Reliability/survival analysis of semi-Markov systems: modeling and estimation

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Plan

Infinite matrices

Semi-Markov kernel and convolution

Markov renewal equation

Applications in reliability/survival analysis

Nonparametric estimation
This talk:

- **semi-Markov processes** in discrete time
- SM processes - important generalization with respect to Markov processes
- the state space - infinitely countable
- need of considering **infinite matrices**
- some domains of application: reliability, survival analysis
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Infinite matrices:

- rather few references and works on the topic
- Cooke (1955), Kemeny, Snell, and Knapp (1976)

Consider: a random system with countable state space, 
\[ E = \{1, \ldots, s\}, \ s < \infty, \text{ or } E = \mathbb{N} - \{0\} \]

Let us denote by:

- \( \mathbb{M}_E \) - set of real matrices on \( E \times E \)
- \( \mathbb{M}_E^b = \{A = (A_{ij})_{i,j \in E} \in \mathbb{M}_E | \exists M, 0 < M < \infty, \text{ such that } \ |A_{ij}| \leq M, \ i,j \in E \} \)
- \( \mathbb{M}_E^{sub} = \{\sum_{i=1}^{n} \lambda_i A_i | A_1, \ldots, A_n \text{ substochastic matrices, } \lambda_1, \ldots, \lambda_n \in \mathbb{R}, n \in \mathbb{N}\} \).
Infinite matrices:

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- \( \mathcal{M}^{\text{sub}}_E = \{ \sum_{i=1}^{n} \lambda_i A_i \mid A_1, \ldots, A_n \text{ substochastic matrices, } \lambda_1, \ldots, \lambda_n \in \mathbb{R}, n \in \mathbb{N} \}\).

Operations with infinite matrices:

1. addition - well defined; denote by \( 0_E \in \mathcal{M}_E \) the zero matrix;
2. multiplication of a matrix by a real number - well defined;
3. for \( A \) and \( B \in \mathcal{M}_E \), the product matrix \( AB \) is defined as usual if \( \sum_{k \in E} A_{ik} B_{kj} \) is well defined and finite for all \( i,j \in E \).
Remarks

- Obviously, if $A, B \in \mathcal{M}_E$ have arbitrary real entries, their product is not always well defined.
- The product of two bounded matrices is not always well defined.
- For $A$ and $B \in \mathcal{M}^{sub}_E$, the product $AB$ is in $\mathcal{M}^{sub}_E$, so it is well defined, with finite entries.
- If $A \in \mathcal{M}^{sub}_E$ and $B \in \mathcal{M}^b_E$, then the product $AB$ is well defined, with finite entries.
- The problem: the associativity of matrix multiplication does not always hold for infinite matrices.
- Note also that the uniqueness of the inverse rests upon associativity.
Remarks

- Obviously, if $A, B \in \mathcal{M}_E$ have arbitrary real entries, their product is not always well defined.
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- For $A$ and $B \in \mathcal{M}^{\text{sub}}_E$, the product $AB$ is in $\mathcal{M}^{\text{sub}}_E$, so it is well defined, with finite entries.
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- Note also that the uniqueness of the inverse rests upon associativity.

Lemma (Kemeny et al., 1976)

1. Nonnegative matrices associate under multiplication.
2. Matrices associate if the product of their absolute values has only finite entries.
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\( Z = (Z_k)_{k \in \mathbb{N}}, \) chain with state space \( E = \{1, 2, \ldots, s\} \)

\( S = (S_n)_{n \in \mathbb{N}}, \) jump times

\( J = (J_n)_{n \in \mathbb{N}}, \) visited states

\( X = (X_n)_{n \in \mathbb{N}}, \) sojourn times of \( Z \)

Figure: A sample path of a semi-Markov chain
If \((J, S)\) verifies:

\[
\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_0, \cdots, J_n; S_1, \cdots, S_n) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_n), j \in E, k \in \mathbb{N}
\]

- \((J, S)\) Markov renewal chain (MRC)
- \(Z = (Z_k)_{k \in \mathbb{N}}\) semi-Markov chain (SMC) associated to the MRC \((J, S)\)
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- \((J, S)\) Markov renewal chain (MRC)
- \(Z = (Z_k)_{k \in \mathbb{N}}\) semi-Markov chain (SMC) associated to the MRC \((J, S)\)

\[Z_k := J_{N(k)} \iff J_n = Z_{S_n}\]

with \(N(k) := \max\{n \in \mathbb{N} \mid S_n \leq k\}, k, n \in \mathbb{N}\)

