TOPOLOGICAL INVARIANTS OF ANOSOV REPRESENTATIONS

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ABSTRACT. We define new topological invariants for Anosov representations and study them in detail for maximal representations of the fundamental group of a closed oriented surface Σ into the symplectic group $\mathrm{Sp}(2n, \mathbf{R})$. In particular we show that the invariants distinguish connected components of the space of symplectic maximal representations. Since the invariants behave naturally with respect to the action of the mapping class group of Σ, we obtain from this the number of components of the quotient by the mapping class group action.

For specific symplectic maximal representations we compute the invariants explicitly. This allows us to construct nice model representations in all connected components. The construction of model representations is of particular interest for $\mathrm{Sp}(4, \mathbf{R})$, because in this case there are $-1 - \chi(\Sigma)$ connected components in which all representations are Zariski dense and no model representations were known so far. Finally, we use the model representations to draw conclusion about the holonomy of symplectic maximal representations.

1. INTRODUCTION

Let Σ be a closed oriented connected surface of negative Euler characteristic, G a connected Lie group. The obstruction to lifting a representation $\rho : \pi_1(\Sigma) \to G$ to the universal cover of G is a characteristic class of ρ which is an element of $\mathrm{H}^2(\Sigma; \pi_1(G)) \cong \pi_1(G)$.

When G is compact it is a consequence of the famous paper of Atiyah and Bott [2] that the connected components of

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

are in one-to-one correspondence with the elements of $\pi_1(G)$. When G is a complex Lie group the analogous result has been conjectured by Goldman [18] and proved by Li [29].

When G is a real non-compact Lie group, this correspondence between connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ and elements of $\pi_1(G)$ fails.

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Obviously characteristic classes of representations still distinguish certain connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, but they are not sufficient to distinguish all connected components.

Here are some examples of this phenomenon:

(i) For $n \geq 3$, the characteristic class of a representation of $\pi_1(\Sigma)$ into $\mathrm{PSL}(n, \mathbf{R})$ is an element of $\mathbf{Z}/2\mathbf{Z}$. But the space

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(n, \mathbf{R}))/\mathrm{PGL}(n, \mathbf{R})$$

has three connected components [24].

(ii) For representations of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2, \mathbf{R})$ the Euler number does distinguish the $4g - 3$ connected components [18]. For representations into $\mathrm{SL}(2, \mathbf{R})$ the Euler number is not sufficient to distinguish connected components, there are $2^{2g+1} + 2g - 3$ components, and in particular there are 2^{2g} components of maximal (or minimal) Euler number, each of which corresponds to the choice of a spin structure on Σ.

(iii) For representations of $\pi_1(\Sigma)$ into $\mathrm{Sp}(2n, \mathbf{R})$ the characteristic class is an integer which generalizes the Euler number. It belongs to $\mathrm{H}^2(\Sigma; \pi_1(\mathrm{Sp}(2n, \mathbf{R}))) \cong \mathbf{Z}$ and is bounded in absolute value by $n(g-1)$. The subspace of representations where it equals $n(g-1)$ is called the space of maximal representations. This subspace decomposes into several connected components, 3×2^{2g} when $n \geq 3$ [14] and $(3 \times 2^{2g} + 2g - 4)$ when $n = 2$ [19]. The space of maximal representations and its connected components are in detail discussed in this article.

We introduce new topological invariants for representations $\rho : \pi_1(M) \to G$, whenever ρ is an *Anosov representation*. Let us sketch the definition (see Section 2.1 for details). Let M be a compact manifold equipped with an Anosov flow. A representation $\rho : \pi_1(M) \to G$ is said to be a (G, H)-Anosov representation if the associated G/H-bundle over M admits a section which is constant along the flow with certain contraction properties.

Theorem 1. *Let $\rho : \pi_1(M) \to G$ be a (G, H)-Anosov representation. Then there is a principal H-bundle over M canonically associated to ρ, whose topological type gives topological invariants of ρ. There is well defined map*

$$\mathrm{Hom}_{H\text{-Anosov}}(\pi_1(M), G) \longrightarrow \mathcal{B}_H(M),$$

where $\mathrm{Hom}_{H\text{-Anosov}}(\pi_1(M), G)$ denotes the subspace of Anosov representations and $\mathcal{B}_H(M)$ the set of gauge isomorphism classes of principal H-bundles over M. This map is natural with respect to:

– *taking covers of M,*
– *certain morphisms of pairs $(G, H) \to (G', H')$ (see Lemma 2.8).*

1.1. **Maximal representations into Sp$(2n, \mathbf{R})$.** Our main focus lies on maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$. Maximal representations into

$\mathrm{Sp}(2n, \mathbf{R})$ are $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$-Anosov representations [9]. More precisely, let M be the unit tangent bundle $T^1\Sigma$ with respect to some hyperbolic metric on Σ, then the geodesic flow is an Anosov flow on M. The fundamental group $\pi_1(M)$ is a central extension of $\pi_1(\Sigma)$ and comes with a natural projection $\pi : \pi_1(M) \to \pi_1(\Sigma)$. Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation, then the composition $\rho \circ \pi : \pi_1(M) \to \mathrm{Sp}(2n, \mathbf{R})$ is a $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$-Anosov representation. The topological invariants obtained by Theorem 1 are the characteristic classes of a $\mathrm{GL}(n, \mathbf{R})$-bundle over M. We only consider the first and second Stiefel-Whitney classes $sw_1(\rho \circ \pi) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$ and $sw_2(\rho \circ \pi) \in \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.

Theorem 2. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation. Then the topological invariants $sw_1(\rho) = sw_1(\rho \circ \pi) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$ and $sw_2(\rho) = sw_2(\rho \circ \pi) \in \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$ are subject to the following constraints:*

(i) *The image of*

$$sw_1 : \mathrm{Hom}_{\max}(\Gamma, G) \longrightarrow \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$$

is contained in one coset of $\mathrm{H}^1(\Sigma; \mathbf{F}_2)$.

- *For n even, $sw_1(\rho)$ is in* $\mathrm{H}^1(\Sigma; \mathbf{F}_2) \subset \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$,
- *for n odd, $sw_1(\rho)$ is in* $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$.

(ii) *The image of*

$$sw_2 : \mathrm{Hom}_{\max}(\Gamma, G) \longrightarrow \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$$

lies in the image of $\mathrm{H}^2(\Sigma; \mathbf{F}_2) \to \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.

In the case when $n = 2$, that is for maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$, let $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ denote the subspace of maximal representations where the first Stiefel-Whitney class vanishes. This means that the $\mathrm{GL}(2, \mathbf{R})$-bundle over $T^1\Sigma$ admits a reduction of the structure group to $\mathrm{GL}^+(2, \mathbf{R})$, equivalently it means that the corresponding $\mathbf{R}^2$-vector bundle is orientable. A reduction of the structure group to $\mathrm{GL}^+(2, \mathbf{R})$ gives rise to an Euler class, but since an orientable bundle does not have a canonical orientation this reduction is not canonical. To circumvent this problem, we introduce an enhanced representation space, which involves the choice of a nontrivial element $\gamma \in \pi_1(\Sigma)$. For pairs (ρ, L_+) consisting of a maximal representation with vanishing first Stiefel-Whitney class and an oriented Lagrangian $L_+ \subset \mathbf{R}^4$ which is fixed by $\rho(\gamma)$, there is a well-defined Euler class (see Section 4.3).

Theorem 3. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation with $sw_1(\rho) = 0$. Let $\gamma \in \pi_1(\Sigma) - \{1\}$ and L_+ an oriented Lagrangian fixed by $\rho(\gamma)$. Then the Euler class $e_\gamma(\rho, L_+) \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$ lies in the image of* $\mathrm{H}^2(\Sigma; \mathbf{Z}) \to \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$.

For every possible topological invariant satisfying the above constraints we construct explicit representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ realizing this invariant (see Section 1.3 and Section 3). From this we deduce a lower bound on the connected components of the space $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ of maximal representations.

Proposition 4. *Let* $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ *be the space of maximal representations.*

(i) *If* $n \geq 3$ *the space* $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ *has at least* 3×2^{2g} *connected components.*
(ii) *The space* $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ *has at least* $3 \times 2^{2g} + 2g - 4$ *connected components.*

Our method (so far) only gives a lower bound on the number of connected components. To obtain an exact count of the number of connected components using the invariants defined here, a closer analysis for surfaces with boundary would be necessary (see [18] for the case when $n = 1$).

Fortunately, the correspondence between (reductive) representations and Higgs bundles allows to use algebro-geometric methods to study the topology of $\mathrm{Rep}(\pi_1(\Sigma), G) := \mathrm{Hom}(\pi_1(\Sigma), G)/G$. These methods have been developed by Hitchin [23] and applied to representations into Lie groups of Hermitian type in [19, 15, 14, 7, 8] leading to the exact count mentioned above (iii).

Combining Proposition 4 with this exact count we can conclude that the invariants defined here distinguish connected components.

More precisely, let $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ be the space of Hitchin representations; by definition it is the union of the connected components of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ containing representations of the form $\phi_{irr} \circ \iota$ where $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ is a discrete embedding and $\phi_{irr} : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Sp}(2n, \mathbf{R})$ is the irreducible representation of $\mathrm{SL}(2, \mathbf{R})$ of dimension $2n$. Hitchin representations are maximal representations.

Theorem 5. *Let* $n \geq 3$. *Then the topological invariants of Theorem 2 distinguish connected components of* $\mathrm{Hom}_{\mathrm{max}} - \mathrm{Hom}_{\mathrm{Hitchin}}$. *More precisely,*

$$\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \xrightarrow{sw_1, sw_2} \mathrm{H}^1(M; \mathbf{F}_2) \times \mathrm{H}^2(M; \mathbf{F}_2)$$

is a bijection onto the set of pairs satisfying the constraints of Theorem 2.

It is easy to see that, when n is even, the first Stiefel-Whitney class of a Hitchin representation vanishes, *i.e.* one has the inclusion $\mathrm{Hom}_{\mathrm{Hitchin}} \subset \mathrm{Hom}_{\mathrm{max}, sw_1 = 0}$.

Theorem 6. *The Euler class defines a map*

$$\mathrm{Hom}_{\mathrm{max}, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \longrightarrow \mathrm{H}^2(M; \mathbf{Z})$$

which is a bijection onto the image of $\mathrm{H}^2(\Sigma; \mathbf{Z})$ *in* $\mathrm{H}^2(M; \mathbf{Z})$. *In particular, the Euler class distinguishes connected components in* $\mathrm{Hom}_{\mathrm{max}, sw_1=0} - \mathrm{Hom}_{\mathrm{Hitchin}}$.

The components of $\mathrm{Hom}_{\max} - \mathrm{Hom}_{\max, sw_1=0}$ *are distinguished by the first and second Stiefel-Whitney classes,* i.e. *the map*

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \xrightarrow{sw_1, sw_2} \big(\mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}\big) \times \mathrm{H}^2(\Sigma; \mathbf{F}_2)$$

is a bijection.

Remark 1. *Hitchin representations are not only* $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$*-Anosov representations, but* $(\mathrm{Sp}(2n, \mathbf{R}), A)$*-Anosov representations, where* A *is the subgroup of diagonal matrices* [27]*. Applying Theorem 1 to the pair* $(G, H) = (\mathrm{Sp}(2n, \mathbf{R}), A)$ *one can define first Stiefel-Whitney classes* $sw_1^A(\rho)$ *in* $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$*, similarly to the above discussion those invariants are shown to belong to* $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ *and distinguish the* 2^{2g} *connected components of* $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$*.*

Remark 2. *The existence of* 2^{2g} *Hitchin components is due to the presence of a center in* $\mathrm{Sp}(2n, \mathbf{R})$*: they all project to the same component in* $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$*. The abundance of non-Hitchin connected components in the space of maximal representations pertains even when we consider representations into the adjoint group, and is in fact explained by the invariants we are defining here. In this special case these invariants can be non-trivial due to the non-trivial topology of* $\mathrm{GL}(n, \mathbf{R})$ *(or* $\mathrm{PGL}(n, \mathbf{R})$*).*

1.2. **The action of the mapping class group.** The first and second Stiefel-Whitney classes of a maximal representation ρ do not change if ρ is conjugated by an element $\mathrm{Sp}(2n, \mathbf{R})$. Thus, they give well defined functions:

$$sw_i : \mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{H}^i(T^1\Sigma; \mathbf{F}_2). \tag{1}$$

The mapping class group $\mathcal{M}od(\Sigma)$ acts by precomposition on $\mathrm{Rep}_{\max}$; this action is properly discontinuous [34, 28] and by Theorem 1 the map (1) is equivariant with respect to this action and the natural action of $\mathcal{M}od(\Sigma)$ on $\mathrm{H}^i(T^1\Sigma; \mathbf{F}_2)$.

For the Euler class e_γ (see Theorem 3) there is a corresponding statement of equivariance for the subgroup of $\mathcal{M}od(\Sigma)$ fixing the homotopy class of γ.

This allows us to determine the number of connected components of $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathcal{M}od(\Sigma)$.

Theorem 7. *If* $n \geq 3$*, the space* $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathcal{M}od(\Sigma)$ *has* 6 *connected components.*

The space $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))/\mathcal{M}od(\Sigma)$ *has* $2g+2$ *connected components.*

1.3. **Model representations.** Given two representations it is in general very difficult to determine whether they lie in the same connected component or not. The invariants defined here can be computed rather explicitly and hence allow us to decide in which connected component of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ a specific representation lies in. We apply this to construct particularly nice model representations in all connected components.

An easy way to construct maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is by composing a discrete embedding $\pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ with a tight homomorphism of $\mathrm{SL}(2, \mathbf{R})$ into $\mathrm{Sp}(2n, \mathbf{R})$ (see [11] for the notion of tight homomorphism). For example, composing with the $2n$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbf{R})$ into $\mathrm{Sp}(2n, \mathbf{R})$ we obtain an *irreducible Fuchsian representation*. Hitchin representations are precisely deformations of such representations. Composing with the diagonal embedding of $\mathrm{SL}(2, \mathbf{R})$ into the subgroup $\mathrm{SL}(2, \mathbf{R})^n < \mathrm{Sp}(2n, \mathbf{R})$ we obtain a *diagonal Fuchsian representation*. These representations have a big centralizer because the centralizer of the image of $\mathrm{SL}(2, \mathbf{R})$ under the diagonal embedding is isomorphic to $\mathrm{O}(n)$. Any representation can be twisted by a representation into its centralizer, thus any diagonal Fuchsian representation can be twisted by a representation $\pi_1(\Sigma) \to \mathrm{O}(n)$, defining a *twisted diagonal representations*. A representation obtained by one of these constructions will be called a *standard maximal representation* (see Section 3.2).

Theorem 8. *Let $n \geq 3$. Then every maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ can be deformed to a standard maximal representation.*

Our computations of the topological invariants in Section 5 give more precise information on when a maximal representation can be deformed to an irreducible Fuchsian or a diagonal Fuchsian representation.

Corollary 9. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation. Then ρ can be deformed either to an irreducible Fuchsian representation or to a diagonal Fuchsian representation if*

(i) *for $n = 2m$, $m > 2$, $sw_1(\rho) = 0$ and $sw_2(\rho) = -m\frac{\chi(\Sigma)}{2} \mod 2$,*
(ii) *for $n = 2m+1$, $sw_2(\rho) = -m\frac{\chi(\Sigma)}{2} \mod 2$.*

Remark 3. *Another corollary of Theorem 8 is that for $n \geq 3$ every maximal representation can be deformed to a maximal representation whose image is contained in a proper closed subgroup of $\mathrm{Sp}(2n, \mathbf{R})$. This conclusion can also be obtained from [14] because the Higgs bundles for standard maximal representation can be described quite explicitly.*

The case of $\mathrm{Sp}(4, \mathbf{R})$ is different. If one combines the count of the connected components of $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ in [19] with results about tight homomorphisms in [11] one can conclude that there are $2g - 3$ exceptional components, in which every representation is Zariski dense (see [6] for a detailed discussion of this).

To construct model representations in these components, we decompose $\Sigma = \Sigma_l \cup \Sigma_r$ into two subsurfaces and define a representation of $\pi_1(\Sigma)$ by amalgamation of an irreducible Fuchsian representation of $\pi_1(\Sigma_l)$ with a deformation of a diagonal Fuchsian representation of $\pi_1(\Sigma_r)$. We call these representations *hybrid representations* (see Section 3.3.1 for details).

We compute the topological invariants of these representations explicitly. Allowing the Euler characteristic of the subsurface Σ_l to vary between $3-2g$ and -1, we obtain $2g-3$ hybrid representations which exhaust the $2g-3$ exceptional components of the space of maximal representations into $\mathrm{Sp}(4,\mathbf{R})$. We conclude

Theorem 10. *Every maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ can be deformed to a standard maximal representation or a hybrid representation.*

Remark 4. *To obtain Theorem 10 it is essential that we are able to compute the topological invariants explicitly. Geometrically there is no obvious reason why different hybrid representations lie in different connected components. In particular, our results on the topological invariants imply that other constructions by amalgamation (see Section 3.3.3) give representations which can be deformed to twisted diagonal representations.*

1.4. **Holonomies of maximal representations.** A direct consequence of the fact that maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R})$ are $(\mathrm{Sp}(2n,\mathbf{R}), \mathrm{GL}(n,\mathbf{R}))$-Anosov representations is that the holonomy $\rho(\gamma)$ is conjugate to an element of $\mathrm{GL}(n,\mathbf{R})$ for every $\gamma \in \pi_1(\Sigma)$. More precisely $\rho(\gamma)$ fixes two transverse Lagrangians, one, L^s, being attractive, the other being repulsive. From this it follows that the holonomy $\rho(\gamma)$ is an element of $\mathrm{GL}(L^s)$ whose eigenvalues are strictly bigger than one.

For representations in the Hitchin components we have moreover that $\rho(\gamma) \in \mathrm{GL}(L^s)$ is a regular semi-simple element [27]. This does not hold for other connected components of $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R}))$. Using the description of model representations in Theorem 8 and Theorem 10 we prove

Theorem 11. *Let $\mathcal{H}$ be a connected component of*

$$\mathrm{Rep}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R})),$$

and let $\gamma \in \pi_1(\Sigma) - \{1\}$ be an element corresponding to a simple curve. If γ is separating, $n = 2$ and the genus of Σ is 2, we require that $\mathcal{H}$ is not the connected component determined by $sw_1 = 0$ and $e_\gamma = 0$. Then there exist

- (i) *a representation $\rho \in \mathcal{H}$ such that the Jordan decomposition of $\rho(\gamma)$ in $\mathrm{GL}(L^s) \cong \mathrm{GL}(n,\mathbf{R})$ has a nontrivial parabolic component.*
- (ii) *a representation $\rho' \in \mathcal{H}$ such that the Jordan decomposition of $\rho(\gamma)$ in $\mathrm{GL}(L^s) \cong \mathrm{GL}(n,\mathbf{R})$ has a nontrivial elliptic component.*

1.5. **Other maximal representations.** Maximal representations $\rho : \pi_1(\Sigma) \to G$ can be defined whenever G is a Lie group of Hermitian type, and they are always (G,H)-Anosov representations [10], where H is a specific subgroup of G (see Theorem 2.15). When G is not locally isomorphic to $\mathrm{Sp}(2n,\mathbf{R})$ there is no analogue of Hitchin representations[1] and we conjecture:

[1]Hitchin components can be defined for any $\mathbf{R}$-split semisimple Lie group (see [24]) and the only simple $\mathbf{R}$-split Lie groups of Hermitian type are the symplectic groups.

Conjecture 12. *Let G be a simple Lie group of Hermitian type. If G is not locally isomorphic to* $\mathrm{Sp}(2n,\mathbf{R})$*, then the topological invariants of Theorem 1 distinguish connected components of* $\mathrm{Rep}_{\mathrm{max}}(\pi_1(\Sigma), G)$*.*

If the real rank of G is n, then there is an embedding of $SL(2,\mathbf{R})^n$ into G, it is unique up to conjugation. Thus, there is always a corresponding diagonal embedding of $\mathrm{SL}(2,\mathbf{R})$ into G. The centralizer of which is a compact subgroup of G. In particular, one can always construct twisted diagonal representations.

Conjecture 13. *Let G be of Hermitian type. If G not locally isomorphic to* $\mathrm{Sp}(2n,\mathbf{R})$*, then every maximal representation* $\rho : \pi_1(\Sigma) \to G$ *can be deformed to a twisted diagonal representation.*

If Conjecture 13 holds the analogue of Theorem 11 will also hold.

1.6. **Comparison with Higgs bundle invariants.** We already mentioned that the correspondence between (reductive) representations and Higgs bundles permits to use algebro-geometric methods to study the structure of $\mathrm{Rep}(\pi_1(\Sigma), G)$, and in particular to count the number of connected components. Where these methods have been applied to study representations into Lie groups of Hermitian type, see [19, 15, 14, 7, 8], the authors associate special vector bundles to the Higgs bundles, whose characteristic classes give additional invariants for maximal representations $\pi_1(\Sigma) \to G$; then they show that for any possible value of the invariants the corresponding moduli space of Higgs bundles is non-empty and connected.

For maximal representations into $\mathrm{Sp}(2n,\mathbf{R})$ the Higgs bundle invariants are first and second Stiefel-Whitney classes taking values in $\mathrm{H}^*(\Sigma; \mathbf{F}_2)$; for $n = 2$ there is also an Euler number taking values in $\mathrm{H}^2(\Sigma; \mathbf{Z})$.

We conclude the introduction with several remarks concerning the relation between the topological invariants defined here and the invariants obtained via Higgs bundles:

(i) The Higgs bundle invariants depend on various choices, *i.e.* they depend on the choice of a complex structure on Σ and on the choice of a square-root of the canonical bundle of Σ (*i.e.* a spin structure on Σ). The Stiefel-Whitney classes we define here are natural and do not depend on any choices. In particular, they are equivariant under the action of the mapping class group of Σ. The invariants defined here are also natural with respect to taking finite index subgroups of $\pi_1(\Sigma)$ and with respect to tight homomorphisms.

(ii) The invariants defined here can be computed for explicit representations, which is very difficult for the Higgs bundle invariants. This computability is essential to determine in which connected components specific representations lie. This is of particular interest for the $2g-3$ exceptional connected components when $n = 2$, because in these connected components no explicit representations were known before.

(iii) The Higgs bundle approach has the feature that the L^2-norm of the Higgs field gives a Morse-Bott function on the moduli space, which allows to perform Morse theory on the representation variety. In fact, for the symplectic structure on the moduli space, this function is the Hamiltonian of a circle action, so that its critical points are exactly the fixed points of this circle action. These fixed points are Higgs bundles of a very special type, called "variations of Hodge structures". An additional information coming from this framework is that one is able to read the index of the critical submanifolds from the eigenvalues of the circle action on the tangent space at a fixed point. This is used to give an exact count of the connected components in many cases, as well as to obtain further important information about the topology of the representation variety. For more details on this strategy we refer the reader to Hitchin's article [24].

For symplectic maximal representations there is a simple relation between the two invariants although they live naturally in different cohomology groups.

Proposition 14. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation. Denote by $w_i(\rho, v) \in \mathrm{H}^i(\Sigma; \mathbf{F}_2)$, $i = 1, 2$ the first and second Stiefel-Whitney classes associated to the Higgs bundle corresponding to ρ, where $v \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ is the chosen spin structure on Σ. Then we have the following equality in $\mathrm{H}^i(T^1\Sigma; \mathbf{F}_2)$:*

$$\begin{aligned} sw_1(\rho) &= w_1(\rho, v) + n \cdot v \\ sw_2(\rho) &= w_2(\rho, v) + sw_1(\rho) \cdot v + (g-1) \mod 2. \end{aligned}$$

When $n = 2$, the Higgs bundle invariant corresponding to the Euler class is the first Chern class of a line bundle on Σ, which we denote by $c(\rho) \in \mathrm{H}^2(\Sigma; \mathbf{Z})$. The suspected relation is

$$c(\rho) = \varepsilon e(\rho) + (g-1)$$

in $\mathrm{Tor}(\mathrm{H}^2(T^1\Sigma; \mathbf{Z}))$, where ε depends on the choices of orientation involved in the definition of $e(\rho)$ and $c(\rho)$.

The existence of such relations is not surprising since the invariants arise basically from the same compact Lie group. Nevertheless, it would be interesting to provide a general proof for these relations for all maximal representations. The relations in Proposition 14 are obtained from case by case considerations for model representations, using the tensor product formulas of Equation (3).

1.7. **Structure of the paper.** In Section 2 we recall the definition and properties of Anosov representations and of maximal representations. Examples of such representations are discussed in Section 3. The topological invariants are defined in Section 4 and computed for symplectic maximal representations in Section 5. Section 6 discusses the action of the mapping class group; Section 7 derives consequences for the holonomy of symplectic maximal representations. In Appendix A we establish several important facts about positive curves and maximal representations; in Appendix B we review some facts about the cohomology of the unit tangent bundle $T^1\Sigma$.

