Algebraic cycles, cobordism and the cohomology of classifying spaces

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Abstract

Let G be an algebraic group. In this thesis we study the classifying space BG and most particularly the Chow ring CH^*BG and the cobordism ring $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$. Previous work of Burt Totaro shows that the former is well-defined, as BG may be approximated by algebraic varieties, and suggests that the two are isomorphic. Totaro also asks for a description of the image of these rings under the cyle map to ordinary cohomology. Very few calculations have been achieved.

We propose to study a large class of groups, namely symmetric groups and certain finite groups of Lie type, and show that their Chow rings and cobordism rings indeed coincide. In passing, we also compute these explicitly. Using the Steenrod algebra, we then give a conjectural prediction for the image of the cycle map modulo a prime p, and check it on the same examples.

There has been little work in the field, and we have had to develop our own tools to explore the properties of these objects. As a consequence, this thesis contains a number of results of independant interest for the reader who wishes to pursue the study of classifying spaces.

Declaration

This dissertation is not substantially the same as any I have submitted for a degree or a diploma or any other qualification at any other university. It is the result of my own work and includes nothing which is the outcome of work done in collaboration.

…and so there are many important theorems whose proof requires nothing more than cleverness and hard work.

Joseph Silverman

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Acknowledgements

When a thing has been said and well, have no scruple. Take it and copy it.
Anatole France (1844 - 1924)

To steal ideas from one person is plagiarism, to steal ideas from many is research.
Anonymous

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Foreword

Le début est autre chose que le commencement. Le début est ce à quoi quelque chose s'accroche, le commencement est ce de quoi quelque chose jaillit. A vrai dire, nous autres hommes, nous ne pouvons jamais commencer par le commencement: cela un Dieu seul le peut. Il nous faut seulement prendre appui sur quelque chose qui soit capable de nous conduire vers le commencement ou de nous l'indiquer.

Martin Heidegger, Gesamtausgabe, Bd. 39, p. 3-4, quoted by Pierre Magnard, in Le livre du néant, Charles de Bovelles, Paris, Vrin, 1983.

Totaro's work on Chow rings

The Chow group $CH^{n-k}X$, where X is an n-dimensional scheme of finite type over a field, is defined to be the free abelian group on k-dimensional reduced subschemes (subvarieties) of X, modulo the so-called linear equivalence relation. The latter is one way of expressing homotopy in algebro-geometric terms, see [16] for details.

If moreover X is smooth, then there are products

$$CH^pX \otimes CH^qX \to CH^{p+q}X$$

which are defined by means of intersection theory, cf *loc cit.* In other words, the classical way in which one can compute cup products in ordinary cohomology by counting intersection numbers is here taken as the very definition of the product.

It is quite remarkable that such an explicit definition actually gives a rather well-behaved theory: a contravariant functor from smooth schemes to graded commutative rings, with transfers for proper morphisms, with the homotopy invariance property that

$$CH^*E = CH^*B$$

when E is a vector bundle over B, and with exact sequences

$$CH^*Y \to CH^{*+c}X \to CH^{*+c}(X - Y) \to 0$$

when Y is a closed subscheme of X of codimension c.

However, the price to pay for concreteness is that Chow rings tend to be very hard to compute.

There are many examples of functors looking more or less like cohomology theories which may be studied via their equivariant counterparts. This being said, what the equivariant version should be is not always the naive definition at hand, but usually involves the *Borel construction.* Here for a minute we let G be a compact Lie group, we denote by EG a universal G-space, and we put $BG = EG/G$, the classifying space of G. The Borel construction, or homotopy orbit space, of a G-space X is

$$X_{hG} = (EG \times X)/G.$$

For example, Quillen ([34]) has used the equivariant cohomology

$$H^*_G X = H^* X_{hG}$$

to very good effect, recovering all results from Smith theory with short, elegant proofs. Also, in K-theory, the Atiyah-Segal theorem ([4]) says that $K^* X_{hG}$ is, for reasonable X, the completion, with respect to some natural topology, of the naive $K^*_G X$ defined by means of the Grothedieck group of equivariant vector bundles – in other words, it is a nicer theory.

These examples hint at the importance of the Borel construction in building nice, equivariant versions of cohomologies. It is not such an artificial process, either: recalling that X/G can be seen as a colimit, we find that X_{hG} is the homotopy colimit of the same functor over the same category. For those with homotopy inclined minds at least, this should eliminate the remaining doubts about homotopy orbit spaces: they are the right objects to consider.

This has been accepted for a number of years now in topology. Nevertheless using the same idea in algebraic geometry is a relatively recent attempt. Here the naive definition to avoid is that obtained by replacing cycles (formal sums of subvarieties) by G-equivariant cycles, where now G is an algebraic group acting on X. Most particularly when the action is not free, this does not give a satisfactory theory.

The Borel construction, on the other hand, complicates matters slightly in that X_{hG} is not a finite CW complex even when X is, and in particular it certainly is not a manifold (working over $\mathbb{C}$), and we are taken outside the scope of algebraic geometry. However, Totaro has shown [47] how to approximate the homotopy orbit space by algebraic varieties, which we explain briefly now. Over any field, one can find a representation V of G such that the action is free outside of a closed subset S of arbitrary large codimension. The idea is to take the "colimit" of the schemes

$$\frac{(V - S) \times X}{G}$$

This can be done literally in Morel-Voevodski's category of models, but it suffices to make the elementary remark that for a fixed i, the group $CH^i((V - S) \times X)/G$ does not depend on the choice of V and S as long as the codimension of S is large enough. Call this $CH^i_G X$, the correct definition of the i-th equivariant Chow group of X. (Note that when the action is free and a geometric quotient X/G exists, then X_{hG} is a vector bundle over X/G and the equivariant Chow ring of X is the ordinary Chow ring of X/G. The point however is that the definition works in general.)

We should point out that in order to get a rigorous and canonical definition, an equivariant Chow ring should be taken as an inverse limit over an appriopriate category, namely that of pairs (V, S) as above with morphisms taken to be equivariant maps

$V - S \to W - T$. For the record there are some subtle (but minor) issues in doing this, see 1.1.5 and [8].

It is reassuring to check what happens over $\mathbb{C}$. In the case when the group is $GL(n, \mathbb{C})$ and X is a point, to start with, one can choose the V's and S's so that $(V - S)$ is a Steifel variety and $(V - S)/GL(n, \mathbb{C})$ is a Grassmannian, that is, a building block in the standard model for $BGL(n, \mathbb{C})$, cf section 1.1.1. For any algebraic group G now, picking an embedding of G into some general linear group gives preferred choices for V and S, namely we can arrange to have $E = \lim V - S$ precisely equal to the infinite Steifel variety, which is contractible, and $E/G = \lim(V - S)/G$ has the homotopy type of BG. (To be precise, these limits are taken in the category of topological spaces, over the evident full subcategory of the one described above.)

Moreover, Totaro's argument can be easily adapted to ordinary cohomology, and we see that $H^i(V - S)/G$ does not depend on V or S, for fixed i. It follows readily that if we take $\lim(V - S)/G$ over *all choices* of representations and not only the particular ones just considered, which we certainly want to do for flexibility, the topological space B thus obtained comes equipped with a map $BG \to B$ which is a homology equivalence. The bottom line is that, if one replaces Chow rings by cohomology and gives the same definitions as Totaro's, one does indeed get the usual cohomology of BG.

The same remark can be made for general X, and as a consequence we see that there is a natural map

$$CH^*_G X \to H^*_G X$$

and in particular

$$CH^* BG \to H^* BG$$

where the left hand side is a notation for $CH^*_G(point)$ (used over any field) and on the right hand side BG is the usual classifying space of G, a topological space.

The equivariant Chow rings thus defined have the same formal properties as usual Chow rings. For example, if G acts on affine space $\mathbb{A}^n$, considering the latter as a vector bundle over a point, the homotopy invariance property says that $CH^*_G \mathbb{A}^n = CH^* BG$.

In fact, the case of a point is more important that might be immediately apparent. For any affine variety X acted on by G admits an equivariant embedding in affine space $\mathbb{A}^n$ for some n, and the usual exact sequence that we have is:

$$CH^*_G X \to CH^{*+c}_G \mathbb{A}^n = CH^{*+c} BG \to CH^{*+c}_G(\mathbb{A}^n - X) \to 0$$

Thus knowledge of $CH^* BG$ will give information on the equivariant Chow ring of any such X.

Our work will focus on $CH^* BG$. But there is more motivation for doing this than is contained in the last remark. We specialise now to varieties over $\mathbb{C}$, and introduce another ring, seemingly quite different.

Among all possible cohomology theories, of particular interest are the so-called complex oriented cohomologies, i.e. those E such that

$$E^* \mathbb{P}^\infty = E^*(pt)[[t]],$$

as this simple condition allows many computations. Oriented theories include among others ordinary cohomology, K-theory, Morava K-theories, and a certain "universal" oriented cohomlogy denoted MU and called complex cobordism. The associated homology is given by:

$$MU_k(X) = \frac{\text{maps } M^k \to X}{\text{bordism}}$$

where M runs over compact, weakly complex manifolds of dimension k. Moreover

$$MU_*(pt) = MU^{-*}(pt) = \mathbb{Z}[x_1, x_2, \cdots]$$

and

$$MU_*(pt) \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \cdots]$$

The universality of MU means that for any oriented cohomology h, there is a natural transformation $MU \to h$.

The explicit definitions of CH^*X and MU^*X when X is a variety (or rather what we know of their Poincaré duals) make it plausible that, as Totaro proved ([46]), there is factorization of the cycle map as follows:

$$CH^*X \to MU^*X \otimes_{MU^*} \mathbb{Z} \to H^*(X, \mathbb{Z})$$

Here the map $MU^* \to \mathbb{Z}$ sends x_i to 0.

There are two remarkable facts to be pointed out now. First, the map

$$CH^*BG \to MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$$

is an isomorphism on all known examples (notice that here, BG not being a finite CW complex, we have to use a completed tensor product). It is striking that a ring arising naturally in the realm of algebraic geometry should coincide with a typical "homotopy object". Also, the ring $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$ is a first approximation to the important but much more complicated ring MU^*BG – the mere computability of $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$ is surprising and is extra motivation for us to investigate these objects, independently of their relation to Chow rings.

Second, the image of this ring in the cohomology seems to be the "simplest possible" subring. We illustrate this modulo a prime p. Assuming p is odd, one has:

$$H^*(B(\mathbb{Z}/p)^n, \mathbb{F}_p) = \mathbb{F}_p[v_1, \cdots, v_n] \otimes \Lambda(u_1, \cdots, u_n)$$

and

$$\begin{aligned} CH^*B(\mathbb{Z}/p)^n \otimes_{\mathbb{Z}} \mathbb{F}_p &= MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p \\ &= \mathbb{F}_p[v_1, \cdots, v_n] \end{aligned}$$

Note that this polynomial part is not the "even" subring of the cohomology ring, but indeed the simplest part of it, in some sense. (In chapter 2, we will justify the simple-minded definition of a mod p Chow ring as above. It is the natural object to consider, and the resulting simplicity of the ring is not a fraud.) Thus, insisting on a more geometric definition of the cycles leads to neater algebraic objects.

The present thesis will be dedicated to the study of CH^*BG and $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$, or their modulo p counterparts, on examples. Let us describe our progress.

Overview of results

Before outlining the contents of each chapter, let us quickly summarize our results.

Let us work over $\mathbb{C}$; let p be a prime and let H^* denote mod p cohomology. The Steenrod algebra $\mathcal{A}_p$ is a central object in this work. We use it, following ideas of Quillen, to make precise the idea of the "niceness" of Chow rings, such as the absence of nilpotent elements. Let $\mathfrak{U}^{ev}$ be the category of unstable modules over $\mathcal{A}_p$ which are concentrated in even degrees; it is a full subcategory of $\mathfrak{U}$, which contains all unstable modules. Typically, a module in $\mathfrak{U}^{ev}$ (resp. $\mathfrak{U}$) arises as $CH^*X \otimes \mathbb{F}_p$ (resp. H^*X) where X is a smooth variety (resp. a space). There is a localization functor $L : \mathfrak{U}^{ev} \to \mathfrak{U}^{ev}$ with respect to the subcategory of nilpotent modules in the sense of Lannes-Schwarz, and similarly for $\mathfrak{U}$ (of course all this will be made precise later!) Call a module M local if $M = L(M)$. Then it is exceptional for H^*BG to be local in $\mathfrak{U}$, but it is very common for $CH^*BG \otimes \mathbb{F}_p$ to be local in $\mathfrak{U}^{ev}$. In practice, this means that $CH^*BG \otimes \mathbb{F}_p$ is completely determined by the elementary abelian subgroups of G and their fusion (and thus has no nilpotent elements). To be precise, we establish that the result (which is trivial for abelian groups) holds true for the symmetric groups and almost all semi-simple Chevalley groups, including Sp_n, $Spin_n$, SL_n when p is prime to n, all exceptional groups when $p > 7$, as well as products of these and certain quotients (such as SO_n^+).

The idea of localization is not only a neat way of expressing the good properties of Chow rings. Let G be a group such that $CH^*BG \otimes \mathbb{F}_p$ is local (there being no counter example of this so far for odd p, and only one for $p = 2$). Then one has

$$CH^*BG \otimes \mathbb{F}_p = L(\tilde{\mathcal{O}}H^*BG)$$

where $\tilde{\mathcal{O}}$ is right adjoint to the forgetful functor $\mathcal{O} : \mathfrak{U}^{ev} \to \mathfrak{U}$. Again, it is highly surprising to find a variety for which the algebraic cycles may be so easily identified in the cohomology ring. In the last chapter we shall say a word on how to apply this to create examples of *projective* varieties with interesting Chow rings.

We also investigate the complex cobordism of Chevalley groups. Some recent techniques from stable homotopy theory may be used to exploit the known structure of the Morava K-theories of these groups and obtain information about MU^*. As it turns out, we are able to compute these rings completely for any Chevalley group (with no semi-simplicity assumption) and any finite field (of char $\neq p$). This gives a lot of examples of groups for which some standard conjectures are verified: the complex cobordism is concentrated in even degrees, is a local module in $\mathfrak{U}^{ev}$, and coincides with the Chow ring whenever this can be checked.

There are many other results to be found in the present work, some intermediate, some having their own interest: in fact we give the reader quite a few ideas and devices to pursue the investigation of Chow rings or cobordism, and this might be the real point. Most of these results are too technical to be explained just now; however let us try and indicate each chapter's content.

- **Chapter 1** contains a number of basic results to be used throughout. It has the twofold purpose of explaining to the reader what tools are at our disposal, and of building new ones. Thus we shall begin with a few explicit computations of CH^*BG when G is a general linear group, a cyclic group, the group of quaternions, and eventually we compute this completely when G is connected and solvable. Then we move on to prove the so-called double coset formula, a very handy equality that is well-known in the cohomology of groups, and which holds for Chow rings as well, though a little care is needed in proving it when working over an arbitrary field. At this point we shall have hopefully answered many of the questions on Chow rings that would spring up in the mind of someone familiar with the computational side of group cohomology (for example groups of p-rank 1 are completely treated).

- **Chapter 2** introduces the Steenrod algebra, and the concept of localization just mentioned, in details. At this point, the symmetric groups, which are at the heart of the very definition of the Steenrod operations, are the natural examples to consider. It is established that they are "local", with a self-contained proof.

  Complex cobordism is also mentioned, and we explain how $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$ seems to be a local object in $\mathfrak{U}^{ev}$ as well, at least whenever we can compute it; however it is not even known in general whether this module is concentrated in even degrees.

- It is in **Chapter 3** that we start to look for more examples of groups such that CH^*BG ($= CH^*BG \otimes_{\mathbb{Z}} \mathbb{F}_p$ from this chapter onwards) and $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$ are local, thus illustrating the importance of the previous chapter. We shall investigate *Chevalley groups*, or groups of $\mathbb{F}_q$-rational points of connected, reductive, split groups schemes – such things as $GL_n(\mathbb{F}_q)$ or $Sp_n(\mathbb{F}_q)$. In chapter 3 we deal with MU, which is easier, and obtain rather complete answers via the Morava K-theories. It might be worth pointing out that an application in group theory is given: an alternative proof to a theorem of Steinberg.

- **Chapter 4** intends to prove that CH^*BG is local for the same Chevalley groups. We cannot prove this in full generality, though we do obtain the result for a large class of groups; some general statements are also given, for example the case when the prime p does not divide the order of the Weyl group is taken care of. This chapter relies on all the others and puts together all the techniques introduced.

- Finally, in **Chapter 5** we take a more modern point of view on the subject and relate Totaro's work and ours to the recent progress of Morel-Voevodsky and Morel-Levine.

Each chapter opens with a more detailed summary of its results.

CHAPTER
ONE

Elementary results

I thought they were kind of trivial. Believe me, nothing is trivial.
The Crow (1994)

In this opening chapter we give a few simple examples of computations with Chow rings of classifying spaces. The aim is for the reader to gain some familiarity with these techniques, and also to establish a few lemmas for later purposes.

Here we point out that this thesis is written with the assumption that the reader knows the material in [47][1]. *However*, we have attempted to write everything in such a way that the explanations given in the introduction and a little good will should be sufficient for understanding all the results and proofs. From the antipodal point of view: even if we start with the very basics of the subject and try to give an exposition that necessitates very few prerequisites, it should not be mistaken for a full treatment à la Bourbaki. For this the reader may turn to the various references given throughout the text.

The chapter is organized as follows. The first section treats the particular cases of GL_n, of a cyclic group, of the group of quaternions, says a word on the Künneth formula, and finishes with remarks on the functoriality of Chow rings. The next section computes the Chow ring of a connected, solvable algebraic group by a direct, geometric method (1.2.4 and 1.2.5). We then turn to the double-coset formula (1.3.4). Finally, all these results are considered again in the context of complex cobordism in the final section.

It might be useful to point out our policy with fields of definition. The subsequent chapters, by nature, have to deal with complex varieties, as it is only in this context that complex cobordism makes sense (see the very last chapter though). Moreover the complex case is often more illuminating, with simpler proofs and topological comparisons to make. Nevertheless, in this first chapter, more precisely in sections 2 and 3, we

[1]the lucky reader who can find the early electronic version of [8] should have a look at the appendices there as well.

sometimes work over an arbitrary field, as we feel the results might be useful for someone to study Chow rings in this general context. We also try to point out when a result is not specific to $\mathbb{C}$, and whenever it is unclear what the field of definition is, it will always mean that it may be chosen arbitrarily.

Finally, a remark on notations. We want to see a Chow ring as a graded module concentrated in even degrees, so we always put $CH^{k/2}X = 0$ when k is an integer, and the degree of $x \in CH^kX$ is $2k$. However, in this situation one finds frequently in the literature the degree k assigned to x, and some ambiguity will be unavoidable when making certain references: this will be of little if any consequence.

§1.1. First computations

1.1.1. Grassmannians. The one example when Totaro's definition of BG is the easiest to understand is certainly $G = GL_n$. For if we take simply

$$V = \mathbf{hom}(\mathbb{A}^m, \mathbb{A}^n)$$

with the action of GL_n induced by the obvious one on $\mathbb{A}^n$, and if S is set to be the closed subset of nonsurjective maps, then both the dimension of V and the codimension of S go to infinity as m increases. Moreover, a point (say, in an algebraically closed field) of $(V - S)/GL_n$ is simply a kernel in $\mathbb{A}^m$. In other words, BGL_n is the variety of Grassmannians.

It follows, over $\mathbb{C}$, that

$$CH^*BGL_n(\mathbb{C}) = H^*(BGL_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1, \cdots, c_n].$$

Indeed, the Grassmannian has an algebraic cell decomposition (see [16]). Since $G \rightsquigarrow CH^*BG$ is functorial as we explain at the end of this section, we have a theory of Chern classes in the Chow ring (which could have been obtained by very different means). It will play a important role later.

1.1.2. Cyclic groups. The next obvious group to look at is probably $\mathbb{Z}/n$. It is possible to make an explicit description of the spaces $(V - S)/G$ and compute their Chow rings directly, but we find it more enlightening to use a couple of theorems of Totaro's. The reader can be assured that the next section will contain some pedestrian calculations.

The first result ([47], theorem 14.1) says that if we embed a group G in GL_n for some n, then CH^*BG is generated, as a module over CH^*BGL_n, by elements of degree less or equal to $2 \cdot \dim GL_n/G$.

Let us try to apply this here, working over the complex numbers, and embedding $\mathbb{Z}/n$ in the obvious way in $\mathbb{C}^*$. We have $BGL_1(\mathbb{C}) = \mathbb{P}^\infty$ and $CH^*BGL_1(\mathbb{C}) = \mathbb{Z}[c_1]$, as observed above. Denote by x the Chern class of the standard character, that is, the image of c_1 in $CH^*B\mathbb{Z}/n$. Then the latter is generated as an abelian group by elements of degree 2 together with the powers of x.

Moreover, another result of Totaro's ([47], theorem 3.3) tells us that $(k-1)!CH^kBG$ is generated by Chern classes; in particular CH^1BG is generated by Chern classes of characters. Since $H^2(BG,\mathbb{Z})$ is isomorphic to the group of characters of G via the first Chern class map, which factors through CH^1BG, we conclude that

$$CH^1BG = H^2(BG,\mathbb{Z}).$$

(This holds for any group G.)

Returning to $\mathbb{Z}/n$, we conclude that $CH^1B\mathbb{Z}/n = \mathbb{Z}/n$ generated by x. Thus $CH^*B\mathbb{Z}/n$ is generated by x as a ring, and

$$CH^*BG = H^{2*}(BG,\mathbb{Z}) = \frac{\mathbb{Z}[x]}{(nx)}$$

In fact CH^*BG has a similar description over any field, see [8] for a nice way of computing this.

1.1.3. The group of quaternions. Let $G = Q_8$ be the quaternion group:

$$G = \{\pm 1, \pm i, \pm j, \pm k\}.$$

This group will be of importance later, so we treat it here. It is also a nice illustration of the methods one can use to compute the Chow ring of a group of small size (or rather, rank).

We have an action of G on $\mathbb{C}^2$ which is free out of $\{0\}$, namely quaternionic multiplication. We can write

$$CH^*_G(\{0\}) \to CH^{*+2}_G\mathbb{C}^2 \to CH^{*+2}_G(\mathbb{C}^2 - \{0\}) \to 0$$

since equivariant Chow rings have the same formal properties as ordinary Chow rings.

For the same reason, since affine space $\mathbb{C}^2$ can be seen as a vector bundle over a point, its equivariant Chow ring is that of a point, i.e. it is CH^*BG. Finally we remark that $CH^*_G(\mathbb{C}^2-0) = CH^*(\mathbb{C}^2-0)/G$ as the action is free on $\mathbb{C}^2-0$; and this is obviously 0 as soon as $* > 2 = \dim(\mathbb{C}^2-0)/G$.

The sequence above thus reduces to

$$CH^*BG \to CH^{*+2}BG \to 0$$

when $* > 0$. In the same way we get a surjective map from $H^*(BG,\mathbb{Z})$ to $H^{*+4}(BG,\mathbb{Z})$ and a commutative diagram between the two. But it is well-known that the cohomology of BG is in fact 4-periodic and it follows by induction that $CH^*BG = H^{2*}(BG,\mathbb{Z})$.

1.1.4. Künneth formula. It is not known in general how to compute the Chow ring of $G\times H$ in terms of the Chow rings of G and H. However in nice cases there is the following result:

Lemma. *Let X and Y be varieties over the complex numbers, and let p be a prime number. We assume that $CH^*(X) \to H^*(X,\mathbb{Z})$ is split injective, and that $CH^*(Y) \to H^*(Y,\mathbb{Z})$ is split injective (resp. that $CH^*(Y)\otimes\mathbb{F}_p \to H^*(Y,\mathbb{Z})\otimes\mathbb{F}_p$ is injective). Then*

$$CH^*(X)\otimes CH^*(Y) \longrightarrow CH^*(X\times Y)$$

(resp.

$$CH^*(X) \otimes CH^*(Y) \otimes \mathbb{F}_p \longrightarrow CH^*(X \times Y) \otimes \mathbb{F}_p)$$

is injective. If moreover X or Y can be partitioned into open subsets of affine spaces, then

$$CH^*(X) \otimes CH^*(Y) \approx CH^*(X \times Y)$$

resp.

$$CH^*(X) \otimes CH^*(Y) \otimes \mathbb{F}_p \approx CH^*(X \times Y) \otimes \mathbb{F}_p$$

In practice this will be sufficient. This is proved in [47], section 6, see also [19] for the minor modification that gives the mod p version. Note that what we call here and elsewhere a "partition into open subsets of affine spaces" for simplicity refers to a *stratification*: that is, a variety X is stratified if it has dimension 0 or inductively, if it possesses a proper subvariety Z which is itself stratified and such that $X - Z$ is an open subset in an affine space.