Remark: \(J = (J_n)_{n \in \mathbb{N}}\) is a Markov chain, called the embedded Markov chain (EMC).
the initial distribution $\alpha(i) := \mathbb{P}(J_0 = i)$
the homogeneous SM kernel $\mathbf{q} = (q_{ij}(\cdot))_{i,j \in E}$

$q_{ij}(k) := \begin{cases} 
\mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i), & k \in \mathbb{N}^* \\
0, & k = 0
\end{cases}$
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0, & k = 0 
\end{cases}$$

the conditional sojourn time distributions $f = (f_{ij}(\cdot))_{i,j \in E}$

$$f_{ij}(k) := \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n+1} = j), \quad f_{ij}(0) := 0$$

the transition matrix of the MC $(J_n)_{n \in \mathbb{N}}$, $p = (p_{ij})_{i,j \in E}$

$$p_{ij} := \mathbb{P}(J_{n+1} = j \mid J_n = i), \quad p_{ii} := 0$$

Note that $q_{ij}(k) = p_{ij} f_{ij}(k)$
\[ M_E(\mathbb{N}) := \{ f : \mathbb{N} \to M_E \} \]
\[ M_E^b(\mathbb{N}) := \{ f : \mathbb{N} \to M_E^b \} \]
\[ M_E^{sub}(\mathbb{N}) := \{ f : \mathbb{N} \to M_E^{sub} \} \]

**Operations with** \( A, B \in M_E(\mathbb{N}) \):

» \( A \ast B = \) the discrete-time matrix convolution product,

\[
AB(k) := \sum_{l=0}^{k} A(k - l) B(l), \quad k \in \mathbb{N},
\]

provided that all the matrix products \( A(k - l) B(l), k \in \mathbb{N}, l = 0, \ldots, k \), are well defined and all their entries are finite.

» \( \delta I = \) the identity element, defined by \( \delta I(0) := I \) and \( I(k) := 0 \) if \( k \neq 0 \)

» \( A^{(n)} = \) the \( n \)-fold convolution of \( A \), provided that \( A \) is self-associative

» \( A^{-1} = \) the left convolution inverse of \( A \) (if it exists)
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Problem: solve equations of the type

\[ L(k) = G(k) + q \ast L(k), \quad k \in \mathbb{N}, \tag{1} \]

with

- \( G \in \mathcal{M}_E(\mathbb{N}) \) known
- \( L \in \mathcal{M}_E(\mathbb{N}) \) unknown.

Theorem

Suppose that \( L \) and \( G \in \mathcal{M}_2(\mathbb{N}) \), where \( \mathcal{M}_2 = \{ A = (A_{ij})_{i,j \in E} \in \mathcal{M}_E \mid (A_{ij})_{i \in E} \in l^2 \text{ for all } j \in E, (A_{ij})_{j \in E} \in l^2 \text{ for all } i \in E \} \),

\( l^2 = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R}, \sum_{n \geq 0} x_n^2 < \infty \} \). Then, the Markov renewal equation (1) has the unique solution

\[ L(k) = (\delta I - q)^{(-1)} \ast G(k) = \left( \sum_{n=0}^{k} q^{(n)} \right) \ast G(k). \]
Theorem (asymptotic behavior)

Let $L$ and $G \in \mathcal{M}_2(\mathbb{N})$ such that $\sum_{i \in E} \sum_{n \in \mathbb{N}} |G_{ij}(n)| < \infty$. Then:

$$L_{ij}(k) = (\psi \ast G)_{ij}(k) \xrightarrow[k \to \infty]{} \sum_{i \in E} \sum_{n \in \mathbb{N}} \frac{1}{\mu_{ii}} G_{ij}(n),$$

where $\mu_{ii}$ is the mean recurrence time of state $i$ for the semi-Markov chain.
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where $\mu_{ii}$ is the mean recurrence time of state $i$ for the semi-Markov chain.