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2. Preliminaries

2.1. **Anosov representations.** Holonomy representations of locally homogeneous geometric structures are very special. The Thurston-Ehresmann theorem [17, 5] states that every deformation of such a representation can be realized through deformations of the geometric structure. The concept of *Anosov structures*, introduced by Labourie in [27], gives a dynamical generalization of this, which is more flexible, but for which it is still possible to obtain enough rigidity of the associated representations.

2.1.1. *Definition.* Let

- M be a compact manifold with an Anosov flow ϕ_t,
- G a connected semisimple Lie group and (P^s, P^u) a pair of opposite parabolic subgroups of G,
- $H = P^s \cap P^u$ their intersection, and
- $\mathcal{F}^s = G/P^s$ (resp. $\mathcal{F}^u = G/P^u$) the flag variety associated to P^s (resp. P^u).

There is a unique open G-orbit $\mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$. We have $\mathcal{X} = G/H$ and as open subset of $\mathcal{F}^s \times \mathcal{F}^u$ it inherits two foliations $\mathcal{E}^s$ and $\mathcal{E}^u$ whose corresponding distributions are denoted by E^s and E^u, *i.e.* $(E^s)_{(f^s,f^u)} \cong T_{f^s}\mathcal{F}^s$ and $(E^u)_{(f^s,f^u)} \cong T_{f^u}\mathcal{F}^u$.

Definition 2.1. *A flat G-bundle P over (M, ϕ_t) is said to have an H-reduction σ that is* flat along flow lines *if:*

- *σ is a section of $\mathsf{P} \times_G \mathcal{X}$;* i.e. *$\sigma : M \to \mathsf{P} \times_G \mathcal{X}$ defines the H-reduction*[2]
- *the restriction of σ to every orbit of ϕ_t is locally constant with respect to the induced flat structure on $\mathsf{P} \times_G \mathcal{X}$.*

The two distributions E^s and E^u on $\mathcal{X}$ are G-invariant and hence define distributions, again denoted E^s and E^u, on $\mathsf{P} \times_G \mathcal{X}$. These two distributions are invariant by the flow, again denoted by ϕ_t, that is the lift of the flow on M by the connection.

[2]For details on the bijective correspondence between H-reductions and sections of $\mathsf{P} \times_G G/H$ we refer the reader to [31, Section 9.4].

To every section σ of $\mathsf{P} \times_G \mathcal{X}$ we associate two vector bundles $\sigma^* E^s$ and $\sigma^* E^u$ on M by pulling back to M the vector bundles E^s and E^u. If furthermore σ is flat along flow lines, so that it commutes with the flow, then these two vector bundles $\sigma^* E^s$ and $\sigma^* E^u$ are equipped with a natural flow.

Definition 2.2. *A flat G-bundle $\mathsf{P} \to M$ is said to be a (G,H)-*Anosov bundle *if:*

(i) *P admits an H-reduction σ that is flat along flow lines, and*
(ii) *the flow ϕ_t on $\sigma^* E^s$ (resp. $\sigma^* E^u$) is contracting (resp. dilating).*

We call σ an Anosov section *or an* Anosov reduction *of P.*

By (ii) we mean that there exists a continuous family of norms $(\|\cdot\|_m)_{m\in M}$ on $\sigma^* E^s$ (resp. $\sigma^* E^u$) and constants $A, a > 0$ such that for any e in $\sigma^*(E^s)_m$ (resp. $\sigma^*(E^u)_m$) and for any $t > 0$ one has

$$\|\phi_t e\|_{\phi_t m} \leq A \exp(-at)\|e\|_m \quad (\text{resp. } \|\phi_{-t} e\|_{\phi_{-t} m} \leq A \exp(-at)\|e\|_m).$$

Since M is compact this definition does not depend on the norm $\|\cdot\|$ or the parametrization of ϕ_t.

Definition 2.3. *A representation $\pi_1(M) \to G$ is said to be (G,H)-*Anosov *(or simply* Anosov*) if the corresponding flat bundle is a (G,H)-Anosov bundle.*

Remark 2.4. *Note that the terminology for Anosov representations is not completely uniform. A (G,H)-Anosov representation is sometimes called a $(G,\mathcal{X})$-Anosov representation or an Anosov representation with respect to the parabolic subgroup P^s or P^u.*

2.1.2. *Properties.* For the definition of topological invariants of Anosov representations in Section 4 the following proposition will be crucial.

Proposition 2.5. *Let $\mathsf{P} \to M$ be an Anosov bundle, then there is a* unique *section $\sigma : M \to \mathsf{P} \times_G \mathcal{X}$ such that properties (i) and (ii) of Definition 2.2 hold. In particular, a (G,H)-Anosov bundle admits a* canonical *H-reduction.*

To prove Proposition 2.5 we will use the following classical fact.

Fact 2.6. *Suppose that $g \in G$ has two fixed points $f^s \in \mathcal{F}^s$, $f^u \in \mathcal{F}^u$ such that*

(i) *f^s and f^u are in general position,* i.e. *(f^s, f^u) belongs to $\mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$; and*
(ii) *the (linear) action of g on the tangent space $T_{f^s}\mathcal{F}^s$ is contracting, the action on $T_{f^u}\mathcal{F}^u$ is expanding.*

Then f^s is the only attracting fixed point of g in $\mathcal{F}^s$. Its attracting set is the set of all flags that are in general position with f^u, and f^u is the only repelling fixed point for g in $\mathcal{F}^u$.

Proof of Proposition 2.5. Let σ be an Anosov section of the flat bundle $\mathsf{P} \to M$. The flow ϕ_t on M is (by hypothesis) an Anosov flow. Because of the density of closed orbits in M, it is enough to show that the restriction of σ to any closed orbit γ is uniquely determined.

We identify γ with $\mathbf{Z}\backslash\mathbf{R}$ and write the restriction of P to γ as $\mathsf{P}|_\gamma = \mathbf{Z}\backslash(\mathbf{R}\times G)$ where $\mathbf{Z}$ acts on $\mathbf{R}\times G$ by $n\cdot(t,g) = (n+t, h_\gamma^n g)$ for some h_γ in G. (The element h_γ is the holonomy of P along γ.)

Therefore, the restriction $\sigma|_\gamma$ is a section of $\mathbf{Z}\backslash(\mathbf{R}\times\mathcal{X})$ (the $\mathbf{Z}$-action being $n\cdot(t,x) = (n+t, h_\gamma^n\cdot x)$). Since σ is locally constant along γ there exists $x = (f^s, f^u)$ in $\mathcal{X}$ such that the lift of $\sigma|_\gamma$ to $\mathbf{R}$ is the map $\mathbf{R}\to\mathbf{R}\times\mathcal{X}$, $t\mapsto(t,x)$. Hence f^s and f^u are h_γ-invariant. Furthermore the restrictions of σ^*E^s, resp. σ^*E^u to γ are in this case $\mathbf{Z}\backslash(\mathbf{R}\times T_{f^s}\mathcal{F}^s)$, resp. $\mathbf{Z}\backslash(\mathbf{R}\times T_{f^u}\mathcal{F}^u)$, so that the contraction property of Definition 2.2(ii) exactly translates into the assumption of Fact 2.6. This implies the uniqueness of (f^s, f^u) and hence the uniqueness of $\sigma|_\gamma$. □

An important feature of Anosov representations is their stability under deformation.

Proposition 2.7. [27, Proposition 2.1.][20] *The set of Anosov representations is* open *in* $\mathrm{Hom}(\pi_1(M), G)$. *Moreover, the H-reduction given by the Anosov section σ depends continuously on the representation.*

2.1.3. *Constructions.* Some simple constructions allow to obtain new Anosov bundles from old ones.

Lemma 2.8.
(i) *Let $\rho : \pi_1(M)\to G$ be a (G,H)-Anosov representation, where $H = P^s\cap P^u$ for two opposite parabolic subgroups in G. Let Q^s, Q^u be opposite parabolic subgroups in G such that $P^s < Q^s$ and $P^u < Q^u$. Then ρ is also a (G,H')-Anosov representation, where $H' = Q^s\cap Q^u$.*
(ii) *Let P be an (G,H)-Anosov bundle over M with canonical H-reduction P_H and E a flat L-bundle over M. Then the fibered product $\mathsf{P}\times E$ is a $(G\times L, H\times L)$-Anosov bundle over M, whose canonical $(H\times L)$-reduction is the fibered product $\mathsf{P}_H\times E$.*
(iii) *Let P be an (G,H)-Anosov bundle over M with canonical H-reduction P_H, where $H = P^s\cap P^u$. Let $f : G\to L$ be a homorphism of Lie groups, let Q^s, Q^u be a pair of opposite parabolic subgroups in L such that $f^{-1}(Q^s) = P^s$, $f^{-1}(Q^u) = P^u$ and they are maximal with respect to this property,* i.e. *for any Q containing Q^s (resp. Q^u) satisfying $f^{-1}(Q) = P^s$ (resp. P^u) one has $Q = Q^s$ (resp. $Q = Q^u$).*
Then $\mathsf{P}\times_G L$ is an (L,M)-Anosov bundle, where $M = Q^s\cap Q^u$ and its canonical M-reduction is the fibered product $\mathsf{P}_H\times_H M$.

2.1.4. *Definition in terms of the universal cover of M.* A (G,H)-Anosov bundle over M can be equivalently defined in terms of equivariant maps from the universal cover of M to $\mathcal{X}\cong G/H$. Let $\widetilde{M}$ be the universal cover of M. Then any flat G-bundle P on M can be written as:

$$\mathsf{P} = \pi_1(M)\backslash(\widetilde{M} \times G), \quad \gamma(\tilde{m}, g) = (\gamma \cdot \tilde{m}, \rho(\gamma)g)$$

for some representation $\rho : \pi_1(M) \to G$.

Let ϕ_t be the lift of the flow on M to $\widetilde{M}$. This flow lifts to $\phi_t(\tilde{m}, g) = (\phi_t(\tilde{m}), g)$, defining a flow on $\widetilde{M} \times G$.

An H-reduction σ is the same as a ρ-equivariant map

$$\tilde{\sigma} : \widetilde{M} \longrightarrow G/H \cong \mathcal{X}.$$

The section σ is flat along flow lines if, and only if, the map $\tilde{\sigma}$ is ϕ_t-invariant.

The contraction property of the flow is now expressed as follows:

(i) There exists a continuous family $(\|\cdot\|_{\tilde{m}})_{\tilde{m}\in\widetilde{M}}$ such that
  - for all $\tilde{m}$, $\|\cdot\|_{\tilde{m}}$ is a norm on $(E^s)_{\tilde{\sigma}(\tilde{m})} \subset T_{\tilde{\sigma}(\tilde{m})}\mathcal{X}$,
  - and $(\|\cdot\|_{\tilde{m}})_{\tilde{m}\in\widetilde{M}}$ is ρ-equivariant, *i.e.* for all $\tilde{m}$ in $\widetilde{M}$, γ in $\pi_1(M)$ and e in $(E^s)_{\tilde{\sigma}(\tilde{m})}$ then $\|\rho(\gamma)\cdot e\|_{\gamma\cdot\tilde{m}} = \|e\|_{\tilde{m}}$.

(ii) The flow ϕ_t is contracting. *i.e.* there exist $A, a > 0$ such that for any $t > 0$ and $\tilde{m}$ in $\widetilde{M}$ and e in $(E^s)_{\tilde{\sigma}(\tilde{m})}$ then $\|e\|_{\phi_t\cdot\tilde{m}} \leq A\exp(-at)\|e\|_{\tilde{m}}$. (This inequality makes sense because, since $\tilde{\sigma}(\phi_t\cdot\tilde{m}) = \tilde{\sigma}(\tilde{m})$, e belongs to $(E^s)_{\tilde{\sigma}(\phi_t\cdot\tilde{m})}$.)

2.1.5. *Specialization to $T^1\Sigma$.* We restrict now to the case when $M = T^1\Sigma$ is the unit tangent bundle of a closed oriented connected surface Σ of negative Euler characteristic and ϕ_t is the geodesic flow on M with respect to some hyperbolic metric on Σ.

Let $\partial\pi_1(\Sigma)$ be the boundary at infinity of $\pi_1(\Sigma)$. Then $\partial\pi_1(\Sigma)$ is a topological circle that comes with a natural orientation and an action of $\pi_1(\Sigma)$.

There is an equivariant identification

$$T^1\widetilde{\Sigma} \cong \partial\pi_1(\Sigma)^{(3+)}$$

of the unit tangent bundle of $\widetilde{\Sigma}$ with the set of positively oriented triples in $\partial\pi_1(\Sigma)$. The orbit of the geodesic flow through the point (t^s, t, t^u) is

$$\mathcal{G}_{(t^s,t,t^u)} = \mathcal{G}_{(t^s,t^u)} = \{(r^s, r, r^u) \in \partial\pi_1(\Sigma)^{(3+)} \mid \ r^s = t^s, r^u = t^u\},$$

and the set of geodesic leaves is parametrized by $\partial\pi_1(\Sigma)^{(2)} = \partial\pi_1(\Sigma)^2 - \Delta$, the complementary of the diagonal in $\partial\pi_1(\Sigma)^2$. (For more details we refer the reader to [21, Section 1.1].)

Let $\pi : T^1\Sigma \to \Sigma$ be the natural projection.

Definition 2.9. *A flat G-bundle P over Σ is said to be* Anosov *if its pull back $\pi^*\mathsf{P}$ over $M = T^1\Sigma$ is Anosov.*

A representation $\rho : \pi_1(\Sigma) \to G$ is Anosov *if the composition $\pi_1(M) \to \pi_1(\Sigma) \xrightarrow{\rho} G$ is an Anosov representation.*

Remark 2.10. *Note that for a (G, H)-Anosov bundle P over Σ, the pull-back $\pi^*\mathsf{P}$ over $T^1\Sigma$ admits a canonical H-reduction, but this H-reduction in* *general does not come from a reduction of* P. *Indeed, the invariants defined in Section 4 also give obstructions for this to happen.*

Let $\rho : \pi_1(\Sigma) \to G$ be an Anosov representation and P the corresponding flat G-bundle over Σ. Using the description in Section 2.1.4 it follows that there exists a ρ-equivariant map:

$$\tilde{\sigma} : T^1\widetilde{\Sigma} \longrightarrow \mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$$

which is invariant by the geodesic flow.

In particular we get a ρ-equivariant map

$$(\xi^s, \xi^u) : \partial\pi_1(\Sigma)^{(2)} \longrightarrow \mathcal{F}^s \times \mathcal{F}^u.$$

In view of the contraction property of $\tilde{\sigma}$ (Definition 2.2(ii)) it is easy to see that $\xi^s(t^s, t^u)$ (resp. $\xi^u(t^s, t^u)$) depends only of t^s (resp. t^u). Hence, we obtain ρ-equivariant maps $\xi^s : \partial\pi_1(\Sigma) \to \mathcal{F}^s$, and $\xi^u : \partial\pi_1(\Sigma) \to \mathcal{F}^u$.

Corollary 2.11. *Let $\rho : \pi_1(\Sigma) \to G$ be a (G,H)-Anosov representation. Then for every $\gamma \in \pi_1(\Sigma) - \{1\}$ the image $\rho(\gamma)$ is conjugate to an element in H, having a unique pair of attracting/repelling fixed points $(\xi^s(t^s_\gamma), \xi^u(t^u_\gamma)) \in \mathcal{X}$, where (t^s_γ, t^u_γ) denotes the pair of attracting/repelling fixed points of γ in $\partial\pi_1(\Sigma)$.*

Remark 2.12. *In the situation when P^s is conjugate with P^u, so that there is a natural identification between $\mathcal{F}^s$ and $\mathcal{F}^u$, it is easy to show the equality $\xi^s = \xi^u$ (see* [27] *or* [20] *for a similar discussion), thus in those cases we will denote the equivariant map simply by $\xi : \partial\pi_1(\Sigma) \to \mathcal{F}$.*

2.2. **Maximal representations.**

2.2.1. *Definition and properties.* Let G be an almost simple noncompact Lie group of Hermitian type, *i.e.* the symmetric space associated to G is an irreducible Hermitian symmetric space of non-compact type. Then $\pi_1(G) \cong \mathbf{Z}$ modulo torsion, and there is a characteristic class $\tau(\rho) \in \mathrm{H}^2(\Sigma; \pi_1(G)) \cong \pi_1(G) \cong \mathbf{Z}$, often called the *Toledo invariant* of the representation $\rho : \pi_1(\Sigma) \to G$. The Toledo invariant $\tau(\rho)$ is bounded in absolute value by a constant $C(G, \Sigma)$ depending only on the real rank of G and the Euler characteristic of Σ:

$$|\tau(\rho)| \leq C(G, \Sigma)$$

Definition 2.13. *A representation $\rho : \pi_1(\Sigma) \to G$ is* maximal *if*

$$\tau(\rho) = C(G, \Sigma).$$

The space of maximal representation is denoted by $\mathrm{Hom}_{\max}(\pi_1(\Sigma), G)$.

The space of maximal representations is a union of connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)$.

Remark 2.14. *In the definition of maximal representations we choose the positive extremal value of the Toledo invariant. The space where the negative extremal value is achieved can be easily seen to be isomorphic to the space of maximal representations by "a change of orientation" (*e.g. *for a standard generating set* $\{a_i, b_i\}_{1\leq i\leq g}$ *of* $\pi_1(\Sigma)$ *the automorphism* σ *of* $\pi_1(\Sigma)$ *defined by* $a_i \mapsto b_{g+1-i}$, $b_i \mapsto a_{g+1-i}$ *precisely changes the sign of* τ: $\tau(\rho\circ\sigma) = -\tau(\rho)$ *for any representation* ρ).

Maximal representations have been extensively studied in the last years [18, 33, 22, 7, 12, 13, 9, 34, 8]. They enjoy several interesting properties, *e.g.* maximal representations are discrete embeddings, but more importantly for our considerations is the following

Theorem 2.15. [9, 10] *A maximal representation* $\rho : \pi_1(\Sigma) \to G$ *is an Anosov representation. More precisely,* ρ *is a* (G, H)*-Anosov representation, where* $H < G$ *is the stabilizer of a pair of transverse points in the Shilov boundary of the symmetric space associated to* G.

We will make use of the following gluing theorem, which follows immediately from additivity properties of the Toledo invariant established in [13, Proposition 3.2.], to construct maximal representations.

Theorem 2.16. [13] *Let* $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ *be the decomposition of* Σ *along a simple closed separating geodesic and* $\pi_1(\Sigma) = \pi_1(\Sigma_1) *_{\langle\gamma\rangle} \pi_1(\Sigma_2)$ *the corresponding decomposition as amalgamated product. Let* $\rho_i : \pi_1(\Sigma_i) \to G$ *be maximal representations which agree on* γ*, then the amalgamated representation*

$$\rho = \rho_1 * \rho_2 : \pi_1(\Sigma) \longrightarrow G$$

is maximal.

2.2.2. *Maximal representations into* $\mathrm{Sp}(2n, \mathbf{R})$. Let $V = \mathbf{R}^{2n}$ be a symplectic vector space and $(e_i)_{1\leq i\leq 2n}$ a symplectic basis, with respect to which the symplectic form ω is given by the anti-symmetric matrix:

$$J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

Let $G = \mathrm{Sp}(2n, \mathbf{R}) = \mathrm{Sp}(V)$.

Let $L_0^s := \mathrm{Span}(e_i)_{1\leq i\leq n}$ be a Lagrangian subspace and $P^s < \mathrm{Sp}(2n, \mathbf{R})$ be the parabolic subgroup stabilizing L_0^s. The stabilizer of the Lagrangian $L_0^u = \mathrm{Span}(e_i)_{n<i\leq 2n}$ is a parabolic subgroup P^u, it is opposite to P^s. The subgroup $H = P^s \cap P^u$ is isomorphic to $\mathrm{GL}(n, \mathbf{R})$:

$$\begin{array}{ccc} \mathrm{GL}(n, \mathbf{R}) & \xrightarrow{\sim} & H \subset G \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix}. \end{array}$$

(Such isomorphisms are in one to one correspondence with symplectic bases $(\epsilon_i)_{1\leq i\leq 2n}$ —*i.e.* $\omega(\epsilon_i, \epsilon_j) = \omega(e_i, e_j)$ for all i, j— for which $(\epsilon_i)_{1\leq i\leq n}$ is a basis of L_0^s). Note that P^u is conjugate to P^s in G, so that the flag variety

$\mathcal{F}^s$ is canonically isomorphic to $\mathcal{F}^u$; we will denote this homogeneous space by $\mathcal{L}$. The space $\mathcal{L}$ is the Shilov boundary of the symmetric space associated to $\mathrm{Sp}(2n, \mathbf{R})$; it can be realized as the space of Lagrangian subspaces in $\mathbf{R}^{2n}$ and the homogeneous space $\mathcal{X} \subset \mathcal{L} \times \mathcal{L}$ is the space of pairs of transverse Lagrangians.

Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation and P the corresponding flat principal $\mathrm{Sp}(2n, \mathbf{R})$-bundle over $T^1\Sigma$ and E the corresponding flat symplectic $\mathbf{R}^{2n}$-bundle over $T^1\Sigma$. Then ρ is an $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$-Anosov representation (Theorem 2.15). The canonical $\mathrm{GL}(n, \mathbf{R})$-reduction of P is equivalent to a continuous splitting of E into two (non-flat) flow-invariant transverse Lagrangian subbundles

$$E = L^s(\rho) \oplus L^u(\rho),$$

Notation 2.17. *We call this splitting the* Lagrangian reduction *of the flat symplectic Anosov $\mathbf{R}^{2n}$-bundle.*

The existence of a Lagrangian reduction is equivalent to the existence of a continuous ρ-equivariant curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$, sending distinct points in $\partial\pi_1(\Sigma)$ to transverse Lagrangians. The construction of a ρ-equivariant limit curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ is actually the first step to prove that a maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is an $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$-Anosov representation, see [9]. This limit curve satisfies an additional positivity property which we now describe.

Let (L^s, L, L^u) be a triple of pairwise transverse Lagrangians in $\mathbf{R}^{2n}$, then L can be realized as the graph of $F_L \in \mathrm{Hom}(L^s, L^u)$.

Definition 2.18. *A triple (L^s, L, L^u) of pairwise transverse Lagrangians is called* positive *if the quadratic form $q(v) := \omega(v, F_L(v))$ on L^s is positive definite.*

Definition 2.19. *A curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ from the boundary at infinity of $\pi_1(\Sigma)$ to the space of Lagrangians is said to be* positive*, denoted by $\xi > 0$, if for every positively oriented triple (t^s, t, t^u) in $\partial\pi_1(\Sigma)^{(3+)}$ the triple of Lagrangians $(\xi(t^s), \xi(t), \xi(t^u))$ is positive.*

Important facts about the space of positive curves in $\mathcal{L}$ are established in Appendix A.1.

Theorem 2.20. [13] *Let $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ be a maximal representation and $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ the equivariant limit curve. Then ξ is a positive curve.*

As a consequence of Proposition 2.7 we have

Fact 2.21. *The positive limit curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ depends continuously on the representation.*

Given a ρ-equivariant limit curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ the construction of a splitting

$$E = L^s(\rho) \oplus L^u(\rho)$$

is immediate: for any triple $v = (t^s, t, t^u) \in \partial\pi_1(\Sigma)^{(3+)} \cong T^1\widetilde{\Sigma}$, we set $(L^s(\rho))_v = \xi(t^s)$ and $(L^u(\rho))_v = \xi(t^u)$.

In the following we will often switch between the three different view points: $\mathrm{GL}(n, \mathbf{R})$-reduction of P, splitting $E = L^s(\rho) \oplus L^u(\rho)$ or (positive) equivariant curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$.

3. Examples of representations

3.1. **Anosov representations.** We give examples of Anosov representations. By Proposition 2.7 every small deformation of one of these representations is again an Anosov representation.

3.1.1. *Hyperbolizations.* Let Σ be a connected oriented closed hyperbolic surface and $M = T^1\Sigma$ its unit tangent bundle equipped with the geodesic flow ϕ_t. Hyperbolizations give rise to discrete embeddings $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{R})$ which are examples of Anosov representations (see [27, Proposition 3.1]). More generally, a discrete embedding of $\pi_1(\Sigma)$ into any finite cover L of $\mathrm{PSL}(2, \mathbf{R})$ is an Anosov representation. Since $\mathrm{PSL}(2, \mathbf{R})$ has rank one there is no choice for the parabolic subgroup.

Later we will be interested in particular in Anosov bundles arising from discrete embeddings $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$. In that case $H = \mathrm{GL}(1, \mathbf{R})$ is the subgroup of diagonal matrices and the H-reduction corresponds to a splitting of the flat $\mathbf{R}^2$-bundle over M into two line bundles $L^s(\iota) \oplus L^u(\iota)$.