The hypotheses of the lemma are satisfied by (the finite dimensional approximations to) the classifying space $B\mathbb{Z}/n$. It follows that the Chow ring of a finite abelian group A can be described very simply as the symmetric algebra on CH^1BA, which as we noticed before is the group of characters of A. In particular the results on elementary abelian p-groups mentioned in the introduction are now proved.

1.1.5. Induced maps between Chow rings. In this section we explain how "the Chow rings are functorial", ie, a map $K \to G$ induces a map $CH^*(BG) \to CH^*(BK)$ in a functorial manner. We also decribe explicitly the maps induced by the most common group homomorphisms: inclusion of a subgroup, conjugations.

We shall skip the proof of proposition (2) below: the only difficulties involved in it are related to the definition of the Chow ring as an inverse limit. The only drawback is that the reader will believe at first sight that the definition of the induced map $CH^*BG \to CH^*BK$ which we provide is not the simplest one – he or she will have to take it on faith that it is.

Let $\phi : K \to G$ be a homomorphism between the groups K and G. Also, let (V, S) (resp (W, T)) be an admissible pair[2] for K (resp G). Via ϕ, W becomes a representation of K, and in fact $(V \oplus W, S \times W \cup V \times T)$ is an admissible pair for K.

Consider the natural maps:

$$\begin{array}{ccc} ((V-S)\times(W-T))/K & \rightarrow & (W-T)/G \\ \downarrow & & \\ (V-S)/K & & \end{array}$$

The vertical map p induces clearly an isomorphism on Chow groups in appropriate dimensions, by standard arguments. Calling the horizontal map α, we define

$$\phi^* = (p^*)^{-1} \circ \alpha^* : CH^*(G) \longrightarrow CH^*(K)$$

[2] in the sense that V is a complex K representation, and S is a closed subset of V such that the action of K is free on $V - S$.

and call it the map induced by ϕ.

Tedious verifications yield:

Proposition (1). *The above construction satisfies the following:*

1. *ϕ^* can be extended naturally to all dimensions, and thus becomes a* ring *homomorphism.*

2. *The construction is functorial, and the map to cohomology is a natural transformation of functors.*

In some cases there is a simpler description of these maps:

Proposition (2). *Let K and H be closed subgroups of G, and let (V, S) be an admissible pair for G (hence for all subgroups of G, by restriction). Put $U = V - S$. Let $\sigma \in G$ and put $H^\sigma = \sigma H \sigma^{-1}$. Then the following statements hold:*

1. *The "restriction" map $CH^*(G) \to CH^*(K)$ induced by the inclusion of K in G is also induced by the natural map*
$$U/K \longrightarrow U/G$$

2. *Let $c_{\sigma^{-1}} : H^\sigma \to H$ denote conjugation by σ^{-1}, and let $i : K \cap H^\sigma \to H^\sigma$ be the inclusion. Define a map*
$$f : U/(K \cap H^\sigma) \longrightarrow U/H$$
by $f(\bar{u}) = \overline{\sigma^{-1} \cdot u}$. This is well defined, and $f^* = (c_{\sigma^{-1}} \circ i)^* = i^* \circ c^*_{\sigma^{-1}}$.

Corollary. *Conjugation by $g \in G$ induces the identity on $CH^*(G)$.*

Proof. Let us prove (1). Let p_1 be the first "projection" map $(U \times U)/K \to U/K$, and define $g(\bar{u}) = \overline{(u, u)}$: this is a well defined map $U/K \to (U \times U)/K$, such that $p_1 \circ g = Id$. On Chow groups p_1^* is an isomorphism, and we deduce $p_1^* = (g^*)^{-1}$. By definition the map induced on Chow rings by the inclusion of K is $(p_1^*)^{-1} \circ \alpha^*$ where $\alpha : (U \times U)/K \to U/G$ is the natural map. Therefore this is also $g^* \circ \alpha^* = (\alpha \circ g)^*$. But $\alpha \circ g$ is the map in the proposition, and thus we have (1).

Turning to (2), we let H act on $U \times U$ in the obvious way on the first factor, and via c_σ (conjugation by σ) on the second one. We have now two maps p_1 and p_2 from $(U \times U)/H$ to U/H and U/H^σ, respectively. As above we introduce a map g from U/H to $(U \times U)/H$ (well) defined by $g(\bar{u}) = \overline{(u, \sigma \cdot u)}$, and again $g^* = (p_1^*)^{-1}$. We let $j : U/(K \cap K^\sigma) \to U/K^\sigma$ be the natural map, and we gather the maps j, p_2, g together with f as in the proposition in the following commutative diagram:

$$\begin{array}{ccc} U/(K \cap H^\sigma) & \longrightarrow & U/H^\sigma \\ \downarrow & & \uparrow \\ U/H & \longrightarrow & (U \times U)/H \end{array}$$

Commutativity is readily checked. On Chow groups, this reads $f^* \circ g^* \circ p_2^* = j^*$. But $g^* \circ p_2^* = (p_1^*)^{-1} \circ p_2^* = c_\sigma^*$ by definition, whereas $j^* = i^*$ by (1). This concludes the proof, noting that $(c_\sigma^*)^{-1} = c^*_{\sigma^{-1}}$. $\square$

Remark (Transfers). If K is a closed subgroup of G of finite index, then the map $U/K \to U/G$, as in the first point of the proposition, is proper, so we have a transfer homomorphism $i_* : CH^*BK \to CH^*BG$. Here i denotes the inclusion. By the proposition and the standard properties of Chow rings, we deduce:

$$i_*i^*(x) = [G : K]x$$

for any x in CH^*BG, where $[G : K]$ is the index of K in G. It follows that *when G is finite, the order of G kills CH^*BG.*

§1.2. The Chow ring of a connected, solvable algebraic group

We now return to a general field k and schemes of finite type over it. Most of the time we use the functor of points to define maps between schemes.

1.2.1. The Chow ring of G_a. In this section we prove that $CH^*(BG_a) = \mathbb{Z}$ in dimension 0, where G_a is the "additive" group scheme Spec $\mathbb{Z}[X]$, and obtain a description of BG_a that will be useful in the sequel.

Let A be a k-algebra. One can identify $G_a(A)$ with the set of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

in $GL_2(A)$, and in this way we see G_a as a closed subscheme of GL_2; in particular, G_a acts on $\mathbb{A}^2$. On A-points this is given by $a \cdot (x, y) = (x, ax + y)$. Taking direct sums, one gets a representation V of G_a of dimension $2n$. If S is the intersection of the hyperplanes $x_{2k+1} = 0$, it is readily seen that the action of G_a is free on $EG_a = V - S$.

Put $\mathbb{A}^* =$ Spec $k[x, x^{-1}]$, the punctured affine line. Then put

$$\Omega_i = \mathbb{A}^1 \times \mathbb{A}^1 \times \cdots \times \mathbb{A}^* \times \mathbb{A}^1 \times \cdots \times \mathbb{A}^1 \times \mathbb{A}^1$$

and

$$U_i = \mathbb{A}^1 \times \mathbb{A}^1 \times \cdots \times \mathbb{A}^* \times pt \times \cdots \times \mathbb{A}^1 \times \mathbb{A}^1$$

with the $\mathbb{A}^*$ in the $2i$-th position. We see Ω_i as an open subscheme of EG_a, and U_i as a closed subscheme of Ω_i (on A-points, U_i is the subset of points $(x_1, y_1, \cdots, x_n, y_n)$ such that $y_i = 0$).

We define an isomorphism $\phi_i : \Omega_i \to G_a \times U_i$ by:

$$(x_1, y_1, \cdots x_n, y_n) \mapsto (x_i^{-1}y_i; x_1, y_1 - x_1x_i^{-1}y_i, \cdots, x_i, \cdots, x_n, y_n - x_nx_i^{-1}y_i)$$

This is clearly an inverse to the obvious map $G_a \times U_i \to \Omega_i$. Also, the action of G_a via this isomophism becomes the obvious one.

It follows that the projection $p_i : G_a \times U_i \to U_i$ is a geometric quotient, and then so also is $\pi_i = p_i \circ \phi_i$. It is then clear how one can glue the schemes U_i together as open subschemes of a geometric quotient $BG_a = EG_a/G_a$.

Thus we conclude that *the map $\pi : EG_a \to BG_a$ is an affine bundle*, clearly. Therefore $CH^*(BG_a) = CH^*(EG_a)$, cf [16]. Finally, EG_a is an open subspace of affine space $\mathbb{A}^{2n}$, so its Chow groups vanish in positive dimensions, which is what we wanted to prove.

In the next few sections we try, in simple cases, to relate the Chow groups of a group scheme G to those of a closed, normal subgroup H.

1.2.2. Extensions by G_a. We assume, to start with, that $G/H = G_a$. The spaces EG_a, BG_a will be as in the previous section, whereas EG will be some open subset of some representation space of G, on which G acts freely; and we put $BG = EG/G$, $EH = EG$, and $BH = EH/H$.

We form the fibre square

$$\begin{array}{ccc} X = EG_a \times_{BG_a} (EG \times EG_a)/G & \longrightarrow & EG_a \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ (EG \times EG_a)/G & \longrightarrow & BG_a \end{array}$$

where G acts on EG_a via the projection $G \to G/H$. Note that, because ϕ is an affine bundle map, so also is π.

Consider the map $EG \times EG_a \times G \to BH$ which on points is $(v, w, g) \mapsto \overline{g^{-1} \cdot v}$. This induces clearly a map $EG \times EG_a \times G/H \to BH$; but $EG_a \times G/H = EG_a \times_{BG_a} EG_a$ because the action of $G_a = G/H$ on EG_a is free.

Now, G acts on $EG \times EG_a$, and hence on $EG \times EG_a \times_{BG_a} EG_a$, and the map $EG \times EG_a \times_{BG_a} EG_a \to BH$ just defined is easily seen to be constant on the G-orbits. Therefore, it factors through the quotient, which by the lemma below exists and is X.

Using the projection we finally end up with a map $X \to BH \times EG_a$. There is an obvious map $BH \times EG_a \to X$, and these are clearly inverses to each other.

Let us now have a look at the Chow rings of the spaces we have. For example, $X = BH \times EG_a$ is an open set in $BH \times V$ where V is some vector space, so in a range of dimensions these two spaces have isomorphic Chow rings. But $BH \times V$ is a vector bundle over BH, so finally

$$CH^*(X) = CH^*(BH)$$

in a range of dimensions. This most standard argument applies also to prove

$$CH^*((EG \times EG_a)/G) = CH^*(BG)$$

However X is an affine bundle over $(EG \times EG_a)/G$, so we have $CH^*(X) = CH^*((EG \times EG_a)/G)$, and we conclude $CH^*(BG) = CH^*(BH)$. To sum up, we have proved the following:

Proposition. *Let H be a closed, normal subgroup of the algebraic group G, and assume that $G/H = G_a$. Then $CH^*(BG) = CH^*(BH)$.*

The only thing left to be proved is the following lemma, which was used above with $A = EG \times EG_a$, $B = EG_a$ and $S = BG_a$:

Lemma. *Let A and B be schemes over S . Suppose that we have a principal bundle $A \to A/G$ with group G, and suppose that the map from A to S factors through A/G. Then there is a principal bundle*

$$A \times_S B \to (A/G) \times_S B$$

Proof. This is simply the base extension $(A/G) \times_S B \to A/G$, because $A \times_{A/G} (A/G) \times_S B = A \times_S B$. □

1.2.3. Extensions by split tori. Now we try the case where the quotient G/H is a split torus T^n, ie a product of n copies of the multiplicative group G_m. The result we obtain is the following:

Proposition. *Suppose H is a closed, normal subgroup of G, and suppose $G/H = T^n$. Then there are elements $x_1, \cdots, x_n \in CH^1(BG)$ such that*

$$CH^*(BG)/(x_1, \cdots, x_n) = CH^*(BH)$$

where $(x_1, \cdots, x_n)$ is the ideal generated by these elements.

Proof. We proceed by induction, there being nothing to say if $n = 0$; so assuming the result for n, we prove it for $n + 1$.

The map $BH \to BG$ is a principal fibre bundle map, with group T^{n+1}. This bundle is Zariski-locally trivial, as is any torus bundle (tori are called "special" because of this). If U is open in BG and if the bundle above it is isomorphic to $U \times T^{n+1}$, we use the open immersion $G_m = \mathbb{A}^* \to \mathbb{A}^1$ to see $U \times T^{n+1}$ as an open subset of $U \times T^n \times \mathbb{A}^1$. Also, we see $U \times T^n$ as a closed subset of $U \times T^{n+1}$.

Using the transition functions of $BH \to BG$, we can glue the spaces $U \times T^n$ (resp. $U \times T^{n+1}$, resp. $U \times T^n \times \mathbb{A}^1$) together to obtain a space X (resp. BH, resp. a line bundle L above X). Moreover BH sits in L as the open subset $L - X$ (recalling that the 0-section allows us to identify X with a closed subset of any vector bundle above it).

The exact sequence for Chow groups reads:

$$CH^i(X) \longrightarrow CH^{i+1}(L) \longrightarrow CH^{i+1}(BH) \longrightarrow 0$$

Now $CH^*(L) = CH^*(X)$, and it is a well-known fact that the map $CH^i(X) \to CH^{i+1}(X)$ thus obtained is multiplication by the (first) Chern class of L, which we call x_{n+1}. Thus $CH^*(BH) = CH^*(X)/(x_{n+1})$.

Now, X is a principal bundle over BG with group T^n, so using the induction hypothesis (which is really a result on T^n-bundles), we deduce

$$CH^*(X) = CH^*(BG)/(x_1, \cdots, x_n).$$

Result follows. □

From now on, as we turn to applications, k is algebraically closed. Thus, tori are split tori, and solvable groups are split solvable, etc...

1.2.4. Connected, unipotent groups. We recall that an algebraic group G is called unipotent if all its elements are unipotent, ie if $G = G_u$. By theorem 4.8 in [6], unipotent groups are nilpotent (ie, can be deduced from the trivial group $\{1\}$ by a finite number of central extensions). However, there are groups (like tori) which are nilpotent without being unipotent. Note that nilpotent groups are solvable.

It is proved in theorem 10.6 in *loc. cit.* that a connected, unipotent group has a chain of closed, connected, normal subgroups such that the successive quotients have dimension 1. The only connected groups of dimension 1 are G_a and G_m (theorem 10.9 in *loc. cit.*), and G_m is certainly not unipotent, being semi-simple. So we have a chain

$$G_0 = \{1\} \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

with $n = \dim G$, such that $G_{i+1}/G_i = G_a$. It follows therefore from (1.2.2, proposition) that:

Proposition. *If G is a connected, unipotent group, then $CH^*(BG) = \mathbb{Z}$ in dimension 0.*

1.2.5. Connected, solvable groups. We can now finish off the computation of the Chow ring of a connected and solvable group:

Proposition. *Let G be a connected, solvable group, and let T^n be a maximal torus. Then the inclusion of T^n in G induces an isomorphism*

$$CH^*(BG) = CH^*(BT^n)$$

so that $CH^*(BG)$ is a polynomial ring in n variables.

Proof. By theorem 10.6 in [6], we have a *split* exact sequence

$$0 \longrightarrow G_u \longrightarrow G \longrightarrow T^n \longrightarrow 0$$

Note that in this proof T^n will denote the quotient G/G_u and not a maximal torus in G. We will prove that the map $G \to G/G_u$ induces an isomorphism on Chow rings – clearly this comes down to what is claimed in the proposition, because the subgroup of G which splits the sequence is a maximal torus.

We already know that the map $CH^*(BT^n) \to CH^*(BG)$ is split injective. The subgroup G_u is connected and unipotent, so that by (1.2.4, proposition) above, $CH^*(BG_u) = \mathbb{Z}$. Using now (1.2.3, proposition), we know that there are elements $x_1, \cdots, x_n$ in $CH^1(BG)$ generating the ideal $I = \ker CH^*(BG) \to CH^*(BG_u)$, so that $CH^*(BG)/I = \mathbb{Z}$. Writing

$$CH^*(BT^n) = \mathbb{Z}[t_1, \cdots, t_n]$$

and likewise for the image of this ring in $CH^*(BG)$, we see that the elements $t_1, \cdots, t_n$ are in I. It will suffice to show that these elements generate I.

But now, $CH^1(BG)$ is the group of characters of G, and the subgroup generated by the t_i's, being the image of $CH^1(BG/G_u) \to CH^1(BG)$, is the subgroup of characters whose restriction to G_u is trivial. And in turn, $CH^1(BG_u) = 0$ as already observed, so that *all* characters of G_u are trivial, and the t_i's generate $CH^1(BG)$ as an abelian group. Because the x_i's are in $CH^1(BG)$, this concludes the proof. □

Example. The Chow ring of the group $\mathbb{T}_n$ of upper triangular matrices is a polynomial ring on n variables, as is that of $\mathbb{D}_n$, the group of diagonal matrices (which is a maximal torus in $\mathbb{T}_n$). The Chow ring of $\mathbb{U}_n$, the group of upper triangular matrices with 1's on the diagonal, is $\mathbb{Z}$ in dimension 0 – in fact $\mathbb{U}_n = (\mathbb{T}_n)_u$.

Remark. If G is solvable but not connected, then we can still use G°, the connected component of 1, and conclude that $CH^*(BG) \otimes \mathbb{Q}$ injects in a polynomial ring. It is in fact well-known ([13]) that $CH^*(BG) \otimes \mathbb{Q} = H^*(BG, \mathbb{Q})$ and that this is a polynomial ring on r generators where r is the rank of G, ie the dimension of a maximal torus . This says nothing about finite groups.

§1.3. The double coset formula for Chow rings

We prove an analog of the "double coset formula", a well-known result about the cohomology of finite groups, in the new context of algebraic groups and Chow rings. We work over an arbitrary field k. If one focuses on the complex numbers, which we do not, a neat simplification can be reached: there is a more or less obvious map of varieties that is a bijection and realizes the identification in 1.3.3, see below, which is the crux of the matter; but over $\mathbb{C}$, this is automatically an isomorphism as soon as we know that the varieties are normal, which they are as étale schemes over BK. The point of this section is to demonstrate this for any field.

1.3.1. Notations. We consider two closed subgroups K and H of G, and we assume that H has finite index in G. Our goal is to find a description of $BK \times_{BG} BH$ (using of course some finite dimensional approximation of BG etc), which is achieved in proposition 1.3.3 below.

Let us fix some notation. We choose representatives σ_i for the double cosets $K\sigma_i H$ (a finite number will do). We put $L_i = K \cap \sigma_i H \sigma_i^{-1}$. Next we choose representatives k_{ij} for the left cosets of L_i in K (which again are finite in number), and we put $K_{ij} = k_{ij} L_i$, a subset of K (we let $k_{i0} = 1$ so that $K_{i0} = L_i$).

We let (V, S) be an admissible pair for G (hence for all its closed subgroups), and we let $U = V - S$. We make the following assumption: for any closed subgroup Γ of G, the map $U \to U/\Gamma$ is a *universal quotient* in the sense of Mumford [30]: there is no loss of generality here, as there are sufficiently many such U's, see [47]. We shall frequently denote U/Γ by $B\Gamma$. Note that the two natural maps $G \times U \to U$ induce an isomorphism $G \times U = U \times_{BG} U$, because the action is free.

Define $\Omega_i = K\sigma_i H \times U$ and $\Omega_{ij} = K_{ij}\sigma_i H \times U$. Since H is open in G, Ω_i and Ω_{ij} are open in $G \times U$. In fact, one checks readily that, for i fixed and variable j, the Ω_{ij}'s are disjoint open sets whose union is Ω_i. Similarly, the disjoint union of the Ω_i's is $G \times U$.

We let $G \times G$ and its subgroups act on $G \times U$ on the left by

$$(\sigma, \tau) \cdot (g, u) = (\sigma g \tau^{-1}, \tau \cdot u)$$

1.3.2. Let us start by describing a few quotient spaces.

Lemma. *The map* $G \times U \longrightarrow BL_i$ *defined by* $(g,u) \mapsto \overline{g \cdot u}$ *is a geometric quotient for the action of* $L_i \times G$. *That is,*

$$(G \times U)/(L_i \times G) = BL_i$$

Proof. Consider the universal quotient map $U \to BG$ and the base extension $BL_i \to BG$; it follows that we have a quotient map

$$U \times_{BG} BL_i \to BL_i$$

(for the action of G). Considering the latter as a base extension, and using that $U \to BL_i$ is a universal quotient, we obtain a quotient map

$$(U \times_{BG} BL_i) \times_{BL_i} U \to U \times_{BG} BL_i$$

(for the action of L_i). But

$$(U \times_{BG} BL_i) \times_{BL_i} U = U \times_{BG} (BL_i \times_{BL_i} U) = U \times_{BG} U = G \times U$$

It is readily checked that the action of $L_i \times G$ on $G \times U$, as defined in the introduction, induces via the last identifications just made the actions on $U \times_{BG} BL_i \times_{BL_i} U$ and $U \times_{BG} BL_i$ obtained by base extensions. Also, the map $G \times U \to BL_i$ thus created is clearly that given in the lemma. It being a composition of two quotient maps, it follows that this morphism is a quotient map, as claimed. □

Lemma. *The map* $L_i\sigma_i H \times U \to BL_i$ *obtained by restricting the previous one is a geometric quotient for the action of* $L_i \times H$. *That is,*

$$L_i\sigma_i H \times U/(L_i \times H) = BL_i$$

Proof. For $(g,u) \in L_i\sigma_i H \times U$ the following is easy to prove:

$$(L_i \times H) \cdot (g,u) = (L_i \times G) \cdot (g,u) \cap L_i\sigma_i H \times U$$

That is, the orbit of (g,u) under $L_i \times H$ is the trace on $L_i\sigma_i H \times U$ of its orbit under $L_i \times G$. It follows that the fibres of the map under consideration are exactly the orbits of $L_i \times H$, and indeed that this map is a quotient map, because the "bigger one" is. □

We construct now a map $\Omega_i \to BL_i$. On Ω_{i0} we take this to be the map of the previous lemma (recall the notations of the introduction). On Ω_{ij}, we use left translation by k_{ij}^{-1} to obtain an isomorphism $\Omega_{ij} \to \Omega_{i0}$, and again we compose with the map of the lemma.

Lemma. *The map* $\Omega_i \to BL_i$ *is a quotient for the action of* $K \times H$. *That is,*

$$K\sigma_i H \times U/(K \times H) = BL_i$$

Proof. Again this is easy from the last lemma, noting that if $(g,u) \in \Omega_{i0}$, then

$$(K \times H) \cdot (g,u) = \coprod_j k_{ij} \cdot ((L_i \times H) \cdot (g,u))$$

Result follows. □

Lemma. *The map $G \times U \to \coprod BL_i$, obtained by putting together the above maps, is a quotient map for the action of $K \times H$.*

This is clear.

We are now able to describe the space $BK \times_{BG} BH$ in terms of the BL_i's:

1.3.3 Proposition. *There is a natural identification*

$$BK \times_{BG} BH = \coprod_i BL_i$$

Proof. Let $X = BK \times_{BG} BH$, and write simply:

$$\begin{aligned} X \times_{BK} U \times_{BH} U &= BK \times_{BG} BH \times_{BK} U \times_{BH} U \\ &= BH \times_{BG} (BK \times_{BK} U) \times_{BH} U \\ &= U \times_{BG} (BH \times_{BH} U) \\ &= U \times_{BG} U \\ &= G \times U \end{aligned}$$

The induced map $G \times U \to X$ is a quotient map, because it is obtained by two successive base extensions of universal quotients, namely $U \to BK$ and $U \to BH$. The action of $K \times H$ that we get is readily seen to be the one given in the introduction. Since we have already described this quotient in 1.3.2, it follows that there exists an isomorphism (unique in the obvious sense) between X and $\coprod BL_i$, which is what we were after. □

1.3.4. Double coset formula. Suppose that H is a closed subgroup of G. Then the restriction map $CH^*(BG) \to CH^*(BH)$ induced by inclusion will be denoted by $i^*_{H\to G}$. If moreover H has finite index in G, then the "push-forward" homomorphism $CH^*(BH) \to CH^*(BG)$ will be written $i_*^{H\to G}$.

