Domains of Discontinuity for Surface Groups

Olivier Guichard [a] Anna Wienhard [b]

[a] *CNRS, Laboratoire de Mathématiques d'Orsay, Orsay cedex, F-91405*
Université Paris-Sud, Orsay cedex, F-91405

[b] *Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544-1000, USA*

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Abstract

Let Σ be a closed connected orientable surface of negative Euler characteristic and G a semisimple Lie group. For any Anosov representation $\rho : \pi_1(\Sigma) \to G$ we construct domains of discontinuity with compact quotient for the action of $\pi_1(\Sigma)$ on flag varieties G/Q. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Quotients compacts et groupes de surfaces. Soit $\pi_1(\Sigma)$ le groupe fondamental d'une surface de Riemann connexe, fermée et de genre supérieur et soit G un groupe de Lie semi-simple. Pour toute représentation Anosov $\rho : \pi_1(\Sigma) \to G$, nous construisons un ouvert de la variété drapeau G/Q sur lequel $\pi_1(\Sigma)$ agit proprement avec quotient compact.

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1. Introduction

In [10] F. Labourie introduced the notion of Anosov structures and their holonomy representations, so called Anosov representations, to study the Hitchin component for $\mathrm{SL}(n, \mathbf{R})$. Anosov representations are in some sense a dynamical analogue of holonomy representations of geometric structures (in the sense of Ehresmann), but the concept of Anosov representations is more flexible. Anosov representations have been proven to be a key tool in the study of higher Teichmüller spaces. In this note we show that Anosov representations of surface groups actually give rise to geometric structures on compact manifolds.

Email addresses: `olivier.guichard@math.u-psud.fr` (Olivier Guichard), `wienhard@math.princeton.edu` (Anna Wienhard).

Theorem 1.1 *Let Σ be a closed connected orientable surface of negative Euler characteristic, and let G be a semisimple Lie group not locally isomorphic to* $\mathrm{SL}(2,\mathbf{R})$.

Suppose that $\rho : \pi_1(\Sigma) \to G$ is an Anosov representation, then there exist a parabolic subgroup $Q < G$ and a non-empty open set $\Omega \subset G/Q$ such that $\rho(\pi_1(\Sigma))$ preserves Ω and acts on it freely, properly discontinuously and with compact quotient.

Note that Anosov representations are easily seen to be faithful with discrete image [10,7]. In particular, Anosov representations into $\mathrm{SL}(2,\mathbf{R})$ are exactly Fuchsian representations, thus their action on the projective line is minimal.

The proof of Theorem 1.1 is constructive, *i.e.* we construct an explicit $Q < G$ and a domain $\Omega \subset G/Q$ (see Section 5 for examples). The construction uses the equivariant curve $\xi : \partial\pi_1(\Sigma) \to G/P$ associated to an Anosov representation (see Proposition 2.2), and the parabolic group Q depends on P.

Note that the projection $\mathrm{pr} : G/P_{min} \to G/Q$ from the full flag variety onto G/Q has compact fibers, therefore the preimage $\widetilde{\Omega} = \mathrm{pr}^{-1}(\Omega)$ is a domain of discontinuity for $\pi_1(\Sigma)$ with compact quotient. Thus, in Theorem 1.1 we could always take $Q = P_{min}$; however, it is useful to keep the dimension of the compact quotients $\Omega/\pi_1(\Sigma)$ as small as possible.

Even though we focus on surface groups here, some results generalize to Anosov representations of fundamental groups of more general manifolds (*e.g.* hyperbolic manifolds).

2. Anosov Representations

Let Σ be a closed connected oriented surface of negative Euler characteristic, $\pi_1(\Sigma)$ its fundamental group, $T^1\Sigma$ its unit tangent with respect to some hyperbolic metric and $\phi_t : T^1\Sigma \to T^1\Sigma$ the geodesic flow. Denote by $\partial\pi_1(\Sigma)$ the boundary at infinity of $\pi_1(\Sigma)$.

Let G be a semisimple real Lie group, let P_+, P_- be a pair of opposite parabolic subgroups of G and denote by $\mathcal{F}^\pm = G/P_\pm$ the flag variety associated to $P_\pm$. There is a unique open G-orbit $\mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$. We have $\mathcal{X} = G/(P_+ \cap P_-)$, and as an open subset of $\mathcal{F}^+ \times \mathcal{F}^-$ it inherits two foliations $\mathcal{E}_\pm$ whose corresponding distributions are denoted by $E_\pm$, $(E_\pm)_{(f_+,f_-)} \cong T_{f_\pm}\mathcal{F}^\pm$.

