Extreme Value Theory for Multivariate Data

Anne-Laure Fougères

Equipe Modal'X

Université Paris Ouest – Nanterre

Assimilex workshop

June 24, 2008

Extreme events with inherent **multivariate** character

Example : Coastal flooding

European Union project *Neptune* 1995–1997

de Haan & de Ronde (1998), **Bruun & Tawn** (1998)

Extreme sea conditions	involve	high wave heights	and	high still water level (surge)
		HmO		SWL

Data : 828 storm events spread over 13 years in front of the Dutch coast (near Petten)

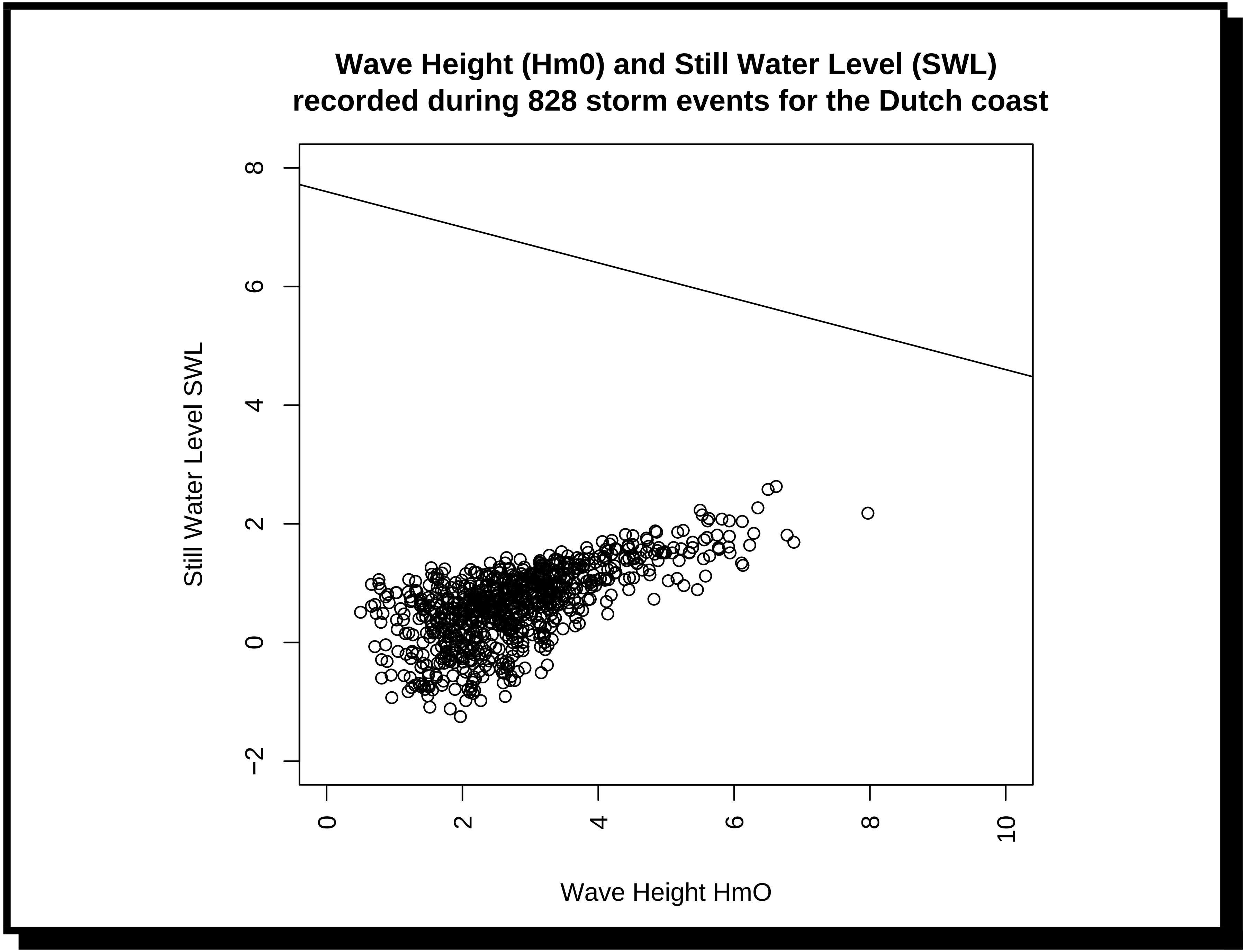
Problem :

Protection of a dike : Focus on a structure variable

$$\Delta(\mathbf{X}, \nu) = 0.3 \text{ HmO} + \text{SWL} - \nu, \quad \text{for } \mathbf{X} = (\text{HmO}, \text{SWL})$$

Evaluation of $P(\mathbf{X} \in A_\nu)$,

$$\text{where } A_\nu = \{\mathbf{x} \in \mathbb{R}^2 : \Delta(\mathbf{x}, \nu) > 0\}.$$


Wave Height (Hm0) and Still Water Level (SWL)
recorded during 828 storm events for the Dutch coast
Still Water Level SWL
−2
0
2
4
6
8
0
2
4
6
8
10
Wave Height HmO

Other examples :

pollutant concentrations - **Joe, Smith & Weissman** (1992)

reservoir safety - **Anderson & Nadarajah** (1993)

rainfall regime - **Coles & Tawn** (1996)

air quality - **Heffernan & Tawn** (2005)

... among others !

