An Introduction to the Differential Geometry of Flat Bundles and of Higgs Bundles

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This chapter collects the notes of the lectures given on that subject during the introductory school of the program "The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles" (7 July - 29 August 2014). The main purpose is to state the correspondence between flat bundles and Higgs bundles and some of its features.

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Partially supported by the European Research Council under the European Community's seventh Framework Programme (FP7/2007-2013)/ERC grant agreement n° FP7-246918.

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1. Introduction

As its title indicates, this chapter aims to give an introduction on vector bundles over Riemannian and Kählerian manifolds. More specifically these notes cover the following topics:

- the necessary backgrounds in differential geometry: Riemannian, symplectic and Kählerian manifolds.
- the theory of flat bundles over (compact) manifolds: representations of the fundamental group; local systems (below hidden under the concept of "flat structures" on a given bundle, this is an ad-hoc definition whose use should be restricted to these notes); connections and curvature.
- the theory of holomorphic bundles over complex (or analytic) manifolds: interpretation in terms of pseudo-connections with vanishing pseudo-curvature.
- the theory of Hermitian bundles over manifolds and, in the particular case when the base manifold is a complex manifold, the correspondence between pseudo-connections and Hermitian connections (via the introduction of Chern connections).

Those necessary tools need to be complemented with the notion of stability of holomorphic vector bundles over Kählerian manifolds. For this, the Chern-Weyl theory for constructing differential forms representing the Chern classes is recalled and afterward come the degree and the slope of vector bundles over Kählerian manifolds.

The slopes at hand, the notion of stability can be explained as well as some of its variants (semi-stability and poly-stability) and the theorem of Narasimhan and Seshadri is stated, i.e. the one-to-one correspondence between stable holomorphic bundles and irreducible projectively flat unitary bundles over a Riemann surface. In fact the more general result due to Uhlenbeck and Yau where the base is a compact Kählerian manifold is given.

The proof of this correspondence, as well as the Hitchin-Kobayashi correspondence, involves some remarkable identities over Kählerian manifolds that lead to automatic cancellations of the curvature of a connection as soon as only "one part" of the curvature is zero and under some topological conditions. For the purpose of this text those cancellations are called "opportune cancellations" and are consequences, among other things, of the Kähler identities. An explanation of the Kähler identities is here accomplished with the help of symplectic geometry and the corresponding "symplectic Kähler identities".

At the heart of the Narasimhan-Seshadri correspondence and of the Hitchin-Kobayashi correspondence, are notions of the "best" Hermitian metrics. In one direction of the correspondence they are called Yang-Mills metrics.

In the other direction, it is the notion of harmonic metrics that appears: it means the best metric in a L^2-sense. A result of Corlette says that harmonic metrics exist on semisimple flat bundles over compact Riemannian manifolds (Donaldson gave also a proof of a particular case of that result, Labourie proved it in a more general setting).

Over a Kählerian manifold, the connection of an harmonic bundle can be decomposed furthermore with respect to the bidegree of forms: the $(0,1)$-component of the unitary connection together with the $(1,0)$-component of the symmetric part give rise to a Higgs bundle. This notion of Higgs bundles is due to Hitchin, a companion definition in a somewhat different setting was introduced by Griffiths under the name "variation of Hodge structure".

The statement of the Hitchin-Kobayashi correspondence is disconcertingly similar to the above correspondence: on a bundle with vanishing Chern classes over a compact Kählerian manifold, there is a one-to-one bijection between the stable Higgs bundle structures and the simple flat bundle structures. Section 9 illustrates the necessity of the (poly)stability and the (semi)simplicity assumptions and explains the strategies for proving the existence of the sought-for metrics (harmonic metrics and Yang-Mills metrics).

Section 8 gives examples of Higgs bundles, and sometimes of the corresponding representations, over Riemann surfaces. The construction of the Hitchin component for $\mathrm{SL}_n(\mathbf{R})$ is given there and also a proof of the Milnor-Wood inequality. This section is moreover the opportunity to illustrates how one can detect representations with values in some subgroups of the general linear groups: $\mathrm{SL}_2(\mathbf{C})$, $\mathrm{SL}_2(\mathbf{R})$, $\mathrm{Sp}_{2n}(\mathbf{R})$ or $\mathrm{U}(p,q)$.

No attemps to build a proper bibliographical section or to account for proper contributions were made in those notes. The unforgivable ommission concerns the moduli spaces aspects of the correspondence (analytical or algebraical structures, hyper-Kählerian structure, $\mathbf{C}^*$-action, etc.). The notes [5] by Richard Wentworth beautifully remedies those lacunas.

The present notes are based on exercises, many proofs are left to the reader in form of exercises. I am also glad to thanks the participants of the summer school, their active participation helped tremendously to improve the following text.

2. Riemannian, Symplectic, Complex and Kählerian Manifolds

This section is mainly here to fix the notations that will be used in the rest of the manuscript. It also introduces differents tools subsequently used through the chapter: L^2-metrics, L^2-adjoints (only on spaces of differential forms) and the Kähler identities (with their proofs using symplectic geometry).

2.1. *Riemannian Manifold*

A Riemannian structure on a n-dimensional manifold M is a smooth section of $\mathrm{Sym}^2_{>0}T^*M \subset \mathrm{Sym}^2T^*M \subset T^*M \otimes T^*M$. More explicitly, the data of a Euclidean scalar product $\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_m$ (or sometimes $\langle \cdot, \cdot \rangle_{TM}$) is given on T_mM, for all m in M, and it varies smoothly with m. This smoothness property can be expressed saying that the function

$$m \longmapsto \langle X_m, Y_m \rangle_m$$

is $\mathcal{C}^\infty$ whenever X and Y are $\mathcal{C}^\infty$ vector fields on M.

In local coordinates $(x^1, \cdots, x^n)$ on M, the Riemannian metric is given by:

$$\sum_{i,j} g_{ij} dx^i \otimes dx^j$$

i.e. $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = g_{ji}$. It is sometimes convenient in calculations to work with:

Definition 2.1: A *normal coordinates system* at m in M is a system of coordinates around m such that

$$\begin{array}{ll} \forall i, j & g_{ij}(m) = 0 \\ \forall i, j, k & \frac{\partial}{\partial x^k} g_{ij}(m) = 0. \end{array}$$

Other times it is even more convenient to work with:

Definition 2.2: A *normal frame* (or sometimes an *orthonormal frame*) is a family $(e_1, \cdots, e_n)$ of vector fields (defined on an open subset $U \subset M$) such that

$$\langle e_i, e_j \rangle = \delta_{i,j} \quad \forall i, j.$$

Remark 2.3: Usually normal coordinates systems are the coordinates obtained by the exponential map $\exp : T_m M \to M$ (hence they are exponential coordinates systems). Here is allowed a little more generality.

An Euclidean structure on a vector space V (here V is the tangent space $T_m M$) induces an Euclidean structure on every vector space constructed from V: V^*, $\bigwedge^p V$, $\bigwedge^p V^* \simeq (\bigwedge^p V)^*$, $\bigotimes^p V$, etc. Hence a Riemannian structure on M induces a scalar product on every bundle constructed from TM: T^*M, $\bigwedge^p T^*M$, etc.

In particular, given α and β two forms of degree p and q respectively, $\langle \alpha, \beta \rangle$ denotes the function $m \mapsto \langle \alpha_m, \beta_m \rangle_m$ with $\langle \cdot, \cdot \rangle_m$ still denoting the scalar product on $\bigwedge^\bullet T^*M = \bigoplus_p \bigwedge^p T^*M$, $\alpha_m \in \bigwedge^p T^*_m M$ and $\beta_m \in \bigwedge^q T^*_m M$. Of course by construction (or by convention), $\langle \alpha, \beta \rangle = 0$ if $p \neq q$.

2.2. *Orientation, Volume Form*

When M is furthermore oriented, there is a preferred volume form vol on M; it is

(1) a top dimensional form: $\mathrm{vol}_m \in \bigwedge^n T^*_m M$ ($\forall m \in M$).
(2) defining the orientation: $(\epsilon_1, \cdots, \epsilon_n)$ is an oriented basis of $T_m M$ if and only if $\mathrm{vol}_m(\epsilon_1, \cdots, \epsilon_n) > 0$.
(3) of norm 1: $\langle \mathrm{vol}, \mathrm{vol} \rangle = 1$.

2.3. *The Hodge Star*

Using the scalar product $\langle \cdot, \cdot \rangle$ on forms together with the volume form (hence the orientation), one defines the Hodge star $\star\alpha$ of a form α as the unique form satisfying[a]:

$$\star\alpha \wedge \beta = \langle \alpha, \beta \rangle \mathrm{vol}, \ \forall \beta.$$

[a] This is not the most standard choice for the Hodge star (different choices differ by signs), it is the one adopted in the book [1].

If the degree of α is p, then the degree of $\star\alpha$ is $n-p$.

For example, $\star\mathbf{1} = \mathrm{vol}$, $\star\mathrm{vol} = \mathbf{1}$ ($\mathbf{1}$ is the constant function equal to 1 in those equalities).

Also

$$\langle\star\alpha, \star\beta\rangle = \langle\alpha, \beta\rangle \text{ and } \star\star\alpha = (-1)^{p(n-p)}\alpha.$$

Remark 2.4: The later equality will be used only when M is even dimensional, it then becomes

$$\star\star\,\alpha = (-1)^{-p^2}\alpha = (-1)^p\alpha.$$

2.4. *Symplectic Manifold*

A symplectic form (or a symplectic structure) on a manifold M is a closed 2-form ω (i.e. $d\omega = 0$) such that $\omega_m \in \bigwedge^2 T_m^* M$ is a nondegenerate symplectic form on the vector space $T_m M$ for every m in M. This forces the dimension of M to be even: $\dim_{\mathbf{R}} M = 2n$.

An important aspect of symplectic manifolds is the uniqueness of the local model:

Theorem 2.5: *[Darboux] At every point m of a symplectic manifold (M, ω) there are local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ such that*

$$\omega = \sum_i dx^i \wedge dy^i.$$

A symplectic manifold has a canonical volume form:

$$\mathrm{vol} = \frac{\omega^n}{n!}$$

(hence a canonical orientation) and the symplectic structure induces a bilinear form on every vector bundle constructed from the tangent space; this bilinear form is symplectic on odd forms and symmetric on even forms.

Exercise 2.1: If ω is a symplectic form on a vector space V ($\dim_{\mathbf{R}} V = 2n$), then ω induces a nondegenerate symmetric bilinear form on $\bigwedge^{2k} V$ ($k \leq n$)[b]. Calculate the signatures.

[b] Often the indications "Show that" or "Prove that" will be missing in the exercises.

2.5. *The Symplectic Star*

With those two ingredients, one can define the symplectic star $\star_s\alpha$ of a form α by the equality:

$$\star_s\alpha \wedge \beta = \omega(\alpha,\beta)\mathrm{vol}, \quad \forall\ \beta.$$

For the symplectic star:

$$\omega(\star_s\alpha, \star_s\beta) = \omega(\alpha,\beta) \quad \forall\alpha,\beta; \quad \star_s\star_s = \mathrm{id}.$$

2.6. *The Operator L*

We denote by L the operator defined on forms by wedging with the symplectic form:

$$\begin{aligned} L : \Omega^\bullet(M) &\longrightarrow \Omega^\bullet(M) \\ \alpha &\longmapsto \omega \wedge \alpha \end{aligned}$$

where $\Omega^\bullet(M) = \bigoplus_p \Omega^p(M)$ is the graded algebra of differential forms.

2.7. *Symplectic Adjoints*

The adjoint (for the symplectic structure) of L is the unique operator:

$$L^{\star_s} : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

such that

$$\int_M \omega(L^{\star_s}\alpha, \beta)\mathrm{vol} = \int_M \omega(\alpha, L\beta)\mathrm{vol}$$

for all α and β.

Similarly the symplectic adjoint of the exterior differential d is denoted by $d^{\star_s}$.

Remark 2.6: Those adjoints (and the ones to be defined below in a Riemannian setting) are called *formal adjoints*. This means that they are defined to be the unique differential operators satisfying the above adjunction formula when evaluated on $\mathcal{C}^\infty$ forms but they are a priori not adjoint operators in the sense of functional analysis.

The symplectic star can be used to give formulas for $L^{\star_s}$ and $d^{\star_s}$:

$$\begin{aligned} L^{\star_s} &= \star_s L \star_s \\ d^{\star_s}\alpha &= -(-1)^{\deg(\alpha)} \star_s d \star_s \alpha. \end{aligned}$$

Exercise 2.2: Derive those identities.

2.8. *Symplectic Kähler Identities*

Theorem 2.7: *Let (M,ω) be a symplectic manifold of dimension $2n$. The following equalities hold:*

$$\begin{aligned}
&[L^{\star_s}, L]\alpha = (L^{\star_s}L - LL^{\star_s})\alpha = (n - \deg(\alpha))\alpha \\
&[L, d^{\star_s}]\alpha = d\alpha \\
&[L^{\star_s}, d]\alpha = -d^{\star_s}\alpha.
\end{aligned}$$

Proof: [sketch] Note first that it is enough to check those equalities locally, hence, by Darboux's theorem, it is enough to prove the result for M a symplectic vector space.

The second observation is that if the equalities hold for (M_1,ω_1) and (M_2,ω_2), they also hold for $(M_1 \times M_2, \omega_1 + \omega_2)$.

The last step is therefore to perform the calculation for $M = \mathbf{R}^2$ and $\omega = dx \wedge dy$, in view of the formulas for the symplectic adjoints it suffices to understand the symplectic star in that case: $\star_s\alpha = \alpha$ if $\deg(\alpha) = 1$, $\star_s f = f\omega$, $\star_s f\omega = f$ for any function f. □

Exercise 2.3: Check the steps in the proof of the symplectic Kähler identities. (You may need to prove the equality $\bigwedge^\bullet(V \oplus W) = \bigwedge^\bullet V \otimes \bigwedge^\bullet W$ where $\bigwedge^\bullet = \bigoplus_k \bigwedge^k$ is the exterior algebra.)
By the same procedure, $\star_s\star_s = \mathrm{id}$ and $\star_s e^\omega = e^\omega$ (i.e. $\forall k$, $\star_s \frac{\omega^k}{k!} = \frac{\omega^{n-k}}{(n-k)!}$).

Exercise 2.4: For κ a differential form and $K : \Omega^\bullet(M) \to \Omega^\bullet(M) : \alpha \mapsto \kappa \wedge \alpha$, $[K,d] = 0$ if and only if $d\kappa = 0$. In particular $[L,d] = 0$, also $[L^{\star_s}, d^{\star_s}] = 0$.

Exercise 2.5: For X a vector field, denote by $\iota_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$ the interior product by X: for α a p-form, $\iota_X\alpha$ is the $(p-1)$-form defined by $\iota_X\alpha(X_2,\ldots,X_p) = \alpha(X, X_2, \ldots, X_p)$. The symplectic gradient of X is then[c] $\xi = -\iota_X\omega$. In this exercice no new notation are introduced for the operator $\xi : \Omega^\bullet(M) \to \Omega^\bullet(M) : \alpha \mapsto \xi \wedge \alpha$.

Then $(\iota_X)^{\star_s} = \xi$, $[L, \iota_X] = \xi$ and $[L^{\star_s}, \xi] = -\iota_X$. For a 1-form ϕ, $[L, \phi^{\star_s}] = \phi$.

2.9. *Complex (Analytic) Manifolds*

A complex manifold is locally modelled on $\mathbf{C}^n$ and the transition functions are biholomorphic. Such a complex manifold is of course a manifold (in the "real" sense) and the (real!) tangent space T_mX, for every $m \in X$, inherits a structure of complex vector space.

[c]The minus sign is chosen so that subsequent formulas have a nicer form. (I.e. less minus signs.)

The multiplication by $\sqrt{-1}$ on this space is usually denoted by J: $J \in \mathrm{End}_{\mathbf{R}}(TX)$, $J^2 = -\mathrm{id}$.

Remark 2.8: The data of such an endomorphism J on the tangent bundle of a (real) manifold with $J^2 = -\mathrm{id}$ is called a pseudo-complex structure. Not every pseudo-complex structure comes from a complex structure.

As before, this endomorphism J of TX induces naturally an endomorphism on every bundle constructed from TX, in particular on the exterior algebra $\bigwedge^{\bullet} T^*X$. However the standard notation for this extension is C, C will be mainly used on forms:

$$C : \Omega^p(X; \mathbf{R}) \longrightarrow \Omega^p(X; \mathbf{R}) \quad C^2 = (-1)^p,$$

and its "complexification":

$$C : \Omega^{\bullet}(X; \mathbf{C}) \longrightarrow \Omega^{\bullet}(X; \mathbf{C}).$$

Remark 2.9: From now on, a distinction will be (hopefully systematically) made between real valued forms (i.e. sections of $\bigwedge^p T^*X$) and complex valued forms (i.e. sections of $(\bigwedge^p T^*X) \otimes \mathbf{C} \simeq \bigwedge^p (T^*X \otimes \mathbf{C})$).

The complexification of J is diagonalizable and the bundle $TX \otimes_{\mathbf{R}} \mathbf{C}$ decomposes according to the eigenspaces:

$$TX \otimes_{\mathbf{R}} \mathbf{C} = T^{1,0}X \oplus T^{0,1}X = \ker(J - \sqrt{-1}) \oplus \ker(J + \sqrt{-1}),$$

also $\overline{T^{1,0}X} = T^{0,1}X$. This decomposition of $TX \otimes_{\mathbf{R}} \mathbf{C}$ induces a decomposition of $T^*X \otimes_{\mathbf{R}} \mathbf{C}$ and its exterior algebra:

$$\begin{aligned} T^*X \otimes_{\mathbf{R}} \mathbf{C} &= T^{*1,0}X \oplus T^{*0,1}X \\ \textstyle\bigwedge^{\bullet}(T^*X) \otimes_{\mathbf{R}} \mathbf{C} &= \textstyle\bigwedge^{\bullet}(T^*X \otimes_{\mathbf{R}} \mathbf{C}) \\ &= \textstyle\bigwedge^{\bullet}(T^{*1,0}X \oplus T^{*0,1}X) \\ &= \textstyle\bigwedge^{\bullet}(T^{*1,0}X) \otimes \bigwedge^{\bullet}(T^{*0,1}X) \\ &= \textstyle\bigoplus_{p,q} \bigwedge^p T^{*1,0}X \otimes \bigwedge^q T^{*0,1}X. \end{aligned}$$

Subsequently, the space of complex differential forms acquires a bi-grading refining the grading by the degree:

$$\Omega^l(X; \mathbf{C}) = \bigoplus_{p+q=l} \Omega^{p,q}(X).$$

The operator C is equal to the multiplication by $(\sqrt{-1})^{p-q}$ in restriction to $\Omega^{p,q}(X)$.

Exercise 2.6: In local holomorphic coordinates $(z^1, \dots, z^n)$ on X, give the expressions of forms in $\Omega^{p,q}(X)$. Use this to show that the "type" (p,q) is independant of the coordinates chart.

With respect to this bigrading, the exterior differential d decomposes (uniquely) as the sum of ∂, of bidegree $(1,0)$ and of $\bar\partial$, of bidegree $(0,1)$:

$$d = \partial + \bar\partial$$
$$\partial(\Omega^{p,q}(X)) \subset \Omega^{p+1,q}(X)$$
$$\bar\partial(\Omega^{p,q}(X)) \subset \Omega^{p,q+1}(X).$$

A function f on X is holomorphic if and only if $\bar\partial f = 0$.

Exercise 2.7: Give conditions on the function f so that the form $\alpha = f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \overline{dz^{j_1}} \wedge \cdots \wedge \overline{dz^{j_q}}$ satisfy $\bar\partial\alpha = 0$. What happens for $q = 0$?

2.10. *Kähler Manifold*

A manifold X is *Kähler* if it is Riemannian, symplectic and complex and if the 3 structures $\langle\cdot,\cdot\rangle$, $\omega(\cdot,\cdot)$ and J cohabite nicely:

$$\langle Jv, Jw\rangle = \langle v, w\rangle,\ \forall v, w \in T_mX$$
$$\omega(Jv, Jw) = \omega(v, w) \text{ (equivalently } C\omega = \omega)$$
$$\omega(v, Jw) = \langle v, w\rangle.$$

Definition 2.10: In this context, an *orthogonal frame* $(\epsilon_i)_{i=1,\ldots,n}$ is an orthogonal basis of the Hermitian vector space $T^{1,0}X$. (Compare with Definition 2.2.)