3.1.2. *Hitchin representations.* A representation of $\pi_1(\Sigma)$ into $G = \mathrm{SL}(n, \mathbf{R})$, $\mathrm{Sp}(2m, \mathbf{R})$ or $\mathrm{SO}(m, m+1)$ is said to be a *Hitchin representation* if it can be deformed into a representation $\pi_1(\Sigma) \xrightarrow{\iota} \mathrm{SL}(2, \mathbf{R}) \xrightarrow{\tau_n} G$, where the homomorphism $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ is a discrete embedding and $\tau_n : \mathrm{SL}(2, \mathbf{R}) \to G$ is the n-dimensional irreducible representation, where $n = 2m$ when $G = \mathrm{Sp}(2m, \mathbf{R})$ and $n = 2m + 1$ when $G = \mathrm{SO}(m, m+1)$. Hitchin representations are (G, H)-Anosov, where H is the subgroup of diagonal matrices in of G [27]. In particular, the H-reduction corresponds to a splitting of the flat $\mathbf{R}^n$-bundle over M into n line bundles.

3.1.3. *Other examples.*

(i) Any quasi-Fuchsian representation $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{C})$ is Anosov.
(ii) Embed $\mathrm{SL}(2, \mathbf{R})$ into $\mathrm{PGL}(3, \mathbf{R})$ as stabilizer of a point and consider the representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R}) \to \mathrm{PGL}(3, \mathbf{R})$. Then special deformations of ρ are $(\mathrm{PGL}(3, \mathbf{R}), H)$-Anosov, where H is the intersection of two opposite minimal parabolic subgroups of $\mathrm{PGL}(3, \mathbf{R})$ [3].
(iii) Let G be a semisimple Lie group, $G' < G$ a rank one subgroup, $\Lambda < G'$ a cocompact torsionfree lattice and $N = \Lambda\backslash G'/K$, where $K < G'$ is a maximal compact subgroup. Let $M = T^1N$ be the unit tangent bundle of N. Then the composition $\pi_1(M) \to \Lambda \to G' < G$ is a (G, H)-Anosov representation, where H is the connected component

of the identity of the centralizer in G of a real split Cartan subgroup in G' (compare with Lemma 2.8), see [27, Proposition 3.1].

(iv) In [30, 4] a notion of quasi-Fuchsian representations for a cocompact lattice $\Lambda < \mathrm{SO}_\circ(1,n)$ into $\mathrm{SO}_\circ(2,n)$ is introduced, and it is shown that these quasi-Fuchsian representations are Anosov representations.

3.2. **Standard maximal representations.** In this section we describe the construction of several maximal representations

$$\rho : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n, \mathbf{R})$$

to which we will refer as *standard representations*. All these representations come from homomorphisms of $\mathrm{SL}(2,\mathbf{R})$ into $\mathrm{Sp}(2n,\mathbf{R})$, possibly twisted by a representation of $\pi_1(\Sigma)$ into the centralizer of the image of $\mathrm{SL}(2,\mathbf{R})$ in $\mathrm{Sp}(2n,\mathbf{R})$. By construction the image of any such representation will be contained in a proper closed Lie subgroup of $\mathrm{Sp}(2n,\mathbf{R})$.[3]

Let us fix a discrete embedding $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$.

3.2.1. *Irreducible Fuchsian representation.* Consider $V_0 = \mathbf{R}_1[X,Y] \cong \mathbf{R}^2$ the space of homogeneous polynomials of degree one in the variables X and Y, endowed with the symplectic form determined by

$$\omega_0(X,Y) = 1.$$

The induced action of $\mathrm{Sp}(V_0)$ on $V = \mathrm{Sym}^{2n-1} V_0 \cong \mathbf{R}_{2n-1}[X,Y] \cong \mathbf{R}^{2n}$ preserves the symplectic form $\omega_n = \mathrm{Sym}^{2n-1}\omega_0$, which is given by

$$\omega_n(P_k, P_l) = 0 \text{ if } k+l \neq 2n-1 \text{ and } \omega_n(P_k, P_{2n-1-k}) = (-1)^k \binom{n}{k}^{-1},$$

where $P_k = X^{2n-1-k}Y^k$.

This defines the $2n$-dimensional irreducible representation of $\mathrm{Sp}(V_0) \cong \mathrm{SL}(2,\mathbf{R})$ into $\mathrm{Sp}(V) \cong \mathrm{Sp}(2n,\mathbf{R})$,

$$\phi_{irr} : \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{Sp}(2n,\mathbf{R}),$$

which, by precomposition with $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$, gives rise to an *irreducible Fuchsian representation*

$$\rho_{irr} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{Sp}(2n,\mathbf{R}).$$

Facts 3.1. (i) *Let $L^s(\iota)$ be the line bundle over $T^1\Sigma$ associated to the embedding $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$, and E_ι, $E_{\rho_{irr}}$ the flat symplectic bundles over $T^1\Sigma$. As $E_{\rho_{irr}} = \mathrm{Sym}^{2n-1}E_\iota$ and $E_\iota = L^s(\iota) \oplus L^u(\iota) = L^s(\iota) \oplus L^s(\iota)^{-1}$, the Lagrangian reduction $L^s(\rho_{irr})$ over $T^1\Sigma$ associated to ρ_{irr} is*

$$L^s(\rho_{irr}) = L^s(\iota)^{2n-1} \oplus L^s(\iota)^{2n-3} \oplus \cdots \oplus L^s(\iota).$$

[3] More precisely, $\rho(\pi_1(\Sigma))$ will preserve a totally geodesic tight disk in the symmetric space associated to $\mathrm{Sp}(2n,\mathbf{R})$ (the notion of tight disk is not used in this paper, the interested reader is referred to [11]).

(ii) *When $n = 2$, let us choose the symplectic identification* $\mathrm{Sym}^3\mathbf{R}^2 \cong \mathbf{R}_3[X,Y] \cong \mathbf{R}^4$ *given by* $X^3 = e_1, X^2Y = -e_2/\sqrt{3}, Y^3 = -e_3, XY^2 = e_4/\sqrt{3}$. *With respect to this identification the irreducible representation* $\phi_{irr} : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{Sp}(4,\mathbf{R})$ *is given by the following formula*

$$\phi_{irr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^3 & -\sqrt{3}a^2b & -b^3 & -\sqrt{3}ab^2 \\ -\sqrt{3}a^2c & 2abc + a^2d & \sqrt{3}b^2d & 2abd + b^2c \\ -c^3 & \sqrt{3}c^2d & d^3 & \sqrt{3}cd^2 \\ -\sqrt{3}ac^2 & 2acd + bc^2 & \sqrt{3}bd^2 & 2bcd + ad^2 \end{pmatrix}.$$

In particular, for $g = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}$ *one has*

$$\phi_{irr}(g) = \begin{pmatrix} e^{3m} & 0 & 0 & 0 \\ 0 & e^m & 0 & 0 \\ 0 & 0 & e^{-3m} & 0 \\ 0 & 0 & 0 & e^{-m} \end{pmatrix}.$$

3.2.2. *Diagonal Fuchsian representations.* Let

$$\mathbf{R}^{2n} = W_1 \oplus \cdots \oplus W_n$$

with $W_i = \mathrm{Span}(e_i, e_{n+i})$ be a symplectic splitting of $\mathbf{R}^{2n}$. Identifying $W_i \cong \mathbf{R}^2$, this splitting gives rise to an embedding

$$\psi : \mathrm{SL}(2,\mathbf{R})^n \longrightarrow \mathrm{Sp}(W_1) \times \cdots \times \mathrm{Sp}(W_n) \subset \mathrm{Sp}(2n,\mathbf{R}).$$

Precomposing with the diagonal embedding of $\mathrm{SL}(2,\mathbf{R}) \to \mathrm{SL}(2,\mathbf{R})^n$ we obtain a diagonal embedding

$$\phi_\Delta : \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{Sp}(2n,\mathbf{R}).$$

By precomposition with $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ we obtain a *diagonal Fuchsian representation*

$$\rho_\Delta : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{Sp}(2n,\mathbf{R}).$$

Facts 3.2. (i) *Let $L^s(\iota)$ be the Lagrangian line bundle over $T^1\Sigma$ associated to $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$, then the Lagrangian reduction $L^s(\rho_\Delta)$ of the flat symplectic $\mathbf{R}^{2n}$-bundle over $T^1\Sigma$ associated to ρ_Δ is given by*

$$L^s(\rho_\Delta) = L^s(\iota) \oplus \cdots \oplus L^s(\iota).$$

(ii) *When $n = 2$ and with respect to the symplectic basis $(e_i)_{i=1,\dots,4}$ the map ψ is given by the following formula*

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ c & 0 & d & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

3.2.3. *Twisted diagonal representations.* We now vary the construction of the previous subsection. For this note that the image $\phi_\Delta(\mathrm{SL}(2,\mathbf{R})) < \mathrm{Sp}(2n,\mathbf{R})$ has a fairly large centralizer, which is a compact subgroup of $\mathrm{Sp}(2n,\mathbf{R})$ isomorphic to $\mathrm{O}(n)$.

Remark 3.3. *For any maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R})$ the centralizer of $\rho(\pi_1(\Sigma))$ is a subgroup of $\mathrm{O}(n)$. This is because the centralizer of $\rho(\pi_1(\Sigma))$ will fix the positive curve in the space of Lagrangians pointwise. In particular, it will be contained in the stabilizer of one positive triple of Lagrangians which is isomorphic to $\mathrm{O}(n)$.*

That the centralizer of $\phi_\Delta(\mathrm{SL}(2,\mathbf{R}))$ is precisely $\mathrm{O}(n)$ can be seen in the following way. Let (W,q) be an n-dimensional vector space equipped with a definite quadratic form q and let again $V_0 = \mathbf{R}_1[X,Y] \cong \mathbf{R}^2$ with its standard symplectic form ω_0. The tensor product $V_0 \otimes W$ inherits a bilinear nondegenerate form $\omega_0 \otimes q$ which is easily seen to be antisymmetric, so that we can choose a symplectic identification $\mathbf{R}^{2n} \cong V_0 \otimes W$. This gives an embedding

$$\mathrm{SL}(2,\mathbf{R}) \times \mathrm{O}(n) \cong \mathrm{Sp}(V_0) \times \mathrm{O}(W,q) \xrightarrow{\phi_\Delta} \mathrm{Sp}(2n,\mathbf{R}),$$

which extends the morphism ϕ_Δ defined above.

Now given $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ and a representation $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$, we set

$$\begin{aligned} \rho_\Theta = \iota \otimes \Theta : \pi_1(\Sigma) &\longrightarrow \mathrm{Sp}(V) \cong \mathrm{Sp}(2n,\mathbf{R}) \\ \gamma &\longmapsto \phi_\Delta(\iota(\gamma), \Theta(\gamma)). \end{aligned}$$

We will call such a representation a *twisted diagonal representation.*

Facts 3.4. (i) *The flat bundle E over Σ associated to $\rho_\Theta : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R}) \cong \mathrm{Sp}(V_0 \otimes W)$ is of the form*

$$E = E_0 \otimes W,$$

where (with a slight abuse of notation) W is the flat n-plane bundle associated to Θ and E_0 the flat plane bundle over Σ associated to ι.

(ii) *Let $L^s(\iota)$ be the line bundle over $T^1\Sigma$ associated to ι. Let $\overline{W}$ denote the flat n-bundle over $T^1\Sigma$ given by the pull-back of W. Then the Lagrangian reduction $L^s(\rho_\Theta)$ is given by the tensor product*

$$L^s(\rho_\Theta) = L^s(\iota) \otimes \overline{W}.$$

3.2.4. *Standard representations for other groups.* Let G be an almost simple Lie group of Hermitian type of real rank n. Then there exist a (up to conjugation by G) unique embedding $\mathrm{SL}(2,\mathbf{R})^n \to G$. We call the precomposition of such an embedding with the diagonal embedding $\mathrm{SL}(2,\mathbf{R}) \to \mathrm{SL}(2,\mathbf{R})^n$ a diagonal embedding

$$\phi_\Delta : \mathrm{SL}(2,\mathbf{R}) \to G.$$

The centralizer of $\phi_\Delta(\mathrm{SL}(2,\mathbf{R}))$ in G is always a compact subgroup K' of G.

The composition $\rho_\Delta = \phi_\Delta \circ \iota : \Gamma \to G$ is a maximal representation. Given a representation $\Theta : \Gamma \to K'$ we can again define a *twisted diagonal representation*

$$\begin{aligned} \rho_\Theta : \pi_1(\Sigma) &\to \mathrm{Sp}(V) \\ \gamma &\mapsto \rho_\Delta(\gamma) \cdot \Theta(\gamma). \end{aligned}$$

Remark 3.5. *In the general case the subgroup K' can be characterized as being the intersection of the maximal compact subgroup K in G with the subgroup $H < G$, which is the stabilizer of a pair of transverse point in the Shilov boundary of the symmetric space associated to G. Equivalently, it is the stabilizer in G of a maximal triple of points in the Shilov boundary. (For the definition of maximal triples see* [13, Section 2.1.3.]*.)*

3.3. **Amalgamated representations.** Due to Theorem 2.16 we can construct maximal representations of $\pi_1(\Sigma)$ by amalgamation of maximal representations of the fundamental groups of subsurfaces.

Let $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ be a decomposition of Σ along a simple closed separating oriented geodesic γ into two subsurfaces Σ_l, lying to the left of γ, and Σ_r, lying to the right of γ. Then $\pi_1(\Sigma)$ is isomorphic to $\pi_1(\Sigma_l) *_{\langle\gamma\rangle} \pi_1(\Sigma_r)$, where we identify γ with the element it defines in $\pi_1(\Sigma)$.

We will call a representation constructed by amalgamation of two representations $\rho_l : \pi_1(\Sigma_l) \to G$ and $\rho_r : \pi_1(\Sigma_r) \to G$ with $\rho_l(\gamma) = \rho_r(\gamma)$ an *amalgamated representation* $\rho = \rho_l * \rho_r : \pi_1(\Sigma) \to G$. By Theorem 2.16, the amalgamated representation ρ is maximal if and only if ρ_l and ρ_r are maximal.[4]

3.3.1. *Hybrid representations.* In this section we describe the most important class of maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ obtained via amalgamation. We call these representation *hybrid representations* to distinguish them from general amalgamated representations.

Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding. The basic idea of the construction of hybrid representations is to amalgamate the restriction of the irreducible Fuchsian representation $\rho_{irr} = \phi_{irr} \circ \iota : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ to Σ_l and the restriction of the diagonal Fuchsian representation $\rho_\Delta = \phi_\Delta \circ \iota : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ to Σ_r. This does not directly work because the holonomies of ρ_{irr} and ρ_Δ along γ do not agree, but a slight modification works.

Assume that $\iota(\gamma) = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}$ with $m > 0$. Set

$$\rho_l := \phi_{irr} \circ \iota : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R}),$$

[4]Note that it is important that the Toledo invariant for both ρ_l and ρ_r are of the same sign. Amalgamating a maximal representation with a minimal representation does not give rise to a maximal representation.

with ϕ_{irr} defined in Facts 3.1(ii). Then $\rho_l(\gamma) = \begin{pmatrix} e^{3m} & 0 & 0 & 0 \\ 0 & e^m & 0 & 0 \\ 0 & 0 & e^{-3m} & 0 \\ 0 & 0 & 0 & e^{-m} \end{pmatrix}$.

Let $(\tau_{1,t})_{t\in[0,1]}$ and $(\tau_{2,t})_{t\in[0,1]}$ be continuous paths of discrete embeddings $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ such that $\tau_{1,0} = \tau_{2,0} = \iota$, and, for all $t \in [0,1]$,

$$\tau_{1,t}(\gamma) = \begin{pmatrix} e^{l_{1,t}} & \\ & e^{-l_{1,t}} \end{pmatrix} \text{ and } \tau_{2,t}(\gamma) = \begin{pmatrix} e^{l_{2,t}} & \\ & e^{-l_{2,t}} \end{pmatrix},$$

where $l_{1,t} > 0$ and $l_{2,t} > 0$, $l_{1,0} = l_{2,0} = m$, $l_{1,1} = 3m$ and $l_{2,1} = m$. The existence of $\tau_{i,t}$ is a classical fact from Teichmüller theory, for the reader's convenience we include the statement we are using in Lemma A.3. Set

$$\rho_r := \psi \circ (\tau_{1,1}, \tau_{2,1}) : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4,\mathbf{R}).$$

Then ρ_r is a continuous deformation of $\phi_\Delta \circ \iota$ which satisfies $\rho_r(\gamma) = \rho_l(\gamma)$.

We now define a *hybrid representation* by

$$(2) \qquad \rho := \rho_l|_{\pi_1(\Sigma_l)} * \rho_r|_{\pi_1(\Sigma_r)} : \pi_1(\Sigma) = \pi_1(\Sigma_l) *_{\langle\gamma\rangle} \pi_1(\Sigma_r) \longrightarrow \mathrm{Sp}(4,\mathbf{R}).$$

Since $\rho_l|_{\pi_1(\Sigma_l)}$ and $\rho_r|_{\pi_1(\Sigma_r)}$ are maximal representations, the representation ρ is maximal (see Theorem 2.16).

Remark 3.6. *The special choices for the embeddings ϕ_{irr} and ψ (Facts 3.1*(ii) *and 3.2*(ii)*) will be important for the calculation of the Euler class of a hybrid representation. Obviously one can always change one of this two embeddings by conjugation by an element of the centralizer of $\rho(\gamma)$,* i.e. *an element, which, with respect to a suitable basis, is of the form* $\mathrm{diag}(a, b, a^{-1}, b^{-1})$.

In order to keep track of this situation we define

Definition 3.7. *Let γ be a curve*[5] *on Σ and let ρ_l and ρ_r two representations of $\pi_1(\Sigma)$ into* $\mathrm{Sp}(4,\mathbf{R})$ *with $\rho_l(\gamma) = \rho_r(\gamma)$ and such that ρ_l is a Hitchin representation and ρ_r is a maximal representation into* $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) < \mathrm{Sp}(4,\mathbf{R})$.

The pair (ρ_l, ρ_r) is said to be positively adjusted with respect to γ *if there exists a symplectic basis $(\epsilon_i)_{i=1,\dots,4}$ and continuous deformations $(\rho_{l,t})_{t\in[0,1]}$ and $(\rho_{r,t})_{t\in[0,1]}$ such that:*

- *$\rho_{l,1} = \rho_l$ and $\rho_{r,1} = \rho_r$,*
- *$\rho_{l,0} = \phi_{irr} \circ \iota$ is an irreducible Fuchsian representation, $\rho_{r,0} = \phi_\Delta \circ \iota$ is a diagonal Fuchsian representation and for each t in $[0,1]$ $\rho_{r,t}$ is a maximal representation into* $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) < \mathrm{Sp}(4,\mathbf{R})$,
- *for each t, $\rho_{l,t}(\gamma)$ and $\rho_{r,t}(\gamma)$ are diagonal in the base (ϵ_i).*

The pair (ρ_l, ρ_r) is said to be negatively adjusted with respect to γ *if the pair $(g\rho_l g^{-1}, \rho_r)$ is positively adjusted where g is diagonal in the base $(\epsilon_i)_{i=1,\dots,4}$ with eigenvalues of different signs,* i.e. $g = \mathrm{diag}(a, b, a^{-1}, b^{-1})$ *with $ab < 0$.*

[5] The curve γ does not need to be separating or simple.

Remark 3.8. *In fact, it is not really necessary here that the representations ρ_l, ρ_r were representations of $\pi_1(\Sigma)$, what matters in the computation is that the representation $\rho_l|_{\pi_1(\Sigma_l)}$ is the restriction of a Hitchin representation of some closed surface, and similarly that $\rho_r|_{\pi_1(\Sigma_r)}$ is the restriction of a deformation of a diagonal maximal representation of some closed surface.*

Since hybrid representations are maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ the associated flat symplectic $\mathbf{R}^4$-bundle E admits a Lagrangian splitting $E = L^s(\rho) \oplus L^u(\rho)$. We are unable to describe the Lagrangian bundles $L^s(\rho)$ and $L^u(\rho)$ explicitly as we did above for standard maximal representations. Nevertheless, computing the topological invariants (see Section 5) we defined for Anosov representations, we will be able to determine the topological type of $L^s(\rho)$. The topological type will indeed only depend on the Euler characteristic of Σ_l.

Definition 3.9. *If $\chi(\Sigma_l) = k$ we call the representation defined by (2) a k-hybrid representation.*

3.3.2. *Hybrid representations: general construction.* In the construction above we decompose Σ along one simple closed separating geodesic, so the Euler characteristic of Σ_l will be odd. To obtain k-hybrid representations of $\pi_1(\Sigma)$ for all $\chi(\Sigma) + 1 \leq k \leq -1$ we have to consider slightly more general decompositions of Σ, in particular Σ_l or Σ_r might not be connected.

Let us fix some notation to describe this more general construction. Let Σ be closed oriented surface of genus g and $\Sigma_1 \subset \Sigma$ be a subsurface with Euler characteristic equal to k.

The (non-empty) boundary $\partial\Sigma_1$ is the union of disjoints circles γ_d for $d \in \pi_0(\partial\Sigma_1)$. We orient the circles such that, for each d, the subsurface Σ_1 is on the left of γ_d. Write the surface $\Sigma - \partial\Sigma_1$ as the union of its connected components:

$$\Sigma - \partial\Sigma_1 = \bigcup_{c \in \pi_0(\Sigma - \partial\Sigma_1)} \Sigma_c.$$

For any d in $\pi_0(\partial\Sigma_1)$ the curve γ_d bounds exactly 2 connected components of $\Sigma - \partial\Sigma_1$: one is included in Σ_1 and denoted by $\Sigma_{l(d)}$ with $l(d)$ in $\pi_0(\Sigma_1)$; the other one is included in the complementary of Σ_1 and denoted by $\Sigma_{r(d)}$ with $r(d)$ in $\pi_0(\Sigma - \Sigma_1)$. Note that $l(d)$ and $r(d)$ are elements of $\pi_0(\Sigma - \partial\Sigma_1)$. In particular, $l(d)$ might equal $l(d')$ for some $d \neq d'$, similarly for $r(d)$ and $r(d')$.

We assume that

– The graph with vertices set $\pi_0(\Sigma - \partial\Sigma_1)$ and edges given by the pairs $\{(l(d), r(d))\}_{d \in \pi_0(\partial\Sigma_1)}$ is a tree.

The fundamental group $\pi_1(\Sigma)$ can be described as the amalgamated product of the groups $\pi_1(\Sigma_c)$, c in $\pi_0(\Sigma - \partial\Sigma_1)$, over the groups $\pi_1(\gamma_d)$, $d \in \pi_0(\partial\Sigma_1)$. The above assumption ensures that no HNN-extensions appear in this description.

With these notations, we can now define general k-hybrid representations.

For each c in $\pi_0(\Sigma_1)$ we choose a representation

$$\rho_c : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4, \mathbf{R})$$

belonging to one of the 2^{2g} Hitchin components. We set with a slight abuse of notation $\rho_c = \rho_c|_{\pi_1(\Sigma_c)}$ for each c in $\pi_0(\Sigma_1)$.

For any d in $\pi_0(\partial\Sigma_1)$, $\rho_{l(d)}(\gamma_d)$ (this makes sense since $l(d) \in \pi_0(\Sigma_1)$) is conjugate to a unique element of the form

$$\rho_{l(d)}(\gamma_d) \cong \epsilon(d) \begin{pmatrix} e^{l_1(d)} & & & \\ & e^{-l_1(d)} & & \\ & & e^{l_2(d)} & \\ & & & e^{-l_2(d)} \end{pmatrix} \in \mathrm{Sp}(4, \mathbf{R})$$

with $\epsilon(d) \in \{\pm 1\}$, $l_1(d) > l_2(d) > 0$.