Multivariate context arises from :

- different **variables**
- one variable, at different **sites**
- one variable, at different **times**

Aim of the talk

Survey of the existing models for multivariate extremes

1. Modeling componentwise maxima

Structure of the Multivariate Extreme Value family

Examples - Inference

Limits of the model

2. Modeling under asymptotic independence

Main references

1. Modeling componentwise maxima

Tiago de Oliveira (1958), **Sibuya** (1960), **de Haan & Resnick** (1977), **Deheuvels** (1978), **Pickands** (1981), ...

Books : **Resnick** (1987, 2007), **Coles** (2001),

Beirlant, Goegebeur, Teugels & Segers (2004)

de Haan & Ferreira (2006), ...

2. Modeling under asymptotic independence

Ledford & Tawn (1996, 1997), **Heffernan & Tawn** (2004), **Balkema & Embrechts** (2007), **Fougères & Soulier** (2008),...

1. Models for componentwise maxima

Main hypothesis :

Existence of a multivariate domain of attraction

Let $\mathbf{X}_i = (X_{i,1}, \ldots X_{i,d}), i = 1, \ldots, n$, be i.i.d. random vectors of dimension d with d.f. F. We assume that

$$P\left\{\frac{\max_i X_{i,1} - b_{n,1}}{a_{n,1}} \le x_1, \ldots, \frac{\max_i X_{i,d} - b_{n,d}}{a_{n,d}} \le x_d\right\}$$

$$= F^n(a_{n,1}\, x_1 + b_{n,1}, \ldots, a_{n,d}\, x_d + b_{n,d}) = F^n(\mathbf{a_n}\, \mathbf{x} + \mathbf{b_n}) \to G(\mathbf{x}),$$

when $n \to \infty$, with G d.f. with non-degenerate margins $G_1, \cdots, G_d$.

Univariate margins : **parametric structure (GEV)**

$$G_j(x) \;=\; \exp\left\{ - \left(1 + \xi \, \frac{x-\mu}{\sigma} \right)_{+}^{-1/\xi} \right\} \qquad j \in \{1, \cdots, d\}.$$

Multivariate law : **no more parametric structure**.

However : The possible limits G are the **max-stable** distributions (with non degenerate margins), and they admit nice representations.

Let assume for simplicity that the univariate EV margins are unit Fréchet distributed ($P\{Y_j \leq y\} = e^{-1/y}, \forall y > 0$).

Denote Ω the unit sphere on $\mathbb{R}_{+}^{2}$.

Representations for the EV d.f. G

(**de Haan & Resnick,** 1977)

(**1**) : $G(\mathbf{x}) = \exp\left(-\mu^*\{(0,\mathbf{x}]^c\}\right)$, with $t\mu^*(tB) = \mu^*(B)$,

for all $t > 0$ and B Borel set of $E = [0,\infty]^d \setminus \{\mathbf{0}\}$.

(**2**) : $G(\mathbf{x}) = \exp\left\{-\int_\Omega \bigvee_{j=1}^{d} \frac{\omega_j}{x_j}\, dS(\mathbf{w})\right\}$, with $\int_\Omega \omega_j\, dS(\mathbf{w}) = 1$.

The *exponent measure* μ^* and the *spectral measure* S are related via :

$$\mu^*\left\{\mathbf{y} \in E \;:\; ||\mathbf{y}|| > r \;;\; \frac{\mathbf{y}}{||\mathbf{y}||} \in A\right\} = \frac{S(A)}{r}. \qquad (1)$$

(1) : $G(\mathbf{x}) = \exp\left(-\mu^*\{(0,\mathbf{x}]^c\}\right)$, with $t\mu^*(tB) = \mu^*(B)$.

(2) : $G(\mathbf{x}) = \exp\left\{-\int_\Omega \bigvee_{j=1}^{d} \frac{\omega_j}{x_j}\, dS(\mathbf{w})\right\}$, with $\int_\Omega \omega_j\, dS(\mathbf{w}) = 1$.