Exercise 2.8: Then $(\bar\epsilon_i)_{i=1,\ldots,n}$ is an orthogonal basis of $T^{0,1}X$.

What is the expression of the symplectic form ω using those 2 basis (and their duals)?

Remark 2.11: The last condition is chosen so that the Hermitian form:

$$(\cdot,\cdot) = \langle\cdot,\cdot\rangle + \sqrt{-1}\omega(\cdot,\cdot)$$

on (TX, J) is $\mathbf{C}$-linear in the second variable.

Note that any two of the above structures determine the third. Also it can be shown that the "integrability" condition on J (i.e. the requirement that it comes from a complex structure) is equivalent to the "integrability" condition on ω (i.e. $d\omega = 0$).

A more pertinent way to assert the compatibility of the 3 structures is to say that the $(0,1)$ part of the (complexified) Levi-Civita connection is equal to the $\bar\partial$ operator.

2.11. *Adjoints and Stars*

Just as in the symplectic setting (see Section 2.7) the Riemannian structure induces a Euclidean structure on $\bigwedge^\bullet T^*X$ as well as a volume form and hence a L^2-scalar product on $\Omega^\bullet(X;\mathbf{R})$ (and also a L^2-Hermitian product on $\Omega^\bullet(X;\mathbf{C})$). The (formal) adjoints of differential operators for that structure will be denoted by a $\star$ in superscript: $d^\star$, $L^\star$, etc. (Those exist already on the more general case of M an oriented Riemannian manifold.)

As suspected, the symplectic star and the Hodge star are compatible on a Kähler manifold, in fact:

$$\star_s = \star C = C\star.$$

Note also that the Riemannian volume form and the symplectic volume form are equal. In turn the symplectic adjoint and the Riemannian adjoint are related:

$$d^\star = -C^{-1}d^{\star_s}C = Cd^{\star_s}C^{-1}.$$

(The last equality follows from $C^2\alpha = (-1)^p\alpha$ and $C^2 d^{\star_s}\alpha = (-1)^{p-1}d^{\star_s}\alpha$ if α is a p-form). Similarly

$$L^\star = C^{-1}L^{\star_s}C = CL^{\star_s}C^{-1} = L^{\star_s}.$$

($L^{\star_s}$ —and hence $L^\star$— commutes with C since $C\omega = \omega$.)

Exercise 2.9: Prove the above formulas, and $\star\star = (-1)^{\deg}$. The adjoint $C^\star$ is equal to C^{-1}.

Exercise 2.10: For A a 1-form, $A^\star = CA^{\star_s}C^{-1} = -C^{-1}A^{\star_s}C$.

For α a $(1,0)$-form, $\sqrt{-1}[L,\alpha^\star] = \bar{\alpha}$. (Hint: exercise 2.5 with $\phi = \alpha + \bar{\alpha}$ and decomposition into "types").

2.12. *Kähler Identities*

From the symplectic Kähler identities and the previous paragraph, one gets

$$[L^\star, d] = C^{-1}d^\star C \quad \text{and} \quad [L, d^\star] = -C^{-1}dC$$

or, decomposing following the bidegree:

$$\begin{aligned} &[L^\star,\bar\partial] = C^{-1}\partial^\star C = \sqrt{-1}\partial^\star && [L,\bar\partial^\star] = -\sqrt{-1}\partial \\ &[L^\star,\partial] = -\sqrt{-1}\bar\partial^\star && [L,\partial^\star] = \sqrt{-1}\bar\partial. \end{aligned}$$

Remark 2.12: It is frequent to see the notation Λ for the operator $L^\star$.

3. Vector Bundles

In this section, the basics of vector bundles are introduced. This inculdes their trivializations and changes of trivializations as well as their spaces of sections and differential forms, the gauge transformation and the gauge group. Real and complex vector bundles will be treated in an uniform way.

3.1. *Trivializations*

A vector bundle is a smooth family of vector spaces. More precisely, if M is a manifold, a vector bundle over M is a manifold E together with a submersion $p : E \to M$ such that every fiber $p^{-1}(m)$, $m \in M$, has a structure of a $\mathbf{K}$-vector space ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) "varying smoothly" with m. The shortest way to make precise this last property is to assume *local triviality*: given $m \in M$, there is an open neighborhood U of m in M and a diffeomorphism (a *trivialization*)

$$\psi_U : p^{-1}(U) \to U \times \mathbf{K}^d,$$

such that

(1) $\mathrm{pr}_1 \circ \psi_U = p$ (where $\mathrm{pr}_1 : U \times \mathbf{K}^d \to U \mid (m', v) \mapsto m'$)
(2) for each m' in U, ψ_U restricts to a linear isomorphism between $p^{-1}(m')$ and $\mathrm{pr}_1^{-1}(m') = \{m'\} \times \mathbf{K}^d$.

3.2. *Changes of Trivializations*

If ψ_U and ψ_V are two trivializations defined on $p^{-1}(U)$ and $p^{-1}(V)$ respectively (and U and V are open subsets of M) then there exists a smooth map (the *change of trivializations*)

$$g_{U,V} : U \cap V \longrightarrow \mathrm{GL}_d(\mathbf{K})$$

such that

$$\begin{aligned} \psi_U \circ \psi_V^{-1} : U \cap V \times \mathbf{K}^d &\longrightarrow U \cap V \times \mathbf{K}^d \\ (m, v) &\longmapsto (m, g_{U,V}(m) \cdot v). \end{aligned}$$

Exercise 3.1: Prove that $\psi_U \circ \psi_V^{-1}$ as the claimed form. Why is it more correct to write $(\psi_U|_{p^{-1}(U\cap V)}) \circ (\psi_V|_{p^{-1}(U\cap V)})^{-1}$?

The change of trivializations $g_{V,U}$ is the inverse of $g_{U,V}$: $\forall m \in M$, $g_{V,U}(m) = g_{U,V}(m)^{-1}$. Also $g_{U,U}(m) = \mathrm{id}$ for all m in U.

If ψ_W is a third trivialization, then the 2 new changes of trivializations $g_{U,W}$ and $g_{V,W}$ are part of an obvious compatibility condition (a "cocycle" condition):

$$g_{U,V}(m)g_{V,W}(m) = g_{U,W}(m) \quad \forall m \in U \cap V \cap W. \qquad \text{(COMP)}$$

This follows from the equality $\psi_U \circ \psi_V^{-1} \circ \psi_V \circ \psi_W^{-1} = \psi_U \circ \psi_W^{-1}$.

3.3. *From Cocyles to Bundles*

Conversely, suppose given an open covering $\mathcal{U}$ of M ($\mathcal{U} \subset \mathcal{P}(M)$ —maybe it would be better to say there is a map $\mathcal{U} \to \mathcal{P}(M)$— every U in $\mathcal{U}$ is open and $\cup_{U\in\mathcal{U}} U = M$) and for every U and V in $\mathcal{U}$ a smooth map $g_{U,V} : U \cap V \to \mathrm{GL}_d(\mathbf{K})$ such that the relation (COMP) is satisfied for all U, V and W in $\mathcal{U}$, then there exists a (essentially unique) vector bundle E over M with trivializations $\{\psi_U\}_{U\in\mathcal{U}}$ such that the changes of trivializations are precisely the maps $g_{U,V}$.

Exercise 3.2: The relation (COMP) implies that

$$\begin{aligned} g_{U,U}(m) &= \mathrm{id} && \forall m \in U \\ g_{V,U}(m) &= g_{U,V}(m)^{-1} && \forall m \in U \cap V. \end{aligned}$$

Exercise 3.3: [Construction of E] Given $\mathcal{U}$ and the $g_{U,V}$ (U, V in $\mathcal{U}$) satisfying (COMP) construct an equivalence relation $\sim$ on the disjoint union $\coprod_{\mathcal{U}} U \times \mathbf{K}^d$ so that $E = (\coprod_{\mathcal{U}} U \times \mathbf{K}^d)/\sim$ is the sought for bundle.

3.4. *Linear Algebra*

Every construction (or maybe most constructions) in linear algebra has its counterpart for vector bundles. We will mainly use:

- E^* the dual of E,
- $\mathrm{End}(E)$, endomorphisms
- $\overline{E}$, the complex conjugate (when $\mathbf{K} = \mathbf{C}$)
- $E \otimes_{\mathbf{R}} \mathbf{C}$ the complexification ($\mathbf{K} = \mathbf{R}$)
- $\overline{E}^* \otimes E^*$ and inside it the subset of Hermitian forms,
- $E \otimes F$ tensor product, etc.

Exercise 3.4: For a complex vector space V, what is $\overline{V}$? [More generally, for V a k-vector space and $\sigma \in \mathrm{Aut}(k)$, what is V^σ?].

Give the changes of trivializations for the bundle $\overline{E}$. What happens when $\overline{E}$ is isomorphic to E?

3.5. *The Gauge Group*

In the category of vector bundles (over a base manifold M say) there are natural notion of morphisms, isomorphisms, etc.

Exercise 3.5: A bundle morphism $E \to F$ is the same thing as a section of the vector bundle $\mathrm{Hom}(E, F)$.

A *gauge isomorphism* is an isomorphism of the bundle E.

Exercise 3.6: Why can a gauge isomorphism be seen either as a map $E \to E$ or as a section of $\mathrm{Aut}(E)$? What is the nature of the bundle $\mathrm{Aut}(E)$? (i.e. is there an additional structure on its fibers? etc.)

The group of gauge automorphisms is denoted by $\mathcal{A}ut(E)$. A gauge isomorphism g induces a gauge isomorphism on any bundle constructed from E but also on the bundles $\bigwedge^p T^*M \otimes E$, it will therefore induce an action on the space of sections of those bundles (see Section 3.6).

Exercise 3.7: In particular the isomorphism induced on $\mathrm{End}(E)$ is

$$\begin{aligned} \mathrm{End}(E) &\longrightarrow \mathrm{End}(E) \\ A &\longmapsto gAg^{-1}. \end{aligned}$$

(What is the meaning of the empty sign for the composition law involved in the notation gAg^{-1}?) Work out other examples: $\mathrm{End}(E^*)$, sesquilinear forms, etc.

Exercise 3.8: Express the trivializations of $\mathrm{Aut}(E)$ and the changes of trivializations in term of those of E. Give a gauge isomorphism in local coordinates (i.e. in the "charts" given by the trivializations.)

Remark 3.1: There is of course a strong link between trivializations and gauge isomorphism. In physics literature and sometimes in mathematics, a trivialization is called a "choice of gauge" or "fixing a gauge" and a change of trivialization is a "change of gauge". This link is quite apparent when comparing the formulas for the action of a gauge transformations with the formulas for changes of trivializations.

3.6. *Spaces of Sections*

A *section* of E is a smooth map $\sigma : M \to E$ such that $p \circ \sigma = \mathrm{id}_M$. The space of sections is a nice topological $\mathbf{K}$-vector space denoted by $\Gamma(M; E)$ or sometimes $\Omega^0(M; E)$. We already met some of theses spaces when we mentioned differential forms:

$$\Omega^p(M; \mathbf{K}) = \Gamma\Big(M; \bigwedge^p T^*M \otimes_{\mathbf{R}} \mathbf{K}\Big),$$

or when X is a complex manifold:

$$\Omega^{p,q}(X) = \Gamma\Big(X; \bigwedge^{p} T^{*1,0}X \otimes \bigwedge^{q} T^{*0,1}X\Big).$$

Generally the sections of $\bigwedge^p T^*M \otimes E$ are called the p-forms with coefficients in E (or sometimes with "values" in E) and their space is denoted by

$$\Omega^p(M; E) = \Gamma\Big(M; \bigwedge^{p} T^*M \otimes E\Big)$$

and for a complex vector bundle E on an analytic manifold X:

$$\Omega^{p,q}(X; E) = \Gamma\Big(X; \bigwedge^{p} T^{*1,0}X \otimes \bigwedge^{q} T^{*0,1}X \otimes E\Big).$$

Exercise 3.9: [$\Omega^\bullet(M; E)$ is a graded $\Omega^\bullet(M; \mathbf{K})$-module] Construct (or define)

$$\begin{aligned} \wedge : \Omega^p(M; \mathbf{K}) \times \Omega^{p'}(M; E) &\longrightarrow \Omega^{p+p'}(M; E) \\ (\alpha, \sigma) &\longmapsto \alpha \wedge \sigma. \end{aligned}$$

This endows $\Omega^\bullet(M; E)$ with the structure of a graded $\Omega^\bullet(M; \mathbf{K})$-module. As a module it is generated by $\Omega^0(M; E)$.

Remark 3.2: It is sometimes a good thing to omit the wedge $\wedge$ symbol to avoid overloaded formulas (or even ambiguous formulas). Therefore this will be yet another composition law denoted by the empty sign.

4. Flat Bundles

In this section flat bundles are introduced using also the point of view of local triviality (hence the objects given here are really "local systems"). The correspondence with linear representations of the fundamental group is detailled. Connections will be presented later.

4.1. *Flat Structures*

A *flat structure* on a vector bundle E (or for short a *flat bundle*) is a family $\{\psi_U\}_{U\in\mathcal{U}}$ of trivializations, where $\mathcal{U}$ is an open cover of M, such that the changes of trivializations $g_{U,V}$ are locally constants.

Note that not every bundle has a flat structure and that the same bundle can have non equivalent flat structures. In a sense the study of equivalence classes of flat structures is the very subject of this paper.

There is a natural notion of isomorphism between flat bundles.

Exercise 4.1: If $(E, \{\psi_U\}_{U\in\mathcal{U}})$ and $(E', \{\psi_{U'}\}_{U'\in\mathcal{U}'})$ are two flat bundles, what is a morphism between E and E'? an isomorphism?

4.2. *Flat Bundles and Representations of the Fundamental Group*

—We suppose from now on that the base manifold M is connected.—

Every flat bundle $(E, \psi_U, g_{U,V})$ gives rise to a representation $\rho : \pi_1(M, m_0) \to \mathrm{GL}_d(\mathbf{K})$ as follow:

- Once for all, fix $U \in \mathcal{U}$ containing m_0.
- for any loop $\gamma : [0,1] \to M$, $\gamma(0) = \gamma(1) = m_0$ there are:
  - a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_N \leq t_{N+1} = 1$ of $[0,1]$,
  - U_0, U_1, ..., U_N belonging to $\mathcal{U}$ such that
    1. $U_0 = U_N = U$
    2. and, for all $i = 0, \ldots, N$, $\gamma([t_i, t_{i+1}]) \subset U_i$.
- We then define:

$$\rho(\gamma) = g_{U_N,U_{N-1}}(\gamma(t_N)) \cdots g_{U_2,U_1}(\gamma(t_2)) g_{U_1,U_0}(\gamma(t_1)).$$

Exercise 4.2: Show that $\rho(\gamma)$ does not depend on the choices of the sequence $(U_0, \ldots, U_N)$ or of the subdivision. I.e. for another subdivision $(t'_i)_{i \leq N'}$ and another sequence $(U'_i)_{i \leq N'}$ as above then the product $g_{U'_{N'},U'_{N'-1}}(\gamma(t'_{N'})) \cdots g_{U'_1,U'_0}(\gamma(t'_1))$ is equal to $\rho(\gamma)$. (Hint: first reduce to the case where the 2 subdivisions are the same, then use the cocycle property (COMP) to change "step by step" the sequence (U'_i) into the sequence (U_i)).

Exercise 4.3: Every flat bundle on $M = [0,1]$ is (isomorphic to) the trivial flat bundle.

It is easy to see that $\rho(\gamma)$ does not change under deformations of γ (this is where the fact that the bundle is flat is involved) and hence depends only on the homotopy class of γ. Therefore, associated with the flat bundle E, is defined a homomorphism:

$$\rho : \pi_1(M, m_0) \longrightarrow \mathrm{GL}_d(\mathbf{K})$$
$$\rho \in \mathrm{Hom}\big(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K})\big).$$

Exercise 4.4: Define the law of composition on $\pi_1(M, m_0)$ so that ρ is a homomorphism.

In fact this representation ρ, called the *holonomy representation*, depends also of the choice of U: changing U to $U' \in \mathcal{U}$ (again with $U' \ni m_0$) amounts to changing the representation ρ to:

$$\begin{aligned} \pi_1(M, m_0) &\longrightarrow \mathrm{GL}_d(\mathbf{K}) \\ \gamma &\longmapsto g\rho(\gamma)g^{-1} \end{aligned}$$

with $g = g_{U,U'}(m_0)$.

More generally if E and E' are two isomorphic flat bundles, the two corresponding holonomy representations are only conjugated.

One can summarize this discussion with the following map:

$$\frac{\{\text{flat bundle of rank } d \text{ over } M\}}{\text{isomorphism}} \xrightarrow{\text{Hol}} \frac{\operatorname{Hom}(\pi_1(M,m_0), \mathrm{GL}_d(\mathbf{K}))}{\text{conjugacy}}.$$

4.3. *From Representations to Flat Bundles*

It turns out that Hol is a bijection. This can be seen via the following construction which provides an inverse to the map Hol. Let $\rho : \pi_1(M, m_0) \to \mathrm{GL}_d(\mathbf{K})$ be a morphism, then the trivial bundle

$$\widetilde{M} \times \mathbf{K}^d$$

over the universal cover $\widetilde{M}$ of M has an obvious flat structure together with an action of $\pi_1(M, m_0)$ preserving the flat structure:

$$\gamma \cdot (\tilde{m}, v) = (\gamma\tilde{m}, \rho(\gamma)v) \quad \forall \gamma \in \pi_1(M, m_0), \tilde{m} \in \widetilde{M}, v \in \mathbf{K}^d.$$

Here $\tilde{m} \to \gamma\tilde{m}$ is the natural action of $\pi_1(M, m_0)$ on the universal cover $\widetilde{M}$. The quotient of the trivial bundle $\widetilde{M} \times \mathbf{K}^d$ by this action is denoted

$$E_\rho = \Gamma \backslash (\widetilde{M} \times \mathbf{K}^d),$$

it is naturally a flat bundle over the base $M = \Gamma \backslash \widetilde{M}$.

Exercise 4.5: E_ρ depends only (up to isomorphism) on the conjugacy class of ρ. The holonomy representation of E_ρ is ρ.

Exercise 4.6: Let E be a flat bundle over M. Then the pull back $\widetilde{E}$ over $\widetilde{M}$ is a flat bundle together with an action of $\pi_1(M, m_0)$. The flat bundle $\widetilde{E}$ is the trivial flat bundle: $\widetilde{E} \simeq \widetilde{M} \times \mathbf{K}^d$ (Hint: use a method similar to the construction of the holonomy representation to extend a fixed trivialization around $\tilde{m}_0$ first along paths, then everywhere by showing the desired homotopy invariance) and the action of $\pi_1(M, m_0)$ has the form $\gamma \cdot (m, v) = (\gamma m, \rho(\gamma)v)$ for some representation ρ. Furthermore $\rho = \operatorname{Hol}(E)$ and at last E is isomorphic to E_ρ.

Conclude that the map Hol is a bijection.

Exercise 4.7: Prove that $\operatorname{Hom}(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K}))$ is in bijection with

$$\frac{\{(E, (\epsilon_i)) \mid E \xrightarrow{p} M \text{ flat bundle, } (\epsilon_i) \text{ is a basis of } p^{-1}(m_0)\}}{\text{isomorphism}}.$$

We will later characterize flat bundles (or flat structures on a given bundle) with the help of connections. We only take note for the moment that there is, for a (non flat!) bundle E, no natural way to differentiate sections of E or more generally forms with coefficients in E. Connections are here to fill this gap.

4.4. *Holomorphic Vector Bundle*

The investigation of the different families of vector bundles continues in this section with the study of vector bundles on a complex manifold. Here the correspondence between holomorphic structures on a given manifold and integrable pseudo-connections (those are a special class of differential operators) is explained. The gauge group acts on the space of holomorphic structures as well as on the space of pseudo-connections and this action is compatible with the mentionned correspondence.