Given a representation $\rho : \pi_1(\Sigma) \to G$ we consider the corresponding flat G-bundle $\mathcal{P}$ over $T^1\Sigma$. Via the flat connection, the flow ϕ_t lifts to $\mathcal{P}$.

Definition 2.1 ([10]) *A representation $\rho : \pi_1(\Sigma) \to G$ is called a P_+-*Anosov representation *(or simply an* Anosov representation*) if the associated bundle $\mathcal{P} \times_G \mathcal{X}$*

(i) admits a section σ that is flat along flow lines, and

(ii) the action of the flow ϕ_t on σ^*E_+ (resp. σ^*E_-) is contracting (resp. dilating), i.e. *there exist constants $A, a > 0$ such that for any e in $\sigma^*(E_\pm)_m$ and for any $t > 0$ one has*

$$\|\phi_{\pm t} e\|_{\phi_{\pm t} m} \leq A \exp(-at) \|e\|_m.$$

The set of P_+-Anosov representations is open in $\mathrm{Hom}(\pi_1(\Sigma), G)$ [10].

Proposition 2.2 ([10]) *Let Σ, G and P_+ be as above. Let ρ be a P_+-Anosov representation. Then*

(i) there are two ρ-equivariant continuous maps $\xi^\pm : \partial\pi_1(\Sigma) \to \mathcal{F}^\pm$,

(ii) for every $t_+ \neq t_- \in \partial\pi_1(\Sigma)$ we have $(\xi^+(t_+), \xi^-(t_-)) \in \mathcal{X}$,

(iii) for every $\gamma \in \pi_1(\Sigma) - \{e\}$, the element $\rho(\gamma)$ is conjugate to an element in $P_+ \cap P_-$, having a unique attracting fix point in G/P_+ and a unique repelling fix point in G/P_-.

Important examples of Anosov representations are Hitchin representations into split real simple Lie groups [9,10,5], maximal representations into Lie groups of Hermitian type [4,3], quasi-Fuchsian representations into $\mathrm{SL}(2,\mathbf{C})$, quasi-Fuchsian representations in the sense of [11,2] and small deformations of embeddings of cocompact lattices in rank one Lie groups into Lie groups of higher rank.

3. A Special Case

Let V be a real vector space and F a non-degenerate bilinear form on V which we assume to be either skew-symmetric or symmetric indefinite of signature (p,q) (with $p \leq q$). Let $G_F = \{g \in \mathrm{GL}(V) \,|\, g^*F = F\}$, let $\mathcal{F}_0 = G_F/Q_0 = \{l \in \mathbb{P}(V) \,|\, F|_l = 0\}$ be the set of isotropic lines and $\mathcal{F}_1 = G_F/Q_1 = \{W \in \mathrm{Gr}_p(V) \,|\, F|_W = 0\}$ be the set of maximal isotropic suspaces ($p = \dim V/2$ when F is skew-symmetric). Let also $\mathcal{F}_{0,1} = \{(l,W) \in \mathcal{F}_0 \times \mathcal{F}_1 \,|\, l \subset W\}$ and $\pi_i : \mathcal{F}_{0,1} \to \mathcal{F}_i$, $i = 0,1$, be the projections. Given a subset $A \subset \mathcal{F}_0$ we define the subset

$$K_A := \pi_1(\pi_0^{-1}(A)) \subset \mathcal{F}_1.$$

For an isotropic line $l \in \mathcal{F}_0$, $K_l \subset \mathcal{F}_1$ is the set of maximal isotropic subspaces containing l, and $K_A = \bigcup_{l \in A} K_l$. Similarily, given $B \subset \mathcal{F}_1$ we define $K_B \subset \mathcal{F}_0$.

Theorem 3.1 *Let Σ be as in Theorem 1.1 and let V, F and G_F as above with $\dim V \geq 4$. Suppose $\rho : \pi_1(\Sigma) \to G_F$ is a Q_i-Anosov representation, with $i = 0$ or 1, and let $\xi_i : \partial\pi_1(\Sigma) \to \mathcal{F}_i$ be the corresponding equivariant map. Define $\Omega_\rho := \mathcal{F}_{1-i} - K_{\xi_i(\partial\pi_1(\Sigma))} \subset \mathcal{F}_{1-i}$.*

Then Ω_ρ is non-empty, open and preserved by $\rho(\pi_1(\Sigma))$. Furthermore, the action of $\rho(\pi_1(\Sigma))$ on Ω_ρ is free, properly discontinuous and the quotient $\Omega_\rho/\rho(\pi_1(\Sigma))$ is compact.