(3) : $G(\mathbf{x}) = P\left(\bigvee_{t_k \le 1} \mathbf{j}_k \le \mathbf{x}\right)$, where $\sum_k \mathbb{1}_{(t_k,\, \mathbf{j}_k) \in\ \cdot}$ is a non homogeneous Poisson process with intensity $\Lambda([0,t] \times B) = t\mu^*(B)$.

Some arguments : Defining $T : \mathbf{y} \mapsto (||\mathbf{y}||, \frac{\mathbf{y}}{||\mathbf{y}||})$,

$$\text{Equation (1) is} \quad \mu^* \circ T^{-1}\{(r, \infty) \times A\} = \frac{S(A)}{r}.$$

Thus $\mu^*\{(0, \mathbf{x}]^c\} \;=\; \mu^* \circ T^{-1}(T\{(0, \mathbf{x}]^c\}) \;=\; \int_{T\{(0,\mathbf{x}]^c\}} \frac{1}{r^2}\, dS(\mathbf{w})dr.$

Moreover, write

$$T((0, \mathbf{x}]^c) = \{(r, \omega) \in (0, \infty) \times \Omega \;:\; rw \in (0, \mathbf{x}]^c\}$$

$$= \{(r, \omega) \in (0, \infty) \times \Omega \;:\; r > \bigwedge_{j=1}^{d} \frac{x_j}{\omega_j}\}.$$

This leads therefore to

$$\mu^*\{(0, \mathbf{x}]^c\} \;=\; \int_{T\{(0,\mathbf{x}]^c\}} \frac{1}{r^2}\, dS(\mathbf{w})dr \;=\; \int_{\Omega} \bigvee_{j=1}^{d} \frac{\omega_j}{x_j}\, dS(\mathbf{w}).$$

Several representations exist for the EV d.f. G.

? **How to formulate them in terms of a d.f. F in the domain of attraction of G** $F \in DA(G)$?

Let consider n i.i.d. observations $\mathbf{X}_i = (X_{i,1}, \ldots X_{i,d})$, $i = 1, \ldots, n$, with unit Fréchet margins.

[In practice : $X_{i,j} \rightsquigarrow Z_{i,j} = 1/\log\{n/(R_{i,j} - 1/2)\}$,

where $R_{i,j}$ is the rank of $X_{i,j}$ among $X_{1,j}, \ldots, X_{n,j}$.]

Representations for $F \in \mathbf{DA}(G)$

(**de Haan & Resnick,** 1977)

(1) : $$\lim_{t\to\infty} \frac{-\log F(t\mathbf{x})}{-\log F(t\mathbf{1})} = \lim_{t\to\infty} \frac{1-F(t\mathbf{x})}{1-F(t\mathbf{1})} = \frac{-\log G(\mathbf{x})}{-\log G(\mathbf{1})} = \frac{\mu^*([\mathbf{0},\mathbf{x}]^c)}{\mu^*([\mathbf{0},\mathbf{1}]^c)}$$

(2) : $$\lim_{t\to\infty} t\, P\left\{ ||\mathbf{X}_i|| > t \;;\; \frac{\mathbf{X}_i}{||\mathbf{X}_i||} \in A \right\} = S(A),$$

(3) : The point process associated with $\{\mathbf{X}_1/n, \ldots, \mathbf{X}_n/n\}$ converges weakly to a non homogeneous Poisson process on E with intensity measure μ^*.

Examples : **(i)** **Bivariate Cauchy** with density

$$f(x,y) = \frac{1}{2\pi}(1+x^2+y^2)^{-3/2}, \ (x,y) \in \mathbb{R}^2.$$

Polar transformation : $\mathbf{X} \mapsto (||\mathbf{X}||, \Theta(\mathbf{X})) := (R, \Theta)$

$$(X,Y) \mapsto \left((X^2+Y^2)^{1/2}, \mathrm{Arctan}(\frac{Y}{X}) \right).$$

Then $R \perp\!\!\!\perp \Theta$, $\Theta \sim \mathcal{U}_{[0,2\pi)}$ and R has density $\dfrac{r}{(1+r^2)^{3/2}}$.

Hence,

$$\lim_{t\to\infty} P[(R/t,\Theta) \in (\xi,\infty)\times(\theta_1,\theta_2)] = \int_{(\xi,\infty)\times(\theta_1,\theta_2)} \frac{dr}{r^2}\frac{d\theta}{2\pi},$$

so that

$$\frac{\mu^*([\mathbf{0},\mathbf{x}]^c)}{\mu^*([\mathbf{0},\mathbf{1}]^c)} = \int_{(x_1,\infty)\times(x_2,\infty)} \frac{dxdy}{(x^2+y^2)^{3/2}}.$$

(ii) **Multivariate normal** d.f. $F_{\mathcal{N}}$, with all univariate margins equal to $\mathcal{N}(0,1)$, and with all its correlations less than 1 ($\mathbb{E}X_iX_j < 1$, for all i, j).