4.5. *Holomorphic Bundles and their Trivializations*

When the base space X is a complex manifold, it makes sense to consider when $p : E \to X$ is a holomorphic map between complex manifolds (where E is a $\mathbf{C}$-vector bundle). One can show that in this situation there are holomorphic trivializations:

$$\psi_U : p^{-1}(U) \longrightarrow U \times \mathbf{C}^d.$$

Consequently, the change of trivializations:

$$g_{U,V} : U \cap V \longrightarrow \mathrm{GL}_d(\mathbf{C})$$

is holomorphic (meaning that every coordinates entry of the matrix $g_{U,V}$ is a holomorphic function on $U \cap V$).

Conversely given an open cover $\mathcal{U}$ of X and holomorphic maps $g_{U,V} : U \cap V \to GL_d(\mathbf{C})$ satisfying the compatibility condition (COMP), then a holomorphic vector bundle can be constructed with those changes of trivializations.

One could also speak of a *holomorphic structure* on a given smooth bundle $p : E \to X$. It is likely that not every smooth bundle has a holomorphic structure (but it may be as well an open question); a given vector bundle can have different holomorphic structure (this is related to the theorem of Narasimhan and Seshadri presented below).

4.6. *Pseudo-Connection*

A holomorphic structure $\mathcal{E}$ on a smooth vector bundle E can be characterize with the help of an operator:

$$\bar\partial^{\mathcal{E}} : \Omega^{\bullet,\bullet}(X;E) \longrightarrow \Omega^{\bullet,\bullet}(X;E)$$

(1) of degree $(0,1)$: $\bar\partial^{\mathcal{E}}\sigma$ is of degree $(p,q+1)$ if σ is of degree (p,q).
(2) satisfying the Leibniz rule: if $\alpha \in \Omega^{p,q}(X)$ and $\sigma \in \Omega^{k,l}(X;E)$

$$\bar\partial^{\mathcal{E}}(\alpha \wedge \sigma) = (\bar\partial\alpha) \wedge \sigma + (-1)^{p+q}\alpha \wedge \bar\partial^{\mathcal{E}}\sigma.$$

(3) A section σ of E (i.e. $\sigma \in \Omega^0(X;E) = \Omega^{0,0}(X;E)$) is holomorphic if and only if $\bar\partial^{\mathcal{E}}\sigma = 0$.

Definition 4.1: An operator $\bar\partial^E : \Omega^{\bullet}(X;E) \to \Omega^{\bullet,\bullet}(X;E)$ satisfying (1) and (2) above is called a *pseudo-connection*.

Exercise 4.8: Express a pseudo-connection in a trivialization.

It is easy to see that a pseudo-connection is entirely determined by the map:

$$\bar\partial^E : \Omega^0(X;E) \longrightarrow \Omega^{0,1}(X;E).$$

When $\mathcal{E}$ is a holomorphic structure on E, the pseudo-connection in degree 0 is defined by the formula

$$\bar\partial^{\mathcal{E}}(f\sigma) = \bar\partial f \wedge \sigma$$

when σ is a holomorphic section of $\mathcal{E}$ and f is any function.

Exercise 4.9: Check the details of the construction of $\bar\partial^{\mathcal{E}}$. Alternatively show the existence of $\bar\partial^{\mathcal{E}}$ by checking compatibility of the constructions in trivializations.

4.7. *Pseudo-Curvature*

It is not true that every pseudo-connection on E comes from a holomorphic structure. Given a pseudo-connection $\bar\partial^E$ on E it means that there are not enough sections σ of E for which $\bar\partial^E\sigma = 0$ (those should be the holomorphic sections). This lack of enough solutions to $\bar\partial^E\sigma = 0$ (often called "default of integrability") is precisely quantified by the pseudo-curvature:

Lemma 4.2: *The operator*

$$\begin{aligned}(\bar\partial^E)^2 : \Omega^{\bullet,\bullet}(X;E) &\longrightarrow \Omega^{\bullet,\bullet+2}(X;E)\\ \sigma &\longmapsto \bar\partial^E(\bar\partial^E\sigma)\end{aligned}$$

is $\mathcal{C}^\infty$*-linear* $((\bar\partial^E)^2(\alpha \wedge \sigma) = \alpha \wedge (\bar\partial^E)^2\sigma$ *for all* $\alpha \in \Omega^\bullet(X;\mathbf{C})$ *and all* $\sigma \in \Omega^\bullet(X;E))$*.*

This implies that there is a (unique) $(0,2)$*-form with coefficients in* $\mathrm{End}(E)$ *—called the* pseudo-curvature *of* $\bar\partial^E$*, denoted by* $G = G(\bar\partial^E)$*— such that*

$$(\bar\partial^E)^2\sigma = G\sigma \quad \forall \sigma \in \Omega^\bullet(X;E).$$

Remark 4.3: The empty sign for the composition law in the notation "$G\sigma$" involves 2 operations: the wedge product on forms and the evaluation of endomorphisms (elements in $\mathrm{End}(E)$) on sections of E. Explicitly if $G = \sum_i \alpha_i A_i$, $\alpha_i \in \Omega^{0,2}(X)$, $A_i \in \Omega^0(X;\mathrm{End}(E))$ and if $\sigma = \sum_j \beta_j\sigma_j$, $\beta_j \in \Omega^{p,q}(X)$, $\sigma_j \in \Omega^0(X;E)$ then $G\sigma = \sum_{i,j} \alpha_i \wedge \beta_j A_i(\sigma_j)$.

Exercise 4.10: Using (3 times) the Leibniz rule —and the fact that $\bar\partial^2 = 0$ on differential forms— prove the "locality" of $(\bar\partial^E)^2$.

Alternatively express $(\bar\partial^E)^2$ in a trivialization and conclude the existence of the pseudo-curvature (checking the compatibility under change of trivializations).

Given a pseudo-connection $\bar\partial^E$ on a smooth vector bundle E, all the vector bundles constructed from E have also a natural pseudo-connection: E^*, $\mathrm{End}(E)$, etc. All those pseudo-connections are such that the natural maps between those bundles must be "$\bar\partial^E$-holomorphic". For example for E^* the evaluation map $E^* \times E \to \mathbf{C} \mid (f,e) \mapsto f(e)$ has to be holomorphic and for $\mathrm{End}(E)$ this is the map $\mathrm{End}(E) \times E \to E \mid (A,e) \to A(e)$ that is required to be holomorphic.

Exercise 4.11: Let $\bar\partial^E$ be a pseudo-connection on E. There is a unique operator $\bar\partial^{E^*}$ such that for every section σ of E and ϕ of E^* (so that $\phi(\sigma)$ is a function) one has

$$\bar\partial(\phi(\sigma)) = (\bar\partial^{E^*}\phi)(\sigma) + \phi(\bar\partial^E(\sigma)).$$

Furthermore $\bar\partial^{E^*}$ is the sought for pseudo-connection on E^*.

Similarly, for A a section of $\mathrm{End}(E)$ and σ a section of E, the formula

$$\bar\partial^E(A(\sigma)) = (\bar\partial^{\mathrm{End}(E)}A)(\sigma) + A(\bar\partial^E\sigma)$$

defines the pseudo-connection on $\mathrm{End}(E)$.

Exercise 4.12: [Bianchi identity for pseudo-connection] $\bar\partial^{\mathrm{End}(E)}G(\bar\partial^E) = 0$.

The following theorem makes the link between holomorphic structure and pseudo-connections with vanishing pseudo-curvature. (see[2] for a proof.)

Theorem 4.4: *Let* $\bar\partial^E$ *be a pseudo-connection on* E *over* X*.*

Then there exists a holomorphic structure $\mathcal{E}$ on E such that $\bar\partial^E = \bar\partial^{\mathcal{E}}$ $\Leftrightarrow$ the pseudo-curvature $G(\bar\partial^E)$ vanishes identically.

Exercise 4.13: If $\mathcal{E}$ is a holomorphic structure on E, then $G(\bar\partial^{\mathcal{E}}) = 0$.

Exercise 4.14: [uniqueness in the above theorem] If $\mathcal{E}$ and $\mathcal{E}'$ are 2 holomorphic structures on E such that $\bar\partial^{\mathcal{E}} = \bar\partial^{\mathcal{E}'}$ then $\mathcal{E} = \mathcal{E}'$.

Hence there is a bijection:

$\{$holomorphic structure on $E\} \leftrightarrow \{$pseudo-connection $\bar\partial^E$ with $(\bar\partial^E)^2 = 0\}$.

It can be checked that this bijection is equivariant with respect to the gauge group action on these 2 spaces.

4.8. *The Space of Pseudo-Connections*

Lemma 4.5: *Let $\bar\partial_1$, $\bar\partial_2$ be 2 pseudo-connections on E.*

Then the operator $A = \bar\partial_2 - \bar\partial_1$ is $\mathcal{C}^\infty$-linear ($A(\alpha \wedge \sigma) = \alpha \wedge A(\sigma)$) and hence can be identified with a $(0,1)$-form with coefficients in $\mathrm{End}(E)$.

Exercise 4.15: Prove the lemma.

Remark 4.6: We will frequently make no distinctions (albeit notational) between a form with coefficients in $\mathrm{End}(E)$ and the corresponding operator acting on $\Omega^\bullet(X;E)$. This introduces further notational ambiguities that can be only overcome with practice. The absence of different signs (in fact the absence of any signs) for the different laws of compositions or the action of operators does not help clearing those ambiguities.

Conversely

Lemma 4.7: *If $\bar\partial_1$ is a pseudo-connection and $A \in \Omega^{0,1}(X; \mathrm{End}(E))$ then $\bar\partial_2 = \bar\partial_1 + A$ is a pseudo-connection.*

Exercise 4.16: Prove the lemma. Give an expression for $G(\bar\partial_1 + A)$. How does this relates to the expression in local trivializations? (see Exercise 4.8.)

Corollary 4.8: *The space of pseudo-connection on E is an affine space over the vector space $\Omega^{0,1}(X; \mathrm{End}(E))$.*

Therefore the space of all pseudo-connections has the most simple topology.

4.9. *The Action of the Gauge Group*

Any gauge transformation $g \in \mathcal{A}ut(E)$ acts on the space of section of E (by composition) as well as on the forms with coefficients in E and also with coefficients in bundles constructed from E. Explicitly, if σ is a section of E then $g\sigma$ is the (fiberwise) image of σ by g, and if $\sum \alpha_i \sigma_i$ is a differential form with coefficients in E then $g(\sum \alpha_i \sigma_i) = \sum \alpha_i g \sigma_i$.

For a pseudo connection $\bar{\partial}^E$ another pseudo-connection $g \cdot \bar{\partial}^E$ is obtained by

$$g \cdot \bar{\partial}^E = \text{“}g\bar{\partial}^E g^{-1}\text{”} : \Omega^\bullet(X;E) \longrightarrow \Omega^\bullet(X;E)$$
$$\sigma \longmapsto g(\bar{\partial}^E(g^{-1}\sigma)).$$

Explanation of the notations:

- $g^{-1}\sigma$ is the action of g^{-1} on $\Omega^\bullet(X;E)$.
- $\bar{\partial}^E(g^{-1}\sigma)$ is the image of $g^{-1}\sigma$ by the operator $\bar{\partial}^E$.
- $g(\bar{\partial}^E(g^{-1}\sigma))$ is again the action of g.

Exercise 4.17: $g \cdot \bar{\partial}^E$ is a pseudo-connection.

This defines an action of $\mathcal{A}ut(E)$ on the space of pseudo-connections.

Lemma 4.9: *One has the relation:*

$$g \cdot \bar{\partial}^E - \bar{\partial}^E = g\big(\bar{\partial}^{\mathrm{End}(E)}(g^{-1})\big) = -g^{-1}\big(\bar{\partial}^{\mathrm{End}(E)}(g)\big).$$

Explanation: We wrote here the equality between an operator (the difference between 2 pseudo-connections) and a $(0,1)$-form with coefficients in $\mathrm{End}(E)$ —c.f. the above remark 4.6. Here g^{-1} is considered as a section of $\mathrm{End}(E)$ and $\bar{\partial}^{\mathrm{End}(E)}$ (the extension of $\bar{\partial}^E$ to $\mathrm{End}(E)$) acts on this section. The “product” $g(\bar{\partial}^{\mathrm{End}(E)}g^{-1})$ is the composition of an element of $\Omega^0(X;\mathrm{End}(E))$ with an element of $\Omega^{0,1}(X;\mathrm{End}(E))$ leading to an element in $\Omega^{0,1}(X;\mathrm{End}(E))$, this of course involves the composition of endomorphisms $\mathrm{End}(E) \times \mathrm{End}(E) \to \mathrm{End}(E)$.

Exercise 4.18: Prove the lemma.

Remark 4.10: This leads, forgetting the superscripts and the parenthesis, to the abysmal identity:

$$g\bar{\partial}g^{-1} = \bar{\partial} + g\bar{\partial}g^{-1}!!!$$

Exercise 4.19: Give explicitly the action $(g,\mathcal{E}) \to g \cdot \mathcal{E}$ of the gauge group onto the space of holomorphic structures on E. Prove that $g \cdot \bar{\partial}^{\mathcal{E}} = \bar{\partial}^{g \cdot \mathcal{E}}$.

Lemma 4.11: $G(g \cdot \bar{\partial}^E) = gG(\bar{\partial}^E)g^{-1}$.

HAving in mind that the space of pseudo-connections is an affine space, the following result is easily proved.

Lemma 4.12: *The tangent space at $\bar{\partial}^E$ to the space of pseudo-connections is*

$$\Omega^{0,1}(X; \mathrm{End}(E)).$$

The tangent space to the orbit $\mathcal{A}ut(E) \cdot \bar{\partial}^E$ is the image of

$$\bar{\partial}^{\mathrm{End}(E)} : \Omega^0(X; \mathrm{End}(E)) \longrightarrow \Omega^{0,1}(X; \mathrm{End}(E)).$$

(As above, the induced pseudo-connection on $\mathrm{End}(E)$ *is denoted $\bar{\partial}^{\mathrm{End}(E)}$.)*

Exercise 4.20: Prove the lemmas.

5. Flat Bundles and Connections

We have characterized (or parametrized) the holomorphic structures with the help of pseudo-connections. Our objective is now to perform the similar program for flat bundles and (flat!) connections. The vanishing of the curvature will be the key notion. Moreover connections (and sometimes non-flat connections) will have their importance as they appear in the Hitchin-Kobayashi correspondence. An instance of this will be the relation between pseudo-connections and unitary connections performed via the Chern connections.

5.1. *Connections*

Let us star with flat bundles or with flat structures on a given bundle. Given a smooth $\mathbf{K}$-vector bundle E over a base M (a manifold) ($\mathbf{K} = \mathbf{R}$ or$\mathbf{C}$), the notation $\mathbb{E}$ will be used for a flat structure on E.

Lemma 5.1: *Let $\mathbb{E}$ be a flat structure on E. Then there exists a (unique) operator:*

$$D^{\mathbb{E}} : \Omega^\bullet(M; E) \longrightarrow \Omega^\bullet(M; E)$$

(1) *of degree 1, $D^{\mathbb{E}}\sigma$ is of degree $p+1$ if $\sigma \in \Omega^p(M; E)$.*
(2) *satisfying the Leibniz rule:*

$$D^{\mathbb{E}}(\alpha \wedge \sigma) = (d\alpha) \wedge \sigma + (-1)^{\deg(\alpha)} \alpha \wedge D^{\mathbb{E}}\sigma.$$

(3) *A section $\sigma \in \Omega^0(M; E)$ is flat (i.e. in the trivializations of the flat structure $\mathbb{E}$, the section σ becomes a (locally) constant map $U \to \mathbf{K}^d$) if and only if $D^{\mathbb{E}}\sigma = 0$.*

Definition 5.2: An operator $D : \Omega^\bullet(M;E) \to \Omega^\bullet(M;E)$ satisfying (1) and (2) of the above lemma is called a *connection*.

Exercise 5.1: Every bundle admits (at least) one connection.

Remark 5.3: In (3) the statement should be: the restriction of σ to some open set $U \subset M$ is flat if and only if the restriction of $D^{\mathbb{E}}\sigma$ to U is zero.

One can check that a connection D is completely determined by its action in degree 0:

$$D : \Omega^0(M;E) \longrightarrow \Omega^1(M;E)$$

(that verifies $D(f\sigma) = df \wedge \sigma + fD\sigma$).

Any connection on E induces a connection on any bundle constructed from E —the relevant bundle will usually be indicated with a superscript, however sometimes this convention could not be respected. Also a connection $D^{E_1} = D_1$ on E_1 and a connection $D^{E_2} = D_2$ on E_2 induces a connection $D^{E_1 \otimes E_2} =$ "$D_1 \otimes D_2$" on $E_1 \otimes E_2$ (see Exercise below).

Exercise 5.2: "Define" the induced connection on E^*, show that

$$d\phi(\sigma) = (D^{E^*}\phi)(\sigma) + \phi(D^E\sigma) \quad \forall \sigma \in \Omega^0(M;E),\ \phi \in \Omega^0(M;E^*).$$

Give formula for the connection $D^{E_1 \otimes E_2}$. Why is the notation $D_1 \otimes D_2$ inappropriate? (Some care is in order here, the operators that will appear here act naturally on $\Omega^\bullet(M;E_1) \otimes \Omega^\bullet(M;E_2)$ but *not* on $\Omega^\bullet(M;E_1 \otimes E_2)$ —of course there is a map $\Omega^\bullet(M;E_1) \otimes \Omega^\bullet(M;E_2) \to \Omega^\bullet(M;E_1 \otimes E_2)$.)

5.2. *Curvature*

The default of integrability ("lack of flatness") of a connection is measured by its curvature.

Lemma 5.4: *Let D^E be a connection on E. Then the operator*

$$(D^E)^2 : \Omega^\bullet(M;E) \longrightarrow \Omega^{\bullet+2}(M;E)$$

is $\mathcal{C}^\infty$-linear ($(D^E)^2(\alpha \wedge \sigma) = \alpha \wedge (D^E)^2\sigma$ for $\alpha \in \Omega^\bullet(M;\mathbf{R})$ and $\sigma \in \Omega^\bullet(M;E)$).

Hence there exists a (unique) 2-form $F = F(D^E)$ with coefficients in $\mathrm{End}(E)$ *so that $(D^E)^2\sigma = F\sigma$ for all σ.*

Exercise 5.3: Give local expressions for D^E and $F(D^E)$.

Exercise 5.4: Express $F(D_1 \otimes D_2)$ with $F(D_1)$ and $F(D_2)$. (see exercise 5.2.)

Exercise 5.5: If $D^{\mathbb{E}}$ is the connection of a flat structure on the bundle E, then $F(D^{\mathbb{E}}) = 0$.

Exercise 5.6: [Bianchi identity] $D^{\mathrm{End}(E)} F(D^E) = 0$.

Theorem 5.5: *A connection D^E is flat (i.e. there exists a flat structure $\mathbb{E}$ on E such that $D^E = D^{\mathbb{E}}$) if and only if $F(D^E) = 0$.*

Exercise 5.7: If $\mathbb{E}_1$ and $\mathbb{E}_2$ are 2 flat structures on E such that $D^{\mathbb{E}_1} = D^{\mathbb{E}_2}$ then $\mathbb{E}_1 = \mathbb{E}_2$.

Therefore there is a one-to-one correspondence:

$$\{\text{flat structures on } E\} \leftrightarrow \{\text{connections } D \text{ such that } F(D) = 0\}$$

and this bijection is equivariant with respect to the gauge group action on these 2 spaces.

Exercise 5.8: Describe the action of $\mathcal{A}ut(E)$ on the space of flat structures as well as on the space of connections. Show the equivariance of the above correspondence.

We have previously seen that the space of isomorphism classes of flat bundles is in bijection with conjugacy classes of representations. To isolate the representations corresponding to the $\mathcal{C}^\infty$-vector bundle E, let us denote by

$$\mathrm{Hom}_E\big(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K})\big)$$

the subset of representations $\rho : \pi_1(M, m_0) \to \mathrm{GL}_d(\mathbf{K})$ such that E_ρ is smoothly isomorphic to E, i.e. there is an isomorphism of *smooth* bundles between E_ρ and E.