The geometric information obtained in the previous section is all we needed to prove:

Theorem. *Let K and H be closed subgroups of G, H having finite index. Then one has:*

$$i^*_{K\to G} \circ i_*^{H\to G} = \sum_i i_*^{L_i\to K} \circ i^*_{L_i\to H^{\sigma_i}} \circ c^*_{\sigma_i^{-1}}$$

Proof. Consider the fibre square:

$$\begin{array}{ccc} \coprod BL_i & \longrightarrow & BH \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BG \end{array}$$

It is clear that the map $\coprod_i BL_i \to BH$ is $x \mapsto (\cdots, i^*_{H^{\sigma_i} \to L_i} \circ c_{\sigma_i^{-1}}(x), \cdots)$ on Chow rings, from proposition 1.1.5. Also the map $\coprod_i BL_i \to BK$ pushes forward as $(\cdots, x_i, \cdots) \mapsto \sum_i i_*^{L_i \to K}(x_i)$, as is easily seen.

Now, the map $BK \to BG$ is flat, and $BH \to BG$ is proper, as can be seen by faithfully flat descent (or directly). Therefore we may apply proposition 1.7 in [16], which yields the result immediately. □

This last formula has a lot of consequences, which we list now without much justification: these are well-known facts in the case of cohomology, and the same demonstrations can be used without any modification. See [45] for example.

From now on all groups are finite.

1.3.5 Proposition. *Let G be a finite group, let G_p be subgroup of index prime to p. Then the p-torsion summand of $CH^k(G)$ restricts isomorphically onto the "stable" subgroup of $CH^k(G_p)$. In other words, defining a map $CH^k(G_p) \to CH^k(G_p \cap G_p^g)$ by*

$$\Psi_g(x) = i^*_{G_p \cap G_p^g \to G_p}(x) - i^*_{G_p \cap G_p^g \to G_p^g} \circ c^*_{g^{-1}}(x)$$

then the following sequence is exact (for $k > 0$):

$$0 \to CH^k(G) \otimes \mathbb{Z}_{(p)} \to CH^k(G_p) \to \prod_{g \in G} CH^k(G_p \cap G_p^g)$$

1.3.6 Remark. Let G and G' be finite groups, and let S and S' be Sylows. The Martino-Priddy conjecture, recently proved by Bob Oliver (see [32] and the references therein), asserts that whenever there is a homotopy equivalence $BG_p^{\wedge} \approx BG'^{\wedge}_p$, then there is an isomorphism of groups $S \approx S'$ which is "fusion preserving", ie it respects the conjugacies in the strongest possible sense. It follows then from the last proposition that there is an isomorphism $CH^*BG \approx CH^*BG'$. Note that any map $G \to G'$ inducing an isomorphism on mod p cohomology yields a homotopy equivalence between the p-completed classifying spaces, and thus *any homomorphism of groups which induces an isomorphism on cohomology also implies the existence of an isomorphism between the Chow rings* (or cobordism rings, or any ring coming from a theory with transfers for that matter).

The proof of the Martino-Priddy "conjecture" is very long and complicated and proceeds by a case-by-case check through the classification of finite simple groups, so we shall refrain from using it. This being said, the application just discussed comes from the "easy half" of the proof.

1.3.7 Proposition (Swan's lemma).

Suppose that the p-Sylow subgroup G_p is abelian. Then if N_p is its normalizer, we have

$$CH^*(G) \otimes \mathbb{Z}_{(p)} = CH^*(G_p)^{N_p}$$

This last proposition shows, for example, that if G has an abelian p-Sylow subgroup, then its p-torsion summand injects into the corresponding cohomology group.

Our last result concerns groups of p-rank 1. Let us first deal with odd p's: in this case having p-rank 1 means that the p-Sylow subgroups are cyclic. Then one has:

1.3.8 Proposition. *Suppose G has p-rank 1, for some odd p. Then*

$$CH^*(G) \otimes \mathbb{Z}_{(p)} = H^{even}(G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)} = H^*(G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}$$

Moreover this ring is generated by Chern classes.

For $p = 2$, having 2-rank 1 means the 2-Sylow subgroups are either cyclic, or generalized quaternion groups. It is likely that such groups have Chow rings mapping isomorphically to cohomology, and therefore that the theorem above holds for $p = 2$. We know for sure that it holds if G_2 is cyclic or if $G_2 = Q_8$.

§1.4. Complex cobordism

At least we should consider ourselves happy that we are not doing Algebraic Geometry.
Frank Adams, *Infinite Loop Spaces.*

All the results of this chapter also hold for complex cobordism, as we proceed to prove. Things are much easier to demonstrate in this case.

And to start with, the **first computations**:

- **Grassmannians** are very easy to deal with: when the cohomology of a space X is torsion-free, it is always isomorphic to $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{Z}$, and so

$$MU^*(BGL_n(\mathbb{C}))\hat{\otimes}_{MU^*}\mathbb{Z} = CH^*BGL_n(\mathbb{C}) = H^*(BGL_n(\mathbb{C}), \mathbb{Z})$$

- **Cyclic groups.** We note that $B\mathbb{Z}/n$ is a circle bundle over BS^1, and so the Gysin exact sequence gives immediately

$$E^*B\mathbb{Z}/n = E^*[[x]]/([n](x))$$

  for any oriented cohomology E, where the (standard) notation $[n]$ refers to the homomorphism $E^*BS^1 \to E^*BS^1$ induced from the map $BS^1 \to BS^1$ which raises a line bundle to its n-th tensor power, or equivalently the map coming from $z \mapsto z^n$ on the circle.

  In the case of MU, we have $[n](x) = nx + \cdots$, and we conclude

$$MU^*(B\mathbb{Z}/n)\hat{\otimes}_{MU^*}\mathbb{Z} = CH^*B\mathbb{Z}/n = H^{2*}(B\mathbb{Z}/n, \mathbb{Z})$$

- **Künneth Formulae.** Suppose all the Morava K-theories of BG_i, at all primes, are concentrated in even degrees, for $i = 1, 2$ (which is extremely often, though not always, the case). Then one has

$$MU^*(BG_1 \times BG_2) = MU^*(BG_1)\hat{\otimes}_{MU^*}MU^*(BG_2)$$

  In fact this is a particular case of a more precise statement regarding BP^*, see 3.2.6 where we also give some details about the Morava K-theories.

- **Quaternions.** We let $E^{p,q} = H^p(BG, MU^q)$ denote the first page of the Atiyah-Hirzebruch spectral sequence converging to MU^*BG, where $G = Q_8$. The integral cohomology of G is concentrated in even degrees, and is periodic, the period being given by (cup) multiplication by an element $z \in H^4(BG, \mathbb{Z})$. This z happens to be a Chern class, in fact the top Chern class of the representation of G used in our computation for Chow rings. Hence it is certainly an infinite cycle in the spectral sequence. Since all differentials are derivations and $d_r(z) = 0$ for all $r \geq 2$, as we have just argued, it follows that multiplication by z commutes with d_r and gives an isomorphism between the columns of the spectral sequences (away from the 0-th column), on every page. We conclude that for $n > 0$, $MU^nBG = MU^{n+4}BG$.

  Now, $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z}$ is 0 in dimension 1 and 3 by [47], theorem 3.3, and the even part $(MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z})^{2i}$ agrees with CH^iBG for $i \leq 2$ by *loc. cit.*, corollary 3.5. By periodicity,

$$CH^*BG = MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{Z} = H^{2*}(BG, \mathbb{Z}).$$

- **Functoriality and transfers:** everything is true and trivial.

We now turn to **connected, solvable groups**, for which it is enough to point out that the propositions in 1.2.2 and 1.2.3 trivially hold for $MU^*(-)\hat{\otimes}_{MU^*}\mathbb{Z}$, by a spectral sequence argument. It follows that the results in 1.2.4 and 1.2.5 also hold.

Finally, the **double coset formula** is really the statement in 1.3.3. From this point onwards, we have a formula as given in 1.3.4 for any theory with transfers having reasonable properties with respect to fibre products. It is certainly the case of MU.

CHAPTER

TWO

The Steenrod algebra

He would make outrageous claims like he invented the question mark. Sometimes he would accuse chestnuts of being lazy. The sort of general malaise that only the genius possess and the insane lament.

Dr Evil

In this chapter appears the Steenrod algebra $\mathcal{A}_p$, the algebra of all stable cohomology operations. Equivalently, $\mathcal{A}_p$ is $H^*(H) = \pi_*(\mathbf{hom}(H, H))$ where H is the mod p Eilenberg Mac-Lane spectrum and the hom object is taken in an appropriate category of spectra. We will denote by $\mathfrak{U}$ the category of unstable modules over $\mathcal{A}_p$, while $\mathfrak{U}^{ev}$ will stand for the full subcategory consisting of modules which are concentrated in even degrees. Brosnan has shown ([8]) how $\mathcal{A}_p$ acts on the Chow ring of a smooth variety, and $CH^*X \otimes \mathbb{F}_p$ thus becomes a typical element in $\mathfrak{U}^{ev}$.

We shall explain how a theorem of Quillen leads one to the study of these two categories after localizing away from a subcategory called $\mathfrak{Nil}$, which roughly consists of modules containing only "nilpotent" elements in an intuitive sense. Localizing away from $\mathfrak{Nil}$ means we want a module in $\mathfrak{Nil}$ to count as zero, and thereby a map whose kernel or cokernel is in $\mathfrak{Nil}$ should count as a monomorphism or epimorphism.

Quillen's theorem then says that after this simplification is done, the cohomology or Chow ring of BG is completely determined by the elementary abelian subgroups of G and their fusion (that is, the conjugacies between them).

Our observation is that in the case of Chow rings, but not that of cohomology, we have very often a much finer result, namely this is true even before localizing. Equivalently, Chow rings are often local (isomorphic to their localizations).

We make all this precise in the first section, and give an illustration in the second, with G taken to be the symmetric group on n letters. We prove that it is indeed local or "$\mathfrak{Nil}$-closed". The symmetric groups are used in the very definition of the Steenrod operations (which together with the Bockstein generate the Steenrod algebra) and this is exploited in our proof: therefore it makes sense to isolate this example from the ones considered in later chapters.

A piece of notation: from now on, we will write simply

$$CH^*X = CH^*X \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

We use the forgetful functor $\mathcal{O} : \mathfrak{U}^{ev} \to \mathfrak{U}$ and its right adjoint $\tilde{\mathcal{O}} : \mathfrak{U} \to \mathfrak{U}^{ev}$. Note then that $\mathcal{O}\tilde{\mathcal{O}}M$ is the largest submodule of M which is concentrated in even degrees, which need not be the even part of M but rather the submodule on which the ideal generated by the Bockstein in $\mathcal{A}_p$ acts trivially.

And p is a prime.

§2.1. Nilpotent modules and localization

2.1.1. Let G be a reductive algebraic group, and let $\mathcal{C}(G)$ be the category whose objects are the elementary abelian p-subgroups of G and whose morphisms are induced by conjugations in G. In [34], Quillen proved that the natural map

$$H^*(BG) \to \varprojlim_{\mathcal{C}(G)} H^*BE$$

is an "F-isomorphism", in the sense that any element in the kernel is nilpotent, and for any x in the target, x^{p^n} is in the image for some n. As it happens, this is a property that can be characterised in $\mathfrak{U}$, because the p-th power is given by a Steenrod operation. Quillen's theorem can be reformulated by saying that the above map becomes an isomorphism upon localizing away from the subcategory of so-called nilpotent modules (we will give more precise definitions below). Rather surprisingly, it is very fruitful to "linearize" the situation in this way ([20], [22]).

Yagita proved a version of Quillen's theorem for Chow rings when G is finite, see [48]: the map

$$CH^*(BG) \to \varprojlim_{\mathcal{C}(G)} CH^*BE$$

is also an F-isomorphism. We will see in this section that we can express this as a localization result in $\mathfrak{U}^{ev}$, parallel to the one for cohomology. If we saw our Chow rings as modules in $\mathfrak{U}$ this would not be possible, which confirms our intuition that Chow rings naturally live in $\mathfrak{U}^{ev}$.

It is natural to ask whether there are any groups for which Quillen's map is actually an isomorphism. For example if $G = S_n$ is the symmetric group, the map is an isomorphism at the prime 2 but not for any other prime, see [20]. However we shall see in the next section that the Quillen-Yagita map for S_n is an isomorphism at all p. Indeed one of the main points of this thesis is to show that this is true for a whole lot of groups.

In this section we make the above statements precise, treating simultaneously the cases of $\mathfrak{U}$ and $\mathfrak{U}^{ev}$. There is essentially nothing new here, cf [22], [20], [21], but we need all this for reference. It is also felt that the reader might appreciate a concise and reasonably self-contained presentation.

2.1.2. Quotient categories; localizations.([18], [17]) Let $\mathfrak{C}$ be an abelian category, and let $\mathfrak{D}$ be a Serre class (a full subcategory with the property that if two objects of a short exact sequence in $\mathfrak{C}$ belong to $\mathfrak{D}$, then so does the third). Then there is a quotient category $\mathfrak{C}/\mathfrak{D}$ which is abelian and an exact functor $r : \mathfrak{C} \to \mathfrak{C}/\mathfrak{D}$ satisfying the obvious universal property. A morphism in $\mathfrak{C}$ induces a monomorphism (resp. epimorphism) in $\mathfrak{C}/\mathfrak{D}$ if and only if its kernel (resp. cokernel) belongs to $\mathfrak{D}$. We will talk about $\mathfrak{D}$-monomorphisms, etc.

The category $\mathfrak{D}$ is said to be *localizing* if r admits a right adjoint, or section functor, $s : \mathfrak{C}/\mathfrak{D} \to \mathfrak{C}$. We put $l = s \circ r$ and we call the natural map $\lambda_M : M \to l(M)$ the *localization of M away from* $\mathfrak{D}$. When this is an isomorphism, we say that M is $\mathfrak{D}$-closed or $\mathfrak{D}$-local.

One can prove easily that the natural transformation $r \circ s(M) \to M$ is an isomorphism. It follows that $\lambda_M : M \to l(M)$ is a $\mathfrak{D}$-isomorphism, that is, $r(\lambda_M)$ is an isomorphism. Hence $\lambda_{l(M)} = l(\lambda_M) : l(M) \to l \circ l(M)$ is an isomorphism, too: in other words, the localization of M is local. More generally, keep in mind that a $\mathfrak{D}$-isomorphism $M \to N$ induces an isomorphism $l(M) \to l(N)$.

Finally, we note that if $\mathfrak{C}$ has enough injectives, then an object M is $\mathfrak{D}$-closed if and only if

$$Ext^i_{\mathfrak{C}}(D, M) = 0$$

for $i = 0, 1$ and all D in $\mathfrak{D}$.

2.1.3. Nilpotent and $\mathfrak{Nil}$-closed modules. Given M in $\mathfrak{U}$ or $\mathfrak{U}^{ev}$, and $x \in M$ of even dimension, we put $P_0 x = P^{|x|/2} x$ (so that if M happens to be an unstable algebra, $P_0 x = x^p$).

We say that M is *nilpotent* if $P_0^N x = 0$ for all x of even dimension and all large N. (See [38], p47). There is an exception if we work with $\mathfrak{U}$ at the prime 2: in this case we put $Sq_0 x = Sq^{|x|} x$ and call a module M nilpotent if $Sq_0^N x = 0$ for all x and all large N.

The subcategory of $\mathfrak{U}$ or $\mathfrak{U}^{ev}$ which is comprised of the nilpotent modules will be denoted by $\mathfrak{Nil}$ (we do not distinguish between $\mathfrak{U}$ and $\mathfrak{U}^{ev}$, hopefully the context will make things clear).

It is easy to see that $\mathfrak{Nil}$ is localizing, see the criterion in [38], prop 6.3.1. Therefore, the results of the previous paragraph apply, and we use the notation $\lambda_M : M \to L(M)$ for the localization away from $\mathfrak{Nil}$ (notation as in [22]).

It is well-known that $\mathfrak{U}$ has enough injectives, and it follows that the same can be said of $\mathfrak{U}^{ev}$. Accordingly, M is $\mathfrak{Nil}$-closed if and only if $Ext^i(N, M) = 0$ for $i = 0, 1$ and for all N nilpotent.

To finish with, we call a module *reduced* if $Hom(N, M) = 0$ for all nilpotent modules N. In $\mathfrak{U}^{ev}$, or in $\mathfrak{U}$ at the prime 2, this is equivalent to demanding that P_0 be injective on M: combine lemma 2.6.4 and equation 1.7.1* in [38] (alternatively, see lemma 4.5 in [21]).

2.1.4 Lemma. *Any reduced module in $\mathfrak{U}^{ev}$ embeds in a reduced $\mathfrak{U}^{ev}$-injective. The tensor product of two reduced $\mathfrak{U}^{ev}$-injectives is a reduced $\mathfrak{U}^{ev}$-injective.*

Proof. We observe immediately that if I is injective in $\mathfrak{U}$, then $\tilde{\mathcal{O}}I$ is injective in $\mathfrak{U}^{ev}$. The first assertion of the lemma follows then from the corresponding statement in $\mathfrak{U}$, which is well-known, and the left exactness of $\tilde{\mathcal{O}}$. It also follows that any reduced injective in $\mathfrak{U}^{ev}$ is a direct summand in such a module $\tilde{\mathcal{O}}I$ with I reduced, and we get the second assertion, again because the analogous result in $\mathfrak{U}$ is well-known (and because $\tilde{\mathcal{O}}(A \otimes B) = \tilde{\mathcal{O}}A \otimes \tilde{\mathcal{O}}B$ if one of the factors is reduced). □

2.1.5 Proposition. *In either $\mathfrak{U}$ or $\mathfrak{U}^{ev}$, we have the following properties:*

1. *If $0 \to M' \to M \to M'' \to 0$ is exact and if M' and M'' are both $\mathfrak{Nil}$-closed, so is M.*

2. *If $0 \to M \to M' \to M''$ is exact and if M' is $\mathfrak{Nil}$-closed while M'' is reduced, then M is $\mathfrak{Nil}$-closed.*

3. *M is $\mathfrak{Nil}$-closed $\iff$ there exists an exact sequence*

$$0 \to M \to I \to J$$

with I and J both reduced and injective.

4. *M_1 and M_2 $\mathfrak{Nil}$-closed $\Rightarrow$ $M_1 \otimes M_2$ $\mathfrak{Nil}$-closed.*

2.1.6 Corollary. *Any product or inverse limit of $\mathfrak{Nil}$-closed modules is $\mathfrak{Nil}$-closed.*

Proof. This is trivial from the lemma. Note that the third point in the proposition is only here to prove the fourth. In case of problems see [20] or [21]. □

2.1.7. Back to Quillen's map. The cohomology of an elementary abelian p-group is reduced and injective ([38], 2.6.5 and 3.1.1), hence is $\mathfrak{Nil}$-closed. Consequently, the target of Quillen's map is $\mathfrak{Nil}$-closed, as an inverse limit of such. Quillen's theorem asserts that his map is a $\mathfrak{Nil}$-isomorphism, so that it becomes an isomorphism upon localizing. In other words

$$L(H^*(BG)) \approx \varprojlim_{\mathcal{C}(G)} H^*BE$$

(as the inverse limit is isomorphic to its own localization). We see in this way that Quillen's map is an isomorphism if and only if H^*BG is $\mathfrak{Nil}$-closed.

Similarly, $CH^*BE = \tilde{\mathcal{O}}H^*BE$ is injective and reduced in $\mathfrak{U}^{ev}$, and the target of the Quillen-Yagita map is $\mathfrak{Nil}$-closed in $\mathfrak{U}^{ev}$. It follows that

$$L(CH^*(BG)) \approx \varprojlim_{\mathcal{C}(G)} CH^*BE$$

and that the Quillen-Yagita map is an isomorphism if and only if CH^*BG is $\mathfrak{Nil}$-closed.

2.1.8 Example. It is instructive to have a look at $G = GL(n,\mathbb{C})$, even if this G is not finite (so only Quillen's map can be mentioned, not Yagita's): in effect $M = H^*BG = CH^*BG$ illustrates well how a module can behave better in $\mathfrak{U}^{ev}$ than in $\mathfrak{U}$, which we see as follows. If E is the subgroup of G of diagonal matrices with p-th roots of unity as entries, then any elementary abelian p-subgroup of G is conjugated to a subgroup of E. If W denotes the Weyl group, it follows that

$$L(M) = (H^*BE)^W \neq H^*BGL(n,\mathbb{C});$$

consequently M is not $\mathfrak{Nil}$-closed in $\mathfrak{U}$. On the other hand

$$(CH^*BE)^W = CH^*BGL(n,\mathbb{C})$$

from which we deduce that M is indeed $\mathfrak{Nil}$-closed in $\mathfrak{U}^{ev}$ (being an inverse limit of $\mathfrak{Nil}$-closed modules).

To get an example with finite groups, it is in fact possible to take a finite field k of characteristic different from p but containing the p-th roots of unity, and to consider $H^*BGL(n,k)$ and $CH^*BGL(n,k)$. The former is reduced but not $\mathfrak{Nil}$-closed in $\mathfrak{U}$ ([22], [35]), while the latter is $\mathfrak{Nil}$-closed in $\mathfrak{U}^{ev}$ (see chapter 4). In the rest of this thesis we shall extend this to other groups of matrices over finite fields.

2.1.9 Example. We cannot hope for CH^*BG to be always $\mathfrak{Nil}$-closed, as is illustrated by the group of quaternions: there is only one elementary 2-subgroup, namely the centre $\{1;-1\}$, and the restriction map is neither injective nor surjective.

However, this example is not too discouraging: the quaternion groups are precisely the 2-groups which have 2-rank 1, together with cyclic groups (recall that the rank of a p-group is the dimension of the largest $\mathbb{F}_p$ vector space contained in it). For odd p on the other hand, this pathology disappears, and a group of p-rank 1 can only be cyclic. So the example above might reflect a purely group-theoretic defect (or subtlety, if you want), and one might still hope that for groups of odd order, the Quillen-Yagita map is "very often" an isomorphism.

2.1.10. Remark. Combining Quillen's and Yagita's result, we obtain of course that the cycle map $CH^*BG \to H^*BG$ is an F-isomorphism. So we can write in $\mathfrak{U}$ that $L(CH^*BG) = L(H^*BG)$ and in $\mathfrak{U}^{ev}$ that $L(CH^*BG) = L(\tilde{O}H^*BG)$. Whenever CH^*BG is $\mathfrak{Nil}$-closed this reads

$$CH^*BG = L(\tilde{O}H^*BG)$$

This is a description of the Chow ring in terms of H^*BG using merely functors between $\mathfrak{U}$ and $\mathfrak{U}^{ev}$ (which do not depend on G).

We end this section with a few more remarks, before starting to give examples of groups G with CH^*BG $\mathfrak{Nil}$-closed, hopefully convincing the reader that there is a fair number of them. When p is odd, we do not know of an example of a group *not* satisfying this property.

2.1.11. Remark. Suppose that G is a group such that H^*BG is reduced. Since the functor $\tilde{\mathcal{O}}$ is left-exact and commutes with inverse limits, we have an injection:

$$\tilde{\mathcal{O}}H^*BG \hookrightarrow \varprojlim \tilde{\mathcal{O}}H^*BE$$

But $\tilde{\mathcal{O}}H^*BE = CH^*BE$, so if we suppose further that the Quillen-Yagita map for Chow rings is surjective, it follows that the image of CH^*BG under the cycle map has to contain all of $\tilde{\mathcal{O}}H^*BG$. We will use this later with $G = S_n$.

2.1.12. Complex cobordism. Most of what we have said so far applies to the theory $MU^*(-)\hat{\otimes}_{MU^*}\mathbb{F}_p$.