The set $K_{\xi_i(\partial\pi_1(\Sigma))}$ is closed and (because $\dim V \geq 4$) of codimension at least 1 in $\mathcal{F}_{1-i}$; by ρ-equivariance of ξ_i it is preserved by $\rho(\pi_1(\Sigma))$, hence Ω_ρ is a $\rho(\pi_1(\Sigma))$-invariant non-empty open subset of $\mathcal{F}_{1-i}$. That the action is free and properly discontinuous follows from the contraction estimates one can deduce from the representation ρ being Q_i-Anosov.

To prove compactness of the quotient $\Omega_\rho/\rho(\pi_1(\Sigma))$, we need to prove that $\mathrm{H}_n(\Omega_\rho/\rho(\pi_1(\Sigma)); \mathbf{F}_2)$ does not vanish. First since the fibration of $E_\rho = \Omega_\rho \times_{\pi_1(\Sigma)} \widetilde{\Sigma}$ over $\Omega_\rho/\rho(\pi_1(\Sigma))$ has contracible fibers, the homology of $\Omega_\rho/\rho(\pi_1(\Sigma))$ is identified with the homology of E_ρ. Then applying the Leray-Serre spectral sequence for the fibration of $E_\rho \to \Sigma$, we deduce $\mathrm{H}_n(\Omega_\rho/\rho(\pi_1(\Sigma)); \mathbf{F}_2) \cong \mathrm{H}_{n-2}(\Omega_\rho; \mathbf{F}_2)$ and this last group is shown to be nonzero by Alexander duality.

4. Reduction to the Special Case

Our strategy to prove Theorem 1.1 is to find a G-module V with a non-degenerate bilinear form F in order to apply Theorem 3.1. Lemmas 4.1, 4.2 and 4.3 show that we can find such a G-module so that the composition $\pi_1(\Sigma) \to G \to G_F$ satisfies the hypothesis of Theorem 3.1.

The next lemma uses standard terminology and notations for decomposition of a G-module V into weight spaces V_χ (see *e.g.* [6]).

Lemma 4.1 *Let $P < G$ be a parabolic subgroup which is conjugated to P^{opp}. Then there exists a real (irreducible) representation $\pi : G \to G_F < \mathrm{GL}(V)$ with one-dimensional highest weight space V_μ such that $P = \mathrm{Stab}_G(V_\mu)$, and where F is a non-degenerate bilinear form as in Section 3.*

Moreover, if $V_+ = \oplus_{\chi > 0} V_\chi$ is the sum of the positive weight spaces, then $V_+ \subset V$ is a maximal F-isotropic subspace and $Q = \mathrm{Stab}_G(V_+)$ is a parabolic subgroup of G.

Note that the parabolic group Q in Theorem 1.1 is determined by this lemma. The existence of the irreducible representation π is classical. That V_+ is a maximal F-isotropic subspace whose stabilizer in G contains a Borel subgroup can be checked by restricting the representation π to $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ associated to the restricted roots.

Lemma 4.2 *Let $\rho : \pi_1(\Sigma) \to G$ be a P-Anosov representation with P being conjugate to P^{opp} and $\pi : G \to G_F$ as in Lemma 4.1, then the composition $\pi \circ \rho : \pi_1(\Sigma) \to G_F$ is Q_0-Anosov.*

Lemma 4.3 *Let $\rho : \pi_1(\Sigma) \to G$ be an Anosov representation, then ρ is also a P-Anosov with P being conjugate to P^{opp}.*

This lemma follows from the fact that any P-Anosov representation is P^{opp}-Anosov, and the fact that a representation that is both P-Anosov and Q-Anosov is also $P \cap Q$-Anosov.

Proposition 4.4 *Let ρ, G be as in Theorem 1.1 and π as in Lemma 4.1. Then the set $\Omega_{\rho,\pi} = \Omega_{\pi\circ\rho} \cap G\cdot[V_+]$ is non-empty in $G\cdot[V_+] \cong G/Q$.*

For this we consider the Bruhat decomposition of G/Q and we show that the set $K_{[V_\mu]} \cap G \cdot [V_+]$ is the union of Bruhat cells of codimension at least 2 in $G/Q \cong G \cdot [V_+]$. In particular, since $\partial\pi_1(\Sigma)$ is one dimensional, the intersection of $K_{\xi_0(\partial\pi_1(\Sigma))}$ with $G \cdot [V_+]$ is of codimension at least one in $G \cdot [V_+] \cong G/Q$.