Exercise 5.9: $\mathrm{Hom}_E(\pi_1(M), \mathrm{GL}_d(\mathbf{K}))$ is open in $\mathrm{Hom}(\pi_1(M), \mathrm{GL}_d(\mathbf{K}))$. Is it closed?
(Hint: you need more tools to do this exercise: (1) $\mathrm{Hom}(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K}))$ is locally path-connected and (2) homotopic bundles are isomorphic: if E is a bundle on $M \times [0, 1]$ then the restrictions of E to $M \times \{0\}$ and $M \times \{1\}$ are isomorphic.)

Then:

$$\frac{\mathrm{Hom}_E(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K}))}{\text{conjugacy}} \leftrightarrow \frac{\{D \text{ connection on } E \text{ with } F(D) = 0\}}{\mathcal{A}ut(E)}.$$

Exercise 5.10: Let $E_0 = p^{-1}(m_0)$, describe the map $\varepsilon : \mathcal{A}ut(E) \to \mathrm{GL}(E_0)$. Let $\mathcal{A}ut(E)_\varepsilon$ be the kernel of this morphism. Prove that

$$\mathrm{Hom}_E(\pi_1(M, m_0), \mathrm{GL}_d(\mathbf{K})) \leftrightarrow \frac{\{D \text{ connection on } E \text{ with } F(D) = 0\}}{\mathcal{A}ut(E)_\varepsilon}.$$

Lemma 5.6:

(1) *The difference between 2 connections is a 1-form with coefficients in* $\mathrm{End}(E)$*.*
(2) *The sum of a connection and (the operator associated with) a 1-form with coefficient in* $\mathrm{End}(E)$ *is a connection.*

This means that the space of all connections on E has a natural structure of an affine space.

Exercise 5.11: Give the expression for $F(D^E + A)$.

The gauge group action is the following:

$$\begin{aligned} g \cdot D : \Omega^\bullet(M; E) &\longrightarrow \Omega^\bullet(M; E) \\ \sigma &\longmapsto g(D(g^{-1}\sigma)). \end{aligned}$$

One has

$$\begin{aligned} g \cdot D^E - D^E &= gD^{\mathrm{End}(E)}(g^{-1}) = -g^{-1}D^{\mathrm{End}(E)}g \\ F(g \cdot D^E) &= gF(D^E)g^{-1}. \end{aligned}$$

6. Chern Connections, Stability, Degree

The expression of the Hitchin-Kobayashi correspondence to be given in the next section will be given "explicitely" in terms of operators (connections, pseudo-connections and operators induced by differentials forms with values in the endomorphisms bundle) and different constructions involving the Hermitian metric (decomposition unitary+Hermitian) and the complex structure (decomposition according to the bidegree). This section explains at length the construction of the Chern connection which (once a Hermitian metric is fixed) can be seen as a bijective correspondence between pseudo-connections and unitary connections on a complex vector bundle E.

Another ingredient is the notion of stability of holomorphic vector bundles. In order to define that notion, the degree and the Chern characters of vector bundles are presented.

6.1. *Hermitian Structure*

An Hermitian structure H on a complex vector bundle E is a smooth family of Hermitian scalar products on E (linear in the second variable and conjugate-linear in the first variable). The Hermitian structure H can be seen as a section of $\overline{E}^* \otimes E^*$, consequently H can be seen as a section of $\mathrm{Hom}(E, \overline{E}^*)$, etc.

Exercise 6.1: The gauge group $\mathcal{A}ut(E)$ acts transitively on the set of Hermitian structures on E.

The subgroup stabilizing H is the unitary gauge group $\mathcal{U}(E,H)$ or $\mathcal{U}(E)$.

Exercise 6.2: There exists (at least) one Hermitian structure on E.

A connection D on (E,H) is called *unitary* if $DH = 0$. In this last equation D is the induced connection on $\overline{E}^* \otimes E^*$. It would be wiser to say that D^E is a connection on E, $D^{\overline{E}^* \otimes E^*}$ its extension to $\overline{E}^* \otimes E^*$and write $D^{\overline{E}^* \otimes E^*} H = 0$.

Exercise 6.3: Let D^E be a connection on E ("extended" to E^*, $\overline{E}$, etc.). For K in $\Omega^0(M; \overline{E}^* \otimes E^*)$ and σ, σ' in $\Omega^0(M;E)$, one has

$$dK(\sigma,\sigma') = D^{\overline{E}^* \otimes E^*} K(\sigma,\sigma') + K(D^E\sigma,\sigma') + K(\sigma, D^E\sigma').$$

D is H-unitary if and only if for all σ, σ' in $\Omega^0(M;E)$

$$dH(\sigma,\sigma') = H(D^E\sigma,\sigma') + H(\sigma, D^E\sigma').$$

Remark 6.1: For sections σ and σ' the function $H(\sigma,\sigma')$ will sometimes be writen $(\sigma,\sigma')_H$ of even (σ,σ') (it is the evaluation of the Hermitian scalar product H on those 2 sections of E). This procedure is also extended to forms with coefficients in E via $H(\alpha\sigma, \alpha'\sigma') = \bar{\alpha} \wedge \alpha' H(\sigma,\sigma')$.

Lemma 6.2: *Any connection D on (E,H) decomposes (uniquely) as the sum of a unitary connection D_H and a H-Hermitian 1-form ψ_H with coefficients in* $\mathrm{End}(E)$ *(i.e. $(\psi_H\sigma,\sigma')_H = (\sigma,\psi_H\sigma')_H$ for all sections σ, σ').*

Exercise 6.4: Prove the lemma. What is the relation between ψ_H and $D^{\overline{E}^* \otimes E^*} H$?

There is always (at least) one unitary connection.

6.2. *Unitary Connections over a Complex Manifold*

Any connection D on E a complex vector bundle over a complex manifold X decomposes ("into types") as the sum of 2 operators

$$\begin{aligned} D : \Omega^0(X;E) &\longrightarrow \Omega^{1,0}(X;E) \oplus \Omega^{0,1}(X;E) \\ \sigma &\longmapsto \partial^E\sigma + \bar{\partial}^E\sigma. \end{aligned}$$

Exercise 6.5: The operator $\bar{\partial}^E$ is automatically a pseudo-connection.

In the presence of an Hermitian structure a converse construction exists.

Lemma 6.3: *Let $E \to X$ be a complex vector bundle equipped with an Hermitian structure H and a pseudo-connection $\bar\partial^E$.*

Then there exists a unique unitary connection D on E whose $(0,1)$ part is $\bar\partial^E$.

Exercise 6.6: Prove the lemma.

The unitary connection produced by this lemma is called the *Chern connection*. As a consequence of this lemma the gauge group $\mathcal{A}ut(E)$ acts on the space of unitary connections on E (the "natural" group acting on the space of unitary connections is $\mathcal{U}(E)$).

6.3. L^2-metrics

When M is a Riemannian manifold and E is an Hermitian vector bundle over M, any of the bundle $\bigwedge^p T^*M \otimes E$ gets equipped with an Hermitian structure. Hence, by integrating on M with respect to the volume form, there are corresponding L^2-scalar products on the spaces $\Omega^p(M;E)$. The (formal) adjoints with respect to those L^2-metrics will be again denoted with a $\star$ in superscript.

6.4. Another Kind of Adjunction

Still the presence of an Hermitian scalar product H on a vector space V leads to the construction of an adjoint for endomorphisms:

$$\begin{aligned} \mathrm{End}(V) &\longrightarrow \mathrm{End}(V) \\ A &\longmapsto A^{*_H} \end{aligned}$$

In term of matrices in an orthonormal frame, this operation is simply the transpose-conjugate. The map $A \to A^{*_H}$ is conjugate-linear ($(\lambda A)^{*_H} = \bar\lambda A^{*_H}$) and involutive ($(A^{*_H})^{*_H} = A$). Performing this operation "in family" we get a similar construction for an Hermitian bundle (E,H), there is a map

$$*_H : \Omega^0(M;\mathrm{End}(E)) \longrightarrow \Omega^0(M;\mathrm{End}(E))$$

(1) characterized by

$$(A\sigma, \sigma')_H = (\sigma, A^{*_H}\sigma')_H$$

for all $A \in \Omega^0(M;\mathrm{End}(E))$, σ, σ' in $\Omega^0(M;E)$.

(2) conjugate-linear: for all f in $\Omega^0(M;\mathbf{C})$ —i.e. f is a function— and for all A in $\Omega^0(M;\mathrm{End}(E))$, $(fA)^{*_H} = \bar{f}A^{*_H}$.
(3) and involutive $(A^{*_H})^{*_H} = A$.

There is only one way to extend $*_H$ to a map:

$$*_H : \Omega^\bullet(M;\mathrm{End}(E)) \longrightarrow \Omega^\bullet(M;\mathrm{End}(E))$$

satisfying the following property: for all α in $\Omega^\bullet(M;\mathbf{C})$ and all A in $\Omega^\bullet(M;\mathrm{End}(E))$, $(\alpha\wedge A)^{*_H} = \bar{\alpha}\wedge A^{*_H}$. This operator is involutive and will appear later in formulas.

Exercise 6.7: Characterize $*_H$ on forms with coefficients in $\mathrm{End}(E)$ with the help of an "adjunction" property.

When the base manifold is X a complex manifold, observe that $\bar{\alpha}\in\Omega^{q,p}(X)$ if $\alpha\in\Omega^{p,q}(X)$. This implies that $A^{*_H}\in\Omega^{q,p}(X;\mathrm{End}(E))$ if $A\in\Omega^{p,q}(X;\mathrm{End}(E))$.

A form A with coefficients in $\mathrm{End}(E)$ is called *Hermitian* if $A^{*_H} = A$ and *antihermitian* if $A^{*_H} = -A$.

Exercise 6.8: Every Hermitian (resp. antihermitian) 1-form with coefficients in $\mathrm{End}(E)$ can be decomposed uniquely as $B + B^{*_H}$ (resp. $A - A^{*_H}$) where B (resp. A) is a $(1,0)$-form.

6.5. *Kähler Manifold*

The analogue of the Kähler identities are now established for (non necessarly unitary) connection on (E,H). Note that the 2-form ω induces again an operator of degree 2

$$L : \Omega^\bullet(X;E) \longrightarrow \Omega^{\bullet+2}(X;E)$$

and its (formal) adjoint is denoted by $L^\star$ —nothing new hence.

Proposition 6.4: *Let X be a Kähler manifold and (E,H) be an Hermitian vector bundle over X. Let also D be a connection on E, decompose now D:*

$$D = D_H + \psi_H$$

as the sum of a unitary connection and an Hermitian 1-form (see Lemma 6.2). Decompose furthermore into type:

$$D_H = \partial_H + \bar{\partial}_H \quad \textit{and} \quad \psi_H = \psi^{1,0} + \psi^{0,1}$$

(see Section 6.2).

Then $(\psi^{1,0})^{*_H} = \psi^{0,1}$ *and setting*

$$D' = \partial_H + \psi^{0,1} \quad D'' = \bar{\partial}_H + \psi^{1,0},$$

one has:

$$\begin{array}{ll} [L, D'^{\star}] = \sqrt{-1}D'' & [L^{\star}, D'] = -\sqrt{-1}D''^{\star} \\ [L, D''^{\star}] = -\sqrt{-1}D' & [L^{\star}, D''] = \sqrt{-1}D'^{\star} \end{array}$$

To prove this proposition it is enough to work locally, hence choosing an orthonormal frame for E one can work with the trivial bundle equipped with the "trivial" Hermitian scalar product (but not the trivial connection). In this trivialization the unitary connection has the form

$$d + A^{1,0} + A^{0,1}$$

and $A^{1,0*_H} = -A^{0,1}$ (see Exercise 6.8).

Since the result is known for $d = \partial + \bar{\partial}$, to prove it for $D = D' + D''$ with $D' = \partial + A^{1,0} + \psi^{0,1}$ and $D'' = \bar{\partial} + A^{0,1} + \psi^{1,0} = \bar{\partial} - A^{1,0*_H} + \psi^{0,1*_H}$ it is enough to establish the following

Lemma 6.5:

(1) If $A \in \Omega^{1,0}(X; \mathrm{End}(E))$ *then*

$$[L, A^{\star}] = -\sqrt{-1}A^{*_H}.$$

(This is an equality between operators.)

(2) If $A \in \Omega^{0,1}(X; \mathrm{End}(E))$ *then*

$$[L, A^{\star}] = \sqrt{-1}A^{*_H}.$$

In fact one can prove this lemma quite easily using the symplectic (formal) adjoints and the operator C (as above, Paragraph 2.9).

Lemma 6.6: $A^{\star_s} = -CA^{\star}C^{-1}$ *(compare with Exercise 2.10).*

If $A \in \Omega^1(X; \mathrm{End}(E))$, *then* $[L, A^{\star_s}] = A^{*_H}$ *(compare with Exercise 2.5).*

6.6. *Chern Characters*

We will define only the first and second Chern characters. They will be given as differential forms and the proof of their properties will be left as exercises.

Let E be a complex vector bundle over a manifold M. The first and second Chern characters of E are the forms:

$$\begin{aligned}\operatorname{ch}_1(E) &= \operatorname{ch}_1(E, D^E) = \frac{\sqrt{-1}}{2\pi}\operatorname{tr}F(D^E) \in \Omega^2(M; \mathbf{C})\\ \operatorname{ch}_2(E) &= \operatorname{ch}_2(E, D^E) = \Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \operatorname{tr}\big(F(D^E)\wedge F(D^E)\big) \in \Omega^4(M; \mathbf{C}).\end{aligned}$$

The notation suggests that they do not depend on the connection D^E whose curvature $F(D^E)$ is involved in the formulas. The 4-form $F(D^E)\wedge F(D^E)$ has coefficients in $\operatorname{End}(E)$; this wedge product involves the wedge product on differential forms and the composition of endomorphisms. Finally for a form with coefficients in $\operatorname{End}(E)$ its trace is the differential form obtained by applying the traces of endomorphisms: $\operatorname{tr}(\sum_i \alpha_i\sigma_i) = \sum_i \alpha_i \operatorname{tr}(\sigma_i)$ if $\alpha_i \in \Omega^\bullet(M; \mathbf{C})$ and σ_i are sections of $\operatorname{End}(E)$.

The needed properties concerning those Chern characters are summarized as follows

Lemma 6.7: *The forms* $\operatorname{ch}_1(E)$ *and* $\operatorname{ch}_2(E)$ *are closed.*

For any 2 connections D *and* D' *on* E*, the difference* $\operatorname{ch}_i(E, D') - \operatorname{ch}_i(E, D)$ *(*$i = 1, 2$*) is exact.*

The first property follows from the Bianchi identity ($DF = 0$) and its consequences (i.e $D(F\wedge F) = 0$) as well as the formula for differentiating traces: $d\operatorname{tr}F = \operatorname{tr}DF$.

The second follows from the relation $F(D') = F(D) + DA + A\wedge A$ (exercise 5.11) and the fact that $\operatorname{tr}A\wedge A = 0$.

Exercise 6.9: Provide the details of the construction of the Chern characters.

Exercise 6.10: Express $\operatorname{ch}_1(E\otimes F)$ and $\operatorname{ch}_1(E\oplus F)$.

The content of the lemma is that the forms $\operatorname{ch}_i(E)$ define uniquely cohomology classes in $H^{2i}(M; \mathbf{C})$. Using (for examples) unitary connections one can show that these cohomology classes belong to $H^{2i}(M; \mathbf{R})$. The relations with the (more common) Chern classes are

$$\begin{aligned}\operatorname{ch}_1(E) &= c_1(E)\\ \operatorname{ch}_2(E) &= c_2(E) - \tfrac{1}{2}c_1(E)^2.\end{aligned}$$

Exercise 6.11: For a vector bundle E of rank d and D a connection on E, express $\bigwedge^d D$ on $\bigwedge^d E$ in term of D and give the relation between $F(\bigwedge^d D)$ and $F(D)$ —you may notice first that for the line bundle $L = \bigwedge^d E$, the bundle $\operatorname{End}(L)$ is the trivial bundle $\operatorname{End}(L) \simeq \mathbf{C}$ so that $F(L)$ as a form with coefficient in $\operatorname{End}(L)$ is in fact a differential form. The equality $\operatorname{ch}_1(\bigwedge^d E) = \operatorname{ch}_1(E)$ holds.

Exercise 6.12: For this exercise you need to know about

- group cohomology $H^\bullet(\Gamma; A)$
- the relation between central extensions $A \to E \to \Gamma$ and $H^2(\Gamma; A)$
- the natural map $\Theta : H^2(\pi_1(M); A) \to H^2(M; A)$.

Let $E \to M$ be a $\mathbf{C}$-vector bundle and $L = \bigwedge^d E$ the corresponding line bundle. Denote by $L - \{0\}$ the complement of the zero section.

(1) $\pi_1(L - \{0\}) \to \pi_1(M)$ is a central extension by a cyclic group A (i.e. there is a surjection $\mathbf{Z} \to A$). Let κ be the class in $H^2(\pi_1(M); A)$ corresponding to this central extension.

(2) The image $\Theta(\kappa)$ in $H^2(M; A)$ is the image of the (reduction of) the Chern class $\mathrm{ch}_1(E)$ in $H^2(M; A)$ (there is a natural map $H^2(M; \mathbf{Z}) \to H^2(M; A)$). [You may need a "functorial" approach of the first Chern class to prove this.]

6.7. *Degree*

For a vector bundle E over a Kähler manifold X and if ω designates the symplectic form on X,

Definition 6.8: The *degree* of E (with respect to ω) is

$$\deg_\omega E = \frac{1}{n!} \int_X \mathrm{ch}_1(E) \wedge \omega^{n-1}$$

6.8. *Stability*

This is a key notion and its relevence will be fully justified later in Paragraph 9.3.

Definition 6.9: A holomorphic vector bundle E of rank $r = r(E)$ is called *stable* (or ω-stable) if for any holomorphic subbundle $F \subsetneq E$ ($F \neq 0$) defined outside an analytic subset $\mathcal{S} \subset X$ of complex codimension 2 one has

$$\mu(F) = \frac{\deg_\omega(F)}{r(F)} < \mu(E) = \frac{\deg_\omega(E)}{r(E)}.$$

The number $\mu(E)$ is called the *slope* of E. The slope is a topological invariant.

Remark 6.10: For F and $\mathcal{S}$ as in the definition, the first Chern character $\mathrm{ch}_1(F)$ belongs a priori to $H^2(X \setminus \mathcal{S}; \mathbf{C})$. The condition on $\mathcal{S}$ precisely insures that $H^2(X \setminus \mathcal{S}; \mathbf{C}) \simeq H^2(X; \mathbf{C})$ and hence the degree $\deg_\omega(F)$ is well defined.

The usual definition of stability involves (coherent) subsheaves of the sheaf of holomorphic sections of E; the above definition makes the use of the fact that such a subsheaf is the sheaf of holomorphic sections of a subbundle defined outside a set of codimension 2.

7. The Correspondence between Flat Bundles and Higgs Bundles

This section starts by stating a generalization of the theorem of Narasimhan and Seshadri using the connections and pseudo-connections. The definition of a Higgs bundle is then given as well as the result of Hitchin and Simpson asserting the correspondence between flat bundles (with conditions) and Higgs bundles (with conditions). The end of this section is dedicated to that correspondence in the case of line bundles: the “classical”Hodge theory is the necessary tool for that case.

7.1. *The Theorem of Narasimhan and Seshadri*

In this more general form it is due to Uhlenbeck-Yau or Donaldson (the version presented here is not the most general).

Theorem 7.1: *Let X be a compact Kähler manifold.*
Then there is a bijective correspondence between

(1) the conjugacy classes of irreducible representations

$$\rho : \pi_1(X) \longrightarrow \mathrm{U}(r),$$

(2) and the isomorphism classes of holomorphic vector bundles of rank r that are stable and with $\mathrm{ch}_1 = \mathrm{ch}_2 = 0$.

This correspondence is implemented by the map which send a flat unitary connection to its $(0,1)$ part.

7.2. *Higgs Bundles*

Definition 7.2: A *Higgs bundle* is a pair of a holomorphic vector bundle $(E, \bar{\partial}^E)$ together with a holomorphic $(1,0)$-form ϕ with coefficients in $\mathrm{End}(E)$ (i.e. $\bar{\partial}^{\mathrm{End}(E)}\phi = 0$) verifying $\phi \wedge \phi = 0$.

Remark 7.3: The conditions are equivalent to saying that $D'' = \bar{\partial}^E + \phi$ (an operator of mixed type but satisfying $D''(f\sigma) = \bar{\partial} f\sigma + fD''\sigma$) verifies the equality $(D'')^2 = 0$.