The construction of $\rho_{c'}$ for c' in $\pi_0(\Sigma - \Sigma_1)$ now goes as follow. By Lemma A.3 one can choose a continuous path

$$\tau_{c',t} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbf{R}),\ t \in [1, 2]$$

such that $\tau_{c',t} : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ are discrete embeddings for all $t \in [1, 2]$ and such that, for any d in $\pi_0(\partial\Sigma_{c'}) \subset \pi_0(\partial\Sigma_1)$, (hence $r(d) = c'$) one has

$$\tau_{c',i}(\gamma_d) \text{ is conjugate to } \epsilon(d) \begin{pmatrix} e^{l_i(d)} & \\ & e^{-l_i(d)} \end{pmatrix},\ \text{for } i = 1, 2.$$

Set

$$\rho_{c'} = \begin{pmatrix} \tau_{c',1}|_{\pi_1(\Sigma_{c'})} & \\ & \tau_{c',2}|_{\pi_1(\Sigma_{c'})} \end{pmatrix} : \pi_1(\Sigma_{c'}) \longrightarrow \mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) \subset \mathrm{Sp}(4, \mathbf{R}).$$

In order to define the amalgamated representation, we need to choose elements g_c in $\mathrm{Sp}(4, \mathbf{R})$ for any c in $\pi_0(\Sigma - \partial\Sigma_1)$ such that, for any d, $g_{l(d)}\rho_{l(d)}(\gamma_d)g_{l(d)}^{-1} = g_{r(d)}\rho_{r(d)}(\gamma_d)g_{r(d)}^{-1}$. As mentioned in Remark 3.6 these elements should be chosen such that

– for any d in $\pi_0(\partial\Sigma_1)$ the pair of representations

$$\left(g_{l(d)}\rho_{l(d)}g_{l(d)}^{-1}, g_{r(d)}\rho_{r(d)}g_{r(d)}^{-1}\right)$$

is positively adjusted with respect to γ_d (Definition 3.7).

Such a family (g_c) always exists by our hypothesis that the graph associated to the decomposition of the surface Σ is a tree. One then constructs the k-hybrid representation

$$\rho : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4, \mathbf{R})$$

by amalgamation of the representations $g_{l(d)}\rho_{l(d)}g_{ld}^{-1}$ and $g_{r(d)}\rho_{r(d)}(\gamma_d)g_{r(d)}^{-1}$

Remark 3.10. *The hypothesis that the dual graph of $\Sigma - \partial\Sigma_1$ is a tree is necessary. For example, if this graph has a double edge, one would try to construct a Hitchin representation whose restriction to the disjoint union of two closed simple curves γ_1 and γ_2 is contained in some* $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) <$

$\mathrm{Sp}(4,\mathbf{R})$. *But it is not difficult to see that the restriction of a Hitchin representation to the subgroup generated by* γ_1 *and* γ_2 *is irreducible and hence cannot be contained in* $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) < Sp(4,\mathbf{R})$.

3.3.3. *Other amalgamated representations.* Let us describe a variant of the construction of hybrid representation. Assume that Σ is decomposed along a simple closed separating geodesic γ into two subsurfaces Σ_l and Σ_r as above. On $\pi_1(\Sigma_l)$ we choose again the irreducible Fuchsian representation $\rho_{irr} = \phi_{irr} \circ \iota$ into $\mathrm{Sp}(4,\mathbf{R})$, for the fundamental group of $\pi_1(\Sigma_r)$ we choose a maximal representation into $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) \subset \mathrm{Sp}(4,\mathbf{R})$ which agrees with the irreducible representation along γ, but sends an element $\alpha \in \pi_1(\Sigma_r)$ corresponding to a non-separating simple closed geodesic to an element of $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ with eigenvalues of different sign. The corresponding amalgamated representation will be maximal. But, the first Stiefel-Whitney class of this representation is non-zero. Using the computations made in Section 5 we can conclude that ρ can be deformed to a twisted diagonal representation.

The analogous constructions can be made to obtain maximal representations into $\mathrm{Sp}(2n,\mathbf{R})$ or also into other Lie groups G of Hermitian type. But in the case when G is not locally isomorphic to $\mathrm{Sp}(4,\mathbf{R})$, we expect that all maximal representations can be deformed to a twisted diagonal representation.

4. Topological invariants

4.1. **Definition.** Let us denote by $\mathrm{Hom}_{H\text{-Anosov}}(\pi_1(M), G)$ the set of (G,H)-Anosov representations and let $\mathcal{B}_H(M)$ the set of gauge isomorphism classes of H-bundles over M. Summarizing Proposition 2.5 and Proposition 2.7 we have

Proposition 4.1. *For any pair* (G,H)*, there is a well-defined locally constant map*

$$\mathrm{Hom}_{H\text{-Anosov}} \longmapsto \mathcal{B}_H(M),$$

associating to an Anosov representation its H*-Anosov reduction.*

This map is natural with respect to taking finite covers of M *and with respect to the constructions described in Lemma 2.8.*

In general $\mathcal{B}_H(M)$ could be rather complicated, so instead of the whole space of gauge isomorphism classes $\mathcal{B}_H(M)$ we will consider the obstruction classes associated to the H-bundle as topological invariants of an Anosov representation, *i.e.* the invariants are elements of the cohomology groups of M, possibly with local coefficients.

Remark 4.2. *The group* H *is the Levi component of* P^s*, therefore* P^s *and* H *are homotopy equivalent. We hence have* $\mathcal{B}_H = \mathcal{B}_{P^s}$*, so instead of working with the* H*-reduction we could equally well work with the corresponding* P^s*-reduction (or similarly with the* P^u*-reduction) of the flat* G*-bundle*

4.2. **First and second Stiefel-Whitney classes.** The inclusion

$$\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \subset \mathrm{Hom}_{\mathrm{GL}(n,\mathbf{R})\text{-Anosov}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})),$$

which is described in Section 2.2.2, allows to apply Proposition 4.1 to associate to a maximal representation into $\mathrm{Sp}(2n, \mathbf{R})$ the first and second Stiefel-Whitney classes of a $\mathrm{GL}(n, \mathbf{R})$-bundles over $T^1\Sigma$.

Proposition 4.3. *Let $G = \mathrm{Sp}(2n, \mathbf{R})$ and $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), G)$ the space of maximal representations. Then the obstruction classes of the canonical $\mathrm{GL}(n, \mathbf{R})$-reduction give maps:*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), G) & \xrightarrow{sw_1} & \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \\ \text{and } \mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), G) & \xrightarrow{sw_2} & \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2). \end{array}$$

The following geometric interpretation makes the first Stiefel-Whitney class $sw_1(\rho)$ easy to compute. Recall that given a closed geodesic γ on Σ there is a natural lift to a closed loop γ on $T^1\Sigma$, this gives rise to a natural map $\pi_1(\Sigma) - \{1\} \to \mathrm{H}_1(T^1\Sigma; \mathbf{Z})$.

Lemma 4.4. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation and $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ the equivariant limit curve. Then*

$$sw_1(\rho)([\gamma]) = \mathrm{sign}(\det \rho(\gamma)|_{\xi(t^s_\gamma)}),$$

where $[\gamma]$ is the class of $\mathrm{H}_1(T^1\Sigma; \mathbf{F}_2)$ represented by γ, $t^s_\gamma \in \partial\pi_1(\Sigma)$ is the attractive fixed point of γ and $\mathbf{F}_2$ is identified with $\{\pm 1\}$.

Proof. The first Stiefel-Whitney class of $sw_1(\rho)([\gamma])$ is the obstruction to the orientability of the bundle $L^s(\rho)|_\gamma \cong \mathbf{Z}\backslash(\mathbf{R} \times \xi(t^s_\gamma))$ over $\gamma \cong \mathbf{Z}\backslash\mathbf{R}$. Thus, $sw_1(\rho)([\gamma]) = 1$ if $\rho(\gamma) \in \mathrm{GL}(\xi(t^s_\gamma))$ lies in the connected component of the identity and $sw_1(\rho)([\gamma]) = -1$ otherwise. □

4.3. **An Euler class.** For $n = 2$, the invariants obtained in Proposition 4.3 do not allow to distinguish the connected components of

$$\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})).$$

In particular, the second Stiefel-Whitney class does not offer enough information to distinguish the connected components of

$$\mathrm{Hom}_{\mathrm{max}, sw_1 = 0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})).$$

The reason for this is that when $sw_1(\rho) = 0$ the Lagrangian bundle $L^s(\rho)$ is orientable. So there should be an Euler class whose image in $\mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$ is the second Stiefel-Whitney class $sw_2(\rho)$. Since an *orientable* vector bundle has no *canonical orientation*, we need to introduce an "enhanced" representation space to obtain a well-defined Euler class.

4.3.1. *Enhanced representation spaces.* Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation. Let $\gamma \in \pi_1(\Sigma) - \{1\}$ and $L^s(\gamma) \in \mathcal{L}$ the attractive fixed Lagrangian of $\rho(\gamma)$. We denote by $\mathcal{L}_+$ the space of *oriented* Lagrangians and by $\pi : \mathcal{L}_+ \to \mathcal{L}$ the projection. Let

$$\mathrm{Hom}^{\mathcal{L},\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) = \\ \big\{(\rho, L) \in \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \times \mathcal{L} \mid L = L^s(\gamma)\big\}$$

and

$$\mathrm{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) = \\ \big\{(\rho, L_+) \in \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \times \mathcal{L}_+ \mid \pi(L_+) = L^s(\gamma)\big\}.$$

The map $\mathrm{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \to \mathrm{Hom}^{\mathcal{L},\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ is a 2-fold cover.

We introduced the space $\mathrm{Hom}^{\mathcal{L},\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$, which is easily identified with $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$, to emphasize the fact that $\rho(\gamma)$ has an attractive Lagrangian.

Lemma 4.5. *The natural map*

$$\begin{array}{ccc} \mathrm{Hom}^{\mathcal{L},\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) & \longrightarrow & \mathcal{L} \\ (\rho, L) & \longmapsto & L \end{array}$$

is continuous.

This lemma follows immediately from the continuity of the eigenspace of a matrix.

4.3.2. *The Euler class.* As a consequence of the following proposition for every element (ρ, L_+) in $\mathrm{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}$ there is a natural associated *oriented* Lagrangian bundle over $T^1\Sigma$.

Proposition 4.6. *Let $\rho \in \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ and let $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ be the corresponding equivariant positive curve. Suppose that n is even. Then there exists a continuous lift of ξ to $\mathcal{L}_+$.*

Let $\xi_+ : \partial\pi_1(\Sigma) \to \mathcal{L}_+$ be one of the two continuous lifts of ξ. Then the map ξ_+ is ρ-equivariant if, and only if, $sw_1(\rho) = 0$. In this case the other lift of ξ is also equivariant.

Proof. The fact that a continuous lift exists depends only on the homotopy class of the curve ξ. Since the space of continuous and positive curves is connected (see Proposition A.1), the existence of a lift can be check for one specific maximal representation. Considering a Fuchsian representation gives the restriction on n.

Let ξ_+ be a continuous lift of ξ and denote by ξ'_+ the other lift of the curve ξ. Since ξ is equivariant, for any γ in $\pi_1(\Sigma)$, the following alternative holds:

$$\gamma \cdot \xi_+ = \xi_+ \text{ or } \gamma \cdot \xi_+ = \xi'_+.$$

Furthermore, given any t in $\partial\pi_1(\Sigma)$, by connectedness, one observes that

$$\gamma \cdot \xi_+ = \xi_+ \Longleftrightarrow \xi_+(\gamma \cdot t) = \rho(\gamma) \cdot \xi_+(t).$$

Using this last equation for the attractive fixed point t_γ^s of γ in $\partial\pi_1(\Sigma)$, the equivalence becomes

$$\gamma \cdot \xi_+ = \xi_+ \Longleftrightarrow \det \rho(\gamma)|_{\xi(t_\gamma^s)} > 0.$$

Using now the equality $sw_1(\rho)([\gamma]) = \operatorname{sign}(\det \rho(\gamma)|_{\xi(t_\gamma^s)})$ (Lemma 4.4), the proposition follows. □

Proposition 4.6 gives a natural way to lift the equivariant curve given any element $(\rho, L_+) \in \operatorname{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \operatorname{Sp}(4,\mathbf{R}))$:

Definition 4.7. *Let* $(\rho, L_+) \in \operatorname{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \operatorname{Sp}(4,\mathbf{R}))$ *and let* $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ *be the* ρ*-equivariant curve.*

The lift (uniquely determined and equivariant) ξ_+ *of* ξ *such that*

$$\xi_+(t_\gamma^s) = L_+$$

is called the canonical oriented equivariant curve *for the pair* (ρ, L_+)*. Here* $t_\gamma^s \in \partial\pi_1(\Sigma)$ *is the attractive fixed point of* γ*.*

This canonical oriented equivariant curve defines a $\operatorname{GL}^+(2,\mathbf{R})$-reduction of the $\operatorname{GL}(2,\mathbf{R})$-Anosov reduction associated to ρ. It also induces a splitting of the flat bundle E associated to ρ

$$E = L_+^s(\rho) \oplus L_+^u(\rho)$$

into two oriented Lagrangians subbundles. Similarly to Notation 2.17 we call this splitting the *oriented Lagrangian reduction* of E.

Definition 4.8. *The Euler class* $e_\gamma(\rho, L_+)$

$$\begin{array}{rccc} e_\gamma : & \operatorname{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \operatorname{Sp}(4,\mathbf{R})) & \longrightarrow & \operatorname{H}^2(T^1\Sigma; \mathbf{Z}) \\ & (\rho, L_+) & \longmapsto & e_\gamma(\rho, L_+) \end{array}$$

is the Euler class of the $\operatorname{GL}^+(2,\mathbf{R})$*-reduction given by Definition 4.7.*

The map e_γ is continuous.

4.3.3. *Connected components.* We consider now a subspace of $\operatorname{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}$ by fixing the oriented Lagrangian. Let $L_{0+} \in \mathcal{L}_+$ be an oriented Lagrangian; we set

$$\operatorname{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max, sw_1=0}(\pi_1(\Sigma), \operatorname{Sp}(4,\mathbf{R})) = \{(\rho, L_+) \in \operatorname{Hom}^{\mathcal{L}_+,\gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \operatorname{Sp}(4,\mathbf{R})) \mid L_+ = L_{0+}\}.$$

Lemma 4.9. *The natural map*

$$\mathrm{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R})) \longrightarrow \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))/\mathrm{Sp}(4,\mathbf{R})$$

is onto. Its fibers are connected.

Proof. Let $\rho \in \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$. Since the first Stiefel-Whitney class of ρ vanishes, there exists an attracting oriented Lagrangian L_+ for $\rho(\gamma)$. Let $g \in \mathrm{Sp}(4,\mathbf{R})$ such that $g \cdot L_+ = L_{0+}$, then $(g\rho g^{-1}, L_{0+})$ belongs to $\mathrm{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$ and projects to $[\rho]$.

Now, let $(\rho, L_{0+}) \in \mathrm{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$. Then the fiber of the projection containing (ρ, L_{0+}) is isomorphic to the quotient:

$$\{g \in \mathrm{Sp}(4,\mathbf{R}) \mid g \cdot L_{0+} = L_{0+}\}/\{z \in Z(\rho) \mid z \cdot L_{0+} = L_{0+}\},$$

where $Z(\rho)$ denotes the centralizer of $\rho(\pi_1(\Sigma))$ in $\mathrm{Sp}(4,\mathbf{R})$. The group $\{g \in \mathrm{Sp}(4,\mathbf{R}) \mid g \cdot L_{0+} = L_{0+}\}$ is connected, thus the fiber is connected. □

Lemma 4.9 implies that $\mathrm{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$ has as the same number of connected components as $\mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$.

Definition 4.10. *Let $\gamma \in \pi_1(\Sigma) - \{1\}$ and $L_{0+} \in \mathcal{L}_+$. The* relative Euler class $e_{\gamma,L_{0+}}(\rho) \in \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$ *of the class of a maximal representation $\rho \in \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$ is defined by being the Euler class of one (any) inverse image of $[\rho]$ in the space $\mathrm{Hom}^{\mathcal{L}_+,\gamma,L_{0+}}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R}))$.*

4.4. **Constraints on invariants.** Since the flat $\mathrm{Sp}(2n,\mathbf{R})$-bundle over $T^1\Sigma$ whose $\mathrm{GL}(n,\mathbf{R})$-reduction gives rise to the invariants of the maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R})$ (Proposition 4.3) is the pull-back of a flat bundle over Σ, one expects that not every cohomology class can arise.

Proposition 4.11. *Let $G = \mathrm{Sp}(2n,\mathbf{R})$. Then*

(i) *The image of*

$$sw_1 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) \longrightarrow \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$$

is contained in one coset of $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$. More precisely:
- *for n even, $sw_1(\rho) \in \mathrm{H}^1(\Sigma;\mathbf{F}_2) \subset \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$,*
- *for n odd, $sw_1(\rho) \in \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2) - \mathrm{H}^1(\Sigma;\mathbf{F}_2)$.*

(ii) *The image of*

$$sw_2 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) \longrightarrow \mathrm{H}^2(T^1\Sigma;\mathbf{F}_2)$$

lies in the image of $\mathrm{H}^2(\Sigma;\mathbf{F}_2) \to \mathrm{H}^2(T^1\Sigma;\mathbf{F}_2)$.

(iii) *Similarly, when $n = 2$, the image of*

$$e_{\gamma,L_{0+}} : \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbf{R})) \longrightarrow \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$$

lies in the image of $\mathrm{H}^2(\Sigma;\mathbf{Z}) \to \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$. This holds for any nontrivial γ in $\pi_1(\Sigma)$ and any oriented Lagrangian $L_{0+} \in \mathcal{L}_+$.

To prove this proposition we use the positivity of the equivariant curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ of a maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ (Definition 2.19).

Proof of Proposition 4.11. For the first property, let $x \in \Sigma$ and consider the exact sequence

$$\mathrm{H}^1(\Sigma; \mathbf{F}_2) \longrightarrow \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \xrightarrow{f_x} \mathrm{H}^1(T^1_x\Sigma; \mathbf{F}_2).$$

We need to check that the image of $sw_1(\rho)$ in $\mathrm{H}^1(T^1_x\Sigma; \mathbf{F}_2) \cong \mathbf{F}_2$ does not depend on ρ, *i.e.* that the gauge isomorphism class of $L^s(\rho)|_{T^1_x\Sigma}$ is independent of ρ.

Let $\tilde{x} \in \widetilde{\Sigma}$ be a lift of x so that $T^1_x\Sigma \cong T^1_{\tilde{x}}\widetilde{\Sigma} \cong \partial\pi_1(\Sigma)$, where the last map is given by the restriction of $\partial\pi_1(\Sigma)^{(3+)} \to \partial\pi_1(\Sigma)/(t^s, t, t^u) \mapsto t^s$ to $T^1_{\tilde{x}}\widetilde{\Sigma} \subset T^1\widetilde{\Sigma} \cong \partial\pi_1(\Sigma)^{(3+)}$. The restriction of the flat G-bundle to $T^1_x\Sigma$ is trivial, thus the restriction of the Anosov section σ to $T^1_x\Sigma$ can be regarded as a map $T^1_x\Sigma \to \mathcal{X}$.

By Remark 4.2, we can work with the P^s-reduction, *i.e.* we consider only the first component of the map $\sigma|_{T^1_x\Sigma} : T^1_x\Sigma \to \mathcal{L} \times \mathcal{L}$. With the above identification $T^1_x\Sigma \cong \partial\pi_1(\Sigma)$ this map is exactly the equivariant limit curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$. Since the space of positive curves is connected (Proposition A.1), $f_x(sw_1(\rho))$ is independent of ρ. The calculation for the diagonal Fuchsian representation (see Section 5) gives the desired statement about $sw_1(\rho)$.

For the second statement, in view of Lemma B.3 it is sufficient to prove that for any closed curve $\eta \subset \Sigma$ the restriction of $sw_2(\rho)$ (or $e_{\gamma, L_{0+}}(\rho)$) to the torus $T^1\Sigma|_\eta$ is zero.

For this we write the torus $T^1\Sigma|_\eta$ as the quotient of $\partial\pi_1(\Sigma) \times \mathbf{R}$ by $\langle\eta\rangle \cong \mathbf{Z}$ where $\eta \cdot (\theta, t) = (\eta \cdot \theta, t + 1)$. With this identification the flat bundle can be written as:

$$\langle\gamma\rangle \backslash \big(\partial\pi_1(\Sigma) \times \mathbf{R} \times G\big) \text{ with } \gamma \cdot (\theta, t, g) = (\gamma \cdot \theta, t + 1, Ag)$$

with $A \in H \cong \mathrm{GL}(n, \mathbf{R})$. The section of the associated $\mathcal{L}$-bundle lifts to $\partial\pi_1(\Sigma) \times \mathbf{R}$:

$$\begin{array}{rcl} \partial\pi_1(\Sigma) \times \mathbf{R} & \longrightarrow & \mathcal{L} \\ (\theta, t) & \longmapsto & \xi(\theta). \end{array}$$

Since the space of pairs

$$\big\{(A, \xi) \mid A \in H, \xi > 0, C^0 \text{ and } A\text{-equivariant}\big\}$$

has exactly two connected components given by the sign of $\det A$ (Proposition A.2), we conclude that $sw_2(L^s(\rho)|_{T^1\Sigma|_\eta})$ depends only on this sign, hence $sw_2(\rho)$ depends only on $sw_1(\rho)$ (see Lemma 4.4). A direct calculation shows that in fact $sw_2(L^s(\rho)|_{T^1\Sigma|_\eta})$ is always zero.

For the Euler class the proof goes along the same lines. □

In view of Proposition 4.11 we make the following

Definition 4.12. *Let $n \in \mathbf{N}$ be even. A pair*

$$(\alpha, \beta) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \times \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$$

is admissible *if α lies in the image of $\mathrm{H}^1(\Sigma; \mathbf{F}_2) \to \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$ and β lies in the image of $\mathrm{H}^2(\Sigma; \mathbf{F}_2) \to \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.*

Let $n \in \mathbf{N}$ be odd . A pair $(\alpha, \beta) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \times \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$ is admissible *if α lies in the coset $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$, and β lies in the image of $\mathrm{H}^2(\Sigma; \mathbf{F}_2) \to \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.*

For $n = 2$, a class $\beta \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$ is called admissible if β lies in the image of $\mathrm{H}^2(\Sigma; \mathbf{Z}) \to \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$.

4.5. **Invariants for other Anosov representations.**

4.5.1. *Maximal representations into* $\mathrm{SL}(2, \mathbf{R})$. Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R}) = \mathrm{Sp}(2, \mathbf{R})$ be a maximal representation. The Anosov structure associated to ρ gives a reduction of the structure group of the flat $\mathrm{SL}(2, \mathbf{R})$-principal bundle P over $T^1\Sigma$ to $\mathrm{GL}(1, \mathbf{R})$. Considering the $\mathbf{R}^2$-bundle E associated to P, this reduction corresponds to a Lagrangian subbundle F of E. As invariants we get the first Stiefel-Whitney class of the line bundle F over $T^1\Sigma$,

$$sw_1(\iota) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2).$$

This first Stiefel-Whitney class is precisely the spin structure on Σ which corresponds to the chosen lift of $\pi_1(\Sigma) \subset \mathrm{PSL}(2, \mathbf{R})$ to $\mathrm{SL}(2, \mathbf{R})$. The invariant $sw_1(\iota)$ can take 2^{2g} different values, distinguishing the 2^{2g} connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbf{R}))$.

4.5.2. *The invariants for Hitchin representations into* $\mathrm{Sp}(2n, \mathbf{R})$. Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a Hitchin representation. Then ρ is indeed a $(\mathrm{Sp}(2n, \mathbf{R}), A)$-Anosov representation, where A is the subgroup of diagonal matrices. The reduction of the structure group of the flat $\mathrm{Sp}(2n, \mathbf{R})$-principal bundle P over $T^1\Sigma$ to A corresponds to a splitting of the associated $\mathbf{R}^{2n}$-bundle E into the sum of $2n$ isomorphic line bundles $F_1 \oplus \cdots \oplus F_{2n}$. The first Stiefel-Whitney class of the line bundle F_1 gives an invariant

$$sw_1^A(\rho) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2),$$

which lies in $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$. This invariant distinguishes the 2^{2g} components of $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$.

5. Computations of the Invariants

5.1. **Irreducible Fuchsian representations.**

Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding and $\rho_{irr} = \phi_{irr} \circ \iota : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ an irreducible Fuchsian representation. We observed in Facts 3.1(i) that the Lagrangian reduction $L^s(\rho_{irr})$ is given by

$$L^s(\rho_{irr}) = L^s(\iota)^{2n-1} \oplus L^s(\iota)^{2n-3} \oplus \cdots \oplus L^s(\iota),$$

where $L^s(\iota)$ is the line bundle associated to ι.

Therefore, by the multiplicative properties of Stiefel-Whitney classes, the first and second Stiefel-Whitney classes are given by

$$sw_1(\rho_{irr}) = sw_1(L^s(\rho_{irr})) = \big(\sum_{i=1}^{n}(2i-1)\big) sw_1(L^s(\iota)) = n sw_1(L^s(\iota)),$$

and

$$\begin{aligned} sw_2(\rho_{irr}) = sw_2(L^s(\rho_{irr})) &= \frac{n(n-1)}{2} sw_1(L^s(\iota)) \cdot sw_1(L^s(\iota)) \\ &= \frac{n(n-1)}{2}(g-1) \mod 2, \end{aligned}$$

where the last equality follows from Proposition 5.13. Note that in particular $sw_1(\rho_{irr}) = 0$ if n is even.