Sibuya, 1960 :

$$F_{\mathcal{N}}^n(a_n\mathbf{x} + b_n) \longrightarrow G(\mathbf{x}) = \prod_{j=1}^{d} \exp\{e^{-x_j}\}.$$

Domain of attraction of the **independence**, with univariate Gumbel margins.

[with $a_n = (2\log n)^{-1/2}$, and
$b_n = (2\log n)^{1/2} - 1/2(\log\log n + \log 4\pi)/(2\log n)^{1/2}$].

Then S is concentrated on $\{e_i, i = 1, \cdots, d\}$, vectors of the canonical basis of $\mathbb{R}^d$.

Inference

From (2) : $\lim_{t\to\infty} t\,P\left\{||\mathbf{X}_i|| > t\ ;\ \frac{\mathbf{X}_i}{||\mathbf{X}_i||} \in A\right\} \;=\; S(A),$

a candidate to estimate S is the **empirical measure** of the $\left(\frac{||\mathbf{X}_i||}{t}\,,\ \frac{\mathbf{X}_i}{||\mathbf{X}_i||}\right)$'s, for $t = t(n)$ ensuring convergence.

Resnick (1986) : $t = \frac{n}{k_n}$, where $k_n \to \infty$ and $\frac{n}{k_n} \to \infty$.

$$\Rightarrow\ S_n(A) = \frac{1}{k_n}\sum_{i=1}^{n} \mathbb{1}\left\{||\mathbf{X}_i|| > \frac{n}{k_n}\,,\ \frac{\mathbf{X}_i}{||\mathbf{X}_i||} \in A\right\}.$$

$$S_n(A) = \frac{1}{k_n} \sum_{i=1}^{n} \mathbb{1} \left\{ ||\mathbf{X}_i|| > \frac{n}{k_n} \, , \, \frac{\mathbf{X}_i}{||\mathbf{X}_i||} \in A \right\} .$$

More convenient in practice, and asympt. equivalent :

$$S_n(A) = \frac{1}{k_n} \sum_{i=1}^{n} \mathbb{1} \left\{ ||\mathbf{X}_i|| > ||\mathbf{X}||_{[k_n]} \, , \, \frac{\mathbf{X}_i}{||\mathbf{X}_i||} \in A \right\} ,$$

where $||\mathbf{X}||_{[k_n]}$ is the $(n - k_n + 1)$th order statistic of the $||\mathbf{X}||_i$'s.

Various *nonparametric* **threshold estimation methods** :

Initiated by **de Haan** (1985).

See for a review **Abdous & Ghoudi** (2005).

Different choices of norm :

$||\mathbf{Z}|| = |Z_1 + Z_2|$: **Joe, Smith & Weissman** (1992),

Capéraà & Fougères (2000) ;

$||\mathbf{Z}|| = (Z_1^2 + Z_2^2)^{1/2}$: **Einmahl, de Haan & Huang** (1993),

de Haan & de Ronde (1998) ;

$||\mathbf{Z}|| = Z_1 \vee Z_2$: **Einmahl, de Haan & Sinha** (1997),

de Haan & de Ronde (1998),

Einmahl, de Haan & Piterbarg (2001).

Various *parametric* threshold estimation methods :

Using parametric families of multivariate EV distributions :

Based on **(3)** : **Coles & Tawn** (1991, 1994),
Joe, Smith & Weissman (1992) ;

Based on **(1)** : **Ledford & Tawn** (1996),
Smith, Tawn & Coles (1997).

Remark : In some situations, observations *that can be directly considered from an EV d.f. G* are available.

$\Rightarrow$ Specific techniques (developed in the bivariate case) :

Pickands (1981), **Tawn** (1988), **Tiago de Oliveira** (1989), **Smith, Tawn & Yuen** (1990), **Deheuvels** (1991), **Coles & Tawn** (1991), **Capéraà, Fougères & Genest** (1997), **Hall & Tajvidi** (2000), **Fils, Guillou & Segers**(2005), among others.

Summary in a very simple case :

Given a sample $(\mathbf{X}_i, i = 1, \ldots, n)$, with d.f. F, how to estimate $P(\mathbf{X} \in A)$, where A is a exceptional set ?

Hyp : $F \in \mathrm{DA}(G)$. $F_j(y) = e^{-1/y}$.