On surfaces, i.e. when $\dim_{\mathbf{C}} X = 1$, the conditions $(\bar\partial^E)^2 = 0$ and $\phi\wedge\phi = 0$ are automatic since there are neither $(0,2)$-forms nor $(2,0)$-forms.

The stability condition for Higgs bundles takes the following form:

Definition 7.4: $(E, \bar\partial^E, \phi)$ is *stable* if for any holomorphic vector bundle $F \subsetneq E$ $(F \neq 0)$ defined outside an analytic subset of codimension 2 and which is ϕ-invariant, one has

$$\mu(F) = \frac{\deg_\omega(F)}{r(F)} < \mu(E) = \frac{\deg_\omega(E)}{r(E)}.$$

Here the meaning of ϕ-invariance is that $\phi(\sigma) \in \Omega^{1,0}(X;F)$ for any $\sigma \in \Omega^0(X;F) \subset \Omega^0(X;E)$.

7.3. *The Hitchin-Kobayashi Correspondence*

It is sometimes under that name that the correspondence between flat bundles and Higgs bundles is found in the litterature (see for example the book [4]). No attempt to retrace the historical progression of the results will be made here. In particular no results of Kobayashi will be cited, however his book on vector bundles [3] is a very nice account on the theory.

Let E be a smooth bundle over a Kähler manifold. Suppose that $\mathrm{ch}_1(E) = \mathrm{ch}_2(E) = 0$. The correspondence is the following

$$\frac{\{\text{irreducible flat connections on } E\}}{\text{isomorphisms}} \leftrightarrow \frac{\{\text{stable Higgs bundle structures on } E\}}{\text{isomorphisms}}.$$

Or, taking any bundle E,

$$\frac{\{\text{irreducible representations of } \pi_1(X,x_0)\}}{\text{conjugacy}} \leftrightarrow \frac{\{\text{stable Higgs bundles } (E,\bar\partial^E,\phi) \text{ with } \mathrm{ch}_1(E)=\mathrm{ch}_2(E)=0\}}{\text{equivalence}}.$$

The details of the constructions involved in the 2 directions of this correspondence are the subject of the next paragraphs.

7.4. *From Flat Bundles to Higgs Bundles*

Both directions involve Hermitian structures on E.

Let D be a (flat) connection on E over X and H a Hermitian structure on E.

Recall the decomposition of D:

$$\begin{aligned} D &= D_H + \psi_H \quad \text{unitary + Hermitian} \\ &= \partial_H + \bar{\partial}_H + \psi^{1,0} + \psi^{0,1} \quad \text{into types.} \end{aligned}$$

Theorem 7.5: *[Donaldson, Corlette, Labourie]*

If D is a flat irreducible connection, then there exists a (unique up to a scalar multiple) Hermitian structure H on E such that $(E, \bar{\partial}_H, \psi^{1,0})$ is a Higgs bundle.

Furthermore this Higgs bundle is stable.

The explanation why the Higgs bundles arising this way are stable will be given in Paragraph 9.3

In fact this statement is rather a translation in terms of Higgs bundles of the theorem of Donaldson, Corlette and Labourie. Their theorem is the existence of harmonic Hermitian metric on (E, D), on a Kähler manifold, an harmonic metric leads to a Higgs bundle.

7.5. *From Higgs Bundles to Flat Bundles*

Let $(E, \bar{\partial}^E, \phi)$ be a Higgs bundle (or more generally a pseudo-connection $\bar{\partial}^E$ and a $(1,0)$-form ϕ with coefficients in $\mathrm{End}(E)$).

Let H be a Hermitian structure on E, call ∂_H the operator of type $(1,0)$ such that $\partial_H + \bar{\partial}^E$ is the Chern connection.

Theorem 7.6: *[Hitchin, Simpson]*

Let $(E, \bar{\partial}^E, \phi)$ be a stable Higgs bundle with $\mathrm{ch}_1(E) = \mathrm{ch}_2(E) = 0$.

Then there exists a unique (up to multiple) Hermitian metric H on E such that the connection

$$D = \partial_H + \bar{\partial}^E + \phi + \phi^{*_H}$$

is flat.

Furthermore the connection D is irreducible.

7.6. *The Case of Line Bundles*

This case does not really shed light on the general theory. However it explains why the Hitchin-Kobayashi correspondence deserves the name of "non-abelian Hodge theory".

First some particularities for line bundles over X (or even over M) are recalled. Let L be a line bundle.

The bundle $\mathrm{End}(L)$ is always trivial:

$$\begin{aligned} X \times \mathbf{C} &\longrightarrow \mathrm{End}(L) \\ f &\longmapsto f\mathrm{id}_L. \end{aligned}$$

(Indeed this bundle is not only trivial it is *trivialized*.)

For any connection D^L on L, the induced connection $D^{\mathrm{End}(L)}$ on $\mathrm{End}(L) \simeq \mathbf{C}$ is always trivial. In view of the above isomorphism it is enough to prove that $D^{\mathrm{End}(L)}\mathrm{id}_L = 0$. In fact this holds in any rank.

Lemma 7.7: *For any bundle with connection* (E, D^E), *one has* $D^{\mathrm{End}(E)}\mathrm{id}_E = 0$.

Proof: The connection $D^{\mathrm{End}(E)}$ is constructed so that for any section σ of E and any section ϕ of $\mathrm{End}(E)$, the following holds:

$$D^E\phi(\sigma) = (D^{\mathrm{End}(E)}\phi)\sigma + \phi D^E\sigma.$$

Applying this to $\phi = \mathrm{id}_E$ only one term remains

$$(D^{\mathrm{End}(E)}\mathrm{id}_E)\sigma = 0 \quad \forall\sigma$$

which says that $D^{\mathrm{End}(E)}\mathrm{id}_E = 0$. □

Even more is true: for any Hermitian structure H on L, the induced Hermitian structure on $\mathrm{End}(L) \simeq \mathbf{C}$ is trivial. For this we need to show that, for the induced norm on $\mathrm{End}(L)$, id_L is of norm one: let e be a section of L of norm one, the dual basis (!) e^* is a section of L^* of norm one (this follows from the definition of the norm on L^*) and hence $e \otimes e^*$ is of norm one. But $\mathrm{id}_L = e \otimes e^*$.

From this discussion one gets that:

- for any connection D on L
- any form ψ in $\Omega^1(M; \mathrm{End}(L)) = \Omega^1(M; \mathbf{C})$
- and any Hermitian structure H on L

the following holds

(1) $F(D + \psi) = F(D) + d\psi$ (since $D\psi = d\psi$ and $\psi \wedge \psi = 0$) and
(2) $\psi^{*_H} = \bar{\psi}$.

7.7. Line Bundles: From Flat to Higgs

Let now D be a flat connection on a line bundle L over a Kähler manifold X.

Let H_0 be one Hermitian metric on L. We will search the Hermitian structure H solution to the theorem 7.5 under the form $H = e^f H_0$.

Some automatic cancellations hold in this case: $(\bar\partial_H)^2 = 0$ and $\phi^{1,0} \wedge \phi^{1,0} = 0$. The second is obvious since $\phi^{1,0}$ is a differential form.

For the first one, recall that

$$\begin{aligned} D &= D_H + \psi_H \quad \text{(unitary + Hermitian)} \\ D_H &= \partial_H + \bar\partial_H \quad \text{(into types)} \end{aligned}$$

since $0 = F(D) = F(D_H) + d\psi_H$, one gets

$$\begin{aligned} F(D_H) &= 0 \quad \text{(antihermitian part)} \\ d\psi_H &= 0 \quad \text{(Hermitian part).} \end{aligned}$$

But $F(D_H) = (D_H)^2 = \partial_H^2 + (\partial_H\bar\partial_H + \bar\partial_H\partial_H) + \bar\partial_H^2$ so that the $(0,2)$-part of $F(D_H)$ is $\bar\partial_H^2$ and $\bar\partial_H^2 = 0$.

For H_0, denote the above decomposition

$$\begin{aligned} D &= D_0 + \psi_0 \\ &= (\partial_0 + \bar\partial_0) + (\phi_0 + \bar\phi_0) \end{aligned}$$

where ϕ_0 is of type $(1,0)$. Decomposing $d\psi_0 = 0$ into type gives that $\partial\phi_0 = 0$, $\bar\partial\phi_0 = -\partial\bar\phi_0$ and $\bar\partial\bar\phi_0 = 0$. In particular $d\phi_0 = \bar\partial\phi_0$ and $\sqrt{-1}\bar\partial\phi_0$ is a real 2-form:

$$\overline{\sqrt{-1}\bar\partial\phi_0} = -\sqrt{-1}\partial\bar\phi_0 = \sqrt{-1}\bar\partial\phi_0 \quad \text{since } \bar\partial\phi_0 = -\partial\bar\phi_0.$$

Application of the classical $\partial\bar\partial$ lemma (this is where Hodge theory is used) implies that there is a real function f on X such that

$$\bar\partial\phi_0 = \bar\partial\partial f \quad \text{i.e. } \bar\partial(\phi_0 - \partial f) = 0.$$

The sought for metric is $H = e^{2f} H_0$. The key is to prove the relation

$$\psi_H = \psi_0 - df$$

so that $\psi_H^{1,0} = \phi_0 - \partial f$ is indeed holomorphic and the Higgs bundle equations are satisfied for $(L, \bar\partial_H, \psi_H^{1,0})$. For this, observe first that for any sections σ and σ' of L:

$$\begin{aligned} dH(\sigma,\sigma') &= H(D_H\sigma,\sigma') + H(\sigma, D_H\sigma') \text{ as } D_H \text{ is unitary} \\ &= H((D-\psi_H)\sigma,\sigma') + H(\sigma,(D-\psi_H)\sigma') \\ &= H(D\sigma,\sigma') + H(\sigma,D\sigma') - 2H(\psi_H\sigma,\sigma') \text{ as } \psi_H \text{ is } H\text{-Hermitian.} \end{aligned}$$

But also $H(\sigma,\sigma') = e^{2f}H_0(\sigma,\sigma')$ so

$$\begin{aligned} dH(\sigma,\sigma') &= 2df e^{2f}H_0(\sigma,\sigma') + e^{2f}dH_0(\sigma,\sigma') \\ &= 2df H(\sigma,\sigma') + e^{2f}(H_0(D\sigma,\sigma') + H_0(\sigma,\sigma') - 2H_0(\psi_0\sigma,\sigma')) \\ &= 2df H(\sigma,\sigma') + H(D\sigma,\sigma') + H(\sigma,\sigma') - 2H(\psi_0\sigma,\sigma') \end{aligned}$$

Comparing these two formulas:

$$2H(\psi_H\sigma,\sigma') = -2df H(\sigma,\sigma') + 2H(\psi_0\sigma,\sigma')$$

therefore $\psi_H = -df + \psi_0$ as claimed.

7.8. *Line Bundles: From Higgs to Flat*

Suppose that $\mathcal{L} = (L, \bar\partial^L)$ is a holomorphic line bundle and ϕ is a holomorphic $(1,0)$-form (seen as an element of $\Omega^{1,0}(X, \mathrm{End}(\mathcal{L}))$). The first Chern class of L is supposed to vanish (the second Chern character is always 0 for line bundles). The stability condition here is empty.

The sought for Hermitian metric H on L satisfies that the associated connection:

$$D = \partial_H + \bar\partial^L + \phi + \bar\phi$$

is flat where $D_H = \partial_H + \bar\partial^L$ is the Chern connection for $(\mathcal{L}, H)$. The curvature of D is

$$F(D) = F(D_H) + d(\phi + \bar\phi).$$

As $(\bar\partial^L)^2 = 0$ one has $(\partial_H)^2 = 0$ (by adjunction) so that $F(D_H) = \partial_H\bar\partial^L + \bar\partial^L\partial_H$ is a form of type $(1,1)$ which is antihermitian (here it means purely imaginary), i.e. $\overline{F(D_H)} = -F(D_H)$.

Also $\bar\partial\phi = 0$ and hence $\partial\bar\phi = 0$ which imply that $d\phi = \partial\phi$ and $d\bar\phi = \bar\partial\bar\phi = \overline{\partial\phi}$ are of type $(2,0)$ and $(0,2)$ respectively.

First the equation $L^\star F = 0$ is solved and then $F = F(D) = 0$ will be a consequence of the cancellation of the Chern class.

Let H_0 be a fixed metric on L and $H = e^f H_0$ the sought for metric. The problem will again be reduced to finding the equation satisfied by the real function f and Hodge theory will provide a solution for this equation.

The difference between ∂_H and ∂_{H_0} is controlled by f:

$$\partial_H = \partial_{H_0} + \partial f.$$

Exercise 7.1: Prove this.

Therefore $\partial_H + \bar{\partial}^L = \partial_{H_0} + \bar{\partial}^L + \partial f$ and $F(\partial_H + \bar{\partial}^L) = F(\partial_{H_0} + \bar{\partial}^L) + d\partial f$ but $d\partial f = \partial^2 f + \bar{\partial}\partial f = \bar{\partial}\partial f$.

Note also that $L^\star(d(\phi + \bar{\phi})) = L^\star\partial\phi + L^\star\bar{\partial}\bar{\phi} = 0$ since $L^\star$ is zero on forms of type $(2,0)$ and $(0,2)$.

Hence $L^\star F = L^\star F_0 + L^\star\bar{\partial}\partial f$ where $F_0 = F(\partial_{H_0} + \bar{\partial}^L)$. Moreover $L^\star\bar{\partial}\partial f = [L^\star, \bar{\partial}]\partial f = \sqrt{-1}\partial^\star\partial f = \sqrt{-1}\Delta_\partial f$ by the Kähler identities (see Section 2.12). The fact that F_0 is antihermitian implies that $L^\star F_0 = \sqrt{-1}g$ with g a real function. We have to find a function f such that $g + \Delta_\partial f = 0$. By classical Hodge theory this is possible if and only if $\int_X g\omega^n = 0$, thus the conclusion will follow from $\int_X L^\star F_0 \omega^n = 0$.

With this lemma at hand the conclusion $F = 0$ follows easily: since $\mathrm{ch}_1(L) = 0$ one knows that $\int F_0 \wedge \omega^{n-1} = 0$ so the wanted cancellation will follow from:

Lemma 7.8: *For any 2-form β:*

$$\int L^\star\beta\omega^n = \int \beta\omega^{n-1}.$$

Proof: By construction $L^\star\beta = \langle L^\star\beta, 1\rangle$ (the notation for scalar products introduced in Section 2.1 are still in use). So $\int L^\star\beta\omega^n = \int\langle L^\star\beta, 1\rangle\omega^n = \int\langle\beta,\omega\rangle\omega\wedge\omega^{n-1}$, the last equality follows from the definition of $L^\star$.

Replacing β by $\beta' = \beta - \langle\beta,\omega\rangle\omega$, it is therefore enough to prove that $\int\beta\omega^{n-1} = 0$ under the condition that $\langle\beta,\omega\rangle = 0$. Using a partition of the unity it is enough to work locally.

Let $\epsilon_1, \ldots, \epsilon_n$ be a (local) orthonormal frame for $T^{1,0}X$, $e_1, \ldots, e_n$ the dual basis of $T^{*1,0}X$ and $\bar{e}_1, \ldots, \bar{e}_n$ the basis of $T^{*0,1}X$.

The form ω is:

$$\omega = \frac{1}{2\sqrt{-1}}\sum e_i \wedge \bar{e}_i.$$

Under the hypothesis $\langle\beta,\omega\rangle = 0$, it is safe to forget the factor $\frac{1}{2\sqrt{-1}}$ in the rest of this proof. Write

$$\beta = \sum a_{ij}e_i \wedge e_j + b_{ij}e_i \wedge \bar{e}_j + c_{ij}\bar{e}_i \wedge \bar{e}_j.$$

One has $\langle\beta,\omega\rangle = \sum b_{ii} = 0$. Also

$$\omega^{n-1} = \sum_i e_1 \wedge \bar{e}_1 \wedge e_2 \wedge \bar{e}_2 \wedge \cdots \wedge \widehat{e_i \wedge \bar{e}_i} \wedge \cdots \wedge e_n \wedge \bar{e}_n,$$

so that

$$\begin{aligned}\beta \wedge \omega^{n-1} &= \sum b_{ii}e_i \wedge \bar{e}_i \wedge e_1 \wedge \bar{e}_1 \wedge e_2 \wedge \bar{e}_2 \wedge \cdots \wedge \widehat{e_i \wedge \bar{e}_i} \wedge \cdots \wedge e_n \wedge \bar{e}_n \\ &= (\textstyle\sum b_{ii})\omega^n = \langle\beta,\omega\rangle\omega^n.\end{aligned}$$

(See also Remark 9.5 for a short proof.) □

Exercise 7.2: Verify the formulas for ω^{n-1} and $\beta \wedge \omega^{n-1}$.

Remark 7.9: An equality between forms was proved and not only between their integrals. There is certainly a symplectic proof.

The last task is now to establish that $F = 0$ knowing that $L^\star F = 0$ and that $\mathrm{ch}_2(L) = F \wedge F = 0$. Write

$$F = F_H + \partial\phi + \overline{\partial\phi}$$

$\partial\phi$ is a form of type $(2,0)$ and $F_H = -\overline{F_H}$ is of type $(1,1)$ and $L^\star F_H = L^\star F = 0$.

Lemma 7.10: *Let β be a $(1,1)$ form, $\beta = -\bar\beta$, $\langle \beta, \omega\rangle = 0$ then $\beta \wedge \beta \wedge w^{n-2} = \frac{1}{2}\|\beta\|^2\omega^n$.*

Let α be a $(2,0)$ form, then $\alpha \wedge \bar\alpha \wedge \omega^{n-2} = \|\alpha\|^2\omega^n$.

Since $F \wedge F \wedge \omega^{n-2} = F_H \wedge F_H \wedge \omega^{n-2} + 2\partial\phi \wedge \overline{\partial\phi} \wedge \omega^{n-2}$ (eliminating the terms that are not of type (n,n)) one get that $0 = \int F \wedge F \wedge \omega^{n-2} = \frac{1}{2}\int \|F_H\|^2\omega^n + 2\int \|\partial\phi\|^2\omega^n$ therefore F_H and $\partial\phi$ are identically 0, consequently F is 0.

Proof: It is a local statement. Take a frame as in the previous proof. Then $\omega = (2\sqrt{-1})^{-1}\sum_i e_i \wedge \bar e_i$ and

$$\omega^{n-2} = \sum_{i<j} \frac{e_1 \wedge \bar e_1}{2\sqrt{-1}} \wedge \cdots \wedge \widehat{\frac{e_i \wedge \bar e_i}{2\sqrt{-1}}} \wedge \cdots \wedge \widehat{\frac{e_j \wedge \bar e_j}{2\sqrt{-1}}} \wedge \cdots \wedge \frac{e_n \wedge \bar e_n}{2\sqrt{-1}}.$$

A form β of type $(1,1)$ is $\beta = \sum_{i,j} b_{ij} \frac{e_i \wedge \bar e_j}{2\sqrt{-1}}$. It satisfies $\bar\beta = -\beta$ if $\bar b_{ji} = b_{ij}$ for all i, j and $\langle \beta, \omega\rangle = 0$ if $\sum b_{ii} = 0$. Its squared norm is $\|\beta\|^2 = \sum |b_{ij}|^2$.

Also

$$\beta \wedge \beta = \sum b_{ij} b_{kl} \frac{e_i \wedge \bar e_j}{2\sqrt{-1}} \wedge \frac{e_k \wedge \bar e_l}{2\sqrt{-1}},$$

wedging with ω^{n-2} gives

$$\begin{aligned} \beta \wedge \beta \wedge \omega^{n-2} &= \textstyle\sum_{i<j} b_{ii} b_{jj} \omega^n - \sum_{i<j} b_{ij} b_{ji} \omega^n \\ &= (-\tfrac{1}{2}\textstyle\sum_{i,j} b_{ij} b_{ji})\omega^n \\ &= \tfrac{1}{2}\textstyle\sum b_{ij}\bar b_{ij}\omega^n = \tfrac{1}{2}\|\beta\|^2\omega^n \end{aligned}$$

where the equality at the second line follows from $\sum b_{ii} = 0$.