We start by explaining why we have Steenrod operations on this theory. First, we have the isomorphism *of graded abelian groups*:

$$MU^*(MU) = MU^*(pt)[[\cdots c_n \cdots]]$$

More generally for any oriented cohomology theory E, the Thom-Dold theorem asserts $E^*MU = E^*BU$, and E^*BU is computed via the Atiyah-Hirzebruch spectral sequence, and is a power series ring as above. *However this is not a ring isomorphism:* we are interested in the multiplication given by composition in $MU^*(MU) = \pi_*(\mathbf{hom}(MU, MU))$. Nevertheless, it is a fact (cf [37],VII,3.1) that any element $x \in MU^*(MU)$ can be written

$$x = \sum_{\omega} a_\omega S_\omega$$

where $a_\omega \in MU^*(pt)$ and S_ω has an (easy) explicit description in terms of the c_i's. The ring structure on $MU^*(pt)$ is the natural one, and the notation $a_\omega S_\omega$ does refer to the "correct" multiplicative structure. (On the other hand multiplying S_ω and $S_{\omega'}$ gives funny results.) From this it follows that the kernel I of the (surjective) map $MU^*(MU) \to H^*MU$, where as always H denotes mod p cohomology, is the ideal generated by $(p, x_i) \subset MU^*(pt)$, in standard notation. In other words, I is generated in $MU^*(MU)$ by the kernel of the map

$$MU^*(pt) = \mathbb{Z}[x_1, x_2, \cdots] \to \mathbb{F}_p = H^*(pt)$$

Now, this I acts trivially on $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ for any space X, of course. (That is, the elements of this ideal act as 0.) Thus H^*MU acts on our theory (again, with some ring structure which is not the one coming from the isomorphism $H^*MU = H^*BU$, but this will not matter). Since the Steenrod algebra acts on H^*MU, it also acts on $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$. Moreover, it is well-known that H^*MU is a free $\mathcal{A}_p/(\beta)$-module, so that the Bocksteins act trivially in this new setting. (Note that H^*MU is thus *not* unstable, this is an example that shows that the Thom-Dold isomorphism is not a map of $\mathcal{A}_p$-modules!)

Is $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ unstable? This appears to be true, and we outline a proof – we shall not need the result in what follows, as we shall only consider some spaces X for which $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ injects into the cohomology ring. To prove the claim, one would construct directly some operations on $MU^*(-)\hat{\otimes}_{MU^*}\mathbb{F}_p$ in the obvious way: starting with a manifold M representing a class in MU^*X, then M^p gives a class in MU^*X^p, and thus it yields one in MU^*Z^pX, the cyclic product; project this onto

$$MU^*(X \times B\mathbb{Z}/p)\hat{\otimes}_{MU^*}\mathbb{F}_p = \big(MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p\big) \otimes \big(MU^*(B\mathbb{Z}/p)\hat{\otimes}_{MU^*}\mathbb{F}_p\big)$$

and define the coefficients of the polynomial in v thus obtained to be your "Steenrod" operations. These act "unstably" for any space, by construction. After taking limits, we extend this to $X = MU$, and the images of $1 \in MU^*(MU)\hat{\otimes}_{MU^*}\mathbb{F}_p$ define elements $\bar{P}^i$ there; these lift to $MU^*(MU)$ and yield operations on $MU^*(-)$ and on $MU^*(-)\hat{\otimes}_{MU^*}\mathbb{F}_p$ which by naturality agree with the ones just described. Thanks to the existence of such lifts, in fact, any element in $MU^*(MU)\hat{\otimes}_{MU^*}\mathbb{F}_p$ acts on $MU^*(-)\hat{\otimes}_{MU^*}\mathbb{F}_p$, and thus any relation between the $\bar{P}^i$'s and the c_i's still holds after it's applied to any x in $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ for any X. By picking some spaces X for which $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ injects into the cohomology H^*X, and using the obvious compatibility of the $\bar{P}^i$'s with the actual Steenrod operations P^i once we have projected to cohomology, we deduce $\bar{P}^i = P^i$.

In any case, if we take this for granted, we see that the modules $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ live naturally in the category $\mathfrak{U}/\beta$ of unstable modules over $\mathcal{A}_p/(\beta)$, which is an intermediate between $\mathfrak{U}^{ev}$ and $\mathfrak{U}$. Considering the odd and even parts of a module M in $\mathfrak{U}/\beta$ gives a decomposition of M as a direct sum of two elements of $\mathfrak{U}^{ev}$. It should be easy from this to extend the localization results obtained for $\mathfrak{U}$ and $\mathfrak{U}^{ev}$ to the category $\mathfrak{U}/\beta$. However, we do not have any application in sight, mostly because all modules of interest to us given as $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$ for some G are concentrated in even degrees and can be seen as objects of $\mathfrak{U}^{ev}$. Therefore we shall not fill in the details here. The interested reader might want to have a look at [33], formula 3.18 in particular.

In any case, it is convenient to use the term "F-isomorphism", and we state for future reference Yagita's result [48]:

Proposition. *Let G be a finite group. The natural map*

$$MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p \to \varprojlim MU^*(BE)\hat{\otimes}_{MU^*}\mathbb{F}_p$$

is an F-isomorphism.

Corollary. *If G is a finite group, then the two maps*

$$CH^*BG \to MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p \to H^*BG$$

are F-isomorphisms.

§2.2. The Symmetric Groups

2.2.1. In this section we prove that CH^*BS_n is $\mathfrak{Nil}$-closed in $\mathfrak{U}^{ev}$. As one could expect, the proof is by induction, by proving that if CH^*BG is $\mathfrak{Nil}$-closed, then so is $CH^*B(S_p \wr G)$. It might seem simpler to use $\mathbb{Z}/p \wr G$ instead, but things turn out to be slghtly easier this way, and also we want the reader to be able to compare our proof with that in [21] which deals with a similar result using $\tilde{\mathcal{O}}H^*BG$ instead of our CH^*BG (and in turn, both treatments follow closely the original one in [20] for mod 2 cohomology). Incidentally, the result in [21] can be recovered from ours, see 2.2.13.

For technical reasons stemming from [47], we will have to restrict attention to a certain class of groups: namely, G will be assumed to have a subgroup H of index prime to p such that BH can be cut into open subsets of affine space. Then $\mathbb{Z}/p \wr H$ is a subgroup of index prime to p in $S_p \wr G$ which has the same property, by [47], lemma 8.1, and the induction may proceed.

We will also assume that the mod p cycle map $CH^*BG \to H^*BG$ is injective. We will prove quickly that the cycle map for $\mathbb{Z}/p \wr G$ and $S_p \wr G$ is injective too.

Here we point out a gap in [47]: even though all the results that we shall refer to are correct and cover a great deal of the work needed to study the Chow ring of S_n, it appears that assertion (3) in lemma 8.1 does not have a correct proof (it is stated for integral coefficients but the proof refers to a paper of Nakaoka which is written mod p). The computation of CH^*BS_n that follows is thus not entirely justified. Therefore our own results partially fill in the gap.

We begin by explaining how Chow rings and cohomology rings are affected by taking wreath products, and we give a few immediate properties, in particular we will end up with an exact sequence which together with 2.1.5,1/, will eventually yield the result.

2.2.2. Cohomology and Chow rings of cyclic products. We shall need the following nice result of Nakaoka[31]:

$$H^*(\mathbb{Z}/p \wr G) = H^*(\mathbb{Z}/p, (H^*G)^{\otimes p})$$

Recall that the cohomology of a cyclic group with coefficients in any ring A is periodic, and the period is given by taking cup-products with an element in $H^2(\mathbb{Z}/p, A)$. Here we denote by $v \in H^2(\mathbb{Z}/p, (H^*G)^{\otimes p})$ an element giving the period (abusing the notation given in the introduction).

For Chow rings of cyclic products, we use results of [47]. There a certain functor F_p from graded abelian groups to graded abelian groups is defined, which comes equipped with a natural map $F_pCH^*X \to CH^*Z^pX$, where X is a variety and Z^pX is it p-fold cyclic product. After changing the grading from dimension to codimension, reducing modulo p, and changing the notations to relate to [8], the definition of F_pA^* is as follows: take the p-fold tensor product $A^* \otimes \cdots \otimes A^*$, take symbols Px in degree $p \cdot |x|$ for $x \in A^*$ of positive degree, take also elements $\alpha_i x$ of degree $p \cdot |x| + 2i$ for all $x \in A^*$ and positive i, and finally divide by the relations:

$$x_1 \otimes \cdots \otimes x_p = x_2 \otimes \cdots \otimes x_p \otimes x_1$$

$$x^{\otimes p} = 0$$

$$P(x+y) = Px + Py + \sum s_1 \otimes \cdots \otimes s_p$$

together with the relations that turn α_i into a homomorphism of groups. Here the sum in the third formula runs over a set of representatives for the $\mathbb{Z}/p$-orbits in the set $\{x,y\}^p - \{(x,\cdots,x),(y,\cdots,y)\}$. To be rigorous, the elements $\alpha_i x$ should be only defined for i small enough and likewise the Px's should only appear when the degree of x is small enough; but in both cases the bound depends on the dimension of the variety X, and when we deal with classifying spaces of groups BG we take a limit of varieties (with "compatible" Chow rings) having their dimensions going to infinity, and we do not need to worry about this complication.

From the construction of F_p as given in [47], it is easy to describe the composition $F_pCH^*BG \to CH^*B(\mathbb{Z}/p\wr G) \to H^*B(\mathbb{Z}/p\wr G)$, for any G: the element $x_1\otimes\cdots\otimes x_p$ is sent to the "norm" $\sum x_{i_1}\otimes\cdots\otimes x_{i_p}$, with the sum running over all cyclic permutations, sitting in $H^0(\mathbb{Z}/p,(H^*G)^{\otimes p})$; the element Px is sent to $x\otimes\cdots\otimes x$ in the same group (note how the relations above relate to the expansion of $(x+y)\otimes\cdots\otimes(x+y)$); and finally the element $\alpha_i x$ goes to $v^i\cdot x\otimes\cdots\otimes x$. This is all clear. In particular this map is injective if the cycle map for G is injective.

For G as in the introduction, $F_pCH^*BG \to CH^*B(\mathbb{Z}/p\wr G)$ is surjective by lemma 8.1 in [47]. Therefore for the G we consider, there is an isomorphism $F_pCH^*BG = CH^*B(\mathbb{Z}/p\wr G)$, the cycle map for $\mathbb{Z}/p\wr G$ is injective, and its image is explicitly described.

We shall denote by τ the transfer from G^p to $\mathbb{Z}/p\wr G$. Its image is spanned by the "norms", clearly.

2.2.3. We will also be interested in wreath product of the form $S_p\wr G$. Nakaoka has established that $H^*(S_p\wr G) = H^*(S_p,(H^*G)^{\otimes p})$ in this case too. Since $\mathbb{Z}/p$ is a p-Sylow of S_p, we deduce:

Lemma. *Let $W = N_{S_p}(\mathbb{Z}/p)/\mathbb{Z}/p = (\mathbb{Z}/p)^*$. Then*

$$H^*(S_p\wr G) = (H^*(\mathbb{Z}/p\wr G))^W$$

and

$$CH^*(S_p\wr G) = (CH^*(\mathbb{Z}/p\wr G))^W$$

Proof. The equality for cohomology groups follows from Nakaoka's results just quoted and Swan's lemma. To get the result for Chow rings, we proceed as follows: there is an obvious inclusion, and the double coset formula tells us that the image of the restriction map is the group of "stable" elements, ie those x such that, putting $K=\mathbb{Z}/p\wr G$, we have $x|_{K\cap gKg^{-1}} = gxg^{-1}|_{K\cap gKg^{-1}}$ (proposition 1.3.5). So we need to show that if x is W-invariant, then x is stable. But $K\cap gKg^{-1}$ is either K or G^p (this is because G^p is normal in $S_p\wr G$, and the order of $\mathbb{Z}/p$ is prime). In either case, the cycle map is injective on this group, and so the result for cohomology implies that for Chow rings. $\square$

2.2.4 Proposition. *There are exact sequences:*

$$0 \to \tau(H^*(BG^p)) \to H^*B(\mathbb{Z}/p\wr G) \to H^*BG\otimes H^*B\mathbb{Z}/p$$

and

$$0 \to \tau(CH^*(BG^p)) \to CH^*B(\mathbb{Z}/p\wr G) \to CH^*BG\otimes CH^*B\mathbb{Z}/p$$

Note that the maps on the right come from the inclusion of $\mathbb{Z}/p \times G$ in $\mathbb{Z}/p \wr G = \mathbb{Z}/p \ltimes G^p$.

Proof. The exact sequence for cohomology follows from results of Steenrod's ([42] chapter VII), as is explained, for example, in [29], theorem II.3.7.

To get the result for Chow rings, we use the cycle map, which here is injective. The only thing to prove is that $CH^*B(\mathbb{Z}/p \wr G) \cap \tau(H^*(BG^p)) = \tau(CH^*(BG^p))$, but this is clear from the explicit description in 2.2.2. $\square$

2.2.5 Corollary. *There are exact sequences:*

$$0 \to (\tau(H^*BG^p))^W \to H^*B(S_p \wr G) \to R_1(H^*BG) \to 0$$

and

$$0 \to (\tau(CH^*BG^p))^W \to CH^*B(S_p \wr G) \to R_1^{ev}(CH^*BG) \to 0$$

where W is as in lemma 2.2.3 and where $R_1(H^*BG)$, resp. $R_1^{ev}(CH^*BG)$, is a submodule of $H^*BG \otimes H^*B\mathbb{Z}/p$, resp. $CH^*BG \otimes CH^*B\mathbb{Z}/p$.

The functorial notations $R_1(H^*BG)$ and $R_1^{ev}(CH^*BG)$ will be justified below.

2.2.6. The point now is to prove that $(\tau(CH^*BG^p))^W$ and $R_1^{ev}(CH^*BG)$ are both $\mathfrak{Nil}$-closed, and to use proposition 2.1.5, 1/. This is where we can only continue the proof for Chow rings, as the result does not hold for cohomology at odd primes. It is proved in [21], however, that if one replaces H^*BK by $\tilde{\mathcal{O}}H^*BK$ for all groups K occuring in Quillen's map, then one still gets an isomorphism.

2.2.7. The functor R_1^{ev}. Let M be a module in $\mathfrak{U}^{ev}$. For any $x \in M$ of degree $2k$, define

$$St_1^{ev}(x) = \sum_{i=0}^{k} (-1)^i v^{i(p-1)} \otimes P^{k-i}x \in CH^*B\mathbb{Z}/p \otimes M$$

where we recall that $P^i = Sq^{2i}$ when $p = 2$. We define $R_1^{ev}M$ to be the $CH^*(BS_p)$-submodule of $CH^*B\mathbb{Z}/p \otimes M$ generated by the elements $St_1^{ev}x$, for $x \in M$. Here we view $CH^*B\mathbb{Z}/p$ as a module over CH^*BS_p via the restriction map.

There is a functor $R_1 : \mathfrak{U} \to \mathfrak{U}$ which is defined in a similar, but more complicated, way (see for example [21], definition 4.2, or the original in [49]). In fact one has $R_1^{ev}M = \tilde{\mathcal{O}}R_1(\mathcal{O}M)$: this can be seen from the explicit description of R_1 given in *loc cit* , and using lemma 2.2.3 which asserts in particular that $CH^*S_p = \mathbb{F}_p[v^{p-1}]$. This proves that $R_1^{ev}M$ is always in $\mathfrak{U}^{ev}$, ie it is stable under the action of the Steenrod algebra. We will not use this seriously, however, and R_1^{ev} can be taken as a functor from $\mathfrak{U}^{ev}$ to graded $\mathbb{F}_p$-vector spaces.

The definition of R_1^{ev} is very explicit and will allow computation. However:

2.2.8 Lemma. *The two definitions of $R_1^{ev}M$ given coincide.*

Proof. We need to show that the image of the map

$$CH^*(S_p \wr G) \to CH^*B\mathbb{Z}/p \otimes CH^*G$$

is $R_1^{ev}M$. The fact that the elements $St_1^{ev}x$ are in this image follows from the very definition of the Steenrod operations on Chow rings: in [8], an element $P(x)$ is constructed in $CH^*(S_p \wr G)$ (prop 4.2 in *loc cit*) which retricts to $St_1^{ev}x$ (definition 7.5 in *loc cit*; note that our $(-1)^i$ sign is on p10, before prop 6.6 there). Therefore, the image certainly contains $R_1^{ev}M$.

To get the reverse inclusion, observe that (with notations as in 2.2.2, our Px being consistent with Brosnan's) the elements in $CH^*(\mathbb{Z}/p \wr G)$ not mapping to 0 in $CH^*G \otimes CH^*B\mathbb{Z}/p$ are of the form Px or $\alpha_i x = v^i \cdot Px$. The latter elements can only be in $CH^*(S_p \wr G) = (CH^*(\mathbb{Z}/p \wr G))^W$ (cf 2.2.3) if i is a multiple of $(p-1)$, and the image of the map above is indeed contained in the $CH^*BS_p = F_p[v^{p-1}]$-module generated by the $St_1^{ev}x$. □

2.2.9. A few properties of R_1^{ev}. Put $P = CH^*B\mathbb{Z}/p$ and $Q = CH^*BS_p$.

Lemma (1). *If $\{x\}$ is an $\mathbb{F}_p$-basis of M, then $\{St_1^{ev}x\}$ is a basis of $R_1^{ev}M$ as a free Q-module.*

Proof. (Almost word for word from [26], proof of 4.2.3, included for the convenience of the reader). Given a relation $\sum \lambda_x St_1^{ev}x = 0$ with $\lambda_x \in P$, put $m = \min |x|$ taken over all x for which $\lambda_x \neq 0$, if there are any. Then project into $P \otimes M^m$, which is a free P-module having a basis containing the elements $1 \otimes x$ for x of degree m. You get, with $m = 2k$, the relation $\sum \lambda_x (-1)^k v^{k(p-1)} \otimes x = 0$ which yields $\lambda_x = 0$. □

Lemma (2). *The functor R_1^{ev} is exact.*

Proof. From the previous lemma, we can say that as a vector space, $R_1^{ev}M$ is $Q \otimes \Phi M$, where here ΦM means M with all degrees mutiplied by p. Result follows. □

Lemma (3). *IF M' is a submodule of M, then $(R_1^{ev}M) \cap (P \otimes M') = R_1^{ev}M'$.*

Proof. This follows from the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_1^{ev}M' & \longrightarrow & R_1^{ev}M & \longrightarrow & R_1^{ev}M/M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P \otimes M' & \longrightarrow & P \otimes M & \longrightarrow & P \otimes M/M' & \longrightarrow & 0
\end{array}$$

□

Our next lemma will involve the functor Φ defined in [38], 1.7. When $M \in \mathfrak{U}^{ev}$, ΦM is M with all degrees multiplied by p, and with an appropriate action of the Steenrod algebra which makes the map $\Phi M \to M, x \mapsto P_0 x$ linear over $\mathcal{A}_p$ (so more formally, $P^i \Phi x = \Phi P^{i/p} x$ if $p|i$ and 0 otherwise). Whenever M is reduced, we identify ΦM with the submodule of M comprised of the elements $P_0 x$.

Lemma (4). *For any $M \in \mathfrak{U}^{ev}$, we have*

$$(R_1^{ev}\Phi M) \cap (\Phi P \otimes M) \subset \Phi R_1^{ev} M$$

Proof. (cf [21], 4.7). Let $x \in M$ have degree $2k$; we can write

$$St_1^{ev}\Phi x = \sum_{j=0}^{k}(-1)^j v^{jp(p-1)} \otimes P^{(k-j)p}\Phi x$$

because P^{kp-i} can only act non-trivially on Φx when p divides i (so we have put $i = pj$). A typical element in $(R_1^{ev}\Phi M) \cap (\Phi P \otimes M)$ is thus

$$y = v^{pm(p-1)} \sum_{j=0}^{k}(-1)^j v^{jp(p-1)} \otimes P^{(k-j)p}\Phi x$$

and this is Φz for

$$z = v^{m(m-1)} \sum_{j=0}^{k}(-1)^j v^{j(p-1)} \otimes P^{(k-j)}\Phi x$$

from Cartan's formula and 1.7.1* in [38]. □

2.2.10 Proposition. *If $M \in \mathfrak{U}^{ev}$ is $\mathfrak{Nil}$-closed, so is $R_1^{ev}M$.*

Proof. We prove that if an element y of $R_1^{ev}M$ is of the form $y = P_o x$ for some $x \in CH^*B\mathbb{Z}/p \otimes M$, then in fact we can choose such an x in $R_1^{ev}M$. It follows that P_0 is injective on the quotient $(CH^*B\mathbb{Z}/p \otimes M)/R_1^{ev}M$, so that this module is reduced (cf 2.1.3). As $CH^*B\mathbb{Z}/p \otimes M$ is $\mathfrak{Nil}$-closed from proposition 2.1.5, 4/, it follows from 2/ of the same propostion that $R_1^{ev}M$ is $\mathfrak{Nil}$-closed.

We have

$$(R_1^{ev}\Phi M) \cap (\Phi P \otimes M) \subset \Phi R_1^{ev} M$$

and

$$R_1^{ev}M \cap (P \otimes \Phi M) = R_1^{ev}\Phi M$$

from (2.2.9, lemmas 3 & 4). Thus

$$R_1^{ev}M \cap \Phi(P \otimes M) \subset \Phi R_1^{ev} M$$

using the fact that on $\mathfrak{U}^{ev}$, Φ commutes with tensor products, just like it does on $\mathfrak{U}$ at the prime 2, so that $\Phi(P \otimes M) = (\Phi P \otimes M) \cap (P \otimes \Phi M)$.

This was what we wanted. □

2.2.11. By contrast, proving that $\tau(CH^*BG^{\otimes p})^W$ is $\mathfrak{Nil}$-closed is straightforward. Put $M = CH^*BG$. The explicit formulae of 2.2.2 and 2.2.3 show that there is an exact sequence

$$0 \to \tau(M^{\otimes p}) \to (M^{\otimes p})^{\mathbb{Z}/p} \to \Phi M \to 0$$

Since M is reduced, direct computation shows that ΦM is reduced; the middle term of the sequence is $\mathfrak{Nil}$-closed by 2.1.5,4/ and 2.1.6; so $\tau(M^{\otimes p})$ is $\mathfrak{Nil}$-closed by 2.1.5, 2/; and finally $(\tau(M^{\otimes p}))^W$ is $\mathfrak{Nil}$-closed by 2.1.6 again.

2.2.12 Theorem. *Let G be a group as in the introduction. If CH^*BG is $\mathfrak{Nil}$-closed, so is $CH^*B(S_p \wr G)$. In particular, CH^*BS_n is $\mathfrak{Nil}$-closed for any n. It follows that*

$$CH^*BS_n = \varprojlim CH^*BE$$

where E runs over the elementary abelian subgroups of S_n.

Proof. The only thing to add is that the Sylow subgroup of S_n is contained in a product of iterated wreath products $S_p \wr S_p \wr \cdots \wr S_p$. □

2.2.13. It has been known since Quillen's paper [34] that the cohomology of S_n is reduced. From 2.1.11, we deduce

$$CH^*BS_n = \tilde{\mathcal{O}}H^*BS_n$$

It follows that

$$\tilde{\mathcal{O}}H^*BG = \varprojlim \tilde{\mathcal{O}}H^*BE$$

which was established in [21], a paper which has been a great source of inspiration.

CHAPTER
THREE

Chevalley groups: cobordism

I have seen men fly bombers with their faces half- blown away. You're going to allow a few algebra formulas to ground you?
Robin Green and Mitchell Burgess, Northern Exposure, Cup of Joe, 1993

Now that we have set up the language of localization of unstable modules and showed that the concept was not vacuous with the example of S_n, we proceed to analyze a large class of groups, namely the finite groups of Lie type, or Chevalley groups. These are essentially the groups of matrices over finite fields which have a counterpart in the realm of Lie groups – e.g. $Sp_n(\mathbb{F}_q)$, $SL_n(\mathbb{F}_q)$, and the like (precise definition below).

This chapter is dedicated to the complex cobordism, while the next one, by no means independent, will deal with the Chow ring.

Briefly, the result we obtain is the following. Let G be a Chevalley group, let p be a prime such that $H^*(BG_{\mathbb{C}}, \mathbb{Z})$ has no p-torsion (where $G_{\mathbb{C}}$ is the associated Lie group), and let k be a finite field of char $\neq p$ containing the p-th roots of unity. Then $MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$ is a $\mathfrak{Nil}$-closed module in $\mathfrak{U}^{ev}$, and there is a natural isomorphism

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p = H^*BG_{\mathbb{C}}.$$

When k does not contain the p-th roots of unity, one has to take a certain Galois action into account, and this is completely described.

We shall take the following route. The first section explains the basic properties of Chevalley groups. Most of these facts are not even needed until the next chapter, but they find a natural niche here, giving the reader a feel for which results of the theory of Lie groups carry over to these finite analogs that we are confronted with. The following two sections do the actual work: section 2 contains a couple of technical lemmas on going back to MU^* from the Morava K-theories, and section 3 applies this, exploiting a theorem of Tanabe [44] on the Morava K-theories of Chevalley groups. The computation is finished just before section 4, where it is interpreted in terms of localizations (they are not mentioned up to that point).