Let $P = (P_{ij}(\cdot))_{i,j \in E}$ be the semi-Markov transition function,

$$P_{ij}(k) := \mathbb{P}(Z_k = j \mid Z_0 = i), \ i,j \in E, \ k \in \mathbb{N}$$
The semi-Markov transition function $P$ verifies

$$P = I - H + q \ast P,$$

the unique solution is given by

$$P(k) = (\delta I - q)^{-1} \ast (I - H)(k), \ k \in \mathbb{N},$$

and

$$\lim_{k \to \infty} P_{ij}(k) = \frac{1}{\mu_{jj}} m_j,$$

where

$$H_i(n) := \mathbb{P}(X_1 \leq n \mid J_0 = i) = \sum_{l=1}^{n} \sum_{j \in E} q_{ij}(l)$$

$$H(n) := \text{diag}(H_i(n); \ i \in E), \ i \in E, \ n \in \mathbb{N}$$

$m_i$ is the mean sojourn time in state $i$. 
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Framework

- Discrete-time semi-Markov system with state space $E$
- Partition $E = U \cup U^c$
- Suppose that the initial state of the chain belongs to $U$.
- **Problems**: compute/estimate the mean time needed to hit $U^c = D$, the mean up/down times, the reliability/survival function, the availability function, the failure rate, etc.
Framework

- Discrete-time semi-Markov system with state space $E$
- Partition $E = U \cup U^c$
- Suppose that the initial state of the chain belongs to $U$.
- **Problems**: compute/estimate the mean time needed to hit $U^c = D$, the mean up/down times, the reliability/survival function, the availability function, the failure rate, etc.
- Reorder $E$ such that all the elements of $U$ precede the elements of $U^c$.
- Partition every matrix or matrix-valued function according to the partition $\{U, U^c\}$.

$$
q(k) = \begin{pmatrix}
U & U^c \\
q_{11}(k) & q_{12}(k) \\
q_{21}(k) & q_{22}(k)
\end{pmatrix}
$$
Consider the equation

\[ V_i = \begin{cases} 
  m_i + (pV)_i & \text{if } i \in U, \\
  0 & \text{if } i \in U^c, 
\end{cases} \tag{2} \]

where \( V = (V_i; \ i \in E) \in \mathbb{R}^E \) is an unknown column vector. In matrix form, this equation can be written

\[ V_1 = m_1 + p_{11} V_1. \]

**Theorem (Kemeny et al., 1976)**

*If the matrix \((I - p_{11})\) is invertible, then \(Nm_1\) is the minimal nonnegative solution of Equation (2), where \(N = \sum_{k \geq 0} p_{11}^k\).*
Mean Time To Failure

Let $T_D$ denote the lifetime of the system. We want to compute $MTTF = \mathbb{E}[T_D]$.

For any state $i \in U$, we introduce:

- $MTTF_i := \mathbb{E}_i[T_D]$ - the MTTF of the system, given that it starts in state $i \in U$;
- $MTTF := (MTTF_i; i \in U)^\top$

We can show that $MTTF$ satisfies equation

$$MTTF = m_1 + p_{11} MTTF.$$ 

**Theorem**

*If the matrix $(I - p_{11})$ is invertible, then the MTTF of the system is given by*

$$MTTF = \alpha_1 (I - p_{11})^{-1} m_1.$$
Reliability/Survival function

\[ T_D := \inf\{n \in \mathbb{N} \mid Z_n \in D\} \]
\[ R(k) := \mathbb{P}(T_D > k) = \mathbb{P}(Z_n \in U, n \in \{0, \ldots, k\}) \]
Reliability/Survival function

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**Theorem**

*The reliability of a discrete-time SM system is given by*

\[ R(k) = \alpha_1 P_{11}(k) \mathbf{1}_U = \alpha_1 \psi_{11} \ast (I - H_{11})(k) \mathbf{1}_U. \]

**Reliability estimator**

\[ \hat{R}(k, M) := \hat{\alpha}_1 \hat{P}_{11}(k, M) \mathbf{1}_U \]
\[ = \hat{\alpha}_1 \left[ \hat{\psi}_{11} (\cdot, M) \ast (I - \hat{H}(\cdot, M)_{11}) \right](k) \mathbf{1}_U \]
Proof 1: direct

\[ Y = (Y_k)_{k \in \mathbb{N}} \] a new semi-Markov chain, of state space
\[ E_Y = U \cup \{\Delta\} \], with \( \Delta \) an absorbing state:

\[ Y_k = \begin{cases} Z_k & \text{if } k < T_D, \\ \Delta & \text{if } k \geq T_D, \end{cases} \quad k \in \mathbb{N}. \]