Similarly a $(2,0)$-form α has the form $\sum_{i<j} a_{ij} e_i \wedge e_j$ and

$$\begin{aligned} \alpha \wedge \alpha &= \textstyle\sum_{i<j,k<l} a_{ij}\bar a_{kl} e_i \wedge e_j \wedge \bar e_k \wedge \bar e_l \\ &= 4\textstyle\sum a_{ij}\bar a_{kl} \frac{e_i \wedge \bar e_k}{2\sqrt{-1}} \wedge \frac{e_j \wedge \bar e_l}{2\sqrt{-1}}. \end{aligned}$$

Thus $\alpha \wedge \bar{\alpha} \wedge \omega^{n-2} = 4 \sum_{i<j} a_{ij}\bar{a}_{ij}\omega^n = \|\alpha\|^2\omega^n$. □

8. Examples

Riemann surfaces are the place where a lot of examples can provided. A few general facts on Riemann surfaces and on their line bundles are first recalled. The examples of Higgs bundles given are first for rank 2 vector bundles, it is also explained when the corresponding representation has image in $\mathrm{SL}_2(\mathbf{C})$ or in $\mathrm{GL}_2(\mathbf{R})$. General statements are also given for detecting in terms of the Higgs bundle when the image of the holonomy representation has determinant 1 or is real valued. The Teichmller component, a remarkable discovery of Hitchin, is also discribed. Doing this other important aspects of the theory of Higgs bundles are eluded: the $\mathbf{C}^*$-action and the fact that stability is an open condition. This section ends with the illustration how to understand Higgs bundles corresponding to representations in $\mathrm{U}(p,q)$ or $\mathrm{Sp}_{2n}(\mathbf{R})$.

8.1. *Every Riemann Surface is Kähler*

If Σ is a complex manifold of dimension 1 and $\langle\cdot,\cdot\rangle$ is a compatible metric then Σ is a Kähler manifold ($d\omega = 0$ since $\dim_{\mathbf{R}} \Sigma < \deg(d\omega)$). Note that in this situation J is entirely determined by $\langle\cdot,\cdot\rangle$ and the orientation of Σ: it is the rotation of angle $\pi/2$.

8.2. *Line Bundles over* Σ

For $\mathcal{L}$ a holomorphic line bundle over Σ, denote by $H^0(\mathcal{L})$ —or $H^0(\Sigma;\mathcal{L})$— the space of holomorphic sections of $\mathcal{L}$, i.e. the elements σ of $\Omega^0(\Sigma,\mathcal{L}) = \Gamma(\Sigma,\mathcal{L})$ satisfying $\bar{\partial}^{\mathcal{L}}\sigma = 0$.

Then $\dim H^0(\mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$ and the Riemann-Roch theorem states that

$$\dim H^0(\mathcal{L}) - \dim H^0(\mathcal{L}^* \otimes \mathcal{K}) = \deg \mathcal{L} - (g-1)$$

where $\mathcal{K}$ is the holomorphic line bundle $T^{*1,0}\Sigma$ whose degree is $2g-2$.

Exercise 8.1: $T^{*1,0}X$ is a holomorphic bundle (any X).

Exercise 8.2: Let L and K be two (smooth) line bundles over Σ. Let $\theta : L \otimes L \to K$ be an isomorphism.

For any holomorphic structure $\bar{\partial}^K$ on K, there is a unique holomorphic structure $\bar{\partial}^L$ on L such that θ becomes a holomorphic isomorphism.

Exercise 8.3: (Corollary) If $\deg \mathcal{K}$ is even, there is a holomorphic line bundle $\mathcal{L}$ such that $\mathcal{L} \otimes \mathcal{L} = \mathcal{K}$. (You need to know the smooth classification of line bundles over Σ.)

$\mathcal{L}$ is denoted by $\mathcal{K}^{1/2}$

8.3. *Some Rank 2 Higgs Bundles*

Let $\mathcal{L}$ be a line bundle over Σ and $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}$ (a negative power of a line bundle means the dual of the corresponding positive power for the tensor product). The bundle $\mathrm{End}(\mathcal{E})$ decomposes as the sum of 4 line bundles

$$\begin{array}{ll} \mathrm{Hom}(\mathcal{L},\mathcal{L}) \simeq \mathbf{C} & \mathrm{Hom}(\mathcal{L}^{-1},\mathcal{L}) \simeq \mathcal{L}^2 \\ \mathrm{Hom}(\mathcal{L},\mathcal{L}^{-1}) \simeq \mathcal{L}^{-2} & \mathrm{Hom}(\mathcal{L}^{-1},\mathcal{L}^{-1}) \simeq \mathbf{C} \end{array}$$

A holomorphic $(1,0)$-form with coefficients in $\mathrm{End}(\mathcal{E})$ is the same thing as a holomorphic section of $\mathrm{End}(\mathcal{E}) \otimes \mathcal{K}$ (compare with exercise 2.7). Therefore the Higgs field ϕ will be given as a matrix

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } \quad \left\{ \begin{array}{ll} a \in H^0(\mathcal{K}), & b \in H^0(\mathcal{L}^2 \otimes \mathcal{K}), \\ c \in H^0(\mathcal{L}^{-2} \otimes \mathcal{K}), & d \in H^0(\mathcal{K}). \end{array} \right.$$

Lemma 8.1: *If $(\mathcal{E},\phi)$ is stable then $|\deg(\mathcal{L})|$ is bounded by $g-1$.*

Proof: If $c = 0$ then $\mathcal{L} \subset \mathcal{L} \oplus \mathcal{L}^{-1} = \mathcal{E}$ is a sub-Higgs bundle hence the stability condition says that $\mu(\mathcal{L}) = \deg(\mathcal{L}) < \mu(\mathcal{E}) = 0$.

If $c \neq 0$ then the bundle $\mathcal{L}^{-2} \otimes \mathcal{K}$ has a non trivial holomorphic section and hence its degree is positive: $\deg(\mathcal{L}^{-2} \otimes \mathcal{K}) = -2\deg(\mathcal{L}) + 2g - 2 \geq 0$ so that $\deg(\mathcal{L}) \leq g-1$.

In any case, $\deg(\mathcal{L}) \leq g-1$.

Similarly, working with b, $\deg(\mathcal{L}) \geq 1-g$. □

Lemma 8.2: *If $(\mathcal{E},\phi)$ is stable then the corresponding representation $\rho : \pi_1(\Sigma) \to \mathrm{GL}_2(\mathbf{C})$ has image in $\mathrm{SL}_2(\mathbf{C})$ if and only if $a + d = 0$.*

Proof: One needs to convince him/herself (i.e. to prove) that the representation $\det(\rho)$ corresponds to the Higgs bundle $(\bigwedge^2 \mathcal{E}, \bigwedge^2 \phi)$. Here $\bigwedge^2 \mathcal{E} = \mathcal{L} \otimes \mathcal{L}^{-1} = \mathbf{C}$ and $\bigwedge^2 \phi = a + d$.

Since the trivial representation corresponds to the trivial Higgs bundle, one has $\det(\rho) = 1$ if and only if $a + d = 0$. □

Proposition 8.3: *If $a = d = 0$ and $(\mathcal{E},\phi)$ is stable then the image of the corresponding representation is in $\mathrm{SL}_2(\mathbf{R})$.*

Proof: Let $g \in \mathcal{A}ut(\mathcal{E})$ given in "matrix"coordinates by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$\begin{aligned} g\phi g^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix} = -\phi. \end{aligned}$$

If H is the Yang-Mills Hermitian metric for $(\mathcal{E}, \phi)$ than H is the metric for $(\mathcal{E}, -\phi)$ and $g \cdot H$ is the metric for $(\mathcal{E}, g \cdot \phi) = (\mathcal{E}, -\phi)$. Hence, by the uniqueness of the solution, $g \cdot H = H$ and g is H-unitary.

This means that the eigenspaces of g are H-orthogonal, thus $H = H_+ \overset{\perp}{\oplus} H_-$ where $H_\pm$ are metrics on $\mathcal{L}^{\pm 1}$.

As $\bigwedge^2 H$ is the Hermitian metric for $\bigwedge^2 \phi = 0$, then $\bigwedge^2 H$ should be the constant metric on $\mathbf{C} = \mathcal{L} \otimes \mathcal{L}^{-1}$. But $\bigwedge^2 H = H_+ \otimes H_-$; this means that H_- is the metric on $\mathcal{L}^{-1}$ induced by $(\mathcal{L}, H_+)$ by duality. More precisely if $\Theta_+ : \mathcal{L} \to \mathcal{L}^{-1}$ is the conjugate-linear isomorphism deduced from H_+ (i.e. $H_+(l, l') = \Theta_+(l) \cdot l'$) and the same for Θ_- and H_-, then Θ_+ and Θ_- are inverse of each other.

Let τ be the conjugate-linear automorphism of $\mathcal{E}$ given by

$$\tau(l_+, l_-) = \big(\Theta_-(l_-), \Theta_+(l_+)\big).$$

Then $\tau^2 = \mathrm{id}_{\mathcal{E}}$ and $H(\tau v, \tau w) = H(w, v)$, this is enough to imply that $D'\tau = \tau D''$ ($D' = \partial_H + \phi^{*_H}$ and $D'' = \bar{\partial} + \phi$) so that τ commutes with D.

The fixed points set of τ is a rank 2 real bundle and, as D commutes with τ, D restricts to a connection of this real bundle.

Consequently the holonomy is a representation in $\mathrm{GL}_2(\mathbf{R})$ —and also in $\mathrm{SL}_2(\mathbf{C})$ by Lemma 8.2. □

Hitchin used this to give a parametrization of the set $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbf{R}))/\sim$. He took advantage of the fact that the divisor of a (non-zero) holomorphic section of a line bundle determines completely the line bundle and the section (up to a multiple). For example if $\deg \mathcal{L} \geq 0$ then the divisor of c (belonging to $\mathrm{Sym}^{2g-2-2\deg \mathcal{L}}\Sigma$) and b (belonging to some vector space) almost completely determine the Higgs bundle.

There is a general procedure to detect real representation.

Proposition 8.4: *Let $(E, D'' = \bar{\partial} + \phi)$ a Higgs bundle (stable, $\mathrm{ch}_1 = \mathrm{ch}_2 = 0$).*

Then the corresponding representation has image in $\mathrm{GL}_d(\mathbf{R})$ *if and only if there is a holomorphic quadratic form* Q *on* E *such that* ϕ *is* Q*-symmetric.*

Of course the conclusion is that the representation has image in a conjugate of $\mathrm{GL}_d(\mathbf{R})$.

Proof: Let H the corresponding Hermitian metric (and always $D' = \partial_H + \phi^{*_H}$). If the holonomy of D is real, there exists a conjugate-linear automorphism τ of E with $\tau^2 = \mathrm{id}_E$ and commuting with D. One can show that this implies that $H(\tau v, \tau w) = \overline{H(v,w)}$: indeed $\overline{H(\tau v, \tau w)}$ is a formula for the Hermitian metric $\tau \cdot H$ and that metric is the solution of Theorem 7.5 for the connection $\tau \cdot D$, since $\tau \cdot D = D$, by uniqueness $\tau \cdot H = H$.

In turns this implies that:

(1) $Q(v,w) = H(\tau v, w)$ is symmetric (hence a quadratic form);
(2) τ commutes with $\partial_H + \bar\partial$ and with $\phi + \phi^{*_H}$;
(3) ϕ is Q-symmetric ($\Leftarrow \tau$ commutes with $\phi + \phi^{*_H}$);
(4) Q is holomorphic ($\Leftarrow \tau$ commutes with $\partial_H + \bar\partial$).

Conversely, let Q be as in the proposition. Define $\varrho : E \to E$ conjugate-linear isomorphism by $Q(v,w) = H(\varrho v, w)$; then $\varrho D'' = D' \varrho$ (and ϱ commutes with D). But ϱ might not be an involution. However:

$$H(\varrho v, \varrho w) = H(v, \varrho^2 w) = H(\varrho^2 v, w)$$

so ϱ^2 is positive definite, self-adjoint and commutes with D, therefore $\varrho^2 = \lambda^2 \mathrm{id}_E$ with $\lambda > 0$ and $\tau = \varrho/\lambda$ is the sought for real structure. □

In fact the converse bijection can be used to prove the uniformization theorem.

Theorem 8.5: *[Hitchin] Let* $(\mathcal{E} = \mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}, \phi = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))$.

Then $(\mathcal{E}, \phi)$ *is stable, let* H *be the Hermitian metric solution of theorem 7.5. Then* H *is diagonal and induces a metric on* $\mathcal{K}^{-1/2}$ *and finally on* $\mathcal{K}^{-1} = T\Sigma$.

Hence a Riemannian metric is obtained on Σ *and the curvature of* h *is equal to* -4.

8.4. *Teichmüller Component*

Les $\mathcal{E} = \mathcal{K}^{-n/2} \oplus \mathcal{K}^{-n/2+1} \oplus \cdots \oplus \mathcal{K}^{n/2}$, holomorphic rank n bundle of degree 0. The bundle of endomorphisms of $\mathcal{E}$ decomposes accordingly:

$$\mathrm{End}(\mathcal{E}) = \begin{pmatrix} \mathbf{C} & \mathcal{K}^{-1} & \mathcal{K}^{-2} & \cdots & \mathcal{K}^{-n+1} \\ \mathcal{K} & \mathbf{C} & \ddots & \ddots & \vdots \\ \mathcal{K}^2 & \mathcal{K} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathcal{K}^{-1} \\ \mathcal{K}^{n-1} & \cdots & \mathcal{K}^2 & \mathcal{K} & \mathbf{C} \end{pmatrix}.$$

For $\underline{\alpha} = (\alpha_2, \ldots, \alpha_n)$, with $\alpha_i \in H^0(\mathcal{K}^i)$ define

$$\phi(\underline{\alpha}) = \begin{pmatrix} 0 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & 0 \end{pmatrix}$$

a holomorphic section of $\mathrm{End}(\mathcal{E}) \otimes \mathcal{K}$.

The results of Hitchin are:

(1) for all $\underline{\alpha}$, $(\mathcal{E}, \phi(\underline{\alpha}))$ is stable;
(2) the corresponding representation $\rho(\underline{\alpha})$ is real;
(3) the map

$$\begin{aligned} \bigoplus_{i=2}^n H^0(\mathcal{K}^i) &\longrightarrow \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_n(\mathbf{R}))/\sim \\ \underline{\alpha} &\longmapsto \rho(\underline{\alpha}) \end{aligned}$$

is a bijection onto a connected component of the character variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_n(\mathbf{R}))/\sim$

Proof: [Proof of (1)] It uses 3 ingredients:

(i) $(\mathcal{E}, \phi(0))$ is stable. Let $(\mathcal{E}_2, \psi) = (\mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))$ be the Higgs bundle of Theorem 8.5, then the corresponding representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbf{R})$ is the uniformisation of Σ, in particular it is Zariski dense.
The Higgs bundle $(\mathcal{E}, \phi(0))$ is the $(n-1)$-th symmetric power $\mathrm{Sym}^{n-1}(\mathcal{E}_2, \psi)$ so that the corresponding representation is $\mathrm{Sym}^{n-1}\rho$, i.e. the composition of ρ with the morphism $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{SL}_n(\mathbf{R})$ given by the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathrm{Sym}^{n-1}\mathbf{R}^2 \simeq \mathbf{R}^n$. This representation is irreducible, thus the corresponding Higgs bundle is stable.

(ii) Stability is an open condition. [Sketch of proof] Let $(E, \bar{\partial}_n, \phi_n)$, $n \in \mathbf{N}$, be a sequence of Higgs bundles converging to $(E, \bar{\partial}^E, \phi)$ (in the sense that $\bar{\partial}_n - \bar{\partial}^E$ and $\phi_n - \phi$ converge to 0 in $\Omega^1(\Sigma; \mathrm{End}(E))$).
Suppose that for all n there is a subbundle $F_n \subset E$ satisfying "$\bar{\partial}_n F_n = 0$" and "$\phi_n F_n = 0$" (the correct equation to be written is $(\bar{\partial}_n + \phi_n)(\Omega^0(\Sigma; F_n)) \subset \Omega^1(\Sigma; F_n)$) with $\mu(F_n) \geq \mu(E)$.
Up to extracting a subsequence, the subbundles F_n can be supposed to have the following properties:

(a) $\mathrm{rank}(F_n) = k$ is constant;
(b) F_n converges to F. F_n can be seen as a (smooth) section of the bundle $\mathrm{Gr}_k(E)$ in Grassmannian manifolds, as the space of sections $\Gamma(\Sigma; \mathrm{Gr}_k(E))$ is compact, there is a converging subsequence.

Now if $\sigma \in \Omega^0(\Sigma; F)$ there are sections $\sigma_n \in \Omega^0(\Sigma; F_n)$ converging to σ and one finds that

$$(\bar{\partial}^E + \phi)\sigma = \lim(\bar{\partial}_n + \phi_n)\sigma_n \in \Omega^1(\Sigma; \lim F_n).$$

Hence F is a Higgs subbundle of $(E, \bar{\partial}^E, \phi)$. Furthermore $\mu(F) = \lim \mu(F_n)$ (in fact $F \simeq F_n$ for n big enough; this can be seen using a projection $E \to F$). Therefore $(E, \bar{\partial}^E, \phi)$ is not stable: non-stability is a closed condition.

(iii) Let $\underline{\alpha}$ in $\bigoplus H^0(\mathcal{K}^i)$. If $(\mathcal{E}, \phi(\underline{\alpha}))$ is not stable, then the same holds for any $(\mathcal{E}, \lambda g \cdot \phi(\underline{\alpha}))$, $\lambda \in \mathbf{C}^*$, $g : \mathcal{E} \to \mathcal{E}$ holomorphic gauge isomorphism (the action is of course $g \cdot \phi = g\phi g^{-1}$). Taking

$$g = g_\lambda = \begin{pmatrix} \lambda & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^n \end{pmatrix},$$

notice that

$$g \cdot \phi(\underline{\alpha}) = \begin{pmatrix} 0 & \lambda^{-1} & & \\ \lambda\alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \lambda^{-1} \\ \lambda^{n-1}\alpha_n & \cdots & \lambda\alpha_2 & 0 \end{pmatrix}$$

so that $\lambda g \cdot \phi(\underline{\alpha}) = \phi(\lambda^2\alpha_2, \ldots, \lambda^n\alpha_n)$. In the limit $\lambda \to 0$, the bundle $(\mathcal{E}, \phi(0))$ would be obtained as a limit of non-stable Higgs bundles, contradicting (i) and (ii).

□

Proof: [Proof of (2)] Take

$$Q = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \\ 1 & & \end{pmatrix}$$

holomorphic quadratic form and apply Proposition 8.4. □

Proof: [Proof of (3)] The map $\underline{\alpha} \mapsto (\mathcal{E}, \phi(\underline{\alpha}), Q)$ is (almost) a section of

$$\begin{aligned} \{(\mathcal{E}, \phi, Q) \text{ stable...}\} &\xrightarrow{P} \bigoplus_{i=2}^{n} H^0(\mathcal{K}^i) \\ (\mathcal{E}, \phi, Q) &\longmapsto (\mathrm{tr}(\phi^i))_{i=2,\dots,n}. \end{aligned}$$

(Since ϕ is a holomorphic morphism $\mathcal{E} \to \mathcal{E} \otimes \mathcal{K}$, ϕ^i is a holomorphic morphism $\mathcal{E} \to \mathcal{E} \otimes \mathcal{K}^i$ and its trace is a holomorphic section of $\mathcal{K}^i$). The above P is well defined since tr is conjugacy invariant.

For all i there is a constant $c_i \neq 0$ such that

$$\mathrm{tr}(\phi(\underline{\alpha})^i) = c_i \alpha_i + \text{polynomial}(\alpha_2, \dots, \alpha_{i-1}).$$

This is what is meant by $\underline{\alpha} \to \phi(\underline{\alpha})$ is almost a section of P.

From this one deduces:

(a) $\underline{\alpha} \to \rho(\underline{\alpha})$ is injective, thus its image is open by dimensionality;
(b) its image is also closed: if a sequence $(\rho(\underline{\alpha}_k))_{k \in \mathbf{N}}$ converges then $P(\rho(\underline{\alpha}_k))$ also converges so that $(\underline{\alpha}_k)_{k \in \mathbf{N}}$ converges equally.