5.2. **Diagonal Fuchsian representations.** Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding and $\rho_\Delta = \phi_\Delta \circ \iota : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ a diagonal Fuchsian representation. We observed in Facts 3.2(i) that the Lagrangian reduction $L^s(\rho_\Delta)$ is given by

$$L^s(\rho_\Delta) = L^s(\iota) \oplus \cdots \oplus L^s(\iota),$$

where $L^s(\iota)$ is the line bundle associated to ι.

Therefore the first and second Stiefel-Whitney classes are given by

$$sw_1(\rho_\Delta) = sw_1(L^s(\rho_\Delta)) = n sw_1(L^s(\iota)),$$

and

$$\begin{aligned} sw_2(\rho_\Delta) = sw_2(L^s(\rho_\Delta)) &= \frac{n(n-1)}{2} sw_1(L^s(\iota)) \cdot sw_1(L^s(\iota)) \\ &= \frac{n(n-1)}{2}(g-1) \mod 2. \end{aligned}$$

Again, $sw_1(\rho_\Delta) = 0$ if n is even.

5.3. **Twisted diagonal representations.** Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding, $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$ an orthogonal representation and $\rho_\Theta = \iota \otimes \Theta : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ the corresponding twisted diagonal representation. We observed in Facts 3.4(ii) that the Lagrangian reduction $L^s(\rho_\Theta)$ is given by

$$L^s(\rho_\Theta) = L^s(\iota) \otimes \overline{W},$$

where $L^s(\iota)$ is the line bundle associated to ι and $\overline{W}$ is the pull-back to $T^1\Sigma$ of the flat n-plane bundle W over Σ associated to Θ.

Since $\overline{W}$ is the pull-back of the flat bundle W over Σ associated to Θ we have $sw_i(\overline{W}) = \pi^* sw_i(W)$, where $\pi^* : \mathrm{H}^i(\Sigma; \mathbf{F}_2) \to \mathrm{H}^i(T^1\Sigma; \mathbf{F}_2)$ is induced by the projection $\pi : T^1\Sigma \to \Sigma$.

Thus, to compute the invariants for twisted diagonal representations we have to study the first and second Stiefel-Whitney classes of orthogonal representations $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$.

5.3.1. *The first Stiefel-Whitney class of an orthogonal representation.* Let $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$ be an orthogonal representation and and W the associated flat orthogonal n-plane bundle over Σ. Let us denote by $sw_1(\Theta) = sw_1(W) \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ the first Stiefel-Whitney class.

Let

$$\det : \mathrm{O}(n) \to \{\pm 1\}$$

be the determinant homomorphism, where we identify $\{\pm 1\}$ with $\mathbf{F}_2$. Then the homomorphism

$$\det \circ \Theta : \pi_1(\Sigma) \to \{\pm 1\}$$

corresponds to the first Stiefel-Whitney class $sw_1(\rho) \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ under the identification $\mathrm{H}^1(\Sigma; \mathbf{F}_2) \cong \mathrm{Hom}(\mathrm{H}^1(\Sigma; \mathbf{Z}), \mathbf{F}_2) \cong \mathrm{Hom}(\pi_1(\Sigma), \mathbf{F}_2)$. In particular, the first Stiefel-Whitney class is zero if the representation has image in $\mathrm{SO}(n)$.

Lemma 5.1. *Let $\alpha \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$. Then there exists a representation $\Theta_\alpha : \pi_1(\Sigma) \to \mathrm{O}(n)$ such that $sw_1(\Theta_\alpha) = \alpha$.*

We postpone the proof of this lemma and first construct an orthogonal representation. For this let us fix a standard presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g \,|\, \Pi_{i=1}^g [a_i, b_i] = 1 \rangle.$$

Then the homology classes corresponding to the curves a_i, b_i, $i = 1, \ldots, g$ generate the first homology group $\mathrm{H}_1(\Sigma; \mathbf{Z})$.

Let us fix an embedding $l : \mathrm{O}(2) \to \mathrm{O}(n)$ as the left upper block. We denote an element of $l(\mathrm{SO}(2)) \subset \mathrm{O}(n)$ by $e^{i\theta}$ and an element of $l(\mathrm{O}(2) - \mathrm{SO}(2)) \subset \mathrm{O}(n)$ by $Re^{i\theta}$, where R is the element of $O(n) - \mathrm{SO}(n)$ with $R^2 = \mathrm{Id}$. Then $\det(e^{i\theta}) = 1$ and $\det(Re^{i\theta}) = -1$.

For every $k \in \mathbf{Z}$ we define the representation $\Theta^k : \pi_1(\Sigma) \to \mathrm{O}(n)$ by

$$\Theta^k(a_i) = \Theta^k(b_i) = 1 \quad \text{for} \quad 2 \le i \le g$$
$$\Theta^k(a_1) = R \quad \text{and} \quad \Theta^k(b_1) = e^{i\pi k}, \text{ where } k \in \mathbf{Z}.$$

Note that

$$Re^{i\theta}R^{-1} = e^{-i\theta}.$$

Thus we have $[\Theta^k(a_1), \Theta^k(b_1)] = \left[R, e^{i\pi k}\right] = e^{-i2\pi k}$. Therefore

$$\Pi_{i=1}^g [\Theta^k(a_i), \Theta^k(b_i)] = e^{-i2\pi k},$$

and Θ^k is indeed a representation.

Lemma 5.2. *The first Stiefel-Whitney class of Θ^k is the homomorphism $sw_1(\Theta^k) \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ determined by*

$$sw_1(\Theta^k)(a_i) = \det \circ \Theta^k(a_i) = 1 \quad \text{if } 2 \le i \le g$$
$$sw_1(\Theta^k)(b_i) = \det \circ \Theta^k(b_i) = 1 \quad \text{if } 1 \le i \le g$$
$$sw_1(\Theta^k)(a_1) = \det \circ \Theta^k(a_1) = -1.$$

Proof of Lemma 5.1. If $\alpha = 0 \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ we can choose the trivial representation $\Theta_0 : \pi_1(\Sigma) \to \mathrm{O}(n)$ which satisfies $sw_1(\Theta_0) = 0 \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$. Hence we can assume that $\alpha \in \mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}$.

Since the mapping class group acts transitively on $\mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}$, we need to construct only one representation Θ such that $sw_1(\Theta) \in \mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}$. Then we obtain a representation with first Stiefel-Whitney class α by precomposing Θ with a suitable element of the mapping class group.

By Lemma 5.2 the representations Θ^k satisfy $sw_1(\Theta^k) \in \mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}$. □

5.3.2. *The second Stiefel-Whitney class of an orthogonal representation.* Let $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$ be an orthogonal representation. The second Stiefel-Whitney class of Θ can be described as the obstruction to lift the representation ρ to the nontrivial double cover $\mathrm{Pin}(n)$.

For this consider the central extension

$$1 \to \{\pm 1\} \to \mathrm{Pin}(n) \to \mathrm{O}(n) \to 1,$$

where we again identify $\{\pm 1\}$ with $\mathbf{F}_2$. Since this extension is central we have a commutator map

$$\widetilde{[\cdot, \cdot]} : \mathrm{O}(n) \times \mathrm{O}(n) \to \mathrm{Pin}(n).$$

Given a representation $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$ the second Stiefel-Whitney class $sw_2(\Theta) \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$ is given by

$$\Pi_{i=1}^{g} \widetilde{[\Theta(a_i), \Theta(b_i)]} \in \mathbf{F}_2.$$

Lemma 5.3. *Let $\Theta^k : \pi_1(\Sigma) \to \mathrm{O}(n)$ be the representation defined above, then the second Stiefel-Whitney class of Θ^k equals*

$$\begin{array}{rll} 1 & \text{if} & k \text{ is even} \\ -1 & \text{if} & k \text{ is odd.} \end{array}$$

Proof. It is enough to do the calculation when $n = 2$. In this case $\mathrm{Pin}(2) \cong \mathrm{O}(2)$ and the restriction to $\mathrm{SO}(2)$ of the cover $\mathrm{Pin}(2) \to \mathrm{O}(2)$ is $\mathrm{SO}(2) \to \mathrm{SO}(2)$, $x \mapsto x^2$. We can hence lift $\Theta^k(b_1) = e^{i\pi k}$ to $\widetilde{\Theta}^k(b_1) = e^{i\frac{\pi}{2}k}$ and $\Theta^k(a_1) = R$ to $\widetilde{\Theta}^k(a_1) = R$. Then we have

$$\Pi_{i=1}^{g} \widetilde{[\Theta^k(a_i), \Theta^k(b_i)]} = e^{-i\pi k}.$$

□

Lemma 5.4. *Let $\alpha \neq 0 \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ and $\beta \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$ arbitrary. Then there exists a representation $\Theta_{\alpha,\beta} : \pi_1(\Sigma) \to \mathrm{O}(n)$ with $sw_1(\Theta_{\alpha,\beta}) = \alpha$ and $sw_2(\Theta_{\alpha,\beta}) = \beta$.*

Proof. As above, by the equivariance under $\mathcal{M}od(\Sigma)$, the problem reduces to finding representations with a non-trivial first Stiefel-Whitney class and arbitrary second Stiefel-Whitney class. Such representations are given by Θ^k, $k = 1, 2$. □

Lemma 5.5. *Let $n \geq 3$. Let $\beta \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$ be arbitrary. Then there exists a representation $\Theta_{0,\beta} : \pi_1(\Sigma) \to \mathrm{O}(n)$ such that $sw_1(\Theta_{0,\beta}) = 0 \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ and $sw_2(\Theta_{0,\beta}) = \beta \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$.*

Proof. It $\beta = 0 \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$ we can choose $\Theta_{0,0}$ to be the trivial representation. Thus we can assume that $\beta \in \mathrm{H}^2(\Sigma; \mathbf{F}_2) - \{0\}$. It is sufficient to construct a representation into $\mathrm{O}(3)$ with the desired properties. For $n \geq 3$ we can then postcompose this representation with the embedding of $\mathrm{O}(3) \to \mathrm{O}(n)$ as upper left block. Thus we are looking for a representation $\Theta_{0,\beta} : \pi_1(\Sigma) \to \mathrm{SO}(3)$ (*i.e.* $sw_1(\Theta_{0,\beta}) = 0$) which does not lift to $\mathrm{Spin}(3)$. Let us realize $\mathrm{Spin}(3) \cong S^3$ as the quaternions of norm one. The covering map $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ is realized by the action by conjugation on the space of imaginary quaternions.

Let us denote by $\{1, i, j, k\}$ the standard basis of the quaternions. We define a representation $\Theta_{0,\beta} : \pi_1(\Sigma) \to \mathrm{SO}(3)$ by sending a_1 to the projection in $\mathrm{SO}(3)$ of i, b_1 to the projection of j and all other generators of $\pi_1(\Sigma)$ to the trivial element. Since $[\Theta_{0,\beta}(a_1), \Theta_{0,\beta}(b_1)]\tilde{} = [i, j] = -1$ this defines a homomorphism into $\mathrm{SO}(3)$ which does not lift to $\mathrm{Spin}(3)$. □

Corollary 5.6. *Let $n \geq 3$. Let $\alpha \in \mathrm{H}^1(\Sigma; \mathbf{F}_2)$ and $\beta \in \mathrm{H}^2(\Sigma; \mathbf{F}_2)$. Then there exists a representation $\Theta_{\alpha,\beta} : \pi_1(\Sigma) \to \mathrm{O}(n)$ such that $sw_1(\Theta_{\alpha,\beta}) = \alpha$ and $sw_2(\Theta_{\alpha,\beta}) = \beta$.*

Remark 5.7. *Note that for Θ in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{O}(2))$ the equality $sw_1(\Theta) = 0$ automatically implies $sw_2(\Theta) = 0$.*

5.4. **Consequences for maximal representations.** Let us now come back and consider maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ and their first and second Stiefel-Whitney classes $sw_1(\rho) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$, $sw_2(\rho) \in \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.

Proposition 5.8. *Let $n \geq 3$. Let $(\alpha, \beta) \in \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \times \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$ be an admissible pair (see Definition 4.12). Then there exists a twisted diagonal representation*

$$\rho_\Theta : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$$

such that $sw_1(\rho_\Theta) = \alpha$ and $sw_2(\rho_\Theta) = \beta$.

Proof. We have to show that we can realize all 2^{2g} different choices for $sw_1(\rho)$ in the fixed coset of $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$ and the 2 choices for $sw_2(\rho)$ in the image of $\mathrm{H}^2(\Sigma; \mathbf{F}_2)$ in $\mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$.

Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ be the fixed discrete embedding, $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$ an orthogonal representation and $\rho_\Theta : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ the corresponding twisted diagonal representation.

We have the following formulas for the first and second Stiefel-Whitney classes [32, Corollary 5.4.]:

$$sw_1(\rho_\Theta) = sw_1(L^s(\iota) \otimes \overline{W}) = n sw_1(L^s(\iota)) + sw_1(\overline{W}) \tag{3}$$

$$sw_2(\rho_\Theta) = sw_2(L^s(\iota)\otimes \overline{W}) = \frac{n(n-1)}{2} sw_1(L^s(\iota))\cdot sw_1(L^s(\iota)) \\ + (n-1) sw_1(L^s(\iota))\cdot sw_1(\overline{W}) + sw_2(\overline{W}),$$

where $\overline{W}$ is the pull-back to $T^1\Sigma$ of the flat n-plane bundle over Σ associated to Θ. Note that $sw_i(\overline{W})$ is the image of $sw_i(\Theta)$ under the natural map in cohomology.

Corollary 5.6 implies that by choosing different $\mathrm{O}(n)$-representations for Θ we can realize $2^{2g}\times 2$ different choices for $sw_1(\Theta)$ and $sw_2(\Theta)$. Since $nsw_1(\iota)$ is fixed, as we vary $sw_1(\overline{W})$ over the 2^{2g} distinct classes in the image of $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ in $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$ we realize all possible 2^{2g} classes in the $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$-coset in $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$ determined by $nsw_1(\iota)$. Similarly for any $sw_1(\overline{W})$ we can realize the two possibilities for $sw_2(\overline{W})$, so we can realize the two possibilities for sw_2 by an appropriately chosen twisted diagonal maximal representation $\rho_\Theta : \pi_1(\Sigma)\to \mathrm{Sp}(2n,\mathbf{R})$. □

Every representation in the Hitchin component is irreducible [27], therefore twisted diagonal representations are never contained in Hitchin components. Proposition 5.8 implies then that for $n\geq 3$ the space

$$\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))$$

consists of at least 2×2^{2g} connected components, and the first and second Stiefel-Whitney classes of the Anosov bundle associated to the representation distinguish them; this proves Theorem 5. The total number of connected components of $\mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))$, if $n\geq 3$, is 3×2^{2g} [14]. In the 2^{2g} Hitchin components any representation can be deformed into an irreducible Fuchsian representation. The remaining 2×2^{2g} connected components all contain a twisted diagonal representation. This gives Theorem 8 of the introduction. Corollary 9 then follows directly from the computations in Section 5.1 and Section 5.2.

When $n=2$ the above computations imply

Corollary 5.9. *Let $n=2$. Let $(\alpha,\beta)\in \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)\times \mathrm{H}^2(T^1\Sigma;\mathbf{F}_2)$ be admissible. Then there exist a twisted diagonal representation $\rho_\Theta:\pi_1(\Sigma)\to \mathrm{Sp}(4,\mathbf{R})$ with $sw_1(\rho_\Theta)=\alpha$ and $sw_2(\rho_\Theta)=\beta$ if and only if $\alpha\neq 0$ or $(\alpha,\beta)=(0,(g-1) \mod 2)$.*

When $n=2$ twisted diagonal representations ρ_Θ with $sw_1(\rho_\Theta)=0$ will always have Euler class $e_{\gamma,L_{0+}}(\rho_\Theta)=(g-1)[\Sigma]\in \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$.

Remark 5.10. *The class $[\Sigma]\in \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$ is the image of the orientation class in $\mathrm{H}^2(\Sigma;\mathbf{Z})$ under the natural map $\mathrm{H}^2(\Sigma;\mathbf{Z})\to \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$; it is a torsion class of order $2g-2$.*

To realize all admissible pairs $\alpha=0\in \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$ and $\beta\in \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$ we have to consider hybrid representations.

5.5. **Hybrid representations.** The goal of this section is to prove the following

Theorem 5.11. *Let $\gamma \in \pi_1(\Sigma) - \{1\}$ and $L_{0+} \in \mathcal{L}_+$ an oriented Lagrangian subspace of $\mathbf{R}^4$. Then the relative Euler class distinguishes the connected components of*

$$\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})).$$

More precisely, given any l in $\mathbf{Z}/(2g-2)\mathbf{Z} \cong \mathrm{Tor}(\mathrm{H}^2(T^1\Sigma;\mathbf{Z}))$,

- *if $l \neq (g-1)$ the set $e_{\gamma,L_{0+}}^{-1}(\{l\})$ is a connected component of $\mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))$. Every representation in $e_{\gamma,L_{0+}}^{-1}(\{l\})$ can be deformed to a k-hybrid representation, where $k = g-1-l \mod 2g-2$;*
- *the set $e_{\gamma,L_{0+}}^{-1}(\{1-g\})$ is the union of the connected component of $\mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))$ consisting of representations which can be deformed to a diagonal Fuchsian representation in $\mathrm{Sp}(4,\mathbf{R})$ and of the Hitchin components $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))$.*

For simplicity we restrict the discussion to the case of k-hybrid representations which are constructed from the decomposition of the surface $\Sigma = \Sigma_l \cup \Sigma_r$ along a simple closed separating geodesic γ (see Section 3.3.1). The computation in the general case is a generalization of this case.

The first step is to calculate the first Stiefel-Whitney class.

Proposition 5.12. *Let $\rho : \pi(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ be a k-hybrid representation as defined in Section 3.3.1, then*

$$sw_1(\rho) = 0.$$

Proof. By Proposition 4.11 we have $sw_1(\rho) \in \mathrm{H}^1(\Sigma;\mathbf{F}_2) \subset \mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$, thus it is sufficient to show that $sw_1(\rho)$ is zero on a basis of the first homology group of Σ.

Let $\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g \,|\, \Pi_{i=1}^g [a_i, b_i] = 1\rangle$ be a standard presentation, then $a_1, b_1, \ldots, a_g, b_g$ form a basis of $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$. We can choose such a standard presentation of $\pi_1(\Sigma)$ such that for any element h of the generating set, either $h \in \pi_1(\Sigma_l)$ or $h \in \pi_1(\Sigma_r)$, where the hybrid representation ρ was constructed with respect to a decomposition $\Sigma = \Sigma_l \cup \Sigma_r$. Then the sign of $\det(\rho(h)|_{L^s(h)})$, where $L^s(h)$ is the attracting Lagrangian for $\rho(h)$, is positive for every element h in the above generating set. This can be checked independently for the irreducible Fuchsian representation ρ_{irr} and (deformations of) the diagonal Fuchsian representation ρ_Δ. In view of Lemma 4.4 this implies $sw_1(\rho) = 0$. □

Proposition 5.13. *Let $\Sigma = \Sigma_l \cup \Sigma_r$ and $\rho = \rho_l * \rho_r : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ be a hybrid representation as defined in Section 3.3.1.*

Let $L_{0+} = \langle e_1, e_2\rangle \subset \mathbf{R}^4$ be an oriented Lagrangian subspace. Let $\gamma \in \pi_1(\partial\Sigma_l)$ and suppose that L_{0+} is an attracting fixed point for $\rho(\gamma)$.

Then

$$e_{\gamma,L_{0+}}(\rho) = (g-1-\chi(\Sigma_l))[\Sigma] \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z}).$$

Remark 5.14. *If the pair (ρ_l, ρ_r) is negatively adjusted with respect to γ (Definition 3.7), then the Euler number is*

$$e_{\gamma,L_{0+}}(\rho) = (g-1+\chi(\Sigma_l))[\Sigma] \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z}).$$

5.5.1. *Reduction to the group $\widehat{\pi_1(\Sigma)}$.* Our strategy to prove Proposition 5.13 is to change the representation ρ slightly.

We denote by $\widehat{\pi_1(\Sigma)}$ the group $\{\pm 1\} \times \pi_1(\Sigma)$; using the morphism $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ there is an embedding $\widehat{\iota} : \widehat{\pi_1(\Sigma)} \to \mathrm{SL}(2,\mathbf{R})$, with $\widehat{\iota}(-1) = -\mathrm{I}_2$ and $\widehat{\iota}|_{\pi_1(\Sigma)} = \iota$. Observe that

$$\widehat{\iota}(\widehat{\pi_1(\Sigma)})\backslash \mathrm{SL}(2,\mathbf{R}) \cong \iota(\pi_1(\Sigma))\backslash \mathrm{PSL}(2,\mathbf{R}) \cong T^1\Sigma,$$

so that the group $\widehat{\pi_1(\Sigma)}$ is a quotient of the group $\pi_1(T^1\Sigma)$ and hence the notion of Anosov representations and their invariants (see Section 2 and Section 4) are well defined for $\widehat{\pi_1(\Sigma)}$.

Let $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\}$ be the projection onto the first factor; for any representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ we define a representation $\varepsilon \otimes \rho$ of $\widehat{\pi_1(\Sigma)}$ into $\mathrm{Sp}(4,\mathbf{R})$ by setting $\varepsilon \otimes \rho(-1) = -\mathrm{Id}_4$ and $\varepsilon \otimes \rho|_{\pi_1(\Sigma)} = \rho$. The relations between the invariants of ρ and of $\varepsilon \otimes \rho$ are discussed in Appendix A.3. In view of these relations Proposition 5.13 follows from:

Proposition 5.15. *Let ρ, Σ_l, Σ_r, γ and L_{0+} be as in Proposition 5.13.*
Then

$$e_{\gamma,L_{0+}}(\varepsilon \otimes \rho) = -\chi(\Sigma_l)[\Sigma].$$

The rest of this section is devoted to the proof of this proposition.

5.5.2. *Constructing sections.* In order to calculate the Euler class for $\varepsilon \otimes \rho$ we will construct a lift of $e_{\gamma,L_{0+}}(\varepsilon \otimes \rho)$ under the connecting homomorphism $\mathrm{H}^1(T^1\Sigma|_\gamma; \mathbf{Z}) \to \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$ appearing in the Mayer-Vietoris sequence (Appendix B) and then calculate this lift in $\mathrm{H}^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathbf{Z}^2$. Such a lift can de constructed from trivializations of the Lagrangian bundle.

Proposition 5.16. *Let ρ, Σ_l, Σ_r, γ and L_{0+} be as in Proposition 5.13. Let $L^s_+(\varepsilon \otimes \rho)$ be the oriented Lagrangian reduction for the flat $\varepsilon \otimes \rho$-flat bundle over $T^1\Sigma$.*

Then the restrictions of $L^s_+(\varepsilon \otimes \rho)$ to both $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$ are trivial.

This proposition is a consequence of the following three lemmas.

Lemma 5.17. *Let $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ be a maximal representation and let $\phi : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{Sp}(4,\mathbf{R})$ be a homomorphism such that $\rho = \phi \circ \iota$ is maximal.*

Then the oriented Lagrangian bundle $L^s_+(\varepsilon \otimes \rho)$ is trivial.

Proof. First we observe that $\varepsilon\otimes\rho=\phi\circ\widehat{\iota}$ with $\widehat{\iota}=\varepsilon\otimes\iota$. In this situation the map defining the oriented Lagrangian bundle is the equivariant map (recall that $T^1\Sigma=\widehat{\pi_1(\Sigma)}\backslash\mathrm{SL}(2,\mathbf{R})$)

$$\begin{array}{rcl}\mathrm{SL}(2,\mathbf{R}) & \longrightarrow & \mathcal{L}_+\\ g & \longmapsto & \phi(g)\cdot L_{0+},\end{array}$$

where L_{0+} is the attracting Lagrangian fixed by $\rho(\gamma)$.

An equivariant map $\mathrm{SL}(2,\mathbf{R})\to\mathrm{Sp}(4,\mathbf{R})$ that trivialize the corresponding bundle is given simply by

$$g\longmapsto\phi(g).$$

□

Lemma 5.18. *Let ρ_0 and ρ_1 be two homotopic maximal representations. Then the Lagrangian bundles $L^s(\rho_0)$ and $L^s(\rho_1)$ are homotopic, hence they are isomorphic.*

If the first Stiefel-Whitney class $sw_1(\rho_0)=sw_1(\rho_1)$ is zero, then the corresponding oriented Lagrangian bundles are also isomorphic.

