG schemes.
- Extension to Shallow-Water/Ripa models and **MHD (main goal)**. For MHD the relaxation it is ok but the splitting is less clear.