(a) and (b) prove that the image of $\underline{\alpha} \to \rho(\underline{\alpha})$ is a connected component of the representation variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_n(\mathbf{R}))/\sim$ and that this connected component is diffeomorphic to a vector space. □

Hitchin proved the existence of such a component in $\mathrm{Hom}(\pi_1(\Sigma), G)/\sim$ for any adjoint simple real Lie group that is **R**-split.

8.5. *Representations in Unitary or Symplectic Groups*

The end of this section illustrates how representations in $\mathrm{U}(p, q)$ or in $\mathrm{Sp}_{2n}(\mathbf{R})$ can be obtained.

8.5.1. *The Groups*

The stabilizer of a Hermitian form of signature (p,q) is the unitary group $\mathrm{U}(p,q)$:

$$\mathrm{U}(p,q)=\left\{g\in\mathrm{GL}_{p+q}(\mathbf{C})\mid {}^t\bar{g}\begin{pmatrix}\mathrm{id}_p & \\ & -\mathrm{id}_q\end{pmatrix}g=\begin{pmatrix}\mathrm{id}_p & \\ & -\mathrm{id}_q\end{pmatrix}\right\}.$$

It is a reductive Lie group of real rank equal to $\min(p,q)$.

Symplectic groups are stabilizers of symplectic forms:

$$\begin{aligned}\mathrm{Sp}_{2n}(\mathbf{C})&=\left\{g\in\mathrm{GL}_{2n}(\mathbf{C})\mid {}^tg\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}\right\}\\ \mathrm{Sp}_{2n}(\mathbf{R})&=\mathrm{Sp}_{2n}(\mathbf{C})\cap\mathrm{GL}_{2n}(\mathbf{R})\\ &=\left\{g\in\mathrm{GL}_{2n}(\mathbf{R})\mid {}^tg\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}\right\}\\ &=\left\{g\in\mathrm{GL}_{2n}(\mathbf{C})\mid g=\bar{g},\ {}^tg\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}\right\}\end{aligned}$$

Exercise 8.4:

$$\begin{aligned}\mathrm{Sp}_{2n}(\mathbf{R})&=\mathrm{Sp}_{2n}(\mathbf{C})\cap\mathrm{U}(n,n)\\ &=\Big\{g\in\mathrm{GL}_{2n}(\mathbf{C})\mid {}^tg\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \mathrm{id}_n\\ -\mathrm{id}_n & \end{pmatrix}\text{ and}\\ &\qquad {}^t\bar{g}\begin{pmatrix} & \sqrt{-1}\mathrm{id}_n\\ -\sqrt{-1}\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \sqrt{-1}\mathrm{id}_n\\ -\sqrt{-1}\mathrm{id}_n & \end{pmatrix}\Big\}.\end{aligned}$$

Exercise 8.5:

$$\begin{aligned}&\mathrm{Sp}_{2n}(\mathbf{R})=\mathrm{GL}_{2n}(\mathbf{R})\cap\mathrm{U}(n,n)\\ &=\left\{g\in\mathrm{GL}_{2n}(\mathbf{C})\mid g=\bar{g}\text{ and }{}^t\bar{g}\begin{pmatrix} & \sqrt{-1}\mathrm{id}_n\\ -\sqrt{-1}\mathrm{id}_n & \end{pmatrix}g=\begin{pmatrix} & \sqrt{-1}\mathrm{id}_n\\ -\sqrt{-1}\mathrm{id}_n & \end{pmatrix}\right\}.\end{aligned}$$

8.5.2. *The Bundles*

Let $\mathcal{V}$ and $\mathcal{W}$ two holomorphic vector bundles of rank p and q respectively.

For α a form in $\Omega^{1,0}(X;\mathrm{Hom}(\mathcal{V},\mathcal{W}))$ holomorphic and β belonging to $\Omega^{1,0}(X;\mathrm{Hom}(\mathcal{W},\mathcal{V}))$ holomorphic, form the following Higgs bundle:

$$\left(\mathcal{E}=\mathcal{V}\oplus\mathcal{W},\phi=\begin{pmatrix} & \beta\\ \alpha & \end{pmatrix}\right).$$

Proposition 8.6: *If* $(\mathcal{E},\phi)$ *is stable and* $\mathrm{ch}_1(\mathcal{E})=\mathrm{ch}_2(\mathcal{E})=0$*, then the corresponding representation* $\rho:\pi_1(X)\to\mathrm{GL}_{p+q}(\mathbf{C})$ *has image in the subgroup* $\mathrm{U}(p,q)$*.*

Conversely if $\rho:\pi_1(X)\to\mathrm{U}(p,q)$, one can show that the associated Higgs bundle has the above form.

Proof: Let H be the Hermitian structure provided by Theorem 7.6. The reasoning is similar to the proof of Proposition 8.3.

The gauge isomorphism $g = \begin{pmatrix} \mathrm{id}_{\mathcal{V}} & \\ & -\mathrm{id}_{\mathcal{W}} \end{pmatrix}$ is holomorphic and anti-commutes with ϕ. This implies (again by the uniqueness of H and the fact that H is also the Hermitian metric for $(\mathcal{E}, -\phi)$) that g is H-unitary, therefore its eigenspaces are H-orthogonal so that $H = H_{\mathcal{V}} \overset{\perp}{\oplus} H_{\mathcal{W}}$ where $H_{\mathcal{V}}$, $H_{\mathcal{W}}$ are Hermitian structure on $\mathcal{V}$ and $\mathcal{W}$.

The consequence will be that the Hermitian form $K = H_{\mathcal{V}} - H_{\mathcal{W}}$ is flat for the connection $D = \partial_H + \bar{\partial}^{\mathcal{E}} + \phi + \phi^{*_H}$, i.e. this connection is K-unitary. This will imply that the holonomy of D is in $\mathrm{U}(p, q)$.

First $\partial_H + \bar{\partial}^{\mathcal{E}}$ is K-unitary: indeed as $\bar{\partial}^{\mathcal{E}} = \begin{pmatrix} \bar{\partial}^{\mathcal{V}} & \\ & \bar{\partial}^{\mathcal{W}} \end{pmatrix}$ and $H = \begin{pmatrix} H_{\mathcal{V}} & \\ & H_{\mathcal{W}} \end{pmatrix}$ one has $\partial_H = \begin{pmatrix} \partial_{H_{\mathcal{V}}} & \\ & \partial_{H_{\mathcal{W}}} \end{pmatrix}$. Since $\partial_H + \bar{\partial}^{\mathcal{E}}$ is H-unitary, the connection $\partial_{H_{\mathcal{V}}} + \bar{\partial}^{\mathcal{V}}$ is $H_{\mathcal{V}}$-unitary and the connection $\partial_{H_{\mathcal{W}}} + \bar{\partial}^{\mathcal{W}}$ is $H_{\mathcal{W}}$-unitary. Therefore $\partial_H + \bar{\partial}^{\mathcal{E}} = \begin{pmatrix} \partial_{H_{\mathcal{V}}} + \bar{\partial}^{\mathcal{V}} & \\ & \partial_{H_{\mathcal{W}}} + \bar{\partial}^{\mathcal{W}} \end{pmatrix}$ is $(H_{\mathcal{V}} - H_{\mathcal{W}})$-unitary.

Second $\phi + \phi^{*_H}$ is K-antihermitian: it is known that $\phi + \phi^{*_H}$ is H-hermitian and that $\phi^{*_H} = \begin{pmatrix} & \alpha^{*_H} \\ \beta^{*_H} & \end{pmatrix}$. For (v, w) and (v', w') sections of $\mathcal{E} = \mathcal{V} \oplus \mathcal{W}$, developing the equality

$$H(\phi \cdot (v, w), (v', w')) = H((v, w), \phi^{*_H}(v', w'))$$

gives

$$\begin{aligned} H_{\mathcal{V}}(\beta w, v') &= H_{\mathcal{W}}(w, \beta^{*_H} v') \\ \text{and } H_{\mathcal{W}}(\alpha v, w') &= H_{\mathcal{V}}(v, \alpha^{*_H} w'). \end{aligned}$$

But

$$\begin{aligned} K(\phi \cdot (v, w), (v', w')) &= H_{\mathcal{V}}(\beta w, v') - H_{\mathcal{W}}(\alpha v, w') \\ &= H_{\mathcal{W}}(w, \beta^{*_H}) - H_{\mathcal{V}}(v, \alpha^{*_H} w') \\ &= -K((v, w), \phi^{*_H} \cdot (v', w')), \end{aligned}$$

equally $K(\phi^{*_H} \cdot (v, w), (v', w')) = -K((v, w), \phi \cdot (v', w'))$ and $\phi + \phi^{*_H}$ is K-antihermitian. So the sum $D = \partial_H + \bar{\partial}^{\mathcal{E}} + \phi + \phi^{*_H}$ is K-unitary. □

Now suppose that $\mathcal{W} = \mathcal{V}^*$, that $\alpha \in \Omega^{1,0}(X; \mathrm{Hom}(\mathcal{V}, \mathcal{V}^*))$ is symmetric (i.e. if v, v' are sections of $\mathcal{V}$ —so that $\alpha(v)$ is a 1-form with coefficient in $\mathcal{V}^*$ thus $\alpha(v)(v')$ is a 1-form— then $\alpha(v)(v') = \alpha(v')(v)$.) and $\beta \in \Omega^{1,0}(X; \mathrm{Hom}(\mathcal{V}^*, \mathcal{V}))$ is symmetric.

Proposition 8.7: *Under these conditions, the representation corresponding to $(\mathcal{E}, \phi)$ has image into $\mathrm{Sp}_{2n}(\mathbf{R})$.*

Proof: In view of the equality $\mathrm{Sp}_{2n}(\mathbf{R}) = \mathrm{Sp}_{2n}(\mathbf{C}) \cap \mathrm{U}(n, n)$ and the preceding proposition it is enough to produce a flat symplectic form for

$D = \partial_H + \bar{\partial}^{\mathcal{E}} + \phi + \phi^{*_H}$. For this it is enough to produce a holomorphic symplectic form such that $\omega(\phi\sigma, \sigma') = -\omega(\sigma, \phi\sigma')$ for all sections σ and σ'.

Exercise 8.6: This is indeed sufficient.

Let us give a formula for ω:

$$\omega((v_1, f_1), (v_2, f_2)) = f_2(v_1) - f_1(v_1).$$

(v_i sections of $\mathcal{V}$, f_i sections of $\mathcal{V}^*$ so that (v_i, f_i) are sections of $\mathcal{E}$). It clearly defines a holomorphic symplectic form and the property of ϕ with respect to ω is a consequence of the symmetry of α and β. □

9. On Hermitian Metrics and More

This section tries to explain how to prove Theorem 7.5 (solving the "harmonic equation") and Theorem 7.6 (solving the "Hitchin equation"). The first thing is to explain why, in reality, these equation are *not* overdetermined. The necessity of the hypothesis in those theorems will be also explained and some considerations on their proofs will be given.

9.1. *Opportune Cancellations: The case of Flat Bundle*

Let D be a flat irreducible connection on a bundle E over a (compact and connected) Kähler manifold X.

Given a metric H on E, the decomposition of D into a unitary connection and a self-adjoint 1-form is again denoted by:

$$D = D_H + \psi_H,$$

and the decomposition into "types" of those are:

$$D_H = \partial_H + \bar{\partial}_H, \quad \psi_H = \phi + \phi^{*_H}.$$

The following operators are often useful:

$$\begin{aligned} D'_H &= \partial_H + \phi^{*_H} \\ D''_H &= \bar{\partial}_H + \phi \\ \text{and } \; D_C &= -D'_H + D''_H. \end{aligned}$$

Proposition 9.1: $G_H = (D''_H)^2 = 0 \Leftrightarrow L^\star G_H = 0 \Leftrightarrow D^\star_H \psi_H = 0.$

The adjunction for the L^2-scalar product is denoted with the $\star$ and L is the operation of wedging with the Kähler form ω.

Remark 9.2: This is the meaning of "opportune cancellations": as soon as $L^\star G_H$ is zero (and of course under the condition $D = 0$) then G_H is also zero.

Before the proof, recall the Kähler identities:

$$[L^\star, \bar\partial_H] = \sqrt{-1}\partial_H^\star \qquad [L^\star, D''_H] = \sqrt{-1}D'^\star_H \qquad [L^\star, D_C] = \sqrt{-1}D^\star$$
$$[L^\star, \partial_H] = -\sqrt{-1}\bar\partial_H^\star \qquad [L^\star, D'_H] = -\sqrt{-1}D''^\star_H \qquad [L^\star, D] = -\sqrt{-1}D_C^\star.$$

Proof:

(1) $G_H = \frac{1}{4}(DD_C + D_C D)$.

Since $D^2 = 0 = D'^2_H + D''^2_H + D'_H D''_H + D''_H D'_H$ one has (separating the Hermitian and antihermitian parts)

$$D'^2_H + D''^2_H = 0 \quad \text{and} \quad D'_H D''_H + D''_H D'_H = 0. \tag{9.1}$$

Thus $D_C^2 = D'^2_H + D''^2_H - D'_H D''_H - D''_H D'_H = 0$. Therefore $G_H = D''^2_H = \left(\frac{D+D_C}{2}\right)^2 = \frac{DD_C + D_C D}{4}$.

(2) $DG_H = D_C G_H = 0$.

Only one of this identities will be proved. Write, for a moment, in superscript the bundle on which a connection (or another operator) acts: the sought for identity is $D^{\mathrm{End}(E)} G_H = 0$.

For any section σ of E, the above holds:

$$\begin{aligned} D^E(G_H\sigma) &= \big(D^{\mathrm{End}(E)} G_H\big)\sigma + G_H D^E\sigma \\ &= \big(D^{\mathrm{End}(E)} G_H\big)\sigma + \tfrac{1}{4}(D^E D^E_C + D^E_C D^E) D^E\sigma \\ &= \big(D^{\mathrm{End}(E)} G_H\big)\sigma + \tfrac{1}{4} D^E D^E_C D^E\sigma \quad \text{since } (D^E)^2 = 0. \end{aligned}$$

But

$$\begin{aligned} D^E(G_H\sigma) &= D^E\Big(\frac{D^E D^E_C + D^E_C D^E}{4}\Big)\sigma \\ &= \frac{1}{4} D^E D^E_C D^E\sigma \quad \text{again since } (D^E)^2 = 0. \end{aligned}$$

Comparing the 2 equations gives $(D^{\mathrm{End}(E)} G_H)\sigma = 0$, for all sections σ, hence $D^{\mathrm{End}(E)} G_H = 0$.

(3) $D(\phi - \phi^{*_H}) = 2G_H$.

The $(0,2)$-part and the $(1,1)$-part of the first term in equation (9.1) are respectively

$$\bar\partial_H^2 + \phi^{*_H} \wedge \phi^{*_H} = 0, \quad \bar\partial_H\phi + \partial_H\phi^{*_H} = 0,$$

also, the $(2,0)$-part and the $(0,2)$-part of the second term in the same equation gives $\partial_H\phi = 0$ and $\bar\partial_H\phi^{*_H} = 0$. On the other hand

$$G_H = \bar\partial_H^2 + \bar\partial_H\phi + \phi \wedge \phi.$$

And the last term to calculate is

$$\begin{aligned} D(\phi - \phi^{*_H}) &= \partial_H \phi + \bar\partial_H \phi - \partial_H \phi^{*_H} - \bar\partial_H \phi^{*_H} + 2\phi \wedge \phi - 2\phi^{*_H} \wedge \phi^{*_H} \\ &= 2\bar\partial_H \phi + 2\phi \wedge \phi + 2\bar\partial_H^2 = 2G_H. \end{aligned}$$

Exercise 9.1: Calculating $G_H\sigma = D''_H(D''_H\sigma)$ and (in 2 different ways) $D((\phi - \phi^{*_H})\sigma)$, validate the above identities for G_H and $D(\phi - \phi^{*_H})$. (Remember that the Leibniz identity involves signs.)

(4) Suppose now that $L^\star G_H = 0$, since $D_C G_H = 0$, by the Kähler identity one get $D^\star G_H = 0$. Thus

$$\begin{aligned} 2\int \langle G_H, G_H\rangle \omega^n &= \int \langle G_H, D(\phi - \phi^{*_H})\rangle \omega^n \\ &= \int \langle D^\star G_H, \phi - \phi^{*_H}\rangle \omega^n = 0, \end{aligned}$$

and hence $G_H = 0$.

(5) $D_H^\star \psi_H = -2\sqrt{-1}L^\star G_H$.

Indeed

$$\begin{aligned} D_H^\star \psi_H &= (\partial_H^\star + \bar\partial_H^\star)(\phi + \phi^{*_H}) \\ &= \bar\partial_H^\star \phi^{*_H} + \partial_H^\star \phi \quad \text{eliminating terms not of type } (0,0) \\ &= -\sqrt{-1}L^\star(\bar\partial_H \phi - \partial_H \phi^{*_H}) \quad \text{by the Kähler identities} \\ &= -2\sqrt{-1}L^\star \bar\partial_H \phi \quad \text{since } \bar\partial_H \phi = -\partial_H \phi^{*_H}. \end{aligned}$$

Also

$$\begin{aligned} L^\star G_H &= L^\star(-\phi^{*_H} \wedge \phi^{*_H} + \bar\partial_H \phi + \phi \wedge \phi) \\ &= L^\star \bar\partial_H \phi \quad \text{since } L^\star = 0 \text{ in degrees } (2,0) \text{ and } (0,2). \end{aligned}$$

This finishes the proof of the proposition. □

9.2. *Opportune Cancellations: Higgs Bundles*

Let $(E, \bar\partial^E, \phi)$ be a Higgs bundle such that

$$\int \mathrm{ch}_1(E)\omega^{n-1} = 0, \quad \text{and} \quad \int \mathrm{ch}_2(E)\omega^{n-2} = 0.$$

Remark 9.3: Those topological assumption are a priori weaker than those stated in Theorem 7.6. Nevertheless, only the cancellations of those 2 integrals are involved in the proof of that result. The present paragraph illustrates why.

For an Hermitian metric H on E, construct:

$$\begin{aligned} D_H &= \partial_H + \bar\partial^E \quad && \text{the Chern connection;} \\ \psi_H &= \phi + \phi^{*_H} && \text{sum of } \phi \text{ and its adjoint,} \end{aligned}$$

and $D'' = \bar{\partial}^E + \phi$, $D'_H = \partial_H + \phi^{*_H}$, $D = D'_H + D'' = D_H + \psi_H$. The curvature of D is

$$\begin{aligned} F = F(D) &= F(D_H) + D_H\psi_H + \psi_H \wedge \psi_H \\ &= D_H'^2 + (D'_H D'' + D'' D'_H) + D''^2. \end{aligned}$$

Proposition 9.4: *If $L^\star F = 0$, then $F = 0$.*

Remark 9.5: The equation $L^\star F = 0$ already implies $\deg_\omega E = 0$. Indeed let β be the 2-form $\operatorname{tr}(F)$, then $L^\star \beta = \operatorname{tr}(L^\star F) = 0$. Thus:

$$\begin{aligned} \int \beta \wedge \omega^{n-1} &= \int \beta \wedge \star\omega && \text{as } \star\omega = \omega^{n-1} \\ &= \int \langle \beta, \omega \rangle \omega^n && \text{by the definition of } \star \\ &= \int \langle \beta, L1 \rangle \omega^n \\ &= \int \langle L^\star \beta, 1 \rangle \omega^n = 0. \end{aligned}$$

But $\deg_\omega E$ is proportional to $\int \beta\omega^{n-1}$.

In the proof of the proposition, the following equivalent of Lemma 7.10 will be involved.

Lemma 9.6: *Let $\beta \in \Omega^{1,1}(X; \operatorname{End}(E))$ such that $\beta^{*_H} = -\beta$ and $L^\star\beta = 0$, then $\operatorname{tr}(\beta \wedge \beta)\omega^{n-2} = \langle \beta, \beta \rangle \omega^n$.*
Let $\alpha \in \Omega^{2,0}(X; \operatorname{End}(E))$ then $\operatorname{tr}(\alpha \wedge \alpha^{*_H})\omega^{n-2} = \langle \alpha, \alpha \rangle \omega^n$.

(There are certainly proportionality constants in the above equalities, no attempt will be made to calculate them).