If $A = (0, n\mathbf{u}]^c$, using for example (1) :

$$P(\mathbf{X} \in A) = 1 - F(n\mathbf{u}) \approx -\frac{1}{n} \log G(\mathbf{u}) = \frac{1}{n} \int_\Omega \bigvee_{j=1}^{d} \frac{\omega_j}{u_j} dS(\mathbf{w}).$$

Making use of the empirical measure S_n, an estimator of $P(\mathbf{X} \in A)$ is then given by

$$\frac{1}{nk_n} \sum_{i=1}^{n} \left(\bigvee_{j=1}^{d} \frac{X_{i,j}}{u_j ||X_{i,j}||} \right) \mathbb{1}\{||\mathbf{X}_i|| > ||\mathbf{X}||_{[k_n]}\}.$$

Remarks :

• The choice of the proportion of data k_n used for the estimation of S is a delicate point in practice.

• Dealing with any form of extreme event A is of course not so straightforward, and needs care !

Refer for example to **de Haan & de Ronde** (1998), **Bruun & Tawn** (1998), or **de Haan & Ferreira** (2006) for complete application and evaluation of failure probabilities.

• Most work done with $d = 2$ or 3.

Limits of the EV model

Asymptotic Independence : $F \in \mathrm{DA}(\text{independence})$.

$\Rightarrow$ • Pb of regularity for MLE.

• Less satisfying results for nonparametric methods.

In this case, the probability mass of **joint tails**

$$\{(X_1 - b_{n,1})/a_{n,1} > x_1 \ , \ (X_2 - b_{n,2})/a_{n,2} > x_2\}$$

is **of lower order** than that for sets like

$$\{(X_1 - b_{n,1})/a_{n,1} > x_1 \ \textbf{or} \ (X_2 - b_{n,2})/a_{n,2} > x_2\}.$$

$\Rightarrow$ **No satisfying way to estimate such joint tails using EV distributions**.

Another case where EV models do not provide a satisfying answer under asympt. independence :

Problem : Evaluate $P\{(X,Y) \in A | X > x\}$, for x large.

An answer is given as soon as there exist a, α, b, β, μ non degenerate such that, when $u \longrightarrow \infty$,

$$P\left\{\frac{X - b(u)}{a(u)} > x; \frac{Y - \beta(u)}{\alpha(u)} \leq y | X > b(u)\right\} \to \mu\{(x, +\infty] \times [-\infty, y]\}.$$

Assume for simplicity that X and Y have same margins.

(MDA) : There exist a non degenerate G, $a_n > 0, b_n$, s.t.

$$\lim_{n \to \infty} P\left\{\frac{\max_i X_i - b_n}{a_n} \leq x; \frac{\max_i Y_i - b_n}{a_n} \leq y\right\} = G(x, y).$$

(MDA) :

$$\lim_{n\to\infty} P\left\{\frac{\max_i X_i - b_n}{a_n} \le x; \frac{\max_i Y_i - b_n}{a_n} \le y\right\} = G(x,y).$$

$$\Leftrightarrow \quad \lim_{n\to\infty} nP\{X > b_n + a_n x \text{ or } Y > b_n + a_n y\} = -\log G(x,y).$$

Reparametrization :

$$n \rightsquigarrow \frac{1}{P(X>u)}, \quad b_n \rightsquigarrow u, \quad \text{and} \quad a_n \rightsquigarrow \psi(u).$$

$$\Leftrightarrow \quad \lim_{u\to\infty} \frac{P(X > u + x\psi(u) \text{ or } Y > u + y\psi(u))}{P(X>u)} = -\log G(x,y)$$

$$\Leftrightarrow \lim_{u\to\infty} P\left(\frac{X-u}{\psi(u)} > x, \frac{Y-u}{\psi(u)} \le y | X > u\right) = \log F^\star(y) - \log G(x,y),$$

Recall the problem : Does there exist a, α, b, β, μ non degenerate such that, when $u \to \infty$,

$$P\left\{\frac{X-b(u)}{a(u)} > x; \frac{Y-\beta(u)}{\alpha(u)} \leq y | X > b(u)\right\} \to \mu\{(x,+\infty]\times[-\infty,y]\}?$$

We obtained via **(MDA)** :

$$\lim_{u\to\infty} P\left(\frac{X-u}{\psi(u)} > x, \frac{Y-u}{\psi(u)} \leq y | X > u\right) = \log F^\star(y) - \log G(x,y),$$

i.e. answer for $a(u) = \alpha(u) = u$ and $b(u) = \beta(u) = \psi(u)$ ... which is unsatisfying under asymptotic independence

$$\textbf{(AI)} \quad G(x,y) = F^\star(x) F^\star(y),$$

since $\mu\{(x,+\infty] \times [-\infty, y]\} = -\log F^\star(x)$ is degenerate in y.