Note the slight twist: to be able to see the result as a localization statement, we need some information on the elementary abelian subgroups of the group G under consideration. In the present case this can be taken care of by a rather strong theorem of Steinberg [43]. However, the computation of $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$ does not depend on any of this (its *interpretation* only does), and in turn, one can deduce Steinberg's result once the cobordism ring is known. So we have a very different proof of a classical group-theoretic fact.

§3.1. Introduction

3.1.1. Chevalley groups. The point of this chapter and the next is to study *Chevalley groups*, ie groups of the form $G(\mathbb{F}_{l^a})$ where G is a connected, reductive, split group scheme over $\mathbb{Z}$, and l is a prime different from p. Recall that "split" means that G has a maximal torus T which is itself "split" in the sense that $T = \mathbb{G}_m^n$ over $\mathbb{Z}$ (or whatever the base is); in this definition $\mathbb{G}_m$ is the multiplicative group scheme Spec $\mathbb{Z}[X, X^{-1}]$. The associated Lie group over $\mathbb{C}$ will be denoted $G_{\mathbb{C}}$ (and in fact the notation will occasionally refer to a maximal compact Lie subgroup, which should not cause any confusion as the inclusion map induces a homotopy equivalence; the two classifying spaces also share the same homotopy type).

3.1.2. The Weyl group. Let us start with some general remarks on group schemes. Recall that normalisers, centralisers, and quotients (at least by diagonalisable subgroups) can be performed over any base scheme, see [1], exposé VIII,6/. It will be important to us that these operations commute with base extensions – for example if G is a group scheme over S and H a subgroup scheme, and if $R \to S$ is any morphism, then $(N_G(H))_R = N_{G_R}(H_R)$ where $X_R = X \times_S R$.

If now G is a reductive group scheme over S with a split maximal torus T, recall that $T = C(T)$ (its own centraliser), cf [2], exposé XIX, 2.8. We (momentarily) put $W_G(T) = N_G(T)/T$. The following is proved in [2], exposé XXII,3/ (in particular proposition 3.4): there exists a finite group W such that $W_G(T)$ is the constant group scheme associated to W. This means in particular that for any ring R above S without idempotents other than 0 and 1, we have $W_G(T)(R) = W$. Furthermore, it is also established in *loc cit* that

$$W \subset \frac{N_G(T)(S)}{T(S)} \subset W_G(T)(S)$$

Using the preceding remark on base extensions, we conclude that the above inclusions are equalities whenever S is replaced by (the spectrum of) a ring without nontrivial idempotents.

Now let G be a Chevalley group as in the previous paragraph. There is thus a finite group W such that

$$\begin{aligned} W &= \tfrac{N_G(T)(\mathbb{Z})}{T(\mathbb{Z})} &= \tfrac{N_G(T)}{T}(\mathbb{Z}) \\ &= \tfrac{N_G(T)(k)}{T(k)} &= \tfrac{N_G(T)}{T}(k) \end{aligned}$$

for any field k.

It follows that W acts on $T(k)$. Note that unless k is algebraically closed, there is no reason for $N_G(T)(k)$ to be the normaliser of $T(k)$ in $G(k)$, it may be a strictly smaller subgroup; similarly $T(k) = C(T)(k)$ may be smaller than the centraliser of $T(k)$ in $G(k)$. However we have the following

Lemma. *Any automorphism of $T(k)$ induced by conjugation by an element of $G(k)$ may also be realised by an element of W. More generally, any isomorphism between two subgroups of $T(k)$ induced by conjugation by an element of $G(k)$ can also be realised by conjugation by an element of W.*

Proof. This is a well-known argument. So let K denote an algebraic closure of k. Suppose $x \in G(k)$ and $xAx^{-1} = B$. Let C be the connected component of 1 in the centraliser of B in $G(K)$. We have then $T(K)$ and $xT(K)x^{-1}$ as maximal tori in C, and therefore they are conjugated by some $c \in C \subset G(K)$. Consider then $n = c^{-1}x$. It is clear that $n \in N_G(T)(K)$, and can be written $n = wt$ with $w \in N_G(T)(k)$ and $t \in T(K)$, according to the equalities above, valid for any field. Thus the elements w and n induce the same automorphism of $T(K)$, and induce the same isomorphism between A and B as x does. □

3.1.3. Notations. Throughout the rest of this thesis, G will be a Chevalley group, T a split maximal torus, N_T will be short for $N_G(T)$, and W will be the finite group just introduced (to be refered to as the Weyl group). We pick a prime number p and assume that $H^*(BG_{\mathbb{C}}, \mathbb{Z})$ has no p-torsion. Some of this will be repeated for emphasis.

We take l to be another prime, and q will be a power of l. The field with q elements will be denoted by $\mathbb{F}_q$, and $\mathbb{F}$ stands for an algebraic closure of $\mathbb{F}_l$. For each field under discussion, μ_p will be the group of p-th roots of unity.

If A is a ring with an action of a group Γ, then $A_\Gamma = A/(a - \gamma \cdot a)$ is the ring of coinvariants – this is not quite standard, as the notation refers usually to the abelian group obtained by dividing out by the *subgroup* spanned by the $a - \gamma \cdot a$, as opposed to the *ideal* that they generate. But we choose to follow Tanabe [44].

On the other hand $A^\Gamma = \{a : \gamma \cdot a = a\}$ is classically the ring (resp. group if A is only a group, etc...) of invariants.

§3.2. Preliminaries on cohomology theories

... utinam intelligere possim rationacinationes pulcherrimas quae e propositione concisa "de quadratum nihilo exaequari" fluunt...
Henri Cartan

3.2.1. Conventions. A *space* will always mean a CW complex with finitely many cells in each dimension, although we will repeat this occasionally for emphasis. If E and F are spectra, then we put $E^*(F) = [F, E]$; if X is a space, we put $E^*(X) = E^*(\Sigma^\infty(X^+))$; if X is a pointed space, we put $\widetilde{E}^*(X) = E^*(\Sigma^\infty(X))$. Here Σ^∞ is the canonical functor from pointed spaces to spectra and X^+ is the disjoint union of X and a base point. Putting $E^* = E^*(point)$, we have thus $E^*(X) = E^* \oplus \widetilde{E}^*(X)$ and $\widetilde{E}^*(X) = \ker(E^*(X) \to E^*)$.

If X is a space and E a spectrum, we will say that "X has no $\lim^1$ term with respect to E" if $E^*(X) = \lim_{\leftarrow} E^*(X^m)$. Here and elsewhere the superscript refers to the m-th skeleton. This terminology is justified by Milnor's exact sequence (cf [37], corollary 4.18).

3.2.2. We shall encounter several (generalised) cohomology theories: complex cobordism MU^* and Brown-Peterson cohomology BP^* (cf [37] chapter 7), Morava K-theories $K(j)^*$ ([37] chapter 9), and we will also mention $P(1)^*$ briefly in the course of one proof (*loc. cit.*). These spectra are to be taken at the prime p. So we have

$$\begin{array}{rcl} MU^* &=& \mathbb{Z}[x_1, x_2, \cdots] \\ BP^* &=& \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \\ P(1)^* &=& \mathbb{F}_p[v_1, v_2, \cdots] \\ K(j)^* &=& \mathbb{F}_p[v_j, v_j^{-1}] \end{array}$$

with x_i of degree $-2i$ and v_i of degree $-2(p^i - 1)$.

3.2.3. Completed tensor products. For each such theory h^* and spaces X_i, $i = 1, 2$ satisfying

$$h^*(X_i) = \lim_{\leftarrow m} h^*(X_i^m)$$

where X_i^m is the m-th skeleton of X_i, we will use the completed tensor products:

$$h^*(X_1)\hat{\otimes}_{h^*}h^*(X_2) = \lim_{\leftarrow m}[im(h^*(X_1) \to h^*(X_1^m)) \otimes_{h^*} im(h^*(X_2) \to h^*(X_2^m))]$$

and $h^*(X)\hat{\otimes}_{h^*}R$ is defined similarly, for a ring R with a map $h^* \to R$ (by considering R as graded but concentrated in dimension zero). Examples in view are the natural maps $MU^* \to \mathbb{Z} \to \mathbb{Z}_{(p)} \to \mathbb{F}_p$ and $BP^* \to \mathbb{Z}_{(p)} \to \mathbb{F}_p$. Recall that

$$MU^*(X) \otimes_{MU^*} \mathbb{Z}_{(p)} = BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)}$$

for, say, a finite dimensional CW complex X so in particular

$$MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p = BP^*(X)\hat{\otimes}_{BP^*}\mathbb{F}_p$$

for a CW complex X.

3.2.4. It is worth making a few comments about this last ring. Intuitively, it is obtained from $MU^*(X)$ (or $BP^*(X)$) by setting to zero all elements in the ideal generated by p and the x_i's, or rather in the closure of this ideal. For some nice spaces we can make this idea precise, namely, for spaces with no $\lim^1$ term with respect to MU (or BP). For future reference we state this as a lemma:

Lemma. *If $MU^*(X) = \lim MU^*(X^m)$, then the natural map*

$$MU^*(X) \to MU^*(X) \hat{\otimes}_{MU^*} \mathbb{F}_p$$

is surjective. It follows that $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$ is obtained from $MU^*(X)$ by dividing out by the closure of the ideal generated by the x_i's and by p.

A similar result holds for BP^*.

This follows from the arguments in [47], section 2. In fact something much stronger is proved there.

3.2.5. An exactness result. We shall need the following rather technical result later on. This is the only time when we shall use $P(1)^*$. It partially generalizes the half-exactness of the usual tensor product functor.

Lemma. *For $i = 1, 2, 3$, let X_i be a space with no $\lim^1$ term with respect to BP, and with even Morava K-theories. Suppose given an exact sequence (where all maps are induced by maps of topological spaces):*

$$\widetilde{BP}^*(X_1) \to \widetilde{BP}^*(X_2) \to \widetilde{BP}^*(X_3) \to 0$$

Then there is an exact sequence

$$\widetilde{BP}^*(X_1)\hat{\otimes}_{BP^*}\mathbb{F}_p \to \widetilde{BP}^*(X_2)\hat{\otimes}_{BP^*}\mathbb{F}_p \to \widetilde{BP}^*(X_3)\hat{\otimes}_{BP^*}\mathbb{F}_p \to 0$$

Proof. Because X_i has even Morava K-theories, the reduction mod p of $BP^*(X)$ is $P(1)^*(X)$ ([36], theorem 1.9). Consider then the following diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker_2 \arrow[r] \arrow[d] & \ker_3 \arrow[r] \arrow[d] & 0 \\
\widetilde{P(1)}^*(X_1) \arrow[r] \arrow[d] & \widetilde{P(1)}^*(X_2) \arrow[r] \arrow[d] & \widetilde{P(1)}^*(X_3) \arrow[r] \arrow[d] & 0 \\
\widetilde{BP}^*(X_1)\hat{\otimes}_{BP^*}\mathbb{F}_p \arrow[r] & \widetilde{BP}^*(X_2)\hat{\otimes}_{BP^*}\mathbb{F}_p \arrow[r] \arrow[d] & \widetilde{BP}^*(X_3)\hat{\otimes}_{BP^*}\mathbb{F}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

By the lemma in 3.2.4, we can define $\ker_i$ to obtain exact columns. We prove now that the rows are all exact.

For the middle one, simply apply $- \otimes_{\mathbb{Z}} \mathbb{F}_p$ to the given exact sequence.

From 3.2.4 again, we see that $\ker_i$ can be described as the closure of the ideal generated by the v_i's in $\widetilde{P(1)}^*(X_i)$. It follows that $\ker_2 \to \ker_3$ has a dense image. But now, since X_i is of finite type, each graded piece of $\widetilde{P(1)}^*(X_i^m)$ is finite, so that $\widetilde{P(1)}^*(X_i) = \lim_{\leftarrow} \widetilde{P(1)}^*(X_i^m)$ (say, by Mittag-Leffler), and the natural topology on $\widetilde{P(1)}^*(X_i)$ is compact Hausdorff. (This is the chief reason for using $P(1)^*$ instead of BP^*.) Thus $\ker_2 \to \ker_3$ is surjective, and the first row is exact.

There remains only a trivial diagram chase to prove that the bottom row is exact, too. □

3.2.6. Künneth formulae. A quick word on the cohomology of a product $X \times Y$ of two spaces. The situation for Morava K-theories is, as usual, very simple, and it is an understatement to say that the following result is well-known; however for lack of a good reference we submit a proof.

Lemma. *Let X, Y be any two spaces. Then:*

$$K(j)^*(X \times Y) = K(j)^*(X)\hat{\otimes}_{K(j)^*}K(j)^*(Y)$$

Proof. (cf [36], proof of theorem 1.11.) Write E for $K(j)$, $\otimes$ for $\otimes_{E^*}$, lim for inverse limits, and $E^*(X)_s$ for $im(E^*(X) \to E^*(X^s))$. We recall that E^* is a graded field in the sense that every graded module over it is free.

Now if Y is a finite complex we have

$$E^*(X) \otimes E^*(Y) = E^*(X \times Y)$$

because if we see both sides of this equation as functors on the category of finite complexes, fixing X, then they are cohomology theories which agree on S^0 and with a natural transformation between them, so the result follows from [37], proposition 3.19(i).

There is never a $\lim^1$ term for Morava K-theory (and for spaces), see [36], corollary 4.8. So in particular for any X and Y we have $\lim_i^1 E^*((X \times Y)^i) = 0$. From the naturality of the Milnor sequence applied to the subcomplexes $(X \times Y)^i$ and $X \times Y^i$ we draw $\lim_i^1 E^*(X \times Y^i) = 0$ and hence

$$E^*(X \times Y) = \lim E^*(X \times Y^i) = \lim E^*(X) \otimes E^*(Y^i)$$

from the previous equation.

Observe that for a finitely generated E^*-module M and any inverse system $\{A_i\}$ of E^*-modules, we have

$$(\lim A_i) \otimes M = \lim(A_i \otimes M)$$

because M being free, this is just the statement that inverse limits commute with finite direct sums. We apply this below with $M = E^*(Y^i)$, $M = E^*(X)_s$ and $M = E^*(Y)_s$. The following computation should now be straightforward:

$$
\begin{aligned}
E^*(X\times Y) &= \lim_j E^*(X)\otimes E^*(Y^j)\\
&= \lim_j(\lim_i E^*(X)_i)\otimes E^*(Y^j)\\
&= \lim_j \lim_i E^*(X)_i\otimes E^*(Y^j)\\
&= \lim_i \lim_j E^*(X)_i\otimes E^*(Y^j)\\
&= \lim_i E^*(X)_i\otimes (\lim_j E^*(Y^j))\\
&= \lim_i E^*(X)_i\otimes (\lim_j E^*(Y)_j)\\
&= \lim_i \lim_j E^*(X)_i\otimes E^*(Y)_j\\
&= \lim_{i,j} E^*(X)_i\otimes E^*(Y)_j\\
&= E^*(X)\hat{\otimes} E^*(Y)
\end{aligned}
$$

□

For BP cohomology, there is the following nice result taken from [36] (theorem 1.11): if X_i is a CW complex of finite type, with even Morava K-theories, and such that $BP^*(X_i) = \lim BP^*(X_i^m)$ (for $i = 1, 2$) then

$$BP^*(X\times Y) = BP^*(X)\hat{\otimes}_{BP^*}BP^*(Y)$$

In particular, one has a map

$$BP^*(X)\otimes_{BP^*} BP^*(Y)\to BP^*(X\times Y)$$

with a dense image, and similarly for Morava K-theory.

We will give some related results for Chow rings in 4.1.3.

3.2.7. More exactness. To finish these preliminaries, we have the following proposition dealing with a certain type of exact sequences. Its formulation might seem cumbersome at first; the reason is that we want to exploit the ring structure on our cohomology groups, and the notion of an exact sequence of rings is delicate to phrase (the image of a homomorphism is always a subring, which is almost never an ideal at the same time). Here we can take advantage of the decomposition $E^*(X) = E^*\oplus \tilde{E}^*(X)$ using reduced cohomology and make some sense (however we shall not use the terminology of "augmented rings").

Proposition. *Suppose given spaces X, Y and Z with no $\lim^1$ term with respect to BP, and suppose that $X\times Y$ has no $\lim^1$ term either. Assume given maps*

$$X \xleftarrow{f,\ g} Y \xleftarrow{h} Z$$

such that $f\circ h$ and $g\circ h$ are homotopic, and giving rise to an exact sequence

$$\widetilde{K(j)}^*(X)\otimes_{K(j)^*} K(j)^*(Y) \xrightarrow{(f^*-g^*)\otimes id} \widetilde{K(j)}^*(Y) \xrightarrow{h^*} \widetilde{K(j)}^*(Z) \longrightarrow 0$$

for all $j>0$. Then the same is true with $K(j)^*(-)$ replaced by $BP^*(-)\hat{\otimes}_{BP^*}\mathbb{F}_p$.

Remark. The exact sequence says that the kernel of h^* is the *ideal* generated by the image of $f^* - g^*$. Thus the multiplicative structure plays the prominent role here.

Proof. Write E for $K(j)$. We use the Kunneth formula for Morava K-theory:

$$\begin{aligned} E^*(X \times Y) &= E^*(X)\hat{\otimes}_{E^*} E^*(Y) \\ &= E^* \oplus \widetilde{E}^*(Y) \oplus \widetilde{E}^*(X)\hat{\otimes}_{E^*} E^*(Y) \end{aligned}$$

Note that the sum of the last two terms above is $\widetilde{E}^*(X \times Y)$, and the last summand is the kernel of the map $\widetilde{E}^*(X \times Y) \to \widetilde{E}^*(Y)$ induced by the inclusion $Y \to X \times Y$. Let $X \propto Y$ be the cone of this map (that is, up to homotopy, $X \times Y / * \times Y$). The cofibration gives an exact sequence on reduced cohomology, and the projection $X \times Y \to Y$ gives a splitting, so that

$$\begin{aligned} \widetilde{E}^*(X \propto Y) &= \ker \widetilde{E}^*(X \times Y) \to \widetilde{E}^*(Y) \\ &= \widetilde{E}^*(X)\hat{\otimes}_{E^*} E^*(Y) \end{aligned}$$

We conclude that the maps $f \propto id : Y \to X \propto Y$ and $g \propto id : Y \to X \propto Y$ induced by (f, id) and (g, id) give rise to an exact sequence

$$\widetilde{K(j)}^*(X \propto Y) \xrightarrow{(f\propto id)^* - (g\propto id)^*} \widetilde{K(j)}^*(Y) \xrightarrow{h^*} \widetilde{K(j)}^*(Z) \longrightarrow 0$$

for all $j > 0$.

Now, after suspending one may find[1] a map $F : SY \to S(X \propto Y)$ such that $F^* = S(f \propto id)^* - S(g \propto id)^*$ (on any cohomology theory), where S is reduced suspension. Moreover, $F \circ Sh$ is inessential. Therefore by theorem 1.18 in [36], we have also

$$\widetilde{BP}^*(X \propto Y) \xrightarrow{(f\propto id)^* - (g\propto id)^*} \widetilde{BP}^*(Y) \xrightarrow{h^*} \widetilde{BP}^*(Z) \longrightarrow 0$$

using (twice) the fact that the effect of reduced suspension on cohomology is only a shift in degrees. Then by (3.2.5, lemma) we have

$$\widetilde{BP}^*(X \propto Y)\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow \widetilde{BP}^*(Y)\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow \widetilde{BP}^*(Z)\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow 0$$

Now there are maps

$$BP^*(X) \otimes_{BP^*} BP^*(Y) \to BP^*(X \times Y) \to BP^*(X \times Y)\hat{\otimes}_{BP^*}\mathbb{F}_p$$

The first map has a dense image by (3.2.6), and the second map is surjective by (3.2.4). Arguing as above we end up with a map

$$\widetilde{BP}^*(X) \otimes_{BP^*} BP^*(Y) \to \widetilde{BP}^*(X \propto Y)\hat{\otimes}_{BP^*}\mathbb{F}_p$$

with a dense image. Clearly we also have a map

$$\left[\widetilde{BP}^*(X)\hat{\otimes}_{BP^*}\mathbb{F}_p\right] \times \left[BP^*(Y)\hat{\otimes}_{BP^*}\mathbb{F}_p\right] \to \widetilde{BP}^*(X \propto Y)\hat{\otimes}_{BP^*}\mathbb{F}_p$$

with a dense image, but here by compactness the map is actually surjective.

Finally, we patch up things together and obtain the desired exact sequence. □

[1] For spectra E and F, $[E, F]$ is always an abelian group, so that given two spaces X and Y one is tempted to consider $E = \Sigma^\infty(X^+)$ and $F = \Sigma^\infty(Y^+)$ if one wants to take the difference between two maps $X \to Y$. However one must be careful, as most results in [36], on which we rely heavily, are strictly unstable, ie do not apply to spectra and their maps.

Remark. During the course of this proof we have implicitly used the fact that the space $X \propto Y$ has no $\lim^1$ term for BP. This follows easily from the injectivity of

$$\widetilde{BP}^*(X \propto Y) \to \widetilde{BP}^*(X \times Y)$$

and the naturality of Milnor's exact sequence.

§3.3. Cobordism of Chevalley groups

3.3.1. The associated Lie group. The study of $G_{\mathbb{C}}$ is quite easy, given the well-known results obtained by Borel for ordinary cohomology. We have:

Proposition. *Let n be the rank of G. There are elements s_i, for $1 \leq i \leq n$, of positive even degree, such that:*

$$BP^*(BG_{\mathbb{C}}) = BP^*[[s_1, \cdots, s_n]]$$

It follows that

$$BP^*(BG_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p = H^*(BG_{\mathbb{C}}, \mathbb{F}_p) = \mathbb{F}_p[s_1, \cdots, s_n]$$

Proof. This follows from the classical result (cf [5])

$$H^*(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[s_1, \cdots, s_n]$$

and similarly with $\mathbb{F}_p$ coefficients, by using the Atiyah-Hirzebruch spectral sequence. □

3.3.2. Brauer lifts. In order to study now the finite groups $G(\mathbb{F}_q)$ we shall make use of "Brauer lifts", a general term meaning that we lift some representations of a group from characteristic l to characteristic 0. A possible approach is described in the proposition below; later in 4.1.4 another point of view will be preferable. Recall that $\mathbb{F}$ denotes the algebraic closure of $\mathbb{F}_q$.

Proposition. *There exists a map*

$$BG(\mathbb{F}) \longrightarrow BG_{\mathbb{C}}$$

natural with respect to maps of group schemes, such that the maps

$$BP^*(BG_{\mathbb{C}}) \longrightarrow BP^*(BG(\mathbb{F}_q))$$

are all surjective, for all choices of q. It follows that

$$BP^*(BG_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow BP^*(BG(\mathbb{F}_q))\hat{\otimes}_{BP^*}\mathbb{F}_p$$

is also surjective.

Proof. The map is described in [15]. In [44], it is shown that it induces surjective maps

$$K(j)^*(BG_{\mathbb{C}}) \longrightarrow K(j)^*(BG(\mathbb{F}_q))$$

for all $j > 0$ (and in this case the kernel is also known). By a result of Ravenel-Wilson-Yagita [36], this implies that the induced map on BP cohomology is surjective as well. The final statement follows by the right exactness of the (completed) tensor product. $\square$

Remark (1). Note that for classifying spaces of compact Lie groups, such as $BG_{\mathbb{C}}$ and $BG(\mathbb{F}_q)$, there is no $\lim^1$ term for BP cohomology (and for the skeletal filtration). This is essential to use the results of [36] in their strong form (ie, with BP itself instead of its completion). This is a major concern which explains why very few of our statements will involve $BG(\mathbb{F})$, for which there might be *a priori* a non-vanishing $\lim^1$.

Remark (2). The kernel of the map above can in fact be described, but we postpone this to 3.3.6, in order to prove as much as possible by elementary means.

3.3.3. Tori. Let K be a finite field of characteristic l which contains the p-th roots of unity and let T be a torus (in our framework, this means that T is split over $\mathbb{Z}$ and hence over K). Then $T(K)$ is a product of cyclic groups whose order is a multiple of p. Therefore

$$BP^*(BT(K))\hat{\otimes}_{BP^*}\mathbb{F}_p = \mathbb{F}_p[\eta_1, \cdots, \eta_n]$$

where n is the dimension of T, and where the η_i's have degree 2. But of course $BP^*(BT_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p$ has a similar description (these are all well-known results). Consequently the surjective map between this two rings that we obtained in (3.3.2, proposition) is in fact an isomorphism.