Proof. This is a consequence of the fact that the equivariant positive curve depends continuously of the representation (Fact 2.21). □

Lemma 5.19. *Let ρ and ρ' be two maximal representations with zero first Stiefel-Whitney class and such that $\rho|_{\pi_1\Sigma'}=\rho'|_{\pi_1\Sigma'}$ for a subsurface $\Sigma'\subset\Sigma$.*

Then the two bundles $L^s_+(\varepsilon\otimes\rho)$ and $L^s_+(\varepsilon\otimes\rho')$ are homotopic when restricted *to $T^1\Sigma|_{\Sigma'}$.*

Proof. Note that if $L^s(\varepsilon\otimes\rho)$ is homotopic to $L^s(\varepsilon\otimes\rho')$ then $L^s_+(\varepsilon\otimes\rho)$ is homotopic to $L^s_+(\varepsilon\otimes\rho')$ There exists a line bundle D_ε such that $L^s(\varepsilon\otimes\rho)=D_\varepsilon\otimes L^s(\rho)$ and $L^s(\varepsilon\otimes\rho')=D_\varepsilon\otimes L^s(\rho')$. Thus it is sufficient to show that $L^s(\rho)|_{M'}\simeq L^s(\rho')|_{M'}$ with $M'=T^1\Sigma|_{\Sigma'}\subset M=T^1\Sigma$.

Let $\widetilde{\Sigma}'$ be the universal cover of Σ'; it can be realized as a $\pi_1(\Sigma')$-invariant subset of $\widetilde{\Sigma}$. More precisely, under an identification $\widetilde{\Sigma}\cong\mathbb{H}_2$ we can set $\widetilde{\Sigma}'=\mathrm{Conv}(\Lambda_{\pi_1(\Sigma')})$ the convex hull of the limit set $\Lambda_{\pi_1(\Sigma')}$ of $\pi_1(\Sigma')$ in the boundary $\partial\mathbb{H}_2$ of the hyperbolic plane.

The manifold $\overline{M}:=T^1\widetilde{\Sigma}$ is a $\pi_1(\Sigma)$-cover of M and we set $\overline{M}':=T^1\widetilde{\Sigma}|_{\widetilde{\Sigma}'}\subset T^1\widetilde{\Sigma}=\overline{M}$. When we identify the unit tangent bundle $\overline{M}$ with $\partial\pi_1(\Sigma)^{(3+)}$, the set of positively oriented triples of $\partial\pi_1(\Sigma)$, we can identify $\overline{M}'$ with the subset of $\partial\pi_1(\Sigma)^{(3+)}$ whose projection to $\widetilde{\Sigma}$ belongs to $\widetilde{\Sigma}'$.

The bundle $L^s(\rho)$ is constructed via the ρ-equivariant map

$$\begin{array}{c}\partial\pi_1(\Sigma)^{(3+)}\xrightarrow{\ p\ }\partial\pi_1(\Sigma)\xrightarrow{\ \xi\ }\mathcal{L}\\ (t^s,t,t^u)\longmapsto t^s\longmapsto\xi(t^s)\end{array}$$

where ξ is the positive ρ-equivariant curve. Similarly $L^s(\rho')$ is constructed from the positive ρ'-equivariant curve ξ'.

This means that the restriction $L^s(\rho)|_{M'}$ is constructed from the $\rho|_{\pi_1(\Sigma')}$-equivariant map

$$\overline{M}' \xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi} \mathcal{L}.$$

And conversely, any $\rho|_{\pi_1(\Sigma')}$-equivariant continuous map $\partial\pi_1(\Sigma) \to \mathcal{L}$ defines a Lagrangian reduction of the symplectic $\mathbf{R}^4$-bundle over M'. Hence we get a homotopy of the two bundles $L^s(\rho)|_{M'}$ and $L^s(\rho')|_{M'}$ once we have a $\rho|_{\pi_1(\Sigma')}$-equivariant homotopy between the two maps

$$\overline{M}' \xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi} \mathcal{L}$$

$$\overline{M}' \xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi'} \mathcal{L}.$$

For this it is sufficient to construct a $\rho|_{\pi_1(\Sigma')}$-equivariant homotopy between the two positive maps ξ and ξ'. This is the content of the next lemma. □

Lemma 5.20. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation and let Σ' be a subsurface. Then the set*

$$\mathcal{C} = \{\xi : \partial\pi_1(\Sigma) \to \mathcal{L} \mid \xi \ > 0, C^0, \rho|_{\pi_1(\Sigma')}\text{-}equivariant\}$$

is connected.

Proof. To simplify notation we assume that the boundary $\partial\Sigma' = \gamma$ consists of one component. Note first that the restriction to the limit set $\Lambda_{\pi_1(\Sigma')}$ of any curve ξ in $\mathcal{C}$ is completely determined by the action of $\pi_1(\Sigma')$. The complement $\partial\pi_1(\Sigma) - \Lambda_{\pi_1(\Sigma')}$ is a countable union of open intervals, which are transitively exchanged by the action of $\pi_1(\Sigma')$. The interval (t^u_γ, t^s_γ), where t^u_γ and t^s_γ are the fixed points of γ in $\partial\pi_1(\Sigma)$, is one such interval. The map ξ is completely determined by its restriction to this interval.

Conversely given a positive $\rho|_{\langle\gamma\rangle}$-equivariant continuous curve $\beta : (t^u_\gamma, t^s_\gamma) \to \mathcal{L}_+$, one obtains a continuous $\rho|_{\pi_1(\Sigma')}$-equivariant positive curve $\partial\pi_1(\Sigma) - \Lambda_{\pi_1(\Sigma')} \to \mathcal{L}$, which one shows to have a continuous extension $\partial\pi_1(\Sigma) \to \mathcal{L}$.

Thus the map:

$$\begin{aligned} \mathcal{C} &\longrightarrow \{\beta : (t^u_\gamma, t^s_\gamma) \to \mathcal{L}, C^0, > 0, (\rho|_\gamma)\text{-equivariant}\} \\ \xi &\longmapsto \xi|_{(t^u_\gamma, t^s_\gamma)} \end{aligned}$$

is a homeomorphism. By Proposition A.2, this space is connected. □

5.5.3. *Calculating the Euler class.* Proposition 5.16 gives precisely what is needed construct a lift of the Euler class $e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$ under the connecting homomorphism $\mathrm{H}^1(T^1\Sigma|_\gamma; \mathbf{Z}) \to \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$. We have (see Appendix B)

$$\mathrm{H}^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(\pi_1(T^1\Sigma|_\gamma), \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}. \tag{4}$$

The last identification sends $\phi \in \mathrm{Hom}_{\mathbf{Z}}(\pi_1(T^1\Sigma|_\gamma), \mathbf{Z})$ to the pair $(\phi(T^1_x\Sigma), \phi(\gamma))$, where the two circles $T^1_x\Sigma$ and γ are naturally considered as loops in $T^1\Sigma|_\gamma$.

Proposition 5.21. *Let ρ, Σ_l, Σ_r, γ and L_{0+} be as in Proposition 5.13. Let $L^s_+(\varepsilon\otimes\rho)$ be the oriented Lagrangian reduction for the flat $\varepsilon\otimes\rho$-bundle over $T^1\Sigma$. Suppose that g_l and g_r are trivializations of the restrictions of $L^s_+(\varepsilon\otimes\rho)$ to $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$.*

Then $g_l\circ g_r^{-1} : T^1\Sigma|_\gamma\times\mathbf{R}^2\to T^1\Sigma|_\gamma\times\mathbf{R}^2$ is a gauge transformation of the trivial oriented $\mathbf{R}^2$-bundle over $T^1\Sigma|_\gamma$ and defines a map $h : T^1\Sigma|_\gamma\to \mathrm{GL}^+(2,\mathbf{R})$. Let $h_*\in\mathrm{Hom}(\pi_1(T^1\Sigma|_\gamma);\pi_1(\mathrm{GL}^+(2,\mathbf{R}))$ denote the map induced by h on the level of fundamental groups.

(i) *The image of $h_*\in\mathrm{Hom}(\pi_1(T^1\Sigma|_\gamma);\pi_1(\mathrm{GL}^+(2,\mathbf{R}))\cong \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z})$ under the connection homomorphism $\mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z})\to\mathrm{H}^2(T^1\Sigma;\mathbf{Z})$ is the Euler class: $\delta(h_*)=e_{\gamma,L_{0+}}(\varepsilon\otimes\rho)$.*

(ii) *Under the identification $\mathrm{H}^1(T^1\Sigma;\mathbf{Z})\cong\mathbf{Z}\times\mathbf{Z}$ in (4), h_* is equal to $(1,0)$, up to the image of the map $\mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z})\oplus\mathrm{H}^1(T^1\Sigma|_{\Sigma_r};\mathbf{Z})\to \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z})$.*

Remark 5.22. *The ambiguity given by the image of the map $\mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z})\oplus \mathrm{H}^1(T^1\Sigma|_{\Sigma_r};\mathbf{Z})\to\mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z})$ accounts for changing the trivializations g_l and g_r of the restrictions of $L^s_+(\varepsilon\otimes\rho)$ to $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$.*

Proof of Proposition 5.15. With the notation of Proposition 5.21 $\delta(h_*)=e_{\gamma,L_{0+}}(\varepsilon\otimes\rho)$. Since h_* is equal to $(1,0)$ in the identification (4), Proposition B.2 precisely says that $\delta(h_*)=(2g(\Sigma_l)-1)[\Sigma]=-\chi(\Sigma_l)[\Sigma]$. □

Proof of Proposition 5.21. The first statement $\delta(h_*)=e_{\gamma,L_{0+}}(\varepsilon\otimes\rho)$ is classical, it follows from the fact that the Euler class is the obstruction to trivialize the oriented Lagrangian bundle $L^s_+(\varepsilon\otimes\rho)$ over $T^1\Sigma$.

For the second statement, let $T^1_x\Sigma$ and γ be the two generators of the fundamental group of the 2-torus $T^1\Sigma|_\gamma$. We have to show, identifying $\pi_1(\mathrm{GL}^+(2,\mathbf{R}))$ with $\mathbf{Z}$, that

$$h_*(T^1_x\Sigma)=1 \text{ and } h_*(\gamma)=0.$$

In both cases our strategy will be the same: we will write the homotopy class we want to calculate as a product of two homotopy classes, the first depending only on the restriction to Σ_l and the second depending on the restriction to Σ_r. With this we will be able to deform the representations and the fiber bundles independently on Σ_l and on Σ_r without changing the homotopy classes we are considering. By construction of the hybrid representations this means that we can reduce the calculation of the homotopy class to the case when the representations are restrictions of the irreducible Fuchsian representation for Σ_l and the diagonal Fuchsian representation for Σ_r.

We start by proving second equality: $h_*(\gamma)=0$.

We identify γ with $\mathbf{Z}\backslash\mathbf{R}$ so that the restriction of the symplectic bundle to γ is identified with $\mathbf{Z}\backslash(\mathbf{R}\times\mathbf{R}^4)$ where $\mathbf{Z}$ acts diagonally on $\mathbf{R}\times\mathbf{R}^4$, $n\cdot(t,v)=(n+t,(A_\gamma)^n v)$, where $A_\gamma=\rho(\gamma)$ is the diagonal element

$\mathrm{diag}(e^{l_\gamma}, e^{k_\gamma}, e^{-l_\gamma}, e^{-k_\gamma})$. Furthermore, the restriction of the oriented Lagrangian reduction $L^s_+(\varepsilon \otimes \rho)$ to the geodesic γ is flat and hence has the form $\mathbf{Z}\backslash(\mathbf{R} \times L_{0+})$.

A trivialization of the symplectic bundle over γ is given by an equivariant map $H : \mathbf{R} \to \mathrm{Sp}(4, \mathbf{R})$, *i.e.* $H(t + 1) = H(t)(A_\gamma)^{-1}$. Such a trivialization induces a trivialization of the Lagrangian bundle $L^s_+(\varepsilon \otimes \rho)|_\gamma$ if furthermore for all t the element $H(t)$ stabilizes L_{0+}. We now provide a "canonical" trivialization in our situation. For this let $A = \mathrm{diag}(e^l, e^k, e^{-l}, e^{-k})$ be a diagonal element and let us denote by L_A the Lagrangian bundle $\mathbf{Z}\backslash(\mathbf{R}\times L_{0+})$ where $n \cdot (t, v) = (n + t, A^n v)$. Then the continuous map

$$H(A, \cdot) : \mathbf{R} \longrightarrow \mathrm{Sp}(4, \mathbf{R}), \quad t \longmapsto H(A, t) = \mathrm{diag}(e^{-tl}, e^{-tk}, e^{tl}, e^{tk})$$

provides such trivialization of L_A.

The homotopy class $h_*(\gamma)$ we are calculating comes from two trivializations $g_l|_\gamma, g_r|_\gamma : L^s_+(\varepsilon \otimes \rho)|_\gamma \to \gamma \times L_{0+}$, and

$$g_l|_\gamma \circ (g_r)^{-1}|_\gamma(t, v) = (t, h(\gamma(t))v),$$

for $(t, v) \in \gamma \times L_{0+}$.

With the trivialization $H(A_\gamma, \cdot)$ of $L^s_+(\varepsilon \otimes \rho)|_\gamma$ given above, we have

$$g_\star|_\gamma \circ H(A_\gamma, \cdot)^{-1}(t, v) = (t, M_\star(t)v), \text{ for } \star = l, r.$$

This means that in $\pi_1(GL^+(2, \mathbf{R})) = \mathbf{Z}$ we have the equality

$$h_*(\gamma) = [M_l] - [M_r].$$

We now prove that both $[M_l]$ and $[M_r]$ are trivial.

By construction of the hybrid representation $\rho = \rho_l * \rho_r$ in Section 3.3.1 we know that $A_\gamma = \rho_l(\gamma) = \rho_{irr}(\gamma) = \phi_{irr} \circ \iota(\gamma)$, where $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ is a discrete embedding and $\phi_{irr} : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Sp}(4, \mathbf{R})$ is the irreducible representation. The trivialization given by Lemma 5.17 and the formula for $H(\cdot, \cdot)$ imply that M_l is the constant map, thus $[M_l] = 0$.

To compute $[M_r] = 0$ we have to consider $A_\gamma = \rho_r(\gamma) = \rho_{r,1}(\gamma)$, where $\rho_{r,s}$, $s \in [0, 1]$, is a continuous path of maximal representations with $\rho_{r,0} = \rho_\Delta = \phi_\Delta \circ \iota$ and, for all s, $\rho_{r,s}(\gamma)$ is diagonal and where ι is as above and $\phi_\Delta : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Sp}(4, \mathbf{R})$ is the diagonal embedding. Thus the family of changes of trivializations $g_{r,s} \circ H(\rho_{l,s}(\gamma), \cdot)^{-1}$, $s \in [0, 1]$, provide a homotopy from the loop $M_r = M_{r,1}$ to the loop $M_{r,0}$, which is the constant map. Therefore $[M_r] = 0$.

We now turn to the proof of the first equality: $h_*(T^1_x\Sigma) = 1$.

In contrast to the previous calculation, here we do not have to satisfy any equivariance properties, but the "standard" trivializations will not be natural.

We identify $T^1_x\Sigma$ with the group $\mathrm{PSO}(2)$, via an orbital map in $T^1\widetilde{\Sigma} \cong \mathrm{PSL}(2, \mathbf{R})$, and identify it also with the boundary $\partial\pi_1(\Sigma)$, via the projection $T^1\widetilde{\Sigma} \cong \partial\pi_1(\Sigma)^{(3+)} \to \partial\pi_1(\Sigma)$, $(t^s, t, t^u) \mapsto t^s$. We can suppose that under these identifications the attractive fixed point t_γ^s of γ is sent to $[\mathrm{Id}_2]$ in $\mathrm{PSO}(2)$ whereas the repulsive fixed point t_γ^u is sent to $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$.

Since we are working with the representation $\varepsilon \otimes \rho$, the flat $\mathrm{Sp}(4, \mathbf{R})$-bundle over $T_x^1\Sigma \cong \mathrm{PSO}(2)$ is the quotient

$$\mathrm{SO}(2) \times_{\{\pm 1\}} \mathrm{Sp}(4, \mathbf{R}) := \{\pm 1\} \backslash (\mathrm{SO}(2) \times \mathrm{Sp}(4, \mathbf{R}))$$

of the trivial bundle over $\mathrm{SO}(2)$ by the group $\{\pm 1\}$, where the action is given by

$$(-1) \cdot (s, g) = (-s, -g).$$

The oriented Lagrangian reduction $L_+^s(\varepsilon \otimes \rho)|_{\mathrm{PSO}(2)}$ is given by the positive continuous curve associated to ρ:

$$\mathrm{SO}(2) \to \mathrm{PSO}(2) \cong \partial\pi_1(\Sigma) \xrightarrow{\xi_+} \mathcal{L}_+$$

into the space of oriented Lagrangians.

A trivialization of the bundle $\mathrm{SO}(2) \times_{\{\pm 1\}} \mathrm{Sp}(4, \mathbf{R})$ is then a $\{\pm 1\}$-equivariant map

$$g : \mathrm{SO}(2) \longrightarrow \mathrm{Sp}(4, \mathbf{R}).$$

This trivialization induces furthermore a trivialization of the Lagrangian reduction $L_+^s(\varepsilon \otimes \rho)|_{\mathrm{PSO}(2)}$ if for all α in $\mathrm{SO}(2)$

$$\xi_+(\alpha) = g(\alpha) \cdot L_{0+}^s.$$

We observe that

$$\xi_+(\mathrm{Id}_2) = L_{0+}^s \text{ and } \xi_+\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = L_{0+}^u,$$

where $L_{0+}^s = \langle e_1, e_2 \rangle$ and $L_{0+}^u = \langle e_3, e_4 \rangle$ with $e_1, \ldots, e_4$ being the standard symplectic basis of $\mathbf{R}^4$.

Lemma-Definition 5.23. *Let $\eta : \mathrm{SO}(2) \to \mathcal{L}_+$ be a continuous, $\{\pm 1\}$-invariant, positive curve defining a Lagrangian reduction L_η of the bundle $\mathrm{SO}(2) \times_{\{\pm 1\}} \mathrm{Sp}(4, \mathbf{R})$.*

Suppose that

$$\eta(\mathrm{Id}_2) = L_{0+}^s \quad \textit{and} \quad \eta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = L_{0+}^s.$$

Then for all α, $\eta(\alpha)$ and $\phi_\Delta(\alpha) \cdot L_{0+}^s$ are transversal Lagrangians. This means that there exists a unique symmetric 2 by 2 matrix $M(\alpha)$ such that:

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \\ M(\alpha) & \mathrm{Id}_2 \end{pmatrix} \phi_\Delta(\alpha)^{-1} \cdot \eta(\alpha) = L_{0+}^s.$$

Then the map

$$\beta_\eta : SO(2) \to \mathrm{Sp}(4, \mathbf{R}),\ \alpha \mapsto \phi_\Delta(\alpha) \begin{pmatrix} \mathrm{Id}_2 & 0 \\ -M(\alpha) & \mathrm{Id}_2 \end{pmatrix}$$

is a trivialization of L_η.

The only point to prove is that $\eta(\alpha)$ and $\phi_\Delta(\alpha)\cdot L^u_{0+}$ are transversal. This is immediate for $\alpha=\pm\mathrm{Id}_2$ and for $\alpha=\pm\begin{pmatrix}0&1\\-1&0\end{pmatrix}$; for other α it follows from the positivity of the 4-tuple $(L^s_{0+},\eta(\alpha),L^u_{0+},\phi_\Delta(\alpha)\cdot L^u_{0+})$ (or the 4-tuple $(L^u_{0+},\eta(\alpha),L^s_{0+},\phi_\Delta(\alpha)\cdot L^u_{0+})$ depending on which interval α lies) which results from the positivity of the triples $(L^s_{0+},\eta(\alpha),L^u_{0+})$ and $(L^u_{0+},\phi_\Delta(\alpha)\cdot L^u_{0+},L^s_{0+})$.

Going back to the proof of Proposition 5.21, the trivializations β_η enable us to write $h_*(T^1_x\Sigma)$ as the difference:

$$h_*(T^1_x\Sigma)=[N_l]-[N_r],$$

where, for $\star=l,r$, $N_\star$ is defined as the change of trivializations $g_\star|_{T^1_x\Sigma}\circ\beta^{-1}_{\xi_+}$.

Again N_l and N_r are in fact homotopic to the corresponding changes of trivializations we obtain from the representations $\phi_{irr}\circ\iota$ and $\phi_\Delta\circ\iota$ respectively. It is then immediate that N_r is homotopic to the constant map and that N_l is homotopic to the map

$$\begin{aligned}\mathrm{PSO}(2)&\longrightarrow \mathrm{Stab}(L^s_{0+})\\ [\alpha]&\longmapsto \phi_{irr}(\alpha)\phi_\Delta(\alpha)^{-1}\begin{pmatrix}\mathrm{Id}_2&0\\-M(\alpha)&\mathrm{Id}_2\end{pmatrix},\end{aligned}$$

where $M(\alpha)$ is the only 2×2 symmetric matrix such that this product belongs to $\mathrm{Stab}(L^s_{0+})$. A direct calculation gives for $\alpha=\begin{pmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{pmatrix}$

$$\phi_{irr}(\alpha)\phi_\Delta(\alpha)^{-1}\begin{pmatrix}\mathrm{Id}_2&0\\-M(\alpha)&\mathrm{Id}_2\end{pmatrix}=\begin{pmatrix}A(\theta)&*\\0&{}^tA(\theta)^{-1}\end{pmatrix},$$

where

$$A(\theta)=\frac{4}{\cos^2(2\theta)+3}\begin{pmatrix}\cos(2\theta)&-\frac{\sqrt3}{2}\sin(2\theta)\\\frac{\sqrt3}{2}\sin(2\theta)&\cos(2\theta)\end{pmatrix}.$$

Hence a path in $\mathrm{GL}^+(2,\mathbf{R})$ representing N_l is $[0,\pi]\to\mathrm{GL}^+(2,\mathbf{R}),\theta\mapsto A(\theta)$. It follows that $h_*(T^1_x\Sigma)=1$. □

Remark 5.24. *The above computation of the Euler class for hybrid representations hints towards a more general gluing formula for topological invariants of representations for surfaces with boundary. We intend to give a rigorous definition of invariants for surfaces with boundary in a subsequent paper.*

6. Action of the mapping class group

It is known that the action of the mapping class group $\mathcal{M}od(\Sigma)$ on

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))=\mathrm{Hom}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))/\mathrm{Sp}(2n,\mathbf{R})$$

is properly discontinuous [28, 34]. Furthermore this group acts naturally on $\mathrm{H}^i(T^1\Sigma;\mathbf{F}_2)$ and the maps

$$sw_i:\mathrm{Rep}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))\longrightarrow \mathrm{H}^i(T^1\Sigma;\mathbf{F}_2)$$

are equivariant.

Therefore, understanding the action of $\mathcal{M}od(\Sigma)$ on the subspaces of $\mathrm{H}^i(T^1\Sigma;\mathbf{F}_2)$ where the Stiefel-Whitney classes take their values in, allows us to determine the number of connected components of the quotient of $\mathrm{Rep}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))$ by $\mathcal{M}od(\Sigma)$.

Lemma 6.1. *Let $A=\mathbf{Z}$ or $\mathbf{F}_2$. The mapping class group of Σ acts trivially on the image of $\mathrm{H}^2(\Sigma;A)$ in $\mathrm{H}^2(T^1\Sigma;A)$.*

Proof. This is immediate from the fact that $\mathcal{M}od(\Sigma)$ preserves the image of $\mathrm{H}^2(\Sigma;A)$ in $\mathrm{H}^2(T^1\Sigma;A)$ and acts trivially on $\mathrm{H}^2(\Sigma;A)$. □

For $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$ the picture is a bit more complicated.

Proposition 6.2. *The action of the mapping class group on $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)$ preserves the two cosets $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ and $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$.*

(i) *On $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ the action of the mapping class group has two orbits, 0 and $\mathrm{H}^1(\Sigma;\mathbf{F}_2)-0$.*
(ii) *On $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ the action of the mapping class group has two orbits.*

Proof. It is obvious that the mapping class group of Σ preserves the two cosets $\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ and $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$.

On $\mathrm{H}^1(\Sigma;\mathbf{F}_2)\cong\mathbf{F}_2^{2g}$ the mapping class group acts by symplectomorphisms and it is a classical fact that it generates the whole group $\mathrm{Sp}(2n,\mathbf{F}_2)$. This action has two orbits, 0 and $\mathbf{F}_2^{2g}-0$.

The action of the mapping class on the space of spin structures $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ has two orbits [26]. The orbits are distinguished by the Arf-invariant of the quadratic form defined by $v\in\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$.[6] □

Corollary 6.3. *Let $n\geq 3$. The space $\mathrm{Rep}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))/\mathcal{M}od(\Sigma)$ has 6 connected components.*

Proof. Let us first consider the Hitchin components $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))$. This subset is invariant by the action of $\mathcal{M}od(\Sigma)$. The 2^{2g} different Hitchin components are indexed by elements $sw_1^A\in\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$ (see Section 4.5.2). The action of the mapping class group has 2 orbits on $\mathrm{H}^1(T^1\Sigma;\mathbf{F}_2)-\mathrm{H}^1(\Sigma;\mathbf{F}_2)$. Therefore the quotient $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))/\mathcal{M}od(\Sigma)$ has 2 connected components.