Theorem 1.1 follows then from Proposition 4.4 and Theorem 3.1

5. Examples

5.1. *Maximal Representations into* $\mathrm{Sp}(2n, \mathbf{R})$

Any maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is P-Anosov where P is the stabilizer of a Lagrangian subspace in $\mathbf{R}^{2n}$ (see [4] for definitions and proofs). Thus Theorem 3.1 applies and gives a domain of discontinuity $\Omega_\rho \subset \mathbf{R}\mathbb{P}^{2n-1}$.

In this case, due to maximality properties of the equivariant curve (see [4]), one can construct a natural $\mathrm{O}(n)$-bundle E over $T^1\Sigma$ and a proper map $\Phi : \widetilde{E} \to \Omega_\rho/\rho(\pi_1(\Sigma))$. Using [8] we can show that the quotient space $\Omega_\rho/\rho(\pi_1(\Sigma))$ is homeomorphic to an $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over the surface Σ.

5.2. *Hitchin Representations into* $\mathrm{SL}(n, \mathbf{R})$

Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}(n, \mathbf{R})$ be a P_{min}-Anosov representation, and let $\xi = (\xi^1, \cdots, \xi^{n-1}) : \partial\pi_1(\Sigma) \to \mathcal{F}(\mathbf{R}^n)$ be the equivariant map into the flag variety. Examples of such representations are Hitchin representations [9,10], but the construction applies also to other such representations.

The trace defines a non-degenerate bilinear form F on $V = \mathrm{End}(\mathbf{R}^n)$. Applying Theorem 3.1 to the Q_1-Anosov representation $\mathrm{Ad} \circ \rho : \pi_1(\Sigma) \to \mathrm{GL}(V)$ we obtain a domain of discontinuity $\Omega_{\mathrm{Ad}\circ\rho}$ in G_F/Q_0 which gives rise to a domain of discontinuity $\Omega_{\rho,\mathrm{Ad}} \subset \mathcal{F}_{1,n-1}(\mathbf{R}^n)$ in the space of partial flags consisting of a line and a hyperplane. $\Omega_{\rho,\mathrm{Ad}}$ is the complement of

$$\left\{(p, H) \in \mathcal{F}_{1,n-1}(\mathbf{R}^n) \,|\, \exists t \in \partial\pi_1(\Sigma),\, \exists\, 1 \le k < n \text{ such that } p \subset \xi^k(t) \subset H\right\}.$$

For $n = 3$ this coincides with the domain of discontinuity defined in [1].

The construction of Section 4, applied to $V = \mathrm{End}(\Lambda^k\mathbf{R}^n)$, gives rise to a domain of discontinuity in $\mathcal{F}(\mathbf{R}^n)$ which is the complement of $\bigcup_{t\in\partial\pi_1(\Sigma)} L_{\xi^k(t),\xi^{n-k}(t)}$, where a flag $(F_1, \ldots, F_{n-1})$ is in $L_{D,E}$ if there exist $(s_i)_{i=1,\ldots,k}$ and $(u_i)_{i=1,\ldots,k}$ such that $\dim(D \cap F_{s_i}) = i$, $\dim(E + F_{u_i-1}) = n - k + i - 1$ and $(s_1, s_2, \ldots, s_k) \le (u_1, u_2, \ldots, u_k)$ with respect to the lexicographic order on k-tuples.

5.3. *Deformations of* $\pi_1(\Sigma) \to \mathrm{SO}(2, 1) \to \mathrm{SO}(n, 1)$

Let $\rho : \pi_1(\Sigma) \to \mathrm{SO}(n, 1)$, $n \ge 3$, be a (small enough) deformation of the embedding $\pi_1(\Sigma) \to \mathrm{SO}(2, 1) \to \mathrm{SO}(n, 1)$. Then the domain of discontinuity Ω_ρ constructed in Section 3 is the complement of the limit set of ρ in S^{n-1} and the quotient $\Omega_\rho/\rho(\pi_1(\Sigma)$ is homeomorphic to an S^{n-3}-bundle over Σ.

Details will appear elsewhere.

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