3.3.4. Quite fortunately, we can easily relate a Chevalley group G as above to its maximal torus, in terms of cohomology. More precisely:

Proposition. *The restriction map*

$$BP^*(BG_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow BP^*(BT_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p$$

is injective. Moreover if the order of the Weyl group is prime to p, then the image of this map is precisely the subring of invariants.

Proof. As observed above (3.3.1, proposition), this is simply the map

$$H^*(BG_{\mathbb{C}}, \mathbb{F}_p) \longrightarrow H^*(BT_{\mathbb{C}}, \mathbb{F}_p)$$

It is well known, since BG has no p-torsion, that the restriction map gives an isomorphism

$$H^*(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) \longrightarrow H^*(BT_{\mathbb{C}}, \mathbb{Z}_{(p)})^W$$

where W denotes the Weyl group. Consider the exact sequence of coefficients:

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{\times p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

This gives rise to exact sequences in cohomology:

$$0 \longrightarrow H^k(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) \xrightarrow{\times p} H^k(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) \longrightarrow H^k(BG_{\mathbb{C}}, \mathbb{F}_p) \longrightarrow 0$$

using either the fact that the rings are concentrated in even degrees, or that multiplication by p is clearly injective here. Apply this to T as well, appeal to naturality, and apply the functor $(-)^W$ to obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^k(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) & \longrightarrow & H^k(BG_{\mathbb{C}}, \mathbb{Z}_{(p)}) & \longrightarrow & H^k(BG_{\mathbb{C}}, \mathbb{F}_p) & \longrightarrow & 0 \\
 & & \downarrow\approx & & \downarrow\approx & & \downarrow & & \\
0 & \longrightarrow & H^k(BT_{\mathbb{C}}, \mathbb{Z}_{(p)})^W & \longrightarrow & H^k(BT_{\mathbb{C}}, \mathbb{Z}_{(p)})^W & \longrightarrow & H^k(BT_{\mathbb{C}}, \mathbb{F}_p)^W & \longrightarrow & H^1(W, H^k(BT_{\mathbb{C}}, \mathbb{Z}_{(p)}))
\end{array}$$

Injectivity is now proved by a quick diagram chase. If the order of W is prime to p, then $H^1(W, H^k(BT_{\mathbb{C}}, \mathbb{Z}_{(p)})) = 0$ and the result follows. □

3.3.5. Big finite fields. We can now finish off the computations for $BG(K)$ if K, as above, is a finite field of char $= l$ containing the p-th roots of unity. Indeed, putting together the information of the previous paragraphs, we get a commutative and exact diagram:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
0 & \longrightarrow & BP^*(BG_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p & \longrightarrow & BP^*(BT_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p \\
 & & \downarrow & & \downarrow \\
 & & BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p & \longrightarrow & BP^*(BT(K))\hat{\otimes}_{BP^*}\mathbb{F}_p \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

From this it follows that *for G as in the introduction and K as above, the map*

$$BP^*(BG_{\mathbb{C}})\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p$$

is an isomorphism.

3.3.6. Galois actions. If now $k = \mathbb{F}_q$ is any field, let $K = k(\mu_p)$ (extension obtained by adding the p-th roots of unity) and let $\Gamma_q = Gal(K/k)$, which is cyclic generated by the Frobenius automorphism γ. Let n be the rank of G and let $r = [K : k]$.

We have a surjective map

$$BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p \longrightarrow BP^*(BG(k))\hat{\otimes}_{BP^*}\mathbb{F}_p$$

from (3.3.2, proposition) and (3.3.5). Moreover the Galois group Γ_q acts on both groups, trivially on $BP^*(BG(k))\hat{\otimes}_{BP^*}\mathbb{F}_p$. Therefore there is actually a surjective map from $(BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p)_{\Gamma_q}$ (coinvariants) to $BP^*(BG(k))\hat{\otimes}_{BP^*}\mathbb{F}_p$ and one has:

Proposition. *This map is an isomorphism, ie*

$$(BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p)_{\Gamma_q} = BP^*(BG(k))\hat{\otimes}_{BP^*}\mathbb{F}_p$$

Proof. Tanabe proved that this was true for Morava K-theory in [44], which can be stated by saying that we have an exact sequence just as in (3.2.7, proposition) with $X = Y = BG(K)$, $Z = BG(k)$, $f^*(a) = a$, and $g^*(a) = \gamma \cdot a$, keeping the same notations. This very proposition gives us the result. □

3.3.7. So let us describe the ring of coinvariants a bit more precisely. Starting with an n-dimensional torus as usual, we have

$$BP^*(BT(K))\hat{\otimes}_{BP^*}\mathbb{F}_p = \mathbb{F}_p[\eta_1, \cdots, \eta_n]$$

and the description of the η_i's as Euler classes makes it clear that γ sends η_i to $q \cdot \eta_i$.

For general G, the ring

$$BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p = \mathbb{F}_p[s_1, \cdots, s_n]$$

injects into the analogous ring for a maximal torus, and s_i is sent to a homogenous polynomial in the η_j's of degree $|s_i|/2$; therefore γ sends s_i to $q^{|s_i|/2} \cdot s_i$. We have then the following lemma (which will be used later on with $R^k = 2k$ -th graded piece of $BP^*(BG(K))\hat{\otimes}_{BP^*}\mathbb{F}_p$):

Lemma. *Suppose $R = \mathbb{F}_p[s_1, \cdots, s_n]$ is a graded polynomial ring with an action of a cyclic group $\Gamma =< \gamma >$ given by $\gamma \cdot s_i = q^{|s_i|} s_i$. Then the ring of coinvariants $R' = R/(a - \gamma \cdot a)$ is isomorphic to $\mathbb{F}_p[s_i : r \text{ divides } |s_i|]$.*

Proof. To see this, observe first that the condition $r|k$ is equivalent to the p-th roots of unity being in $\mathbb{F}_{l^k}$, and so in turn this is equivalent to $q^k = 1 \bmod p$. Suppose now that $k = |s_i|$ is *not* a multiple of r, then $1 - q^k$ is invertible in $\mathbb{F}_p$, and we can write

$$s_i = s_i \cdot \frac{1 - q^k}{1 - q^k} = \frac{s_i}{1 - q^k} - \gamma \cdot \frac{s_i}{1 - q^k}$$

so that s_i maps to 0 in R'. Hence there is a surjective map $\mathbb{F}_p[s_i : r \text{ divides } |s_i|] \to R'$. We leave to the reader the easy task of proving the injectivity of this map. □

3.3.8. To summarize, we have proved the following (casually replacing BP with MU, cf (3.2.3)):

Theorem. *Let G be a reductive, connected group scheme (over $\mathbb{Z}$). Let p be a prime number such that $H^*(G_{\mathbb{C}}, \mathbb{Z})$ has no p-torsion. Let K be a finite field of characteristic $l \neq p$ containing the p-th roots of unity, and let n be the rank of G. Then there are elements s_i of even degree such that*

$$\begin{aligned} MU^*(BG(K))\hat{\otimes}_{MU^*}\mathbb{F}_p &= MU^*(BG_{\mathbb{C}})\hat{\otimes}_{MU^*}\mathbb{F}_p \\ &= H^*(BG_{\mathbb{C}}, \mathbb{F}_p) \\ &= \mathbb{F}_p[s_1, \cdots, s_n] \end{aligned}$$

If k is any finite field of char $\neq p$, let $K = k(\mu_p)$ and let $r = [K : k]$. Then

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p = \mathbb{F}_p[s_i : 2r \text{ divides } |s_i|]$$

3.3.9. Example: the symplectic group. Let us have a look at $G = Sp_{2n}$. We have

$$H^*(Sp_{2n}(\mathbb{C}), \mathbb{Z}) = \Lambda(e_1, e_2, \cdots, e_n)$$

and

$$H^*(BSp_{2n}(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[q_1, q_2, \cdots, q_n]$$

where $|e_i| = 4i - 1$ and $|q_i| = 4i$. Say for example that we choose $p = 5$ and that we look at fields of characteristic 3. If we take $q = 81$, the 5-th roots of unity are already in $\mathbb{F}_{81}$, and we have

$$MU^*(BSp_{2n}(\mathbb{F}_{81}))\hat{\otimes}_{MU^*}\mathbb{F}_5 = \mathbb{F}_5[q_1, \cdots, q_n]$$

If we take $q = 9$ we need to go to $\mathbb{F}_{81}$ and $r = 2$, but all degrees appearing above are divisible by $2r = 4$. So we also obtain that

$$MU^*(BSp_{2n}(\mathbb{F}_9))\hat{\otimes}_{MU^*}\mathbb{F}_5 = \mathbb{F}_5[q_1, \cdots, q_n]$$

However for $q = 3$ we have $r = 4$, and also for $q = 27$ (the 5-th roots of unity appear only in $\mathbb{F}_{531441=27^4}$) and we drop the "odd" generators, whose degree is not divisible by $2r = 8$. Hence

$$MU^*(BSp_{2n}(\mathbb{F}_3))\hat{\otimes}_{MU^*}\mathbb{F}_5 = MU^*(BSp_{2n}(\mathbb{F}_{27}))\hat{\otimes}_{MU^*}\mathbb{F}_5 = \mathbb{F}_5[q_2, q_4, \cdots]$$

§3.4. Localization

The elementary abelian subgroups of Lie groups have been studied extensively by Borel [6], among others. He proves the following amazing theorem:

Theorem. *Let G be a compact connected Lie group. Then the following three conditions are equivalent:*

1. *$H^*(G, \mathbb{Z})$ has no p-torsion,*
2. *$H^*(BG, \mathbb{Z})$ has no p-torsion,*
3. *any elementary abelian subgroup of G is contained in a maximal torus.*

The reader might find it amusing to see how the implication $2/ \Rightarrow 3/$ can be proved very quickly using some recent results in homotopy theory: namely, if E is elementary abelian, then conjugacy classes of maps $E \to G$ are precisely the same as maps $H^*BG \to H^*BE$ of unstable algebras ([25], corollaire 3.1.4); the assumption on p implies, classically, that $H^*BG \to H^*BT$ is injective, where T is any maximal torus, and thus what we need to prove is that we can extend the map $H^*BG \to H^*BE$ induced by the inclusion of E into G to a map $H^*BT \to H^*BE$ of unstable algebras. In turn, this follows from the "non-linear injectivity of H^*BE", [38], 3.8.7.

We prove now a variation on Borel's theorem.

Proposition. *Let G be a Chevalley group with split maximal torus T, let p be a prime number such that $H^*(BG_{\mathbb{C}}, \mathbb{Z})$ has no p-torsion, and let k be a finite field of characteristic $\neq p$ which contains the p-th roots of unity. Then any elementary abelian p-subgroup of $G(k)$ is conjugated to a subgroup of $T(k)$.*

Proof. We have just seen that for such k, we have

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p = H^*BG_{\mathbb{C}}$$

for any G (hence for T as well). By choice of p, the map $H^*BG_{\mathbb{C}} \to H^*BT_{\mathbb{C}}$ is injective, so that

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p \to MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$$

is injective. Combining this with (2.1.12, proposition) tells us that the natural map

$$\varprojlim_{\mathcal{C}(G)} MU^*(BE)\hat{\otimes}_{MU^*}\mathbb{F}_p \to \varprojlim_{\mathcal{C}'(G)} MU^*(BE)\hat{\otimes}_{MU^*}\mathbb{F}_p$$

is an F-monomorphism, where $\mathcal{C}'(G)$ is the subcategory of $\mathcal{C}(G)$ (definition in section 2.1) consisting of those E which are conjugated to a subgroup of $T(k)$. Note that we write G for $G(k)$ for simplicity. (In fact this is simply a monomorphism, as both its source and target are $\mathfrak{Nil}$-closed modules in $\mathfrak{U}^{ev}$.)

Suppose that there is a $V \in \mathcal{C}(G)$ which is not in $\mathcal{C}'(G)$. Choose a V maximal with respect to this property, and note that V is then maximal in $\mathcal{C}(G)$. We construct an element x in $MU^*(BV)\hat{\otimes}_{MU^*}\mathbb{F}_p$ which restricts to 0 in any proper subgroup of V: for this, choose for each such subgroup E a non-zero element x_E in $MU^*(BV)\hat{\otimes}_{MU^*}\mathbb{F}_p$ which restricts to 0 in $MU^*(BE)\hat{\otimes}_{MU^*}\mathbb{F}_p$, for example the first Chern class of a non-trivial 1-dimensional representation of V which is trivial on E; and then take x to be the product of all the different x_E's. By symmetry, x is invariant under the action of the normaliser of V. (In fact, in this way we end up taking x to be the product of *all* non-zero elements of degree 2, but we prefer to phrase it this way.)

Given this, define $\alpha(E) = 0$ if $E \in \mathcal{C}(G)$ is not conjugated to V and $\alpha(E) = x$ otherwise, with an obvious abuse of notation. By maximality of V and choice of x, this defines an element α in the inverse limit on the left hand side above (we need such an x because there could be a group W which is not conjugated to V but having a subgroup E conjugated to a subgroup of V).

This is a contradiction, as α is non-zero (ie non-nilpotent) but it lies in the kernel of the (F-) monomorphism above. Hence $\mathcal{C}(G) = \mathcal{C}'(G)$. □

It follows in particular that

$$\varprojlim_{\mathcal{C}(G)} CH^*BE = (CH^*BT(k))^W$$

and

$$\varprojlim_{\mathcal{C}(G)} MU^*(BE)\hat{\otimes}_{MU^*}\mathbb{F}_p = (MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p)^W$$

We may then state:

Corollary (1). *The localization of $CH^*BG(k)$ away from $\mathfrak{Nil}$ is*

$$L(CH^*BG(k)) = (CH^*BT(k))^W$$

Corollary (2). *The module $MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$ is in $\mathfrak{U}^{ev}$ and its localization away from $\mathfrak{Nil}$ is $(MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p)^W$. If moreover p is odd, this module is $\mathfrak{Nil}$-closed, i.e.*

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p = (MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p)^W$$

Proof. This corollary is a summary of everything proved so far, together with the fact that $H^*BG_{\mathbb{C}} = (H^*BT_{\mathbb{C}})^W$ when G has no p torsion and p is odd. A good reference for this is ([11], 2.11), see also ([27], lemma 7.1). □

Remark (1). Here's a variant of the proposition. Keep the same hypotheses but do not assume that k contains the p-th roots of unity; then each elementary abelian p-subgroup of $G(k)$ is contained in a maximal torus, possibly not split. To see this, let E be such a subgroup, and let K be an algebraic closure of k. Let $Z(E, G)$ be the centraliser of E in $G(K)$, and let Z^0 be the connected component of 1 – a reductive group. From the result obtained when k is big enough, we deduce that E is contained in Z^0. Being central, it is contained in any maximal torus of Z^0, and there is one which is defined over k.

Remark (2). This proposition is classically seen as an application of a theorem of Steinberg, though our proof is so different that it seemed worth giving it. One normally proceeds as follows. Let E be an elementary p-subgroup of $G(k)$ and let K be an algebraic closure of k. Then theorem 2.28 in [43] says that E is toral over K, ie there is a $g \in G(K)$ such that $gEg^{-1} \subset T(K)$ where T is the fixed, split maximal torus. Considering the assumption on k, gEg^{-1} is in fact contained in $T(k)$. Now consider the set X of $x \in G(K)$ such that $xex^{-1} = geg^{-1}$ for all e in E. It is a principal homogeneous space under Z, the centraliser of E in $G(K)$. This Z is defined over k and connected (theorem 2.28 in *loc cit* again), and by Lang, it has a k-rational point. In other words there is an $x \in G(k)$ such that xEx^{-1} is contained in $T(k)$, which was what we wanted.

Note. The two remarks above are due to J.P. Serre.

§3.5. A word on ordinary cohomology

The computation of the cobordism ring of a space X, or rather the simplified ring $MU^*(X)\hat{\otimes}_{MU^*}\mathbb{F}_p$, will give some information about the ordinary cohomology ring of X. In the context of Chevalley groups, it is easy to see that the map

$$MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p \longrightarrow H^*(BG(k), \mathbb{F}_p)$$

is injective. Indeed, if k contains the p-th roots of unity, this is obvious for tori, and one concludes from (3.3.4) and (3.3.5). Alternatively, the cycle map is an F-monomorphism as observed in the previous chapter, and here the source has no nilpotent elements, so the map is actually injective for any k. So we know that $H^*(BG(k), \mathbb{F}_p)$ contains a polynomial ring, which is also the image of $H^*(BG_{\mathbb{C}}, \mathbb{F}_p)$ under the evident map (cf (3.3.2)); incidentally this map is injective when $k = k(\mu_p)$. Moreover the p^m-th power of any element in the cohomology ring, for m large enough, is in this polynomial subring.

In fact we can expect the cohomology ring to be always obtained from the cobordism ring by tensoring with an exterior algebra. The cohomology rings of Chevalley groups have been investigated, see [14] and [24]. Using more or less case-by-case considerations, we see from these references that the cohomology ring is indeed very often the tensor product of some polynomial ring and some exterior algebra. Moreover the polynomial part is abstractly isomorphic to the cobordism ring (as graded rings, ie we have the same number of generators in the same degrees). So for most choices of G, n, p and l the conjecture above will be true. However it would still be interesting to have a direct, neat proof of this. The geometry beyond this problem involves comparing usual complex cobordism and "cobordisms with singularities" – ordinary cohomology being the example in view.

CHAPTER

FOUR

Chevalley groups: Chow ring

There are no shortcuts to any place worth going.
Anonymous.

We proceed to the investigation of $CH^*BG(k)$ for a Chevalley group G and a finite field k. We shall not obtain such complete results as we did in the previous chapter, but we deal with most semi-simple groups over a field containing the p-th roots of unity. We can also make a few general statements, for example $CH^*BG(k)$ is $\mathfrak{Nil}$-closed when G is semi-simple and p does not divide the order of the Weyl group.

The strategy is simple: deal with the general linear group first, which always gives information about other groups via representation theory; then reduce the computation to the case of simply-connected groups; finally make the calculations that you can in the simply-connected case. The three sections of this chapter follow these steps.

We will say that G satisfies

- condition (1) when $CH^*BG(k) \to (CH^*BT(k))^W$ is *injective*, and
- condition (2) when $CH^*BG(k) \to (CH^*BT(k))^W$ is *surjective.*

We have seen in the previous chapter that $CH^*BG(k)$ is $\mathfrak{Nil}$-closed if and only if (1) and (2) both hold.

§4.1. The general linear group

4.1.1. In order to compute now $CH^*(BGL(n,k))$, we shall follow closely Quillen's paper [35] where the cohomology ring is investigated. The argument is greatly simplified in the case of Chow rings since abelian groups have such nice Chow rings.

As usual, k will be a finite field of characteristic l and K will denote, as above, the extension of k obtained by adding the p-th roots of unity. Again put $r = [K : k]$, and write the long division $n = rm + e$ with $0 \leq e < r$. We shall assume that $m \geq 1$, noting that in the trivial case when $m = 0$ (ie, $n < r$), then the order of $GL(n,k)$ is prime to p: to see this, recall first that this order is

$$q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-1)$$

and note that r is the smallest integer such that p divides $q^r - 1$.

Hence in this case $CH^*(BGL(n,k)) = \mathbb{F}_p$ in dimension 0 (and 0 in other dimensions).

A last notation: write $q^r - 1 = p^a h$, with h prime to p. By choice of r, we have $a \geq 1$.

4.1.2. A subgroup of $GL(n,k)$. We let $C = K^\times$, a cyclic group of order $q^r - 1$. We have an obvious representation of C on K which is 1-dimensional. If we let $\pi = Gal(K/k)$, then we can extend the action of C to one of the semi-direct product $C \rtimes \pi$; the representation we obtain is faithful. Taking direct sums, we now have a faithful representation of $(C \rtimes \pi)^m$ on K^m.

In turn, we can form the semi-direct product with S_m to obtain the wreath product $(C \rtimes \pi) \wr S_m = (C \rtimes \pi)^m \rtimes S_m$, and we can again extend the above representation to this new group. It is still faithful.

Regarding K as an r-dimensional vector space over k, and adding an e-dimensional trivial representation, we end up with an embedding of $(C \rtimes \pi)^m \rtimes S_m$ in $GL(n,k)$. It has the advantage of providing an embedding of C^m in $GL(n,k)$ such that the actions of π^m and S_m are realized by inner automorphisms of $GL(n,k)$.

The subgroup C^m enjoys the following property:

Lemma. *Any abelian subgroup of $GL(n,k)$ of exponent dividing p^a is conjugate to a subgroup of C^m.*

A proof can be found in [35], lemma 12. The reader can check that when k contains the p-th roots of unity, we do not need this lemma in what follows: it is enough to study abelian groups of exponent p (i.e. elementary abelian p-groups) and to use our previous results to the effect that they are, up to conjugation, contained in a maximal split torus. The considerations above are thus a refinement for other fields, which is very specific to the general linear group.

4.1.3. Detecting families. We also need a refinement of the statement that $CH^*B(G \wr \mathbb{Z}/p)$ is reduced when CH^*BG is.

Let G be a finite group. We shall call (*) the following condition: BG can be cut into open subspaces of affine spaces, and the map $CH^*(BG) \to H^*(BG, \mathbb{Z})$ is split injective, where exceptionally CH^*BG is not the mod p Chow ring but the integral one.

Lemma (1). *Let G be a finite group satisfying (*). Then the map*

$$CH^*(B(G \wr \mathbb{Z}/l))/l \to CH^*(BG^l)/l \oplus CH^*(B(G \times \mathbb{Z}/l))/l$$

is injective. Moreover, $G \wr \mathbb{Z}/l$ satisfies (*).

This lemma is essentially taken from [47]; the result is stated in this form at the end of section 9 for a special case, but it is clear from the material contained in sections 8 and 9 that it holds as we have stated it. See lemma 8.1 in particular for the last statement.

A family of subgroups $(G_i)_{i\in I}$ of G will be called a *p-detecting family* if the map

$$CH^*(BG) \to \bigoplus_{i\in I} CH^*(BG_i)$$

is injective. Throughout, the indexing sets I will be finite.

Lemma (2). *Suppose that G satisfies (*), and let l be a prime number. If G has an p-detecting family of abelian subgroups whose exponent divides p^a, then $G \wr S_n$ has the same property.*

Proof. (cf [45], corollary of 9.4) We proceed by induction, the result being trivial for $n = 1$. We have two cases:

1. $n = n'p$. In this case the subgroup $(G\wr\mathbb{Z}/p)\wr S_{n'}$ has index prime to p, so it suffices to prove the existence of a p-detecting family for this subgroup. By the first part of (4.1.3, lemma (1)), $G \wr \mathbb{Z}/p$ has such a family; by the second part of the same lemma, it satisfies (*), so that we may simply appeal to the induction hypothesis.

2. n is prime to p. The subgroup $G \times (G \wr S_{n-1})$ has index prime to p, so it detects the mod p Chow rings, and it is enough to exhibit a p-detecting family for this subgroup. Now by the induction hypothesis there is such a family $(H_i)_{i\in I}$ for $H = G \wr S_{n-1}$. Consider the following commutative diagram

$$\begin{array}{ccc} CH^*(BH) & \longrightarrow & \bigoplus_i CH^*(BH_i) \\ \downarrow & & \downarrow \\ H^*(H,\mathbb{Z})\otimes\mathbb{F}_p & \longrightarrow & \bigoplus_i H^*(H_i,\mathbb{Z})\otimes\mathbb{F}_p \end{array}$$

   The top map is injective, as is the right vertical one (because H_i is abelian). We deduce that the left vertical map is injective as well.

   From (1.1.4, lemma (2)), we conclude that

$$CH^*(B(G\times H)) \approx CH^*(BG)\otimes CH^*(BH)$$

   and the same holds with H_i in place of H. It follows that the map

$$CH^*(B(G\times H)) \longrightarrow \bigoplus_i CH^*(B(G\times H_i))$$

   is none other than the map on the top of the last diagram above, tensored by $CH^*(BG)$. It is therefore injective.