Consequence : Under **Asymptotic Independence**, using extreme value rates yields degenerate distributions for the conditional events :

$$\lim_{u\to\infty} P(Y \leq u + y\psi(u)|X > u) = 1,$$

$$\lim_{u\to\infty} P(X > u + x\psi(u), Y > u + y\psi(u)|X > u) = 0.$$

Conclusion : **Checking presence or absence of Asymptotic Independence might be important !**

See additional details in **de Haan & Ferreira** (2006) and **Resnick** (2007).

Short numerical data excursion

Wave data set : 828 storm events on Dutch coast.

X= Water height (HmO), Y= Still water level (SWL).

Two steps : – Marginal analysis

– Analysis of the dependence structure.

See **De Haan & Ferreira** (2006) for more!

A. Marginal analysis : Univariate extreme fit

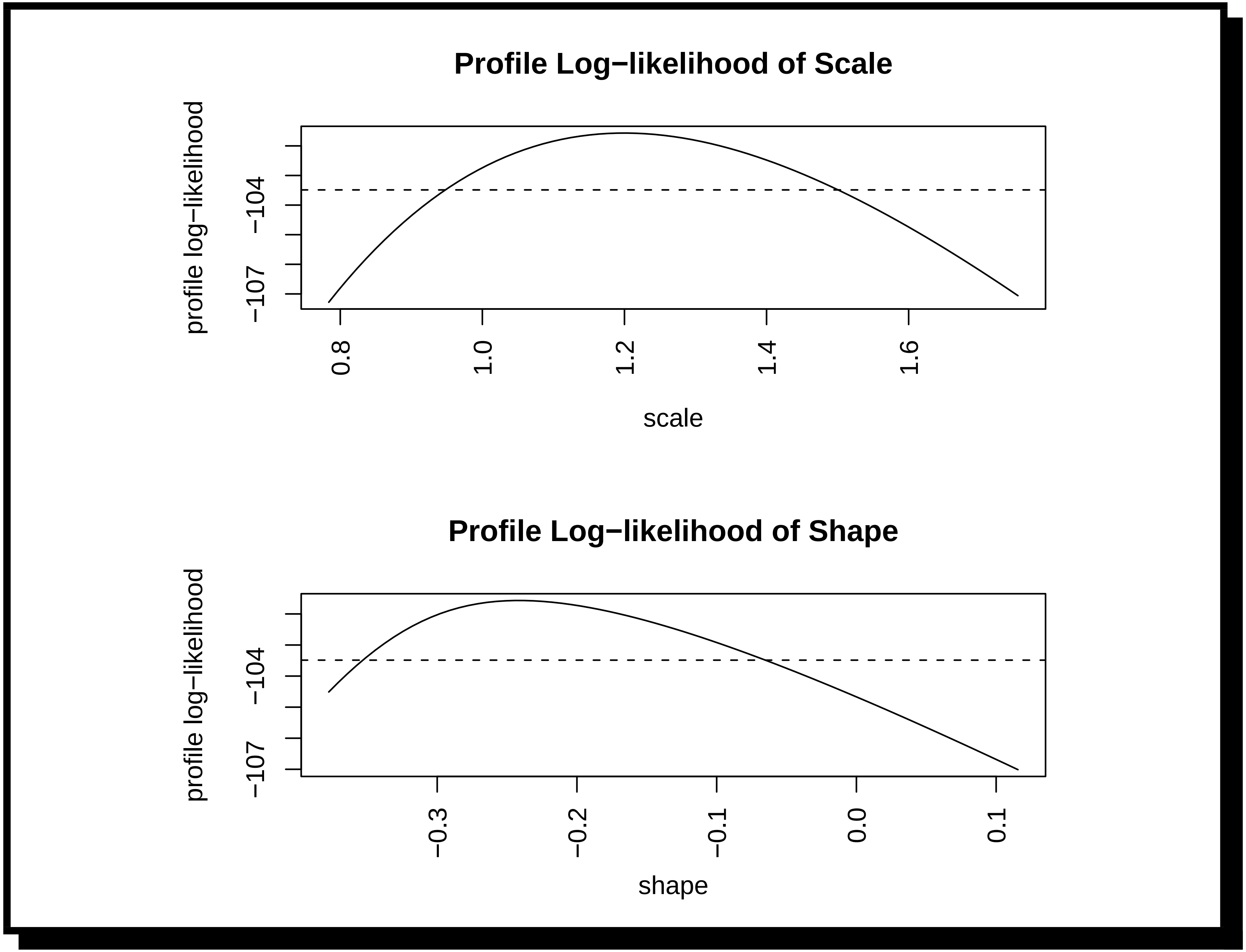
Wave data set (n=828)