The 2×2^{2g} connected components of

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))-\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbf{R}))$$

are indexed by the first and second Stiefel-Whitney classes. The mapping class group acts trivially on the image of $\mathrm{H}^2(\Sigma;\mathbf{F}_2)$ in $\mathrm{H}^2(T^1\Sigma;\mathbf{F}_2)$ and has

[6] This Arf-invariant is the same as the invariant of the corresponding spin structure defined by Atiyah in [1].

two orbits in the coset of $\mathrm{H}^1(\Sigma; \mathbf{F}_2)$ in $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$. This implies that it has 4 orbits in the set of admissible pairs (Definition 4.12), hence the quotient

$$\big(\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))\big)/\mathcal{M}od(\Sigma)$$

has 4 connected components. □

6.1. **The case of Sp$(4, \mathbf{R})$.** To define the Euler class of a representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ with $sw_1(\rho) = 0$ we had to make several choices (see Section 4.3). In particular, we fixed a nontrivial element $\gamma \in \pi_1(\Sigma)$ to define the space $\mathrm{Hom}^{\mathcal{L}_+, \gamma, L_{0+}}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ and the Euler class $e_{\gamma, L_{0+}}$. Denoting $\mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ the quotient of $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ by the action of $\mathrm{Sp}(4, \mathbf{R})$, the equivariance of

$$e_{\gamma, L_{0+}} : \mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \longrightarrow \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$$

only holds for the subgroup $\mathrm{Stab}(\gamma) \subset \mathcal{M}od(\Sigma)$ of the mapping class group stabilizing the homotopy class of γ:

Lemma 6.4. *Let $\gamma \in \pi_1(\Sigma) - \{1\}$ and $L_{0+} \in \mathcal{L}^+$ an oriented Lagrangian. Let $\overline{\psi}$ be an element of $\mathrm{Stab}(\gamma) \subset \mathcal{M}od(\Sigma) \cong \mathrm{Out}(\pi_1(\Sigma))$ fixing the homotopy class of γ and let ψ be an automorphism of $\pi_1(\Sigma)$ representing $\overline{\psi}$ and fixing γ.*

Then ψ acts on $\mathrm{Hom}^{\mathcal{L}_+, \gamma}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ and preserves the subspace $\mathrm{Hom}^{\mathcal{L}_+, \gamma, L_{0+}}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$. Furthermore, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}^{\mathcal{L}_+, \gamma, L_{0+}}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) & \xrightarrow{\psi_*} & \mathrm{Hom}^{\mathcal{L}_+, \gamma, L_{0+}}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \\ \downarrow & & \downarrow \\ \mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) & \xrightarrow{\overline{\psi}_*} & \mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})). \end{array}$$

Consequently, we have the the equalities $e_{\gamma, L_{0+}} \circ \overline{\psi}_* = \overline{\psi}_* \circ e_{\gamma, L_{0+}} = e_{\gamma, L_{0+}}$.

This implies

Proposition 6.5. *Let $\mathcal{C}$ be a connected component of $\mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$. Then $\mathcal{C}$ is sent to itself by the action of the mapping class group of Σ.*

Proof. It is a classical fact that the mapping class group $\mathcal{M}od(\Sigma)$ is generated by Dehn twists along simple closed curves (see for example [25]). As a consequence $\mathcal{M}od(\Sigma)$ is generated by $\{\mathrm{Stab}(\gamma)\}_{\gamma \in \pi_1(\Sigma) - \{1\}}$. With this said it is enough to show the invariance of $\mathcal{C}$ under every $\overline{\psi}$ belonging to some $\mathrm{Stab}(\gamma)$. However we know that $\mathrm{Hom}_{\max, sw_1=0} - \mathrm{Rep}_{\mathrm{Hitchin}}$ is invariant under $\mathcal{M}od(\Sigma)$ and that

$$e_{\gamma, L_{0+}} : \mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \longrightarrow \mathrm{Tor}(\mathrm{H}^2(T^1\Sigma; \mathbf{Z}))$$

is a bijection. The invariance property given in Lemma 6.4 implies the claim. $\square$

Corollary 6.6. *The space* $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))/\mathcal{M}od(\Sigma)$ *has* $2g+2$ *connected components.*

Proof. Let us write $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))$ as a union of subspaces

$$\begin{aligned}&\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})) = \\ &\quad \mathrm{Rep}_{\max, sw_1\neq 0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})) \bigcup \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})) \bigcup \\ &\qquad \big((\mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))\big).\end{aligned}$$

The mapping class group action preserves this composition. Because the mapping class group has two orbits on $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) - \mathrm{H}^1(\Sigma; \mathbf{F}_2)$, the space $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))/\mathcal{M}od(\Sigma)$ has 2 connected components. The mapping class group acts transitively on $\mathrm{H}^1(\Sigma; \mathbf{F}_2) - \{0\}$ and trivially on the image of $\mathrm{H}^2(\Sigma; \mathbf{F}_2) \in \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2)$, thus the space

$$\mathrm{Rep}_{\max, sw_1\neq 0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbf{R}))$$

has 2 connected components. By Proposition 6.5 the mapping class group stabilizes the others components. This gives a total of $2g+2$ connected components. $\square$

7. Holonomy of maximal representations

Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R})$ be a maximal representation and $\xi : \partial\pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbf{R})$ the ρ-equivariant limit curve. Then, for every nontrivial $\gamma \in \pi_1(\Sigma)$, the image $\rho(\gamma)$ fixes the two transverse Lagrangians $\xi(t_\gamma^s)$, $\xi(t_\gamma^u)$, where $t_\gamma^s, t_\gamma^u \in \partial\pi_1(\Sigma)$ are the fixed points of γ. Moreover, $\xi(t_\gamma^s)$ is the unique attracting Lagrangian for $\rho(\gamma)$. This readily implies that $\rho(\gamma)$ is an element of $\mathrm{GL}(\xi(t_{\gamma^+}))$ whose eigenvalues are bigger than 1 (see Corollary 2.11). For representations in the Hitchin components $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma, \mathrm{Sp}(2n,\mathbf{R})) \subset \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R}))$, Corollary 2.11 implies moreover that $\rho(\gamma) \in \mathrm{GL}(\xi(t_\gamma^s))$ is a semi-simple element. The following statement shows that we cannot expect anything similar for maximal representations in general.

Theorem 7.1. *Let* $\mathcal{H}$ *be a connected component of*

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R})) - \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbf{R})),$$

and let γ *be an element in* $\pi_1(\Sigma) - \{1\}$ *corresponding to a simple curve.*

If γ *is separating,* $n = 2$ *and the genus of* Σ *is 2, we require that* $\mathcal{H}$ *is not the connected component determined by* $sw_1 = 0$ *and* $e_\gamma = 0$.

Then there exist

(i) *a representation* $\rho \in \mathcal{H}$ *such that the Jordan decomposition of* $\rho(\gamma)$ *in* $\mathrm{GL}(n,\mathbf{R})$ *has a nontrivial parabolic component.*

(ii) *a representation* $\rho' \in \mathcal{H}$ *such that the Jordan decomposition of* $\rho'(\gamma)$ *in* $\mathrm{GL}(n,\mathbf{R})$ *has a nontrivial elliptic component.*

We first establish some preliminary results towards the proof of the theorem.

Lemma 7.2. *Let Σ be a surface with one boundary component $\gamma = \partial\Sigma$. Let G be a semisimple Lie group and $\rho_0 : \pi_1(\Sigma) \to G$ a representation whose centralizer in G is finite.*

Then the differential of the map

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(\Sigma), G) & \xrightarrow{\tau_\gamma} & G \\ \rho & \longmapsto & \rho(\gamma) \end{array}$$

at the point ρ_0 is surjective.

Proof. One can always find a set $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ freely generating $\pi_1(\Sigma)$ and such that $\gamma = [a_1, b_1] \cdots [a_k, b_k]$. The result is then simply a reformulation of [16, Proposition 3.7.]. □

Lemma 7.3. *Let γ be a simple closed separating curve on a closed surface Σ; denote by Σ_1 and Σ_2 the components of $\Sigma - \gamma$. Let $\mathcal{H}$ be a component of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ such that the conditions of Theorem 7.1 are satisfied.*

Then there exists ρ in $\mathcal{H}$ such that

- *$\rho(\gamma)$ (considered as an element of $\mathrm{GL}(n, \mathbf{R}) < \mathrm{Sp}(2n, \mathbf{R})$) is a multiple of the identity,*
- *the restriction of ρ to $\pi_1(\Sigma_1)$ (resp. $\pi_1(\Sigma_2)$) has finite centralizer.*

Proof. By Theorem 8 and Theorem 10 we only need to prove that there are representations satisfying the conclusions of the lemma in a neighborhood of a model representation (*i.e.* a standard maximal representation or a hybrid representation).

First consider a diagonal Fuchsian representation $\rho_0 = \phi_\Delta \circ \iota$. Using standard Fricke-Klein theory of the Teichmüller space, there exists deformations $\iota_{1,t}, \ldots, \iota_{n,t}$ ($t \in [0, 1]$) such that $\iota_{i,t}(\gamma) = \iota(\gamma)$ and $\iota_{i,0} = \iota$ and, for all $t > 0$, the representation $\rho_t = (\iota_{1,t}, \ldots, \iota_{n,t})$ sends $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ into a Zariski dense subgroup of $\mathrm{SL}(2, \mathbf{R})^n < \mathrm{Sp}(2n, \mathbf{R})$ (this construction was already used in [13, Section 9]). As the centralizer of $\mathrm{SL}(2, \mathbf{R})^n$ inside $\mathrm{Sp}(2n, \mathbf{R})$ is finite, the statement of the lemma follows.

Now consider a twisted diagonal representation $\rho_0 = \iota \otimes \Theta$ which cannot be deformed to a diagonal Fuchsian representation. The analysis in Section 5.3 implies that it is sufficient to consider the case when Θ takes values in $\mathrm{O}(2) < \mathrm{O}(n)$ and has finite image or the case when Θ takes values in $\mathrm{SO}(3)$ and has finite image. In the first case, the representation ρ_0 takes values in $\mathrm{Sp}(4, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})^{n-2} < \mathrm{Sp}(2n, \mathbf{R})$; we write $\rho_0 = (\iota \otimes \Theta, \iota, \ldots, \iota)$. As above, we can find a deformation $\rho_t = (\iota_{1,t} \otimes \Theta, \iota_{2,t}, \ldots, \iota_{n-2,t})$ of ρ_0 such that, fot all t, $\rho_t(\gamma) = \rho(\gamma)$ and, for all $t > 0$, the Zariski closure of $\rho_t(\pi_1(\Sigma_1))$ (resp. $\rho_t(\pi_1(\Sigma_2))$) contains $\phi_\Delta(\mathrm{SL}(2, \mathbf{R})) \times \mathrm{SL}(2, \mathbf{R})^{n-2} <$

$\mathrm{Sp}(4, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})^{n-2} < \mathrm{Sp}(2n, \mathbf{R})$. This already means that the centralizer of $\rho_t(\pi_1(\Sigma_1))$ is contained in $\mathrm{O}(2) < \mathrm{O}(n) < \mathrm{Sp}(2n, \mathbf{R})$; it also implies that the image $\Theta(\pi_1(\Sigma_1))$ is contained in the Zariski closure of $\rho_t(\pi_1(\Sigma_1))$.

Suppose now that the image of $\pi_1(\Sigma_1)$ by Θ is not contained in $\mathrm{SO}(2)$. Then there exists a reflection $R \in \mathrm{O}(2) - \mathrm{SO}(2)$ that belongs to the Zariski closure $\overline{\rho_t(\pi_1(\Sigma_1))}^Z$. Therefore the centralizer of $\overline{\rho_t(\pi_1(\Sigma_1))}^Z$ (which equals the centralizer of $\rho_t(\pi_1(\Sigma_1))$) will be contained in the centralizer of R in $\mathrm{O}(2)$ which is finite.

If the restriction of Θ to $\pi_1(\Sigma_1)$ is contained in $\mathrm{SO}(2)$, we can suppose that this restriction is the trivial representation. In this situation we write ρ_0 as amalgamated representation $\rho_0 = \rho_1 * \rho_2$, where $\rho_i = \rho_0|_{\pi_1(\Sigma_i)}$. Then the restriction of Θ to $\pi_1(\Sigma_2)$ is not contained in $\mathrm{SO}(2)$ (otherwise ρ_0 would be in the same connected component as a diagonal Fuchsian representation). We then deform ρ_0 as an amalgamated representation: $\rho_t^{(1)} * \rho_t^{(2)}$, where $\rho_t^{(1)}$ is a deformation of ρ_1 considered as a diagonal Fuchsian representation (the first case we investigated) and $\rho_t^{(2)}$ is a deformation of the twisted diagonal representation ρ_2. The centralizer of $\pi_1(\Sigma_2)$ is finite by the same argument as above.

The case when Θ is in $\mathrm{SO}(3)$ with finite image can be treated in a very similar fashion and is left to the reader.

Let us now assume that $n = 2$ and and ρ_0 is a hybrid representation in $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$. The definition of a hybrid representation ρ involves the choice of a subsurface Σ' in Σ, for whose fundamental group we choose an irreducible Fuchsian representation. Since we want the holonomy around γ to be a multiple of the identity in $\mathrm{GL}(2, \mathbf{R})$ the curve γ has to be contained in $\Sigma - \Sigma'$ and not homotopic to a boundary component of Σ'. This requires the Euler characteristic $\chi(\Sigma')$ to be different from $3 - 2g$.

The way hybrid representations are constructed in Section 3.3, and with the hypothesis on γ and Σ', one can ensure that $\rho(\gamma)$ is a multiple of the identity in $\mathrm{GL}(2, \mathbf{R}) < \mathrm{Sp}(4, \mathbf{R})$ and that the Zariski closure $\overline{\rho(\pi_1(\Sigma_1))}^Z$ (resp. $\overline{\rho(\pi_1(\Sigma_2))}^Z$) contains $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ or an irreducible $\mathrm{SL}(2, \mathbf{R})$ in $\mathrm{Sp}(4, \mathbf{R})$. Therefore, the hybrid representation ρ satisfies all the desired conclusions. Considering not only hybrid representation constructed from positively adjusted pairs as in Section 3.3.1, but also such constructed from negatively adjusted pairs (Definition 3.7) we get representations with Euler class $g - 1 \pm \chi(\Sigma') \in \mathbf{Z}/(2g-2)\mathbf{Z}$ (see Proposition 5.13 and Remark 5.14). Varying the subsurface Σ', every Euler characteristic $\chi(\Sigma')$ different form $3 - 2g$ can be attained, hence we obtain representations in any connected component of

$$\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) - \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})),$$

except if $g = 2$, when we cannot obtain representations with Euler class $e_\gamma = 0$ by the above construction. □

Proof of Theorem 7.1. Suppose that γ is a separating curve and let ρ_0 in $\mathcal{H}$ be a representation satisfying the conclusions of Lemma 7.3. Let us denote by Σ_1 and Σ_2 the components of $\Sigma - \gamma$. We call τ_1 and τ_2 the two evaluation maps:

$$\begin{array}{rcl} \tau_i : \mathrm{Hom}(\pi_1(\Sigma_i), \mathrm{Sp}(2n, \mathbf{R})) & \longrightarrow & \mathrm{Sp}(2n, \mathbf{R}) \\ \rho & \longmapsto & \rho(\gamma). \end{array}$$

The representations space for $\pi_1(\Sigma)$ is the fiber product of the representations spaces for $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ over $\mathrm{Sp}(2n, \mathbf{R})$:

$$\begin{aligned} &\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) = \\ &\quad \big\{(\rho_1, \rho_2) \in \mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{Sp}(2n, \mathbf{R})) \times \mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{Sp}(2n, \mathbf{R})) \mid \\ &\qquad\qquad \tau_1(\rho_1) = \tau_2(\rho_2)\big\}. \end{aligned}$$

By Lemma 7.2 the map τ_1 (resp. τ_2) is locally surjective in a neighborhood of $\rho_0|_{\pi_1(\Sigma_1)}$ (resp. $\rho_0|_{\pi_1(\Sigma_2)}$). This implies that the map

$$\begin{array}{rcl} \tau : \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) & \longrightarrow & \mathrm{Sp}(2n, \mathbf{R}) \\ \rho & \longmapsto & \rho(\gamma) \end{array}$$

is locally surjective in a neighborhood of ρ_0. As $\tau(\rho_0) = \rho_0(\gamma)$ is a multiple of the identity in $\mathrm{GL}(n, \mathbf{R})$ we obtain representations ρ and ρ' in a neighborhood of ρ_0 having the desired properties.

When γ is not separating, the proof follows a similar strategy, with the arguments being a little easier. We leave this proof to the reader. □

Appendix A. Maximal representations

A.1. **The space of positive curves.** In this section we establish certain connectedness properties of the space of positive curves into the Lagrangian Grassmannian.

We will use the notation from Section 2.2: $\mathbf{R}^{2n}$ is a symplectic vector space, with symplectic basis $(e_i)_{i=1,\dots,e_{2n}}$; $\mathcal{X} \subset \mathcal{L} \times \mathcal{L}$ is the space of pairwise transverse Lagrangian subspaces of V, $L_0^s = \mathrm{Span}(e_i)_{1 \leq i \leq n}$, $L_0^u = \mathrm{Span}(e_i)_{n+1 \leq i \leq 2n}$ are two transverse Lagrangian subspaces of $\mathbf{R}^{2n}$, $P^u, P^s \subset \mathrm{Sp}(2n, \mathbf{R})$ their stabilizer. The unipotent radical of P^s is

$$U^s = \left\{ u^s(M) = \begin{pmatrix} \mathrm{Id}_n & M \\ 0 & \mathrm{Id}_n \end{pmatrix} \mid M \in \mathrm{M}(n, \mathbf{R}),\ {}^tM = M \right\}.$$

A Lagrangian L can be written as $u^s(M) \cdot L_0^u$ for some M if and only if L and L_0^s are transverse, in which case M is uniquely determined by L. The triple of Lagrangians (L_0^s, L, L_0^u) is positive (Definition 2.18) if and only if the symmetric matrix M such that $L = u^s(M) \cdot L_0^u$ is positive definite.

Recall that a curve $\xi : S^1 \to \mathcal{L}$ is said *positive* if it sends every positive triple of S^1 to a positive triple of Lagrangians.

Proposition A.1. *The space $\mathcal{P}$ of* continuous and positive *curves from S^1 to $\mathcal{L}$ is connected. In fact, fixing two points $x^s \neq x^u$ in S^1, the fibers of the map*

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{X} \subset \mathcal{L} \times \mathcal{L} \\ \xi & \longmapsto & (\xi(x^s), \xi(x^u)) \end{array}$$

are contractible.

Proof. Consider the set

$$\mathcal{P}_0 = \{\xi \in \mathcal{P} \mid \xi(x^s) = L_0^s, \xi(x^u) = L_0^u\}.$$

Since $\mathrm{Sp}(2n, \mathbf{R})$ acts transitively on $\mathcal{X}$, it is enough to show that $\mathcal{P}_0$ is contractible. For every ξ in $\mathcal{P}_0$ and every $x \neq x^s$, $\xi(x)$ and $\xi(x^s)$ are transverse, therefore we can regard $\mathcal{P}_0$ as a subset of the space of maps from $S^1 - \{x^s\}$ to $\mathrm{Sym}(n, \mathbf{R})$. It is precisely the set of maps $\tilde{\xi} : S^1 - \{x^s\} \to \mathrm{Sym}(n, \mathbf{R})$ such that

- $\lim_{x \to x^s} u^s(\tilde{\xi}(x)) \cdot L_0^u = L_0^s$
- $\tilde{\xi}$ is continuous
- $x \to u^s(\tilde{\xi}(x)) \cdot L_0^u$ is positive (*i.e.* it sends positive triples to positive triples).
- $\tilde{\xi}(x^u) = 0$.

This set of maps is a convex subset of the space of all maps from $S^1 - \{x^s\}$ into $\mathrm{Sym}(n, \mathbf{R})$ (this follows from the fact that the set of positive definite matrices is convex). The contractibility of $\mathcal{P}_0$ follows. □

Proposition A.2. *Let γ be a nontrivial element of $\pi_1(\Sigma)$. Then the space of pairs*

$$\mathcal{P}^\gamma = \big\{(\rho, \xi) \in \mathrm{Hom}(\langle\gamma\rangle, \mathrm{Sp}(2n, \mathbf{R})) \times \mathcal{P} \mid \xi \textit{ is } \rho\textit{-equivariant}\big\}$$

has two connected components. If t_γ^s and t_γ^u are the fixed points of γ in $\partial\pi_1(\Sigma)$, then the connected components are detected by which element of

$$\pi_0\big(\mathrm{Stab}(\xi(t_\gamma^s)) \cap \mathrm{Stab}(\xi(t_\gamma^u))\big) \cong \mathbf{F}_2$$

contains $\rho(\gamma)$.

Proof. There are at least two connected components since the sign of $\det(\rho(\gamma)|_{\xi(t_\gamma^s)})$ varies continuously.

It is sufficient to understand the connected components of the fibers of the map

$$\begin{array}{cccc} \phi : & \mathcal{P}^\gamma & \longrightarrow & \mathcal{X} \\ & (\rho, \xi) & \longmapsto & (\xi(t_\gamma^s), \xi(x^u)). \end{array}$$

Again it is enough to calculate the components of $\mathcal{P}_0^\gamma := \phi^{-1}(L_0^s, L_0^u)$. The points t_γ^s and t_γ^u divide the circle $\partial\pi_1(\Sigma)$ in two intervals I_{su} and I_{us}: they are chosen so that x belongs to I_{su} (respectively I_{us}) if, and only if, the triple $(t_\gamma^s, x, t_\gamma^u)$ (respectively $(t_\gamma^u, x, t_\gamma^s)$) is positively oriented. These two intervals are homeomorphic to $\mathbf{R}$ and isomorphisms are chosen so that the action of γ is conjugate to $t \mapsto t+1$ on $\mathbf{R}$.

It is not difficult to show that a curve $\xi : \partial\pi_1(\Sigma) \to \mathcal{L}$ such that $\xi(t_\gamma^{s/u}) = L_0^{s/u}$ is positive if, and only if, the following conditions are satisfied:

- for all x in I_{su} the triple $(L_0^s, \xi(x), L_0^u)$ is positive.
- for all x in I_{us} the triple $(L_0^u, \xi(x), L_0^s)$ is positive.
- the restriction of ξ to I_{su} is positive (*i.e.* it sends positive triples to positive triples).
- the restriction of ξ to I_{us} is positive.

Therefore we can consider the two intervals I_{su} and I_{us} separately. Using the parametrization by symmetric matrices as above, it is sufficient to show that the set

$$\mathcal{S} = \big\{(A, \tilde{\xi}) \in \mathrm{GL}(n, \mathbf{R}) \times C^0(\mathbf{R}, \mathrm{Sym}_{>0}(n, \mathbf{R})) \mid \\ \tilde{\xi}(t+1) = A\tilde{\xi}(t)\, {}^t\!A, \forall s < t\ \tilde{\xi}(t) - \tilde{\xi}(s) > 0\big\}$$

has two connected components that are distinguished by the sign of $\det A$. (Note that the ρ-equivariance of ξ guarantees that $\lim_{t\to\infty} u^s(\tilde{\xi}(t)) \cdot L_0^u = L_0^s$ and $\lim_{t\to-\infty} u^s(\tilde{\xi}(t)) \cdot L_0^u = L_0^u$). Taking into account the natural action of $\mathrm{GL}(n, \mathbf{R})$ on $\mathcal{S}$ reduces the question to determining the connected components of the subset $\mathcal{S}_0 := \{(A, \tilde{\xi}) \in \mathcal{S} \mid \tilde{\xi}(0) = \mathrm{Id}_n\}$. The map

$$\begin{array}{ccc} \mathcal{S}_0 & \longrightarrow & \mathrm{GL}(n, \mathbf{R}) \\ (A, \tilde{\xi}) & \longmapsto & A. \end{array}$$

has convex, hence contractible fibers. Its image is

$$\big\{A \in \mathrm{GL}(n, \mathbf{R}) \mid A\,{}^t\!A - \mathrm{Id}_n \in \mathrm{Sym}_{>0}(n, \mathbf{R})\big\}.$$

Using the Cartan decomposition of $\mathrm{GL}(n, \mathbf{R})$, it is easy to show that this set has precisely two connected components given by the sign of $\det A$. □

A.2. **Deforming maximal representations in $\mathbf{SL}(2, \mathbf{R})$.** The following fact follows from classical Fricke-Klein theory for Teichmüller space.