Let $(K_j)_{j\in J}$ be a p-detecting family for G, and repeat the argument with some H_i in place of G, and the K_j's in place of the H_i's: we end up with an injective map:

$$CH^*(B(G\times H_i)) \longrightarrow \bigoplus_j CH^*(B(K_j\times H_i))$$

Composing, we finally obtain an injective map

$$CH^*(B(G\times H)) \longrightarrow \bigoplus_{i,j} CH^*(B(K_j\times H_i))$$

Thus $(K_j\times H_i)_{(i,j)\in I\times J}$ is a p-detecting family of abelian groups for $G\times H$.

This completes the induction step. □

Remark. This lemma does not only refine our previous results (for $a=1$), its proof also shows that $G\wr S_n$ possesses a subgroup of index prime to p satisfying (*), which will be used to later in order to apply the Künneth formula.

4.1.4. More Brauer lifts. We recall another way of lifting representations from characteristic l to characteristic 0, thus creating interesting new characters. From chapter 18 of Serre's book [39] we keep the following ingredient:

Proposition. *Let G be a finite group, and let $\rho: G\to GL(n,\mathbb{F})$ be a representation of G over a finite field $\mathbb{F}$. Let m be the least common multiple of the orders of the regular elements of G, let $\mathbb{E}$ be the extension of $\mathbb{F}$ obtained by adjoining the m-th roots of unity, and suppose given an embedding $\psi:\mathbb{E}^*\to\mathbb{C}^*$. Put*

$$\chi(g) = \sum_\lambda \psi(\lambda)$$

where the sum runs over all eigenvalues of $\rho(g)$ counted with multiplicities.

Then $\chi\in R(G)$, that is, χ is a virtual character of G.

Remark. Recall that the *regular* elements of G are those whose order is prime to l. Note also that such an embedding ψ clearly always exists, but is not unique. Most of what follows depends on this choice. By abuse of language we refer to χ simply as the Brauer lift of ρ (thus ignoring the dependance on ψ).

4.1.5. In the case of $G=GL(n,k)$ there is an obvious candidate for such a lift to characteristic 0, and it turns out that the Chern classes of the virtual character thus obtained satisfy an important property:

Proposition. *Write $CH^*(BC^m)/l=\mathbb{F}_p[\eta_1,\cdots,\eta_m]$. Let ρ be the lift, as defined above, of the obvious representation of $GL(n,k)$ on k^n, and let $\bar{c_i}$ be the restriction to C^m of the Chern class $c_i(\rho)$. Then $\bar{c_{ir}}=0$ if $i>m$, and otherwise $\bar{c_{ir}}$ is (up to sign) the i-th symmetric function on the variables η_j^r, for $1\le j\le m$.*

Proof. Consider first the cyclic group C and its natural representation θ on $\mathbb{C}$. Its first Chern class $\eta = c_1(\theta)$ is such that $CH^*(BC) = \mathbb{F}_p[\eta]$. Consider also the representation

$$W = \theta \oplus \theta^{\otimes q} \cdots \oplus \theta^{\otimes q^{r-1}}$$

The homomorphism $x \mapsto qx$ of C sends θ to $\theta^{\otimes q}$ and hence it sends η to $q\eta$. The ring of fixed elements in the Chow ring is $\mathbb{F}_p[\eta^r]$ (see the argument in the proof of theorem 4.1.6 below). Because W is unchanged by this transformation, its Chern classes are necessarily 0 in dimensions other than r (and 0). In dimension r, by the sum formula, $c_r(W) = q^{r(r-1)/2}\eta^r$. But note that $q^{r(r-1)/2} = (-1)^{r-1}$ modulo p.

Next consider the group C^m whose Chow ring is $\mathbb{F}_p[\eta_1, \cdots, \eta_m]$. We let W_i be the lift of the above representation of C via the i-th projection, and we now use the letter W to denote $W = \bigoplus_i W_i$. By the formula giving the Chern class of a sum, we deduce that c_{ir} is the i-th symmetric function in the η_j^r (and is 0 if $j > m$), up to a sign.

The proof will therefore be complete if we show that the restriction of ρ to C^m is isomorphic to W. To see this, let x be a generator of C. The action of x on K, considered as a k-linear map, has by definition the trace

$$tr_{K/k}(x) = \sum_{\sigma \in Gal(K/k)} \sigma(x) = \sum_{i=1}^{r-1} x^{q^i}$$

Result follows. □

4.1.6. Main result. We are now in position to prove:

Theorem. *Let $k = \mathbb{F}_q$ where q is a power of the prime l. Let p be an odd prime different from l, put $r = [k(\mu_p) : k]$, and put $m = [n/r]$. Let ρ be the Brauer lift of the natural action of $GL(n,k)$ on k^n, and let c_i be its i-th Chern class. Then:*

$$CH^*(BGL(n,k)) = \mathbb{F}_p[c_r, c_{2r}, \cdots, c_{mr}]$$

Note that if $m = 0$, this means $CH^*(BGL(n,k)) = \mathbb{F}_p$ in dimension 0, as announced in the introduction.

Proof. Let $N = \pi \wr S_m$, a subgroup of $GL(n,k)$ which normalizes C^m. We have a restriction map:

$$CH^*(BGL(n,k)) \longrightarrow (CH^*(BC^m))^N$$

which we will prove to be an isomorphism.

Let us write $CH^*(BC^m) = \mathbb{F}_p[\eta_1, \cdots, \eta_m]$. Recall that π is cyclic generated by the Frobenius map $x \mapsto x^q$; this element acts on η_i by multiplication by q, and hence on η_i^s by multiplication by q^s. It follows that an invariant element of the Chow ring has to be a polynomial in the η_i^r (because $q^s = 1$ modulo p if and only if the p-th roots of unity are in $\mathbb{F}_{q^s}$, ie if and only if $r|s$). Furthermore S_m acts by permuting the indices, so that finally the ring of fixed elements is the polynomial algebra on the s_i, the symmetric functions in the η_j^r.

It is then immediate from (4.1.5, proposition) that the above map is surjective. If we can show that it is injective as well, then $CH^*(BGL(n,k))$ will have the desired description.

First consider the group $C \wr S_m$. Now, C has a cyclic subgroup of order p^a, which is a p-detecting family in itself, so that by (4.1.3, lemma), $C \wr S_m$ has also a p-detecting family of abelian groups whose exponent divides p^a (noting that a cyclic group satisfies (*)). Observe now that the index of $C \wr S_m$ in $GL(n,k)$ is prime to p (cf proof of proposition 4 in [35], so it detects the modulo p Chow ring of $GL(n,k)$, and therefore the family of subgroups just considered works for $GL(n,k)$ itself. But now from (4.1.2, lemma), it is clear that the restriction map above is injective. □

Corollary. *If k contains the p^b-th roots of unity for some integer b, then*

$$CH^*(BGL(n,k))/p^b = \mathbb{Z}/p^b[c_1, \cdots, c_n]$$

Proof. For $b = 1$ this is just a particular case of the above theorem. For general b, it is always true that $CH^*(BC^n)/p^b = \mathbb{Z}/p^b[\eta_1, \cdots, \eta_n]$. Since the Galois group acts now trivially, the ring of fixed elements remains easy to describe: it is $\mathbb{Z}/p^b[s_1, \cdots, s_n]$ (notation as above). Therefore the restriction map already considered is still surjective even if we take the coefficients in $\mathbb{Z}/p^b$. One concludes using induction and a cardinality argument. □

4.1.7. Back to cobordism. Since we have seen that $MU^*(BGL(n,k))\hat{\otimes}_{MU^*}\mathbb{F}_p$ and $CH^*(BGL(n,k))$ have the same description, it is quite easy to see (at least) that the cycle map is an isomorphism: indeed consider the diagram:

$$\begin{array}{ccc} CH^*(BGL_n(\mathbb{C})) & \longrightarrow & CH^*(BGL(n,k)) \\ \approx\downarrow & & \downarrow cl^0 \\ MU^*(BGL_n(\mathbb{C}))\hat{\otimes}_{MU^*}\mathbb{F}_p & \longrightarrow & MU^*(BGL(n,k))\hat{\otimes}_{MU^*}\mathbb{F}_p \end{array}$$

The map on the left is an isomorphism because the variety of Grassmannians, ie $BGL_n(\mathbb{C})$, has an algebraic cell decomposition, so $CH^*(BGL_n(\mathbb{C})) = H^*(BGL_n(\mathbb{C}), \mathbb{Z})$ (cf [16]), while on the other hand $H^*(BGL_n(\mathbb{C}), \mathbb{Z}) = MU^*(BGL_n(\mathbb{C}))\hat{\otimes}_{MU^*}\mathbb{Z}$ because the cohomology ring has no torsion. This would be enough to conclude that cl^0 is surjective and hence an isomorphism if we could only be sure that the diagram is commutative. However this is not clear: the horizontal maps are essentially coming from the two different maps $BGL(n,k) \to BGL_n(\mathbb{C})$ constructed in 3.3.2 and 4.1.4, and it would require some work to prove that they coincide, if they do at all.

Instead we take a different route, which does not even use the computation *a priori* of $MU^*(BGL(n,k))\hat{\otimes}_{MU^*}\mathbb{F}_p$. It has the virtue of involving an interesting lemma, a souped-up version of the results of Ravenel-Wilson-Yagita [36] and Hopkins-Kuhn-Ravenel [23]:

Lemma. *Let G be a finite group such that $K(m)^*(BG)$ is generated by transferred Euler classes as a $K(m)^*$-module, for all m. Then $G \wr S_n$ has the same property, for any $n \geq 0$.*

Proof. We shall make use of the fact that if A is a subgroup of the finite group H with index prime to p, then the map $K(m)^*(BH) \to K(m)^*(BA)$ is injective whereas the transfer is surjective (where, as always, we use Morava K-theory at the prime p). Indeed, if you consider the spectra of BA and BH localized at p, then by looking at cohomology with $\mathbb{Z}_{(p)}$ coefficients shows that $BH_{(p)}$ is a retract of $BA_{(p)}$ (we omit the symbol Σ^∞). But localizing preserves $\mathbb{Z}_{(p)}$ cohomology, hence preserves mod p cohomology (Five Lemma), hence preserves Morava K-theory (Atiyah-Hirzebruch spectral sequence).

We use the same type of induction as before.

1. $n = n'p$. In this case the subgroup $(G \wr \mathbb{Z}/p) \wr S_{n'}$ has index prime to p, and the corestriction (transfer) map is surjective, so it is enough to prove the result for this group. But $G \wr \mathbb{Z}/p$ satisfies the same property as G by [23], and we only have to apply the induction hypothesis.

2. n is prime to p. The subgroup $G \times (G \wr S_{n-1})$ has index prime to p, so again it is enough to prove the result for this subgroup. The Künneth formula for Morava K-theories gives
$$K(m)^*(B(G \times G \wr S_{n-1})) = K(m)^*(BG)\hat{\otimes}_{K^*(m)}K(m)^*(B(G \wr S_{n-1}))$$
from which the result follows at once.

This completes the induction step. □

Corollary (1). *Let G be as above. Then $MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$ is generated by transfered Euler classes, and so is $MU^*(B(G \wr S_n))\hat{\otimes}_{MU^*}\mathbb{F}_p$.*

Proof. By corollary 2.2.1 in [36], $BP^*(BG)$ is generated by transfers of Euler classes as a BP^*-module, so $BP^*(BG)\hat{\otimes}_{BP^*}\mathbb{F}_p$ is generated by such classes as an abelian group, and the result follows for G. But by the lemma $G \wr S_n$ can replace G. □

Corollary (2). *For G as above (for example, G abelian), the two maps*
$$CH^*(BG) \to MU^*(BG)\hat{\otimes}_{MU^*}\mathbb{F}_p$$
and
$$CH^*(B(G \wr S_n)) \to MU^*(B(G \wr S_n))\hat{\otimes}_{MU^*}\mathbb{F}_p$$
are surjective.

Remark. In this section “generated” means “topologically generated”, and “surjective” means “with a dense image”. The reader will check that this is of no consequence in what follows.

4.1.8 Theorem. *Let k be a finite field of characteristic $l \neq p$ and assume that p is odd. Then the map*
$$CH^*(BGL(n,k)) \to MU^*(BGL(n,k))\hat{\otimes}_{MU^*}\mathbb{F}_p$$
is an isomorphism.

Proof. Recall that $GL(n,k)$ has a subgroup C^m (product of m cyclic groups) which is normalized by some subgroup N, such that the restriction map induces an isomorphism $CH^*(BGL(n,k)) \to (CH^*(BC^m))^N$, the subring of fixed elements. Now consider the following commutative diagram:

$$\begin{array}{ccccc}
CH^*(BGL(n,k)) & \xrightarrow{i_0^*} & CH^*(B(C \wr S_m)) & \xrightarrow{j_0^*} & CH^*(BC^m) \\
\downarrow & & cl\downarrow & & \phi\downarrow\approx \\
MU^*(BGL(n,k))\hat{\otimes}_{MU^*}\mathbb{F}_p & \xrightarrow{i_1^*} & MU^*(B(C \wr S_m))\hat{\otimes}_{MU^*}\mathbb{F}_p & \xrightarrow{j_1^*} & MU^*(BC^m)\hat{\otimes}_{MU^*}\mathbb{F}_p
\end{array}$$

We have just indicated the image of the composition $j_0^* \circ i_0^*$.

The maps i_0^* and i_1^* are injective for index reasons, and j_0^* is known to be injective. By (4.1.7, corollary (2)), the map cl is surjective, and it follows that j_1^* is injective.

Also, because the isomorphism ϕ above (C^m being abelian) is natural with respect to maps induced by homomorphisms of groups, it induces also an isomorphism between the subrings of fixed elements under the action of N. This describes the image of $j_1^* \circ i_1^*$ (namely, all of the fixed subring), and concludes the proof. □

§4.2. Reducing to the simply-connected case

We turn our attention again to a general Chevalley group G. From now on, p is assumed to be odd. Our first task is to explain how to go from semi-simple to simply-connected.

So assume that G is semi-simple and consider now the universal cover $\tilde{G}$ of G, ie the unique Chevalley group such that $\tilde{G}_{\mathbb{C}}$ is the universal cover of $G_{\mathbb{C}}$. The centre of a semi-simple Chevalley group is finite and contained in any maximal torus, hence is a product of groups μ_a (a-th roots of unity). These groups have the peculiarity that all subgroups of $\mu_a(\mathbb{C})$ are defined over $\mathbb{Z}$, and the same can be said thus of the subgroups of the centre $\mathcal{Z}(\mathbb{C})$ of $\tilde{G}_{\mathbb{C}}$. In particular $\pi_1(G_{\mathbb{C}})$ can be seen as a finite and central subgroup π of $\tilde{G}$, defined over $\mathbb{Z}$. Now, the groups $\tilde{G}/\pi$ and G agree over $\mathbb{C}$, and hence agree over $\mathbb{Z}$, by the structure theorem for Chevalley groups, cf [2], exposé XXV, théorème 1.1.

The map $\tilde{G}(k) \to G(k)$, where as always k is finite, may not be as nice as one would expect it to be (eg it is almost never surjective). Its kernel, at least, is certainly a subgroup of $\pi(k)$, so it is central. The order of $\mu_a(k)$ divides that of $\mu_a(\mathbb{C})$, and it follows that $|\pi(k)|$ divides $|\pi(\mathbb{C})| = |\pi_1(G_{\mathbb{C}})|$ (this is a result that holds for more general groups, but it is trivial to check it directly here). Now let $Tors(G)$ denote the set of prime numbers l such that $H^*(BG_{\mathbb{C}}, \mathbb{Z})$ has l-torsion. All prime numbers dividing the order of $\pi_1(G_{\mathbb{C}})$ are in $Tors(G)$, in fact $Tors(G) = Tors(\tilde{G}) \cup \{l : l \text{ divides } |\pi_1(G_{\mathbb{C}})|\}$: see [40], 1.3.1 to 1.3.4, and the references there, in particular [6]. So our standing assumption that $H^*(BG_{\mathbb{C}}, \mathbb{Z})$ have no p-torsion implies that the order of $\pi(k)$, and thus that of the kernel of $\tilde{G}(k) \to G(k)$, is prime to p.

We also have to note that $|\tilde{G}(k)| = |G(k)|$: this is very specific to finite fields, and is proved in [7], proposition 16.8. It follows that the image of $\tilde{G}(k)$ in $G(k)$ has index prime to p.

We note:

Lemma. *Let Γ be given as a quotient $\tilde{\Gamma}/C$ where C is a central subgroup of the finite group $\tilde{\Gamma}$. Assume that the order of C is prime to p. Then $H^*\Gamma = H^*\tilde{\Gamma}$ and $CH^*B\Gamma = CH^*B\tilde{\Gamma}$.*

Proof. A spectral sequence argument gives the result immediately for cohomology. The result for Chow rings follows from remark 1.3.6, but of course it is preferable to find an elementary argument (even if we only use the easier part of 1.3.6). We let $\tilde{S}$ and S denote Sylow subgroups of $\tilde{\Gamma}$ and Γ respectively. Clearly $\tilde{S}$ maps isomorphically to (a conjugate of) S under the quotient map; we now use proposition 1.3.5. To prove that the stable subgring for S is sent isomorphically onto the stable subring for $\tilde{S}$ under the isomorphism $CH^*BS \to CH^*B\tilde{S}$, things come down to checking the following: if $\tilde{x} \in \tilde{S}$ and if $\tilde{g} \in \tilde{G}(k)$, then $\tilde{g}\tilde{x}\tilde{g}^{-1}$ is in $\tilde{S}$ if and only if gxg^{-1} is in S, where x and g correspond to $\tilde{x}$ and $\tilde{g}$ under the projection $\tilde{\Gamma} \to \Gamma$. To see the non-trivial part of this, assume that $gxg^{-1} \in S$ and write $\tilde{g}\tilde{x}\tilde{g}^{-1} = c \cdot s$ with $c \in \pi(k)$ and $s \in \tilde{S}$. Writing $o(\sigma)$ for the order of an element σ, we have $o(c \cdot s) = o(c)o(s)$ because c and s commute and have coprime orders. But $o(c \cdot s)$ is a power of p, as this element is conjugate to $\tilde{x}$. Therefore $o(c) = 1$ and $c = 1$. $\square$

Applying this with Γ taken to be the image of $\tilde{G}(k)$ in $G(k)$ shows that $CH^*BG(k)$ is a direct summand in $CH^*B\tilde{G}(k)$ (in $\mathfrak{U}^{ev}$). So if $CH^*B\tilde{G}(k)$ is reduced or $\mathfrak{Nil}$-closed, so is $CH^*BG(k)$. In fact, we can say something more precise.

Letting Γ be as above, we claim that $G(k)$ is the product of $T(k)$ and Γ (although these subgroups have a non-trivial intersection, of course). To see this, let K be an algebraic closure of k; we may assume to have chosen a maximal torus $\tilde{T}$ in $\tilde{G}$ such that $p(\tilde{T}(K)) = T(K)$, where $p : \tilde{G} \to G$ is the natural map. Since $\tilde{T}(k)$ contains the kernel C of the map $\tilde{G}(k) \to G(k)$, it follows easily that the image E of $\tilde{T}(k)$ in $T(k)$ is all of $\Gamma \cap T(k)$. Moreover, since $\tilde{T}(k)$ and $T(k)$ have the same order (by the result already quoted, or directly), it follows that $[T(k) : E] = [G(k) : \Gamma]$ (in turn these indices equal the order of C). Consequently, the number of products $\gamma \cdot t$ with $\gamma \in \Gamma$ and $t \in T(k)$ is

$$\frac{|\Gamma| \cdot |T(k)|}{|\Gamma \cap T(k)|} = |\Gamma| \cdot [T(k) : E] = |\Gamma| \cdot [G(k) : \Gamma] = |G(k)|$$

Thus $G(k) = \Gamma \cdot T(k)$, as claimed. We apply now the double coset formula (see the appendix), which gives here:

$$i^*_{T(k)\to G(k)} \circ i_*^{\Gamma\to G(k)} = i_*^{E\to T(k)} \circ i^*_{E\to\Gamma}$$

Since E has index prime to p in the abelian group $T(k)$, it is clear that $i_*^{E\to T(k)}$ is an isomorphism. The transfer $i_*^{\Gamma\to G(k)}$ is surjective for index reasons. Thus we see from the formula above that the surjectivity of $CH^*BG(k) \to (CH^*BT(k))^W$ is equivalent to that of $CH^*B\Gamma \to (CH^*BE)^W$. From the last lemma applied to Γ and E, this is also equivalent to $\tilde{G}$ satisfying (2). Summarizing:

Proposition. *Let $\tilde{G}$ be the universal cover of the semi-simple Chevalley group G. If $\tilde{G}$ satisfies (1), resp. (2), then G satisfies (1), resp. (2). Moreover if G satisfies (2), then so does $\tilde{G}$.*

Remark. It is clear from the discussion above that the proposition holds a bit more generally: if $\tilde{G} \to G$ is an isogeny (over $\mathbb{Z}$) whose kernel C is central and has $|C(k)|$ prime to p, then the conclusion of the proposition holds.

Remark (on notation). The simply-connected group $Spin_n$ has a central subgroup μ_2, and the quotient $Spin_n/\mu_2$ is the orthogonal group SO_n. Note that the group we call $SO_n(k)$ is usually denoted $SO_n^+(k)$, while $SO_{2n}(\mathbb{R})$ sometimes goes under the name $SO(n,n)$ (and $SO_{2n+1}(\mathbb{R})$ can be $SO(n+1,n)$). The familiar group of orthogonal motions of $\mathbb{R}^n$ with determinant 1, normally denoted $SO(n)$, is the compact form of the group we consider.

§4.3. Examples

4.3.1. When p does not divide $|W|$. We assume that G is semi-simple and as usual that the characteristic of k is not p. Under these hypotheses, a theorem of Springer-Steinberg ([41] 5.19) asserts that a p-Sylow S of $G(k)$ normalises a maximal torus T' defined over k. This T' does not have to be split; however if we suppose further that p does not divide the order of the Weyl group of G, it follows that $S \subset T'(k)$, and in particular S is abelian. If we denote its normaliser in $G(k)$ by N_S, we have by Swan's lemma (1.3.7)

$$CH^*BG(k) = (CH^*BS)^{N_S}$$

Now, let S' be the maximal p-elementary subgroup of S. We have $CH^*BS = CH^*BS'$. Let $N_{S'}$ denote the normaliser of S' in $G(k)$. Clearly $N_S \subset N_{S'}$ (since S' is precisely the subgroup of S of elements of order p, it is preserved by any automorphism of S) and thus $(CH^*BS')^{N_{S'}} \subset (CH^*BS')^{N_S}$. However since there are isomorphisms

$$CH^*BG(k) \to (CH^*BS)^{N_S} \to (CH^*BS')^{N_S}$$

we obtain the reverse inclusion, and $CH^*BG(k) = (CH^*BS')^{N_{S'}}$.

Now, S' being a p-elementary subgroup of S of maximal rank, it is also of maximal rank in $G(k)$; since (by 3.4) we know that S' is (conjugated to) a subgroup of $T(k)$ where T is our fixed, split maximal torus, we deduce that S' is precisely the maximal elementary abelian subgroup of $T(k)$ and that $CH^*BT(k) = CH^*BS'$.

Finally, the group of automorphisms of CH^*BS' induced by $N_{S'}$ is the same as that induced by $N_T(k)$ (or W), by (3.1.2, lemma). Thus:

Proposition. *Suppose G is semi-simple. If p does not divide $|W|$, then $CH^*BG(k)$ is $\mathfrak{Nil}$-closed, ie*

$$CH^*BG(k) = (CH^*BT(k))^W$$

Example. Consider the five exceptional Lie groups G_2, F_4, E_6, E_7 and E_8. Their Weyl groups have orders 12, $2^7.3^2$, $2^7.3^4.5$, $2^{10}.3^4.5.7$ and $2^{14}.3^5.5^2.7$ respectively. The primes which give torsion in their integral cohomology are $\{2\}$ for G_2, $\{2,3\}$ for F_4, E_6 and E_7, and $\{2,3,5\}$ for E_8 (see [40]). In other words, we can say that $CH^*BG(k)$ is always $\mathfrak{Nil}$-closed when G is exceptional and does not have p-torsion, except possibly in finitely many cases, namely when (G, p) is either $(G_2, 3)$, $(E_6, 5)$, $(E_7, 5)$, $(E_7, 7)$, or $(E_8, 7)$. Restricting attention to $p > 7$ rules out these cases.