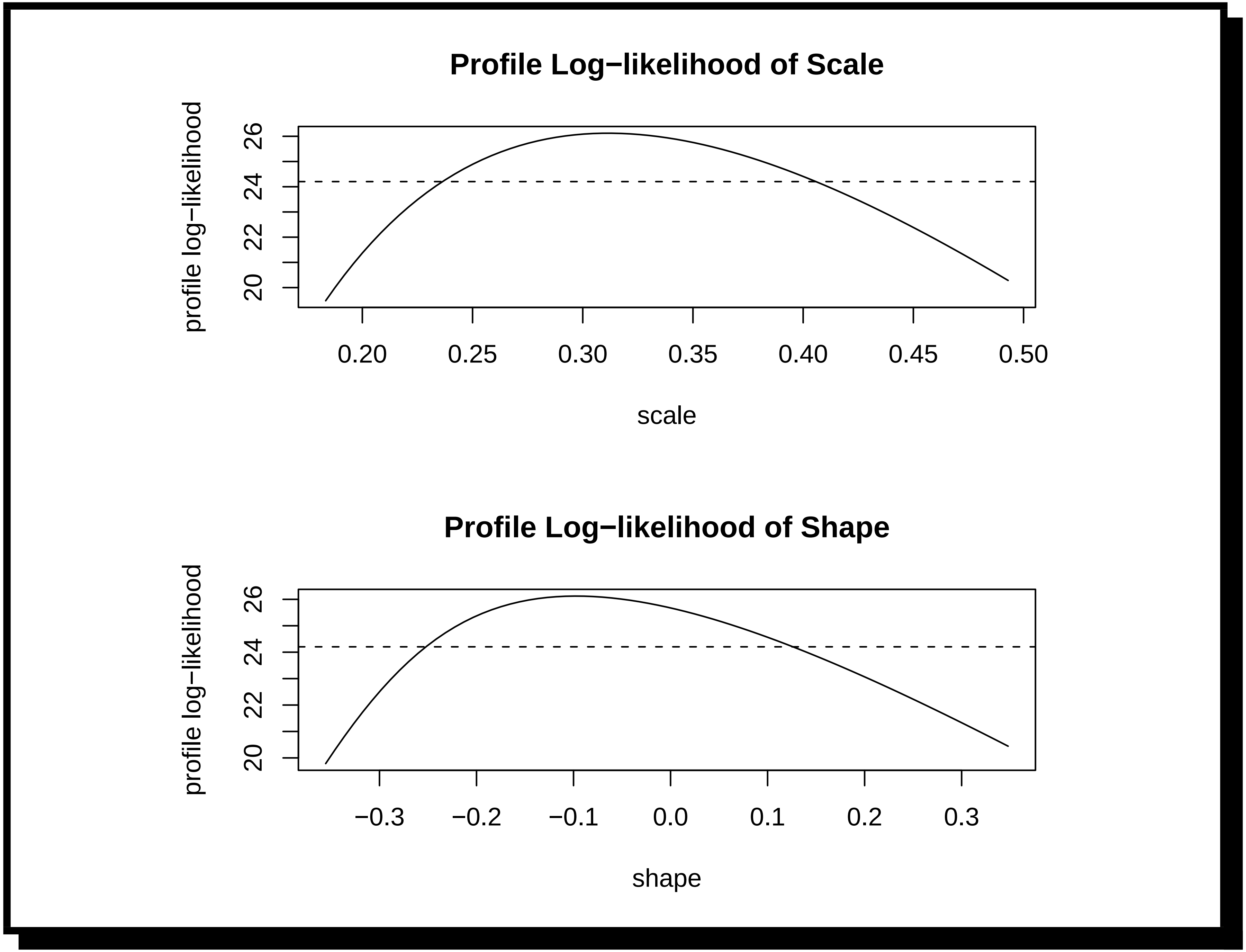
MLE in GPD model and Moment-type estimator both give :

$$\hat{\gamma}_{HmO} = -0.22 \ , \quad \hat{\gamma}_{SWL} = 0.$$

Next pages :

Profile log-likelihood and 95%-confidence intervals for the maximum likelihood parameters of the GPD distributions of HmO and SWL respectively.