The SM chain \( Y \): kernel \( q_Y \), transition matrix \( P_Y \)

\[ q_Y(k) = \begin{bmatrix} q_{11}(k) & q_{12}(k) \mathbf{1}_D \\ 0_{1,s_1} & 0 \end{bmatrix}, \quad k \in \mathbb{N}. \]
The reliability of the semi-Markov system:

\[ R(k) = \mathbb{P}(Z_l \in U, \forall l \in \{0, \ldots, k\}) \]
\[ = \mathbb{P}(Y_k \in U) \]
\[ = \sum_{j \in U} \sum_{i \in U} \mathbb{P}(Y_k = j \mid Y_0 = i) \mathbb{P}(Y_0 = i) \]
\[ = \sum_{j \in U} \sum_{i \in U} (P_Y)_{ij}(k) \mathbb{P}(Y_0 = i) \]

For \( i, j \in U \)

\[ (P_Y)_{ij}(k) = \mathbb{P}(Z_k = j, Z_l \in U, l = 1, \ldots, k - 1 \mid Z_0 = i) = (P_{11})_{ij}(k) \]
Proof 2: Markov renewal equation

Define:

1. \( R_i(k) := \mathbb{P}(T_D > k \mid Z_0 = i), \; i \in U \)
2. \( \mathbf{R}(k) := (R_1(k), \ldots, R_{s_1}(k))^\top \Rightarrow R(k) = \alpha_1 \mathbf{R}(k) \)
Proof 2: Markov renewal equation

Define:

- \( R_i(k) := \mathbb{P}(T_D > k \mid Z_0 = i), \ i \in U \)
- \( \mathbf{R}(k) := (R_1(k), \ldots, R_{s_1}(k))^\top \Rightarrow \mathbf{R}(k) = \alpha_1 \mathbf{R}(k) \)

For all \( i \in U \)

\[
R_i(k) = \mathbb{P}_i(T_D > k, S_1 > k) + \mathbb{P}_i(T_D > k, S_1 \leq k)
\]

\[
\vdots
\]

\[
= 1 - H_i(k) + \sum_{j \in U} \sum_{m=1}^{k} q_{ij}(m)R_j(k-m)
\]

\[
\mathbf{R}(k) = (\mathbf{I} - \mathbf{H}_{11})(k) + \mathbf{q}_{11}(k) \ast \mathbf{R}(k); \ \mathbf{R}(k) = (\delta \mathbf{I} - \mathbf{q}_{11})^{-1} \ast (\mathbf{I} - \mathbf{H}_{11})(k)
\]

\( \blacksquare \)
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Two estimation procedures

1. adapted to **reliability analysis**
   - consider one SM sample path of length $M$
   - estimate the quantities of interest
   - look for the asymptotic properties as $M \to \infty$

2. adapted to **survival analysis**
   - consider $K$ SM sample paths
   - estimate the quantities of interest
   - look for the asymptotic properties as $K \to \infty$
Consider
\[ \mathcal{H}(M) := (J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M), \]
where \( u_M := M - S_{N(M)}, M \in \mathbb{N}. \)

The associated likelihood function is
\[ L(M) = \alpha(J_0) \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k) \overline{H}_{J_{N(M)}}(u_M), \quad (3) \]

with \( \overline{H}_{J_{N(M)}}(u_M) = \mathbb{P}(X_{N(M)+1} > u_M \mid J_{N(M)}). \)
Consider

$$\mathcal{H}(M) := (J_0, X_1, \ldots, J_{N(M)} - 1, X_{N(M)}, J_{N(M)}, u_M),$$

where $u_M := M - S_{N(M)}$, $M \in \mathbb{N}$.

The associated likelihood function is

$$L(M) = \alpha(J_0) \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k) \overline{H}_{J_{N(M)}}(u_M), \quad (3)$$

with $\overline{H}_{J_{N(M)}}(u_M) = \mathbb{P}(X_{N(M)} + 1 > u_M \mid J_{N(M)})$.