4.3.2. Chern classes. Another case of interest is that of a group G such that $H^*BG_{\mathbb{C}}$ is generated by Chern classes of (finitely many) representations of $G_{\mathbb{C}}$. In this case we have:

Proposition. *Suppose that $H^*BG_{\mathbb{C}}$ is generated by Chern classes, and let p be odd. Then the map*

$$CH^*BG(k) \to MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$$

is surjective, as is

$$CH^*BG(k) \to (CH^*BT(k))^W$$

Proof. By assumption there is a surjective map

$$\bigotimes_i H^*BGL(n_i, \mathbb{C}) \to H^*BG_{\mathbb{C}}$$

The representations of $G_{\mathbb{C}}$ must be defined over $\mathbb{Z}$ (essentially because irreducible representations of G over $\mathbb{Z}$ are classified by their highest weights, just like those of $G_{\mathbb{C}}$ over $\mathbb{C}$; and as G is split, the weights are the same over $\mathbb{Z}$ or $\mathbb{C}$. Alternatively, appeal to [2], XXV, 1.1.). Consider the following commutative diagram:

$$\begin{tikzcd}
\bigotimes CH^*BGL(n_i, k) \arrow[r] \arrow[d, "a"', "\approx"] & CH^*BG(k) \arrow[r, "c"] \arrow[d, "b"'] & (CH^*BT(k))^W \arrow[d, "\approx"] \\
\bigotimes MU^*(BGL(n_i, k))\hat{\otimes}_{MU^*}\mathbb{F}_p \arrow[r, "d"] & MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p \arrow[r, "e"] & (MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p)^W
\end{tikzcd}$$

The map a is an isomorphism, as we have seen. Moreover d is surjective by the remark just made and the results of *loc cit.* It follows that b is surjective. Finally, e is an isomorphism by (3.4), so c is surjective. □

Example. Groups such that $H^*BG_{\mathbb{C}}$ is generated by Chern classes include SL_n, Sp_n, and SO_{2n+1} for odd p's. Here again we point out that SO_{2n+1} has 2-torsion in its integral cohomology, so the restriction that p be odd in the above proposition is not important anyway; however for SL_n and Sp_n, we are leaving out cases where the conclusion of the proposition might still be true.

4.3.3. The orthogonal groups. We have already established that, with some restrictions on p, all simple, simply connected groups satisfy condition (2), with the notable exception of $Spin_{2n}$. Replacing this group by SO_{2n} as we may, we can use an *ad hoc* argument.

We have seen in the previous chapter that a certain continuous map $BSO_{2n}(k) \to BSO_{2n}\mathbb{C}$ induces isomorphisms

$$MU^*(BSO_{2n}\mathbb{C})\hat{\otimes}_{MU^*}\mathbb{F}_p \xrightarrow{\approx} MU^*(BSO_{2n}(k))\hat{\otimes}_{MU^*}\mathbb{F}_p \xrightarrow{\approx} (MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p)^W$$

Each of these rings is a polynomial ring over $\mathbb{F}_p$ on variables c_{2i} $(1 \leq i \leq n-1)$ and e. By definition then, these classes are the Pontryagin classes and the Euler class of the oriented bundle $E \to BSO_{2n}(k)$ induced by the classifying map $BSO_{2n}(k) \to BSO_{2n}\mathbb{C}$. Now if this bundle were algebraic, it would have Pontryagin and Euler classes in $CH^*BSO_{2n}(k)$ which would restrict to the elements with the same name in $CH^*BT(k)$, proving that (2) holds (recall that $CH^*BT(k) = MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$, of course, and that the cobordim modules are $\mathfrak{Nil}$-closed, cf 3.4). However E need not be algebraic.

The Pontryagin classes are Chern classes and one can argue as in 4.3.2 to prove that they exist in $CH^*BSO_{2n}(k)$. We need more work for the Euler class.

So let S be a p-Sylow of $SO_{2n}(k)$. It is proved in [10] that any map $BP \to BG$, where p is a finite p-group and G a compact Lie group, comes from a homomorphism $P \to G$, up to homotopy. Therefore the composition $BS \to BSO_{2n}(k) \to BSO_{2n}\mathbb{C}$ is homotopic to a map coming from a homomorphism $S \to SO_{2n}\mathbb{C}$. As S is finite, this is automatically a map of algebraic groups, and the map $BS \to BSO_{2n}\mathbb{C}$ is algebraic. Using this map to pull back the universal vector bundle over $BSO_{2n}\mathbb{C}$, we get an algebraic vector bundle over BS carrying a quadratic form and which is equivalent to E (or rather, to the pull back of E over BS). By [12], this bundle has an Euler class in the Chow ring, ie there is a class in CH^*BS mapping under the cycle map to the Euler class in $MU^*(BS)\hat{\otimes}_{MU^*}\mathbb{F}_p$ (up to scalar multiplication by a power of 2, but we are still assuming that p is odd). Call this $x \in CH^*BS$, put $e' = i_*^{S\to SO_{2n}(k)}(x)$, and finally let $e = i^*_{T(k)\to SO_{2n}(k)}(e')$. To prove that this is the Euler class defined above (and justify the notation), it is enough to check this in $MU^*(BT(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$, that is, after applying the cycle map. But in cobordism we have Euler classes for topological bundles, and $x = i^*_{S\to SO_{2n}(k)}(y)$ for some $y \in MU^*(BSO_{2n}(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$, so that $e' = y$ up to a nonzero scalar (we abuse the notations x and e' here). It follows that e is indeed the Euler class that we want.

4.3.4. Condition (1) and wreath products. The Weyl groups of the simple, simply-connected Lie groups have a lot in common. Namely, for the classical groups at least, W always contains a copy of the symmetric group acting on the torus by permuting the eigenvalues. This often gives $N_T(k)$ a tractable group structure. The following lemma is trivial, but it is remarkable to note how often it applies.

Lemma. *Suppose that $G(k)$ contains a subgroup of index prime to p which is of the form $S_r \ltimes T(k) = S_r \wr k^*$ (here r is the rank of G). Then the restriction map:*

$$CH^*BG(k) \to CH^*BT(k)$$

is injective.

Proof. (Compare with 2.2.12.) The restriction to $S_r \wr k^*$ is injective for index reasons, and it follows from (4.1.3, lemma 2), that $S_r \wr k^*$ possesses a collection of abelian p-groups which detect the Chow ring. Thus the Chow ring is also detected by the family of elementary abelian p-subgroups. As observed above, these are conjugated to subgroups of $T(k)$. □

Remark. The reader will have noticed that an easy induction based on 2.2.12 will suffice to prove that $S_r \wr k^*$ has a reduced Chow ring. However, as remarked at the end of 4.1.3, the argument above also shows that this group possesses a subgroup H of index prime to p whose Chow ring is reduced and which satisfies (*). Since two such groups H_1 and H_2 have a Künneth formula, in the sense that

$$CH^*BH_1 \times BH_2 = CH^*BH_1 \otimes CH^*BH_2,$$

it follows that if the lemma applies for two Chevalley groups G_1 and G_2, then their product also has a reduced Chow ring.

Example (1). It is easily seen that the lemma applies, for odd p, for the groups Sp_n and $Spin_n$.

Example (2). In some cases we can also deal with SL_n. Here $N_T(k)$ is of the form $T(k) \rtimes S_n$, but the action of S_n is trickier: if we see $T(k)$ as $\{(x_1, \cdots, x_n) \in (k^*)^n : x_1 \cdots x_n = 1\}$ (that is, as a subgroup of the maximal torus for GL_n), then S_n acts by permutation of the x_i's. However, when n is prime to p, then we can have a look at the subgroup $T \rtimes S_{n-1}$ where S_{n-1} sits in S_n as the subgroup which fixes, say, the n-th coordinate. This subgroup is indeed a wreath product $S_{n-1} \wr k^*$ of index prime to p in $N_T(k)$, and when p is also assumed to be odd then $N_T(k)$ has order prime to p in $SL_n(k)$.

4.3.5. Conclusion.

Ye shall know the truth, and the truth shall make you mad.
Aldous Huxley

Let us put together the information gathered so far. First of all, there is the

Theorem. *Let G be a Chevalley group, let p be a prime number, and let k be a finite field of characteristic $\neq p$ containing the p-th roots of unity. Assume that G has no p-torsion. Then $CH^*BG(k)$ is $\mathfrak{Nil}$-closed in $\mathfrak{U}^{ev}$ if G is locally isomorphic to a product of the following groups:*

- *Type A_{n-1}: GL_n for $p > 2$; SL_n if n is prime to p and $p > 2$;*
- *Type B_n or D_n: $Spin_n$ for $p > 2$;*
- *Type C_n: Sp_n for $p > 2$;*
- *Exceptional type: G_2 for $p > 3$; F_4 for $p > 3$; E_6 for $p > 5$; E_7 for $p > 7$; E_8 for $p > 7$.*

Here "locally isomorphic" refers to the situation in 4.2 (see the remark following the proposition there: this is why it makes sense to include GL_n, which is not simply-connected, in the list).

Our main omission with simply-connected groups is SL_n when p divides n. If one could prove the result in this case as well, we would be able to say that $CH^*BG(k)$ is $\mathfrak{Nil}$-closed for *any* semi-simple group (and say, for $p > 7$).

Nevertheless, we have

Proposition. *Let G be any semi-simple Chevalley group having no p-torsion, and let k be any finite field of characteristic $\neq p$. Let K be the finite field obtained from k by adjoining the p-roots of unity and let $r = [K : k]$.*

Then the map

$$CH^*BG(k) \to MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$$

is surjective. Moreover for a certain Brauer lift $BG(k) \to BG_{\mathbb{C}}$, the image of $H^*BG_{\mathbb{C}} \to H^*BG(k)$ coincides with the image of the Chow ring under the cycle map. Writing $H^*BG_{\mathbb{C}} = \mathbb{F}_p[s_1, \cdots, s_n]$, this image is

$$\mathbb{F}_p[s_i : 2r \textit{ divides } |s_i|].$$

Proof. The point is that we have established (2) for any simply-connected group, hence for any semi-simple group by 4.2. The first statement follows immediately from this and 3.4, at least for K; but the map $MU^*(BG(K))\hat{\otimes}_{MU^*}\mathbb{F}_p \to MU^*(BG(k))\hat{\otimes}_{MU^*}\mathbb{F}_p$ is always surjective.

Everything else in the proposition has already been proved, see in particular 3.5. $\square$

CHAPTER

FIVE

Perspectives

It is not necesssary to understand things in order to argue about them.
Caron de Beaumarchais

§5.1. $\mathbb{A}^1$ homotopy

Morel and Voevodsky have defined the $\mathbb{A}^1$ homotopy of schemes, which allows one to "do homotopy" in the category of schemes. In this setting the classifying space of an algebraic group shows up naturally, whereas as far as we are concerned in this thesis, CH^*BG is not the Chow ring of an object BG. The way in which classifying spaces arise in Morel-Voevodsky's category has a subtle aspect, and it is probably worth giving a few details.

Let $\mathcal{V}$ be the category of manifolds and smooth maps, or any category of locally contractible topological spaces and continuous maps. We describe an alternative treatment of homotopy in $\mathcal{V}$, which will have the advantage of applying in great generality. Let $\mathfrak{M}$ (for models) be the category of simplicial sheaves on $\mathcal{V}$ – contravariant functors from $\mathcal{V}$ to the category of simplicial sets satisfying the "sheaf condition", a crucial point which we shall discuss below. We can embed $\mathcal{V}$ into $\mathfrak{M}$ by viewing a manifold V as the sheaf $U \mapsto Hom_{\mathcal{V}}(U, V)$.

There is a notion of homotopy in $\mathfrak{M}$ which uses the unit interval Δ^1 seen as a constant sheaf; call this simplicial homotopy. Two manifolds V and W are never simplicially homotopic unless they are equal, as they are "simplicially constant" sheaves in $\mathfrak{M}$. To circumvent this, the next idea is thus to force $\mathbb{A}^1 = \mathbb{R}$ to be homotopic to a point, as follows. There is a closed model category structure on $\mathfrak{M}$, the weak equivalences being maps of sheaves inducing weak equivalences of simplicial sets on the stalks. We then take the associated homotopy category $\mathcal{H}_s$ and localize further with respect to the class of maps $V \times \mathbb{A}^1 \to V$ (see for example [9]). Call the resulting category $\mathcal{H}(\mathcal{V})$.

Now if $\mathcal{V}$ is reduced to a point (and forgetting about $\mathbb{A}^1$ which in this case is not even in $\mathcal{V}$), then it is clear that $\mathcal{H}(point)$ is the usual homotopy category (of simplicial sets). But in general now, one can see as an easy consequence of the definitions that two homotopic spaces V and W in $\mathcal{V}$ are isomorphic in $\mathcal{H}(\mathcal{V})$; based on this simple observation, since we assume that the spaces in $\mathcal{V}$ are built of pieces which are contractible (homotopic to points), one can show without too much trouble ([28], prop 3.3.3) that $\mathcal{H}(\mathcal{V})$ is *always* isomorphic to the usual homotopy category.

But there is more. It is in fact the case, if we denote by $[V, W]$ the usual homotopy classes of maps between V and W, that

$$[V, W] = Hom_{\mathcal{H}(\mathcal{V})}(V, W).$$

A priori this is not obvious, and we only know that

$$[V, W] = Hom_{\mathcal{H}(\mathcal{V})}(V', W')$$

for some spaces (simplicial sets) V' and W'. But if one sends $V \in \mathcal{V}$ to $\mathcal{H}(\mathcal{V})$ not as above but via $Sing(V) \in \mathcal{H}(point)$ (the singular simplicial set associated to V) and then via the equivalence of categories $\mathcal{H}(point) \to \mathcal{H}(\mathcal{V})$, one gets an object isomorphic to V, or so we claim. Granting this, we have reduced the statement to the classical result that the functor $Sing$ induces an equivalence of homotopy categories.

Briefly, one proves the claim as follows. For any open cover $\mathcal{U}$ of V, one considers the simplicial sheaf $\check{C}(\mathcal{U})$ which in degree n is (represented by) the disjoint union of all n-fold intersections of open sets in $\mathcal{U}$. It is easy to see that for any simplicial sheaf $\mathcal{X}.$ over V:

$$Hom(\mathcal{X}., \check{C}(\mathcal{U})) = Hom(\mathcal{X}_0, V)$$

where the first Hom is in the category of simplicial sheaves over V, and the second is for sheaves. Hence the object $\check{C}(\mathcal{U})$ converts the complexity of V as a sheaf into a simplicial sheaf that might be represented by very simple (contractible) spaces, while the equality above also ensures that $\check{C}(\mathcal{U})$ and V are isomorphic in $\mathcal{H}(\mathcal{V})$. Now, for a carefully chosen cover $\mathcal{U}$ (related to a smooth triangulation), $\check{C}(\mathcal{U})$ is essentially the same as the singular simplicial set associated to V.

What is the point of all this? Quite simply, the idea is to replace $\mathcal{V}$ by any category in which there is no notion of homotopy at hand, and perform the same construction as above to get one. In particular, Morel and Voevodsky consider the category Sm_k of smooth schemes over the field k, and build $\Delta^{op}(Shv(Sm_k))$, the category of simplicial sheaves over Sm_k, where they consider simplicial homotopy and localizations with respects to the maps $V \times \mathbb{A}^1 \to V$. The resulting homotopy category is denoted by $\mathcal{H}(k)$. The discussion above shows that it is the "right" thing to do since in the case of smooth manifolds one obtains the right homotopy category.

Now here comes a delicate point. A sheaf is not only a contravariant functor, but also satisfies a certain glueing property which is supposed to be the analog of the familiar condition for sheaves in the naive sense (on the category of open sets of a topological space). To make this precise it is necessary to introduce a Grothendieck topology on our category, ie a notion of covering (the category is then called a site). For $\mathcal{V}$ one may just take "covering" in the usual sense. For Sm_k though, we can choose between Zariski coverings, étale coverings, or Nisnevich coverings. Accordingly, we obtain a category $\mathcal{H}_\tau(k)$ with $\tau = Zar$, Nis, or et. (In $\mathcal{V}$ one may try to introduce these topologies, but the local inversion theorem says that they are equivalent.)

It turns out that the Nisnevich topology is technically much more convenient than the others. One example is that Postnikoff towers only have the desirable properties when $\tau = Nis$ (see the discussion in [28], section 3). The bottom line is that the right definition of the homotopy category of schemes is $\mathcal{H}_{Nis}(k)$.

However, it is hard to deny that the étale topology is a more natural thing to consider, and if we have indeed the right category in which to do algebraic geometry, then it should have something to say about étale coverings. Consider for example group sheaves – the definition should be obvious – and simplicial principal bundles or torsors – likewise. Again "sheaves" will depend on the topology, and étale torsors are certainly interesting objects. For any τ then, we may define an object BG_τ in the appropriate category exactly as one does for topological groups: in degree n, BG_τ is (represented by) G^n, with the usual simplicial structure (in other words, the simplicial structure is the same as that of the nerve of the category with one object and a morphism for each $g \in G$). With this definition, BG_τ is easy to handle, and as one would hope, it classifies G-torsors in the sense that for any smooth scheme X the set of isomorphism classes of G-torsors above X is in bijection with $Hom_{\mathcal{H}_\tau(k)}(X, BG_\tau)$.

Now, it is perfectly legitimate to study étale torsors in $\Delta^{op}(Shv_{Nis}(Sm_k))$. This is achieved by means of the morphism of sites π from the category Sm_k endowed with the étale topology to itself equipped with the Nisnevich topology and the corresponding push-forward functor π_* from étale sheaves to Nisnevich sheaves (similarly, a morphism $f : X \to Y$ of schemes defines a push-forward homomorphism f_* which relies solely on the operation f^{-1} pulling back open sets in Y to open sets in X; in the abstract setting, a morphism of sites π is defined in terms of an operation π^{-1}, in our case the forgetful functor from Nisnevich coverings to étale coverings). This functor does not preserve homotopy and we have to use its total right derived functor $R\pi_*$ to define $B_{et}G = R\pi_*(BG_{et})$. It is an object in $\Delta^{op}(Shv_{Nis}(Sm_k))$ which can be seen to classify *étale* G torsors; in symbols, $Hom_{\mathcal{H}_{Nis}(k)}(X, B_{et}G) = H^1_{et}(X, G)$. (To be perfectly rigorous we should consider pointed objects and maps).

There is less control over the object $B_{et}G$ than there is over BG_τ. However, it is proved in [28] that $B_{et}G$ is in fact homotopic to Totaro's definition of BG. That is, if one performs in $\Delta^{op}(Shv_{Nis}(Sm_k))$ the construction that we did in the introduction in the category of topological spaces, namely by embedding G into a general linear group and dividing the Steifel variety by the action of G (which is Totaro's idea for some particular representations), one gets an object isomorphic to $B_{et}G$ in $\mathcal{H}_{Nis}(k)$. In particular, any space in $\mathcal{H}_{Nis}(k)$ having a well-defined Chow ring, the ring CH^*BG we have been studying is $CH^*B_{et}G$.

There are a number of natural questions related to $B_{et}G$. For example, the $\mathbb{A}^1$ homotopy groups $\pi_n^{\mathbb{A}^1}(B_{et}G)$ are not known, and the 0-th group might be the most interesting (if only because it is not trivial which is perhaps counterintuitive). For instance when G is finite, $\pi_0^{\mathbb{A}^1}(B_{et}G)$ is the sheafification of $U \mapsto H^1_{et}(U, G)$.

Also, there is a natural map $BG_{Nis} \to B_{et}G$ and Morel conjectures that this map is the inclusion of the connected component corresponding to 0 (the trivial torsor).

§5.2. Algebraic cobordism

Morel and Levine have defined algebraic cobordism, which provides a different, perhaps more enlightening proof of the factorization

$$CH^*X \to MU^*X \otimes_{MU^*} \mathbb{Z} \to H^*(X, \mathbb{Z})$$

for a complex variety X.

They start by introducing algebraic cohomologies, which are functors from smooth schemes over a field k to graded rings satisfying a few axioms similar to the Steenrod-Eilenberg axioms, although weaker. In fact they are essentially the axioms satisfied by the Chow ring functor – such as the homotopy invariance as described in the Foreword or the existence of well-behaved Chern classes. We may call these axioms weaker than the usual ones because if one works over the complex numbers and considers a cohomology theory h^*, then it is also an algebraic cohomology; but not conversely, of course.

Then Morel and Levine proceed to prove the existence of a universal algebraic cohomology Ω^*, which they call algebraic cobordism in reference to the universality of MU^*. In a few words, what they do is associate a formal group to an algebraic cohomology, much as one does for oriented cohomologies, and then define Ω^* by forcing its formal group to be Lazard's universal example. As it turns out, the formal groups contain enough information about the cohomologies themselves for this construction to work.

In particular, over $\mathbb{C}$, one has a natural transformation $\Omega^*(X) \to MU^*(X)$, which is *not* an isomorphism as MU^* is not universal among algebraic cohomologies, only among oriented ones. Nonetheless, there is also a natural homomorphism

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \to MU^*X \otimes_{MU^*} \mathbb{Z}.$$

Finally, Morel and Levine prove that

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} = CH^*X$$

and also

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}[t, t^{-1}] = K^0(X)[t, t^{-1}]$$

which proves that the Chow ring (resp. localized K-theory) is the universal example of a cohomology whose formal group is a module over the additive (resp. multiplicative) formal group. (The homomorphism $\Omega^* \to \mathbb{Z}$ (resp. $\Omega^* \to \mathbb{Z}[t, t^{-1}]$) is Lazard's map classifying the additive (resp. multiplicative) formal group.)

In particular, these results together imply Totaro's factorization of the cycle map.

§5.3. Chow rings of projective varieties

The Chow ring of a general projective variety is far from being understood. There are examples for which the Chow ring is not countable, or not even parametrized by an algebraic variety. Thus it is worth getting one's hands on projective varieties with a Chow ring that is nontrivial but computable.

There is a conjecture of Totaro's ([47], conjecture 5.1) which asserts that certain complete intersections in projective space have the same Chow rings as classifying spaces of given (arbitrary) algebraic groups. Given this, our computations in this thesis could be used to exhibit projective varieties with interesting Chow rings.

The conjecture amounts to (or at least would follow from) a version of the Lefschetz theorem for Chow rings, which seems far out of reach even with the tools of $\mathbb{A}^1$ homotopy. However it is fairly reasonable to hope for it to be true, in view of other conjectures (Hartshorne, Nori), and because Totaro in *loc. cit.* is able to prove a weaker variant.

It is also strongly related to the paper of Atiyah-Hirzebruch ([3]) which disproves the Hodge conjecture over $\mathbb{Z}$. Namely, there it is shown that given a group G, there is a particular complete intersection X (originally considered by Serre) that has the same n-homotopy type as $BG \times BS^1$ for n less than the dimension of X. Totaro conjectures that the Chow ring of X also coincides with that of $BG \times BS^1$ in small codimensions. In turn, if one could prove this, one would disprove thereby a variant of the Hodge conjecture, as explained by Soulé. In the 2003 conference in Edinburgh on Hodge theory, Soulé pointed out that the example X used by Atiyah and Hirzebruch, which carried a p-torsion cohomology class not coming from the Chow ring (unlike what Hodge predicted), had a very large dimension compared to p. He suggested that for projective varieties of a given dimension, the property described by Hodge might be true for all large p. However he also remarked that Totaro's conjecture would imply the existence of a counterexample to this.

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Epilogue

An expert is someone who knows more and more about less and less, until eventually he knows everything about nothing.

Anonymous

Là, se rencontrent encore plus de causes pour la destruction physique et morale que partout ailleurs. Ces gens vivent, presque tous, en d'infectes études, en des salles d'audience empestées, dans de petits cabinets grillés, passent le jour courbés sous le poids des affaires, se lèvent dès l'aurore pour être en mesure, pour ne pas se laisser dévaliser, pour tout gagner ou pour ne rien perdre, pour saisir un homme ou son argent, pour emmancher ou démancher une affaire, pour tirer parti d'une circonstance fugitive, pour faire pendre ou acquitter un homme. [...] Quelle âme peut rester grande, pure, morale, généreuse, et conséquemment quelle figure demeure belle dans le dépravant exercice d'un métier qui force à supporter le poids des misères publiques, à les analyser, les peser, les estimer, les mettre en coupe réglée? Ces gens-là déposent leur coeur, où?... je ne sais; mais ils le laissent quelque part, quand ils en ont un, avant de descendre tous les matins au fond des peines qui poignent les familles.

Honoré de Balzac, *La fille aux yeux d'or.*

I don't want to achieve immortality through my work. I want to achieve it through not dying.

Woody Allen

Good-bye. I am leaving because I am bored.

George Saunders, last words