Remark 9.7: These equalities are probably related with the equation $\star(\omega \wedge \omega) = \omega^{n-2}$.

Proof: [Proof of Proposition 9.4]
In the Hermitian/antihermitian decomposition of F:

$$\begin{aligned} F &= (F(D_H) + \psi_H \wedge \psi_H) + (D_H \psi_H) \\ &= (D'_H D'' + D'' D'_H) + (D_H'^2 + D''^2), \end{aligned}$$

the antihermitian part $\beta = F(D_H) + \psi_H \wedge \psi_H$ is of type $(1,1)$ whereas the hermitian part as the form $\alpha + \alpha^{*_H}$ with $\alpha = \partial_H \phi$ of type $(2,0)$.

This already implies that $L^\star\beta = L^\star F$ since $L^\star$ is zero on forms of degrees $(2,0)$ and $(0,2)$. By the lemma (and the hypothesis $L^\star F = 0$)

$$\begin{aligned} \int \operatorname{tr}\beta \wedge \beta \omega^{n-2} &= \|\beta\|^2_{L^2} \\ \int \operatorname{tr}\alpha \wedge \alpha^{*_H} \omega^{n-2} &= \|\alpha\|^2_{L^2}. \end{aligned}$$

The hypothesis on E says that $\int \mathrm{tr} F \wedge F\omega^{n-2} = 0$. Developping $F = \beta + \alpha$ in this equality and canceling out the terms not of degree (n, n):

$$\begin{aligned} 0 = \int \mathrm{tr} F \wedge F\omega^{n-2} &= \int \mathrm{tr}\beta \wedge \beta\omega^{n-2} + \int \mathrm{tr}\alpha \wedge \alpha^{*_H}\omega^{n-2} + \int \mathrm{tr}\alpha^{*_H} \wedge \alpha\omega^{n-2} \\ &= \|\beta\|^2_{L^2} + \|\alpha\|^2_{L^2}. \end{aligned}$$

Consequently $\beta = \alpha = 0$ and $F = 0$. □

9.3. *Complete Reducibility and Poly-Stability*

We illustrate in this paragraph the necessity of the hypothesis in Theorems 7.5 and 7.6.

Proposition 9.8: *Let (E, D) be a flat bundle over a Riemannian manifold M. Suppose there exists an harmonic metric H (i.e. such that $D_H^\star \psi_H = 0$ always using the decomposition $D = D_H + \psi_H$ as the sum of a unitary connection and an Hermitian 1-form).*

The flat bundle (E, D) is then a direct orthogonal sum of irreducible connections.

One sometimes says that D is completely reducible or semisimple.

Proof: By a recurrence on $\dim E$, it is enough to prove that $F^\perp$ (orthogonal with respect to H) is D-stable as soon as the subbundle F is D-stable.

In the direct sum $E = F \oplus F^\perp$, the connection D has the form

$$D = \begin{pmatrix} D_1 & 2\beta \\ 0 & D_2 \end{pmatrix}$$

with $\beta \in \Omega^1(M; \mathrm{Hom}(F^\perp, F))$ (the factor 2 here is to avoid factors 1/2 in the subsequent formulas).

The unitary part of D is the connection:

$$D_H = \begin{pmatrix} D_{H_1} & \beta \\ -\beta^{*_H} & D_{H_2} \end{pmatrix}.$$

Let $S = \left(\begin{smallmatrix} -\mathrm{id}_F & \\ & id_{F^\perp} \end{smallmatrix}\right) \in \Omega^0(M; \mathrm{End}(E))$, then

$$0 = \int \langle D_H^\star \psi_H, S\rangle \mathrm{vol}_M = \int \langle \psi_H, D_H S\rangle \mathrm{vol}_M.$$

But

$$\begin{aligned} D_H S &= \left(\begin{smallmatrix} D_{H_1} & \\ & D_{H_2} \end{smallmatrix}\right) S + \left[\left(\begin{smallmatrix} 0 & \beta \\ -\beta^{*_H} & 0 \end{smallmatrix}\right), S\right] \\ &= 0 + 2\left(\begin{smallmatrix} 0 & \beta \\ \beta^{*_H} & 0 \end{smallmatrix}\right) \end{aligned}$$

Also

$$\psi_H = \begin{pmatrix} \psi_1 & \beta \\ \beta^{*_H} & \psi_2 \end{pmatrix}$$

so $\langle \psi_H, D_H S\rangle$ is the function equal to (local calculation in an orthonormal frame $(e_i)_{i=1,\dots,n}$ of TM are understood here):

$$\begin{aligned}\langle \psi_H, D_H S\rangle &= \sum \mathrm{tr}(\psi_H(e_i)(D_H S(e_i))^{*_H}) \\ &= \sum 2\mathrm{tr}\begin{pmatrix} & \beta \\ \beta^{*_H} & \end{pmatrix}^2(e_i) = 4\sum \mathrm{tr}(\beta(e_i)\beta^{*_H}(e_i)) \\ &= 4\sum \langle \beta(e_i), \beta(e_i)\rangle = 4\langle \beta, \beta\rangle.\end{aligned}$$

The conclusion is that $\int \langle \beta, \beta\rangle \mathrm{vol}_M = 0$, i.e. $\beta = 0$ and the flat bundle splits. □

Proposition 9.9: *Let $(E, \bar\partial^E, \phi)$ be a Higgs bundle. Suppose that there is a Hermitian metric H and a scalar $\lambda \in \mathbf{C}$ such that $L^\star F = \lambda \mathrm{id}_E$ (with $F = F(D)$, $D = D'_H + D''$, etc.).*

Then any Higgs subbundle $\mathcal{F}$ (defined outside an analytic subset $\mathcal{S}$ of complex codimension 2) satisfies $\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\mathrm{rank} \mathcal{F}} \le \mu(\mathcal{E})$. In case of equality $\mu(\mathcal{F}) = \mu(\mathcal{E})$, the subbundle $\mathcal{F}^\perp$ is holomorphic and ϕ-invariant (i.e. it is a Higgs subbundle) (and $\mathcal{S}$ is empty).

The consequence of this proposition is that $\mathcal{E}$ is poly-stable: it is a direct sum of stable Higgs bundles of the same slope (the slope of $\mathcal{E}$ is $\mu(\mathcal{E})$).

Remark 9.10: The complex number λ is determined by the topological (or $\mathcal{C}^\infty$) vector bundle E:

$$\mathrm{rank}(E)\frac{\sqrt{-1}}{2\pi}\lambda \mathrm{vol}(X) = \deg(E).$$

Proof: (The proof is done only for a bundle $\mathcal{F}$ defined *everywhere*).

In the orthogonal decomposition $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}^\perp = \mathcal{F} \oplus \mathcal{Q}$, the operator $\bar\partial^E$ and the form ϕ are triangular:

$$\bar\partial^E = \begin{pmatrix} \bar\partial^F & \alpha^{*_H} \\ & \bar\partial^Q \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_S & \beta \\ & \phi_Q, \end{pmatrix}$$

$$\begin{aligned}\text{with}\quad & \beta \in \Omega^{1,0}(X; \mathrm{Hom}(Q, S)) \\ & \alpha \in \Omega^{1,0}(X; \mathrm{Hom}(S, Q)).\end{aligned}$$

One has

$$D = \begin{pmatrix} D_F & \alpha^{*_H} + \beta \\ -\alpha + \beta^{*_H} & D_Q \end{pmatrix} = \begin{pmatrix} D_F & \\ & D_Q \end{pmatrix} + \begin{pmatrix} & \alpha^{*_H} + \beta \\ -\alpha + \beta^{*_H} & \end{pmatrix},$$

so that

$$F(D) = \begin{pmatrix} F_F & \\ & F_Q \end{pmatrix} + \begin{pmatrix} & D(\alpha^{*_H} + \beta) \\ D(-\alpha + \beta^{*_H}) & \end{pmatrix} + \begin{pmatrix} (\alpha^{*_H} + \beta) \wedge (-\alpha + \beta^{*_H}) & \cdots \\ \cdots & \cdots \end{pmatrix}$$

($\cdots$ are for the terms that are not calculated), looking at the component in $\mathrm{End}(F)$ and applying $L^\star$ give:

$$\lambda \mathrm{id}_F = L^\star F_F + L^\star(-\alpha^{*_H} \wedge \alpha + \beta \wedge \beta^{*_H}).$$

Multiplying by $\frac{\sqrt{-1}}{2\pi}$, taking the trace, wedging with ω^{n-1} and integrating over X gives:

$$\begin{aligned} \tfrac{\sqrt{-1}}{2\pi}\lambda \mathrm{vol}(X)\mathrm{rank}(F) - \deg(F) &= -\tfrac{\sqrt{-1}}{2\pi}\int \mathrm{tr}\alpha^{*_H} \wedge \alpha \wedge \omega^{n-1} + \tfrac{\sqrt{-1}}{2\pi}\int \mathrm{tr}\beta \wedge \beta^{*_H} \wedge \omega^{n-1} \\ &= \tfrac{-1}{\pi}\int \|\alpha\|^2\omega^n - \tfrac{1}{\pi}\int \|\beta\|^2\omega^n. \end{aligned}$$

Exercise 9.2: Prove the last equalities.

Hence $\mu(\mathcal{E}) - \mu(\mathcal{F}) \leq 0$ and if this last difference is zero, α and β are zero and the Higgs bundle splits. □

9.4. *Minimizing Functionals*

The two equations under concern here (the harmonic equation or the Hitchin equation) are in fact related to minimizing functionals.

9.5. *The Energy*

For a flat connection D on $E \to M$ and a Hermitian metric H, let us define:

$$\mathscr{E}(H) = \mathscr{E}(D, H) = \int \langle \psi_H, \psi_H \rangle \mathrm{vol}_M = \langle \psi_H, \psi_H \rangle_{L^2}.$$

The point is now to understand the minimum of $\mathscr{E}$ when H varies and D is fixed. However, for any gauge automorphism $g \in \mathcal{A}ut(E)$, the following holds:

$$\mathscr{E}(g \cdot D, g \cdot H) = \mathscr{E}(D, H).$$

Exercise 9.3: Recall the action $H \mapsto g \cdot H$ and verify this equality.

Hence, as the action of $\mathcal{A}ut(E)$ is transitive on the space of Hermitian structures on E, an equivalent point of view consists in fixing H and leting D vary in a gauge orbit. The benefits of this change are that the L^2 metrics are not varying anymore. (The definition of the L^2-metric on $\mathrm{End}(E)$ involves H).

Lemma 9.11: *If H is a minimum for the functional $\mathscr{E}$, then H is a solution of the harmonic equation: $D_H^\star \psi_H = 0$.*

Remark 9.12: The energy functional $\mathscr{E}$ is in fact convex so that any critical point is a minimum.

Proof: A formula for the first variation of the energy $\frac{d}{dt}|_{t=0}\mathscr{E}(D, H_t)$ (with $H_0 = H$) is sought for.

As there is $g_t \in \mathcal{A}ut(E)$ with $g_0 = \mathrm{id}_E$ and $H_t = g_t \cdot H$, it is equivalent to understand

$$\mathscr{E}(g_t^{-1} \cdot D, H) = \mathscr{E}(D, g_t \cdot H)$$

up to first order.

Since $g_t^{-1} \cdot D - D = g_t^{-1} D^{\mathrm{End}(E)} g_t$, one has

$$\begin{aligned} \tfrac{d}{dt}|_{t=0} g_t^{-1} \cdot D &= \lim_{t\to 0} \tfrac{g_t^{-1}\cdot D - D}{t} \\ &= \lim_{t\to 0} g_t^{-1} \tfrac{D^{\mathrm{End}(E)} g_t}{t} \\ &= \lim_{t\to 0} g_t^{-1} \tfrac{D^{\mathrm{End}(E)}(g_t - g_0)}{t} \qquad \text{as } Dg_0 = D\mathrm{id}_E = 0 \\ &= D\dot{g} \quad \text{with } \dot{g} = \tfrac{d}{dt}|_{t=0} g_t \\ &= D_H \dot{g} + [\psi_H, \dot{g}] \end{aligned}$$

Let ψ_t the Hermitian part of $g_t^{-1} \cdot D$. The derivative $\frac{d}{dt}|_{t=0}\psi_t$ is the Hermitian part of $\frac{d}{dt}|_{t=0} g_t^{-1} \cdot D$. Writing $\dot{g} = A + S$ as the sum of an antihermitian and a Hermitian endomorphism and observing that the operator D_H respects these decompositions whereas the operator $[\psi_H, \cdot]$ permutes them, one obtain:

$$\frac{d}{dt}|_{t=0}\psi_t = D_H S + [\psi_H, A].$$

Consequently

$$\frac{d}{dt}|_{t=0}\mathscr{E}(D, H_t) = 2 \int \langle \psi_H, D_H S + [\psi_H, A] \rangle \mathrm{vol}_M.$$

It is not difficult to establish that $\langle \psi_H, [\psi_H, A] \rangle = 0$. Hence the following formula is obtained:

$$\frac{d}{dt}|_{t=0}\mathscr{E}(D, H_t) = 2 \langle D_H^\star \psi_H, S \rangle_{L^2}.$$

As $D_H^\star \psi_H$ is Hermitian, taking $S = D_H^\star \psi_H$ in this last equation implies that the critical values of $\mathscr{E}$ must satisfy $D_H^\star \psi_H = 0$. □

9.6. *Why is this the Energy?*

To establish this, let us write $E = E_\rho = \Gamma\backslash(\widetilde{M} \times \mathbf{C}^d)$ (as in paragraph 4.3) for $\rho : \Gamma = \pi_1(M) \to \mathrm{GL}_d(\mathbf{C})$.

A Hermitian metric H is the same thing as a ρ-equivariant metric on $\widetilde{M} \times \mathbf{C}^d$. But the space of metrics on $\mathbf{C}^d$ is

$$N = \{S \in M_d(\mathbf{C}) \mid {}^t\overline{S} = S,\ S > 0\}$$

the space of positive definite Hermitian matrices. Hence an equivariant metric on $\widetilde{M} \times \mathbf{C}^d$ is a map

$$f : \widetilde{M} \longrightarrow N$$

satisfying the equivariance relation:

$$f(m) = {}^t\overline{\rho(\gamma)} f(\gamma \cdot m)\rho(\gamma), \quad \forall\, \gamma \in \Gamma,\, m \in \widetilde{M}.$$

Lemma 9.13: $\psi_H = -\frac{1}{2} f^{-1} df$.

The first observation is that this equality makes sense: $\alpha = f^{-1}df \in \Omega^1(\widetilde{M}; \mathrm{End}(\mathbf{C}^d))$ is equivariant:

$$\alpha_{\gamma \cdot m} = \rho(\gamma)\alpha_m \rho(\gamma)^{-1}$$

and hence descends to M: it "belongs" to $\Omega^1(M; \mathrm{End}(E))$.

Proof: By the definitions of d (the exterior derivative), D_H and ψ_H and since $H(\sigma, \sigma') = {}^t\bar{\sigma} f \sigma'$ for σ and σ' sections of $\widetilde{M} \times \mathbf{C}^d$, one has

$$\begin{aligned} dH(\sigma, \sigma') &= H(D_H\sigma, \sigma') + H(\sigma, D_H\sigma') \\ d &= D_H + \psi_H, \end{aligned}$$

(Here the equivariant decomposition of the trivial connection d on $\widetilde{M} \times \mathbf{C}^d$ is introduced with the same notations as on M). Therefore

$$\begin{aligned} dH(\sigma, \sigma') &= d({}^t\bar{\sigma} f \sigma') \\ &= {}^t\overline{d\sigma} f \sigma' + {}^t\bar{\sigma} df \sigma' + {}^t\bar{\sigma} f d\sigma' \\ &= H(d\sigma, \sigma') + {}^t\bar{\sigma} df \sigma' + H(\sigma, d\sigma') \\ &= H(D_H\sigma, \sigma') + H(\sigma, D_H\sigma') + H(\psi_H\sigma, \sigma') + H(\sigma, \psi_H\sigma') + {}^t\bar{\sigma} df \sigma' \\ &= dH(\sigma, \sigma') + 2H(\sigma, \psi_H\sigma') + {}^t\bar{\sigma} df \sigma', \end{aligned}$$

thus $2f\psi_H + df = 0$. □

The open subset $N \subset \mathrm{Herm}(\mathbf{C}^d) = \{S \in M_d(\mathbf{C}) \mid {}^t\overline{S} = S\}$ has a structure of a Riemannian manifold:

$$\text{if } X,\, Y \in T_A N = \mathrm{Herm}(\mathbf{C}^d), \text{ then } \quad \langle X, Y\rangle_A = \mathrm{tr} A^{-1} X A^{-1} Y.$$

Using this and the above formula in the lemma:

$$\mathscr{E}(H) = \int_M \langle \psi_H, \psi_H \rangle \mathrm{vol}_M = \frac{1}{4} \int_M \langle df, df \rangle \mathrm{vol}_M,$$

where $\langle df, df\rangle$ uses the scalar products from the Riemannian structures on M and N. One notes that $\langle df, df\rangle : \widetilde{N} \to \mathbf{R}$ is $\pi_1(M)$-invariant and hence is (or descends to) a function on M as the notation $\int_M$ already suggested.

The conclusion of this discussion is that $\mathscr{E}$ represents the energy of the function f.

9.7. *The Heat Flow*

One of the possible methods to finding a solution of the harmonic equation is to use the associated gradient flow. Here this flow is called the heat flow, and in term of the metric H, it is given by the equation:

$$H_t^{-1} \frac{d}{dt} H_t = -D_H^\star \psi_H.$$

This is still a non-linear PDE, but its linearization is a parabolic linear equation. Using this (and after a significant amount of work) the existence of solutions for short period of time can be shown.

Of course the existence of H_t is equivalent to the existence of a family $(f_t)_{t\in I}$ of ρ-equivariant functions $f_t : \widetilde{M} \to N$ satisfying the corresponding heat equation.

An important observation is that the sectional curvature of N is non-positive. Zen a Bochner type formula (and another significant amount of work) shows that. for any two solutions f_t and f'_t of the heat equation, the function $t \mapsto \sup_{m\in M} d_N(f_t(m).f'_t(m))$ is bounded.

This in turn implies that solutions are defined for all $t \in \mathbf{R}_+$ but it also implies that for all $\gamma \in \pi_1(M)$ there is a constant $c(\gamma)$ such that, for all $t \geq 0$, $d_N(f_t(m_0), \rho(\gamma) f_t(m_0)) \leq c(\gamma)$.

The case when the solution f_t "converges to infinity" needs to excluded. This would means (maybe up to extraction) that the family of points $(f_t(m_0))_{t\geq 0}$ converges to some ξ in the visual boundary at infinity of N. Using the above bound, one deduces that $\rho(\gamma)\xi = \xi$ for all γ in $\pi_1(M)$. This means precisely that ρ is not irreducible.

9.8. *The Yang-Mills Functional*

In order to solve the Hitchin equation for a Higgs bundle (E, D''), a natural approach is to try to minimize the Yang-Mills functional $\mathscr{Y}\mathscr{M}(H) = \|F(D'_H + D'')\|_{L^2} = \|L^\star F(D'_H + D'')\|_{L^2}$ (in fact the equality of the 2 last terms is only up to an additive constant depending on the topological bundle E). The gradient flow for the Yang-Mills functional leads to an evolution equation involving fourth order derivatives of H and it is not the route pursued. Instead of the Yang-Mills flow is the following evolution equation:

$$H_t^{-1}\frac{d}{dt}H_t = -L^\star F_t,$$

where F_t is the curvature of $D'_H + D''$. The justification for considering this equation is as follow (this discussion is borrowed from REF SIU). One can consider the map $H \mapsto L^\star F(D'_H + D'')$ as a vector field on the space $\mathrm{Herm}(E)$ of Hermitian metric on E and one searches for a zero of that vector field. A possible approach to find such a zero is to follow the gradient flow of the vector field (its opposite here). It turns out that in the present situation, the norm of the vector field is indeed decreasing along its gradient flow.

It is again a significant effort to prove the existence of the flow for all positive time and its convergence to a solution of the Hitchin equation under the stability hypothesis. In its full generality, this result is due to Simpson, based on an important result by Uhlenbeck and Yau. Earlier results have been obtained by Donaldson and by Hitchin.

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