We neglect $\overline{H}_{J_{N(M)}}(u_M)$ in (3) $\Rightarrow$ maximize:

$$L_1(M) = \alpha(J_0) \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k).$$
Theorem

Let \((Z_n)_{n \in \mathbb{N}}\) be an irreducible and aperiodic SMC, with finite mean sojourn times, \(m_i := \mathbb{E}_i(X_1) < \infty\). Let \(\mathcal{H}(M)\) be a censored sample path of the chain. Then, the approached maximum likelihood estimators of \(p_{ij}\), \(f_{ij}(k)\) and \(q_{ij}(k)\) are:

- \(\hat{p}_{ij}(M) = \frac{N_{ij}(M)}{N_i(M)}\)
- \(\hat{f}_{ij}(k, M) = \frac{N_{ij}(k, M)}{N_{ij}(M)}\)
- \(\hat{q}_{ij}(k, M) = \frac{N_{ij}(k, M)}{N_i(M)}\)

where:

\[
N_i(M) := \sum_{n=0}^{N(M)-1} 1\{J_n=i\}
\]
\[
N_{ij}(M) := \sum_{n=1}^{N(M)} 1\{J_{n-1}=i, J_n=j\}
\]
\[
N_{ij}(k, M) := \sum_{n=1}^{N(M)} 1\{J_{n-1}=i, J_n=j, X_n=k\}.
\]

\(\Rightarrow\) obtain plug-in estimators for the quantities of interest
Theorem (Asymptotic normality)

For fixed $i,j \in E$ and $k \in \mathbb{N}$, the estimator of $q_{ij}(k)$ is asymptotically normal:

$$\sqrt{M}[\hat{q}_{ij}(k, M) - q_{ij}(k)] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{q_{ij}(k)}),$$

$$\sigma^2_{q_{ij}(k)} := \mu_{ii}q_{ij}(k)[1 - q_{ij}(k)].$$
Proof 1: main steps

- Use the central limit theorem for Markov renewal chains (Pyke and Schaufele, 1964; Moore and Pyke, 1968)
- There exists a measurable function $f : E \times E \times \mathbb{R} \rightarrow \mathbb{R}$ such that

\[
\sqrt{M}[\hat{q}_{ij}(k, M) - q_{ij}(k)] = \frac{M}{N_i(M)} \frac{1}{\sqrt{M}} \sum_{n=1}^{N(M)} f(J_{n-1}, J_n, X_n)
\]

\[
f(m, l, u) := 1_{\{m=i, l=j, u=k\}} - q_{ij}(k) 1_{\{m=i\}}
\]

- For a MRC:

\[
\frac{N_i(M)}{M} \xrightarrow{a.s.} \frac{1}{\mu_{ii}} \quad \text{as} \quad M \rightarrow \infty
\]
Proof 2: main steps

- Use the Lindeberg-Lévy CLT for martingales (Billingsley, 1961; 1995)

\[
\sqrt{M}[\hat{q}_{ij}(k, M) - q_{ij}(k)] = \frac{M}{N_i(M)} \frac{1}{\sqrt{M}} \sum_{n=1}^{N(M)} [\mathbf{1}_{J_n=j, X_n=k} - q_{ij}(k)]\mathbf{1}_{\{J_{n-1}=i\}}
\]

- \(F_n := \sigma(J_l, X_l; l \leq n)\) \(Y_n := \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}} - q_{ij}(k)\mathbf{1}_{\{J_{n-1}=i\}}\)

- \((Y_n)_{n \in \mathbb{N}}\) is a \(F_n\)-difference martingale
\[ \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(Y_l^2 \mathbf{1}_{\{|Y_l|>\epsilon\}}) \xrightarrow{n \to \infty} 0 \]

By the Lindeberg-Lévy theorem, we have
\[ \frac{1}{\sqrt{n}} \sum_{l=1}^{n} Y_l \xrightarrow{D} \mathcal{N}(0, \sigma^2); \quad \frac{1}{\sqrt{N(M)}} \sum_{l=1}^{N(M)} Y_l \xrightarrow{D} \mathcal{N}(0, \sigma^2) \]

\[ \sigma^2 \text{ is given by} \]
\[ \sigma^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(Y_l^2 \mid \mathcal{F}_{l-1}) \]
\[
\frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(Y_l^2 \mathbf{1}_{\{|Y_l|>\epsilon\}}) \xrightarrow{n \to \infty} 0
\]

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\[
\frac{1}{\sqrt{n}} \sum_{l=1}^{n} Y_l \xrightarrow{D} \mathcal{N}(0, \sigma^2); \quad \frac{1}{\sqrt{N(M)}} \sum_{l=1}^{N(M)} Y_l \xrightarrow{D} \mathcal{N}(0, \sigma^2)
\]

\[
\sigma^2 \text{ is given by}
\]

\[
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(Y_l^2 \mid \mathcal{F}_{l-1})
\]

\[
\Rightarrow \text{asymptotic normality of } \hat{\psi}_{ij}(k, M) \text{ and } \hat{P}_{ij}(k, M)
\]
Reliability/survival function estimation