Lemma A.3. *Let $(\gamma_i)_{i=1,\dots,k}$ be a family of pairwise disjoint simple closed geodesics on Σ and denote by $\gamma_i \in \pi_1(\Sigma)$ the corresponding elements of the fundamental group. Let $\iota_0 : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$ a discrete embedding with $\iota_0(\gamma_i) = \begin{pmatrix} e^{\lambda_{i0}} & 0 \\ 0 & e^{-\lambda_{i0}} \end{pmatrix}$. Let $(\lambda_{it})_{t\in[0,1]}$ be continuous paths in $\mathbf{R}$.*

Then there exists a continuous path of discrete embeddings $\iota_t : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$, $t \in [0, 1]$ of ι such that, for any $t \in [0, 1]$, $\iota_t(\gamma_i) = \begin{pmatrix} e^{\lambda_{it}} & 0 \\ 0 & e^{-\lambda_{it}} \end{pmatrix}$.

A.3. **Twisting representations.** In this section we explain the strategy which we used to calculate the topological invariants for maximal representations.

A.3.1. *The group* $\widehat{\pi_1(\Sigma)}$. We fix a discrete embedding of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2,\mathbf{R})$: $\pi_1(\Sigma) < \mathrm{PSL}(2,\mathbf{R})$. Let $\pi : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{PSL}(2,\mathbf{R})$ be the projection.

We set $\widehat{\pi_1(\Sigma)} = \pi^{-1}(\pi_1(\Sigma)) \subset \mathrm{SL}(2,\mathbf{R})$. The group $\widehat{\pi_1(\Sigma)}$ is a two-to-one cover of $\pi_1(\Sigma)$, which is isomorphic to $\{\pm 1\}\times\pi_1(\Sigma)$. The isomorphism can be chosen so that it intertwines π with the second projection $\{\pm 1\} \times \pi_1(\Sigma) \to \pi_1(\Sigma)$. Any choice of such an isomorphism amounts to choosing a lift of $\pi_1(\Sigma) < \mathrm{PSL}(2,\mathbf{R})$ to $\mathrm{SL}(2,\mathbf{R})$; such lifts are in one-to-one correspondence with spin-structures on Σ.

For the rest of this section we fix such an isomorphism $\widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma)$.

A.3.2. *Maximal representation of* $\widehat{\pi_1(\Sigma)}$.

Definition A.4. *A representation* $\widehat{\rho} : \widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ *is said to be* maximal *if the restriction* $\widehat{\rho}|_{\pi_1(\Sigma)}$ *is maximal (see Definition 2.13).*

The set of maximal representation is denoted by $\mathrm{Hom}_{\max}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4,\mathbf{R}))$.

Let $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4,\mathbf{R})$ be a maximal representation and $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\}$ be any representation, then the representation $\varepsilon \cdot \widehat{\rho}$, defined by $\gamma \mapsto \varepsilon(\gamma)\widehat{\rho}(\gamma)$, is also maximal.

If $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbf{R})$ is a maximal representation, then $\widehat{\rho} = \rho \circ pr_2 : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4,\mathbf{R})$ is a maximal representation, where $pr_2 : \widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma) \to \pi_1(\Sigma)$ denotes the projection onto the second factor.

Since $T^1\Sigma \cong \widehat{\pi_1(\Sigma)}\backslash \mathrm{SL}(2,\mathbf{R})$ the notion of *Anosov representations* and *Anosov reductions* (see Section 2.1) can be easily extended to representation of $\widehat{\pi_1(\Sigma)}$. The following lemma is an immediate consequence of Theorem 2.15.

Lemma A.5. *Every maximal representation* $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4,\mathbf{R})$ *is Anosov.*

Similarly to the discussion in Section 4, the Anosov reduction leads to a map

$$\mathrm{Hom}_{\max}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4,\mathbf{R})) \xrightarrow{sw_1} \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2)$$

Fixing a nontorsion element $\widehat{\gamma}$ of $\widehat{\pi_1(\Sigma)}$, we introduce the space

$$\mathrm{Hom}^{\mathcal{L}_+,\hat{\gamma}}_{\max, sw_1=0}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4,\mathbf{R}))$$

of pairs (ρ, L_+) consisting of a maximal representation ρ of $\widehat{\pi_1(\Sigma)}$ whose first Stiefel-Whitney class is zero and an attracting oriented Lagrangian L_+ for $\rho(\widehat{\gamma})$. For those pairs, following the discussion in Section 4.3.2, we define an Euler class

$$\mathrm{Hom}^{\mathcal{L}_+,\hat{\gamma}}_{\max, sw_1=0}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4,\mathbf{R})) \xrightarrow{e_{\widehat{\gamma}}} \mathrm{H}^1(T^1\Sigma; \mathbf{Z})$$

A.3.3. *Relations between the invariants.* In this section we describe the relations between topological invariants of maximal representations of $\pi_1(\Sigma)$ and $\widehat{\pi_1(\Sigma)}$. More precisely:

- If $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation, we compare the invariants of ρ and $\widehat{\rho} = \rho \circ pr_2$.
- If $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation and $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\}$ a homomorphism, we compare the invariants of $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$.

Lemma A.6. *Let ρ be a maximal representation of $\pi_1(\Sigma)$ and let $\widehat{\rho} = \rho \circ pr_2$. Then*

$$sw_1(\rho) = sw_1(\widehat{\rho}).$$

When $sw_1(\rho) = 0$, let γ be a nontrivial element of $\pi_1(\Sigma)$, $\widehat{\gamma}$ one of the two elements of $\widehat{\pi_1(\Sigma)}$ projecting to γ and let L_+ be an attracting oriented Lagrangian for $\rho(\gamma) = \widehat{\rho}(\widehat{\gamma})$. Then

$$e_\gamma(\rho, L_+) = e_{\widehat{\gamma}}(\widehat{\rho}, L_+).$$

Proof. The (oriented) Lagrangian reductions associated to ρ and for $\widehat{\rho}$ are exactly the same, hence their characteristic classes coincide. □

Lemma A.7. *Let $\widehat{\rho}$ be a maximal representation of $\widehat{\pi_1(\Sigma)}$ and $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\}$ a representation. Then the first Stiefel-Whitney class of $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$ coincide:*

$$sw_1(\widehat{\rho}) = sw_1(\varepsilon \cdot \widehat{\rho}).$$

Proof. Let L_1 and L_2 be the two Lagrangians reductions over $T^1\Sigma$ associated to $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$ respectively, then

$$sw_1(\widehat{\rho}) = sw_1(L_1) \quad \text{and} \quad sw_1(\varepsilon \cdot \widehat{\rho}) = sw_1(L_2).$$

Let D_ε be the flat real line bundle (over $T^1\Sigma$) associated to the representation ε. Then the flat bundle associated with $\varepsilon \cdot \widehat{\rho}$ is the tensor product of D_ε and the flat bundle associated with $\widehat{\rho}$, therefore we have

$$L_2 = D_\varepsilon \otimes L_1.$$

In particular, using Equation (3), we get

$$sw_1(\varepsilon \cdot \widehat{\rho}) = sw_1(L_2) = 2sw_1(D_\varepsilon) + sw_1(L_1) = sw_1(\widehat{\rho}).$$

□

Proposition A.8. *Let $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation with $sw_1(\widehat{\rho}) = 0$. Let $\widehat{\gamma} \in \widehat{\pi_1(\Sigma)}$ be a nontorsion element and L_+ an attracting oriented Lagrangian for $\widehat{\rho}(\widehat{\gamma})$. Let $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\}$ be a homomorphism. Then L_+ is an attracting oriented Lagrangian for $(\varepsilon \cdot \widehat{\rho})(\widehat{\gamma})$ and the Euler class (relative to $\widehat{\gamma}$) for the pair $(\widehat{\rho}, L_+)$ is*

$$e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho}) = e_{\widehat{\gamma}}(\widehat{\rho}) \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z}) \text{ if } \varepsilon(-1) = 1,$$

and

$$e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho}) = e_{\widehat{\gamma}}(\widehat{\rho}) + (g-1)[\Sigma] \in \mathrm{H}^2(T^1\Sigma; \mathbf{Z}) \text{ if } \varepsilon(-1) = -1.$$

Proof. Let L_1 and L_2 be the Lagrangian reductions associated to $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$. Denote by L_{1+} and L_{2+} the corresponding oriented Lagrangian bundles determined by the choice of $\widehat{\gamma}$ and L_+. If D_ε is the flat real line bundle over $T^1\Sigma$ associated to the representation ε, we have

$$L_{2+} = D_\varepsilon \otimes L_{1+}.$$

(Because L_1 has even dimension there is a canonical orientation on $D_\varepsilon \otimes L_{1+}$, even if D_ε is not oriented nor necessarily orientable)

Let S_1 and S_2 be the associated S^1-bundles corresponding to L_{1+} and L_{2+} and let S_ε be the flat S^1-bundle associated to the representation $\varepsilon : \widehat{\pi_1(\Sigma)} \to \{\pm 1\} \subset S^1$, then the above equality can be restated as

$$S_2 = S_\varepsilon \times_{S^1} S_1.$$

This implies for the Euler classes:

$$e(S_2) = e(S_\varepsilon) + e(S_1).$$

And since $e(S_2) = e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho})$ and $e(S_1) = e_{\widehat{\gamma}}(\widehat{\rho})$, the proposition will follow from the following lemma. □

Lemma A.9. *Let $\varepsilon : \widehat{\pi_1(\Sigma)} \to S^1$ be a representation and let S_ε be the associated flat S^1-bundle. Then*

$$e(S_\varepsilon) = 0 \text{ if } \varepsilon(-1) = 1,$$

and

$$e(S_\varepsilon) = (g-1)[\Sigma] \text{ if } \varepsilon(-1) = -1.$$

Proof. First we note that $e(S_\varepsilon)$ varies continuously with ε. Hence e only depends on which connected component of $\mathrm{Hom}(\widehat{\pi_1(\Sigma)}, S^1)$ ε belongs to. Since $\mathrm{Hom}(\widehat{\pi_1(\Sigma)}, S^1) = \mathrm{Hom}(\{\pm 1\} \times \pi_1(\Sigma), S^1) = \mathrm{Hom}(\{\pm 1\} \times \mathbf{Z}^{2g}, S^1) \cong \{\pm 1\} \times (S^1)^{2g}$, this space has two connected components distinguished precisely by the value of $\varepsilon(-1)$.

We only need to calculate the Euler class for two specific representations. The first is the trivial representation for which the result is obvious. The second one is the projection $pr_1 : \widehat{\pi_1(\Sigma)} \cong \{\pm 1\} \times \pi_1(\Sigma) \to \{\pm 1\}$ onto the first factor. Consider the (non-flat) S^1-bundle over the surface: $\pi_1(\Sigma) \backslash \mathrm{SL}(2, \mathbf{R}) \to \pi_1(\Sigma) \backslash \mathbb{H}$, its Euler class is given by the Toledo number of the injection $\pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{R})$, which is $(g-1)$. The S^1-bundle S_ε is the pullback of this S^1-bundle by the natural projection $\widehat{\pi_1(\Sigma)} \backslash \mathrm{SL}(2, \mathbf{R}) \cong T^1\Sigma \to \pi_1(\Sigma) \backslash \mathbb{H} \cong \Sigma$. This implies the claim. □

Appendix B. The cohomology of $T^1\Sigma$

In this section we compute the cohomology of the unit tangent bundle $T^1\Sigma$ with $\mathbf{Z}$ and $\mathbf{F}_2$ coefficients and study the connecting homomorphism in the Mayer-Vietoris sequence. The results are used in Section 4.4 and Section 5.5.

Proposition B.1. *Let Σ be a closed, connected, oriented surface of genus $g > 1$. The cohomology groups of $T^1\Sigma$ with $\mathbf{Z}$ coefficients are:*

- $\mathrm{H}^0(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}$,
- $\mathrm{H}^1(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}^{2g}$,
- $\mathrm{H}^2(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}^{2g} \times \mathbf{Z}/(2g-2)\mathbf{Z}$,
- $\mathrm{H}^3(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}$.

The cohomology groups of $T^1\Sigma$ with $\mathbf{F}_2$ coefficients are:

- $\mathrm{H}^0(T^1\Sigma; \mathbf{F}_2) = \mathbf{F}_2$,
- $\mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) = \mathbf{F}_2^{2g+1}$,
- $\mathrm{H}^2(T^1\Sigma; \mathbf{F}_2) = \mathbf{F}_2^{2g+1}$,
- $\mathrm{H}^3(T^1\Sigma; \mathbf{F}_2) = \mathbf{F}_2$.

Proof. We prove only the statement about the cohomology with $\mathbf{Z}$ coefficients, the proof for $\mathbf{F}_2$ coefficients is similar.

The unit tangent bundle $T^1\Sigma \to \Sigma$ is a principal S^1-bundle whose Euler class e is $(2-2g)$ in $\mathbf{Z} \cong \mathrm{H}^2(\Sigma; \mathbf{Z})$. The Gysin exact sequence for this bundle is

$$0 \longrightarrow \mathrm{H}^0(\Sigma; \mathbf{Z}) \longrightarrow \mathrm{H}^0(T^1\Sigma; \mathbf{Z}) \longrightarrow 0 \longrightarrow \mathrm{H}^1(\Sigma; \mathbf{Z})$$
$$\longrightarrow \mathrm{H}^1(T^1\Sigma; \mathbf{Z}) \longrightarrow \mathrm{H}^0(\Sigma; \mathbf{Z}) \xrightarrow{\cup e} \mathrm{H}^2(\Sigma; \mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\Sigma; \mathbf{Z})$$
$$\longrightarrow \mathrm{H}^1(\Sigma; \mathbf{Z}) \longrightarrow 0 \longrightarrow \mathrm{H}^3(T^1\Sigma; \mathbf{Z}) \longrightarrow \mathrm{H}^2(\Sigma; \mathbf{Z}) \longrightarrow 0.$$

The conclusion for H^0 and H^3 follows immediately from this. Since $\mathrm{H}^0(\Sigma; \mathbf{Z}) \xrightarrow{\cup e} \mathrm{H}^2(\Sigma; \mathbf{Z})$ is injective, we get the exact sequences:

$$0 \longrightarrow \mathbf{Z}^{2g} \longrightarrow \mathrm{H}^1(T^1\Sigma; \mathbf{Z}) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbf{Z}/(2g-2)\mathbf{Z} \longrightarrow \mathrm{H}^2(T^1\Sigma; \mathbf{Z}) \longrightarrow \mathbf{Z}^{2g} \longrightarrow 0.$$

From this the result for H^1 and H^2 follows easily. □

The above proof gives a *canonical* isomorphism

$$\mathrm{Tor}(\mathrm{H}^2(T^1\Sigma; \mathbf{Z})) \cong \mathbf{Z}/(2g-2)\mathbf{Z}$$

between the torsion of $\mathrm{H}^2(T^1\Sigma; \mathbf{Z})$ and $\mathbf{Z}/(2g-2)\mathbf{Z}$. In particular, $[\Sigma]$ is the *canonical* generator of $\mathrm{Tor}(\mathrm{H}^2(T^1\Sigma; \mathbf{Z}))$.

Let γ be a simple closed oriented separating curve on the surface Σ, *i.e.* $\Sigma - \gamma$ has two connected components, Σ_l denotes the component on the left of γ and Σ_r the component on the right (this uses the orientations of γ and Σ). This induces a decomposition of the unit tangent bundle: $T^1\Sigma$ is the union of $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$ identified along $T^1\Sigma|_\gamma$. The Mayer-Vietoris sequence for this decomposition reads as

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(T^1\Sigma;\mathbf{Z}) \longrightarrow \mathrm{H}^0(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^0(T^1\Sigma|_{\Sigma_r};\mathbf{Z}) \\ &\longrightarrow \mathrm{H}^0(T^1\Sigma|_\gamma;\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma;\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^1(T^1\Sigma|_{\Sigma_r};\mathbf{Z}) \\ &\longrightarrow \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\Sigma;\mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^2(T^1\Sigma|_{\Sigma_r};\mathbf{Z}) \\ &\qquad\qquad\longrightarrow \mathrm{H}^2(T^1\Sigma|_\gamma;\mathbf{Z}) \longrightarrow \mathrm{H}^3(T^1\Sigma;\mathbf{Z}) \longrightarrow 0. \end{aligned}$$

This sequence can also be used to compute the cohomology of the unit tangent bundle. We concentrate on the connecting morphism

$$\delta : \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\Sigma;\mathbf{Z})$$

and its kernel.

We realize γ as a simple closed geodesic on Σ, it then has a natural lift to the unit tangent bundle $T^1\Sigma$ which we denote again by γ. This lift induces a trivialization $T^1\Sigma|_\gamma \cong S^1 \times \gamma$ and hence isomorphisms

$$\mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}) \cong \mathrm{H}^1(S^1;\mathbf{Z}) \oplus \mathrm{H}^1(\gamma;\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

The first identification is the map in cohomology corresponding to the projections $T^1\Sigma|_\gamma \to S^1$ and $T^1\Sigma|_\gamma \to \gamma$ whereas the second identification involves the orientations on S^1 and γ (the orientation on $S^1 \cong T^1_x\Sigma$ is induced by the orientation on Σ).

Proposition B.2. *Let γ be an oriented closed simple separating geodesic on the surface Σ.*

Then the orientation class $o_\gamma \in \mathrm{H}^1(\gamma;\mathbf{Z}) \cong \mathbf{Z}$ is sent to $[\Sigma]$ by the connecting homomorphism of the Mayer-Vietoris sequence:

$$\mathrm{H}^1(\gamma;\mathbf{Z}) \subset \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}) \xrightarrow{\delta} \mathrm{H}^2(T^1\Sigma;\mathbf{Z}).$$

The kernel of δ is generated by the elements:

$$(1, 1-2g(\Sigma_l)) \text{ and } (-1, 1-2g(\Sigma_r)) \in \mathbf{Z}\times\mathbf{Z} \cong \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}).$$

Proof. The connecting homomorphisms for the decompositions of the surface and the unit tangent bundle fit in a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(\gamma;\mathbf{Z}) & \xrightarrow{\delta} & \mathrm{H}^2(\Sigma;\mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(T^1\Sigma|_\gamma;\mathbf{Z}) & \xrightarrow{\delta} & \mathrm{H}^2(T^1\Sigma;\mathbf{Z}) \end{array}$$

So the first result follows from the equality $\delta(o_\gamma) = o_\Sigma$, where o_Σ is the orientation class in $\mathrm{H}^2(\Sigma;\mathbf{Z})$. This equality is easy to establish. In fact the Mayer-Vietoris sequence for the surface:

$$\mathrm{H}^1(\gamma;\mathbf{Z}) \longrightarrow \mathrm{H}^2(\Sigma;\mathbf{Z}) \longrightarrow \mathrm{H}^2(\Sigma_l;\mathbf{Z}) \oplus \mathrm{H}^2(\Sigma_r;\mathbf{Z})$$

already shows that $\delta : \mathrm{H}^1(\gamma;\mathbf{Z}) \to \mathrm{H}^2(\Sigma;\mathbf{Z})$ is surjective so that $\delta(o_\gamma) = \pm o_\Sigma$. The signs conventions are precisely arranged so that $\delta(o_\gamma) = o_\Sigma$.

Due to the exactness of the Mayer-Vietoris sequence the kernel of δ is the image of

$$\mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^1(T^1\Sigma|_{\Sigma_r};\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}).$$

It is therefore enough to show that the image of

$$\mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}) \cong \mathbf{Z}\oplus\mathbf{Z} \tag{5}$$

is generated by $(1, 1-2g(\Sigma_l))$. (The calculation for Σ_r is similar.) The commutative square

$$\begin{array}{ccc} \mathrm{H}^1(\Sigma_l;\mathbf{Z}) & \longrightarrow & \mathrm{H}^1(\gamma;\mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) & \longrightarrow & \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}) \end{array}$$

implies that the composition $\mathrm{H}^1(\Sigma_l;\mathbf{Z}) \subset \mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \to \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z})$ is zero, because the map $\mathrm{H}^1(\Sigma_l;\mathbf{Z}) \to \mathrm{H}^1(\gamma;\mathbf{Z})$ is zero as γ is a boundary in Σ_l. The restriction $T^1\Sigma|_{\Sigma_l}$ is the trivial bundle $S^1\times\Sigma_l$ so that

$$\mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \cong \mathrm{H}^1(S^1;\mathbf{Z}) \times \mathrm{H}^1(\Sigma_l;\mathbf{Z}).$$

This means that the image of the above map (5) has rank 1 and is the image of

$$\mathbf{Z} \cong \mathrm{H}^1(S^1;\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}) \cong \mathbf{Z}\times\mathbf{Z}.$$

The first component of this last map is the identity $\mathbf{Z} \cong \mathrm{H}^1(S^1;\mathbf{Z}) \to \mathrm{H}^1(S^1;\mathbf{Z}) \cong \mathbf{Z}$ so that the image of (5) is generated by $(1,n)$ for some integer n. To calculate this integer n let us consider the closed surface $\overline{\Sigma}_1 = \Sigma_l \cup_\gamma D^2$ obtained by gluing a disk along γ. The genus of $\overline{\Sigma}_1$ is $g(\overline{\Sigma}_1) = g(\Sigma_l)$ and the two S^1-bundles $T^1\Sigma|_{\Sigma_l}$ and $T^1\overline{\Sigma}_{1|\Sigma_l}$ are isomorphic. From the Mayer-Vietoris sequence for the decomposition of $T^1\overline{\Sigma}_1$ we get

$$\begin{aligned} \mathrm{H}^1(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^1(T^1\overline{\Sigma}_{1|D^2};\mathbf{Z}) \longrightarrow \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\overline{\Sigma}_1;\mathbf{Z}) \\ \longrightarrow \mathrm{H}^2(T^1\Sigma|_{\Sigma_l};\mathbf{Z}) \oplus \mathrm{H}^2(T^1\overline{\Sigma}_{1|D^2};\mathbf{Z}) \longrightarrow \mathrm{H}^2(T^1\Sigma|_{\gamma};\mathbf{Z}). \end{aligned}$$

A small calculation shows that

$$\mathbf{Z} \cong \mathrm{H}^1(T^1\overline{\Sigma}_{1|D^2};\mathbf{Z}) \to \mathrm{H}^1(T^1\Sigma|_{\gamma};\mathbf{Z}) \cong \mathbf{Z}\times\mathbf{Z}$$

sends 1 to $(-1,1)$. The exact sequence reduces to

$$0 \longrightarrow \mathbf{Z}^2/\langle(1,n),(-1,1)\rangle \longrightarrow \mathrm{H}^2(T^1\overline{\Sigma}_1;\mathbf{Z}) \longrightarrow \mathbf{Z}^{2g(\Sigma_l)} \longrightarrow 0.$$

This exact sequence implies that the torsion of $\mathrm{H}^2(T^1\overline{\Sigma}_1;\mathbf{Z})$ is isomorphic to $\mathbf{Z}/(n+1)\mathbf{Z}$. This torsion being isomorphic to $\mathbf{Z}/(2g(\Sigma_l)-2)\mathbf{Z}$ by Proposition B.1 the value of n is $1-2g(\Sigma_l)$ or $2g(\Sigma_l)-3$.

Doubling Σ_l along γ to obtain a closed surface, a similar argument shows that the torsion of the double of Σ_l is isomorphic to $\mathbf{Z}/(2n)\mathbf{Z}$. Thus the only possible value for n is $1-2g(\Sigma_l)$ because the genus of the double of Σ_l is $2g(\Sigma_l)$). □

Lemma B.3. *Let $\{\eta_i\}$ be curves in Σ whose images generate the homology group $\mathrm{H}_1(\Sigma;\mathbf{Z})$; we denote by $f_i : \eta_i \to \Sigma$ the corresponding map. With a slight abuse of notation we write $T^1\Sigma|_{\eta_i}$ for the pulled back circle bundle $f_i^* T^1\Sigma$ and denote again by $f_i : T^1\Sigma|_{\eta_i} \to T^1\Sigma$ the corresponding map.*

Let c be a class in $\mathrm{H}^2(T^1\Sigma; A)$ such that, for all i,

$$c|_{T^1\Sigma|_{\eta_i}} := f_i^*(c) = 0.$$

Then c belongs to $\mathrm{Im}(\mathrm{H}^2(\Sigma; A) \to \mathrm{H}^2(T^1\Sigma; A))$.

Proof. Using the Gysin exact sequence with A coefficients we only need to show that the image a of c in $\mathrm{H}^1(\Sigma; A)$ is zero. As $\{\eta_i\}$ generates the homology it will be the case if $f_i^*(a) = 0$ for all i. Observe that the Gysin sequences for $T^1\Sigma$ and for $T^1\Sigma|_{\eta_i}$ fit in a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(T^1\Sigma; A) & \longrightarrow & \mathrm{H}^1(\Sigma; A) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(T^1\Sigma|_{\eta_i}; A) & \longrightarrow & \mathrm{H}^1(\eta_i; A). \end{array}$$

So the property follows from the hypothesis $f_i^*(c) = 0$. □

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