**Theorem**

*For a discrete-time SM system, for fixed $k \in \mathbb{N}$:

1. $\hat{R}(k, M) := \hat{\alpha}_1 \left[ \hat{\psi}_{11}(\cdot, M) \ast \left( I - \hat{H}(\cdot, M)_{11} \right) \right] (k) \mathbf{1}_U$ is strongly consistent, as $M \to \infty$

2. $\sqrt{M}[\hat{R}(k, M) - R(k)] \xrightarrow{D} \mathcal{N}(0, \sigma^2_R(k))$

$$
\sigma^2_R(k) = \sum_{i=1}^{s} \mu_{ii} \left\{ \sum_{j=1}^{s} \left[ D_{ij}^U - \mathbf{1}_{i \in U} \left( \sum_{t \in U} \alpha(t) \Psi_{ti} \right) \right]^2 * q_{ij}(k) 
- \left[ \sum_{j=1}^{s} \left( D_{ij}^U * q_{ij} - \mathbf{1}_{i \in U} \left( \sum_{t \in U} \alpha(t) \psi_{ti} * Q_{ij} \right) \right)^2 (k) \right] \right\}
$$

$$
D_{ij}^U := \sum_{n \in U} \sum_{r \in U} \alpha(n) \psi_{ni} * \psi_{jr} * \left( I - \text{diag}(Q \cdot \mathbf{1}) \right)_{rr}
$$
Figure: Confidence interval of reliability
References


Let \((J, S)\) be a MRC and \(f : E \times E \times \mathbb{R} \to \mathbb{R}\) measurable.

\[
W_f(M) := \sum_{i,j=1}^{s} \sum_{n=1}^{N(M)} f(i, j, X_{ijn}) = \sum_{n=1}^{N(M)} f(J_{n-1}, J_n, X_n), M \in \mathbb{N}.
\]

Put:

\[
A_{ij} := \sum_{x=1}^{\infty} f(i, j, x)q_{ij}(x), \quad A_i := \sum_{j=1}^{s} A_{ij},
\]

\[
B_{ij} := \sum_{x=1}^{\infty} f^2(i, j, x)q_{ij}(x), \quad B_i := \sum_{j=1}^{s} B_{ij},
\]
\[ n_i := \sum_{j=1}^{s} A_j \frac{\mu_{ii}^*}{\mu_{jj}^*}, \quad m_f := \frac{n_i}{\mu_{ii}}, B_f := \frac{\sigma_i^2}{\mu_{ii}}, \]

\[ \sigma_i^2 := -n_i^2 + \sum_{j=1}^{s} B_j \frac{\mu_{ii}^*}{\mu_{jj}^*} + 2 \sum_{r=1}^{s} \sum_{l \neq i} \sum_{k \neq i} A_{rl} A_{k} \mu_{ii}^* \frac{\mu_{li}^* + \mu_{ik}^* - \mu_{lk}^*}{\mu_{rr}^* \mu_{kk}^*}, \]

where \( \mu_{ii}^* \) is the mean return time in state \( i \) for the chain \( J \).

**Theorem (Pyke and Schaufele, 1964; Moore and Pyke, 1968)**

For an irreducible Markov renewal chain, with finite mean sojourn times, such that all the above sums are finite, we have:

\[
\frac{1}{\sqrt{M}} \left[ W_f(M) - M m_f \right] \xrightarrow{D} \mathcal{N}(0, B_f). \]

return
The Lindeberg-Lévy CLT for martingales

Theorem (Billingsley, 1961 ; 1995)

Let $X_n, n \in \mathbb{N}^*$, be a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ and let $Y_n := X_n - X_{n-1}, n \in \mathbb{N}^*$, be the difference martingale (with $Y_1 := X_1$). If:

1. $\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[Y_k^2 \mid \mathcal{F}_{k-1}] \xrightarrow{P} \sigma^2$, 
2. $\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[Y_k^2 1\{|Y_k|>\epsilon\}] \xrightarrow{n \to \infty} 0$, for all $\epsilon > 0$,

then

$$\frac{1}{\sqrt{n}} X_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \xrightarrow{D} \mathcal{N}(0, \sigma^2).$$