Profile Log−likelihood of Scale
profile log−likelihood
−107
−104
0.8
1.0
1.2
1.4
1.6
scale
Profile Log−likelihood of Shape
profile log−likelihood
−107
−104
−0.3
−0.2
−0.1
0.0
0.1
shape



Profile Log−likelihood of Scale
profile log−likelihood
20
22
24
26
0.20
0.25
0.30
0.35
0.40
0.45
0.50
scale
Profile Log−likelihood of Shape
profile log−likelihood
20
22
24
26
−0.3
−0.2
−0.1
0.0
0.1
0.2
0.3
shape

B. Analysis of the dependence structure

Some tools :

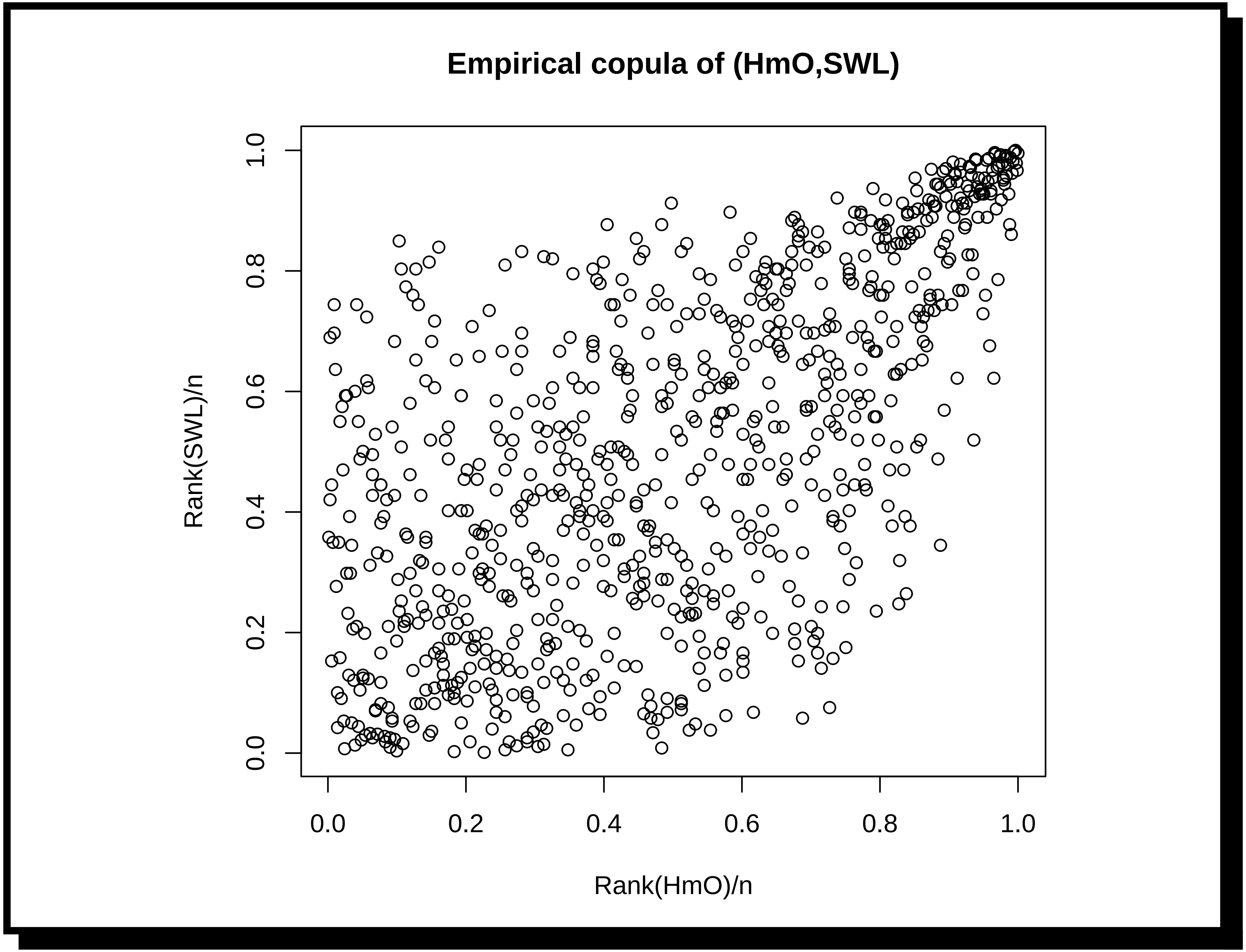
– Empirical copula

– Independence test (Genest & Rémillard, 2004) ; see R package "Copula", subroutine "empcopu.test".

– Coles, Heffernan & Tawn (1999) functions $\chi(u)$ and $\bar{\chi}(u)$.

$$\chi(u) = 2 - \frac{\log C(u,u)}{\log u} \quad , \quad \bar{\chi}(u) = \frac{2\log(1-u)}{\log \bar{C}(u,u)} - 1$$

– Estimation of the spectral measure.


Empirical copula of (HmO,SWL)
Rank(SWL)/n
0.0
0.2
0.4
0.6
0.8
1.0
Rank(HmO)/n

– **Genest & Rémillard (2004) independence test** :

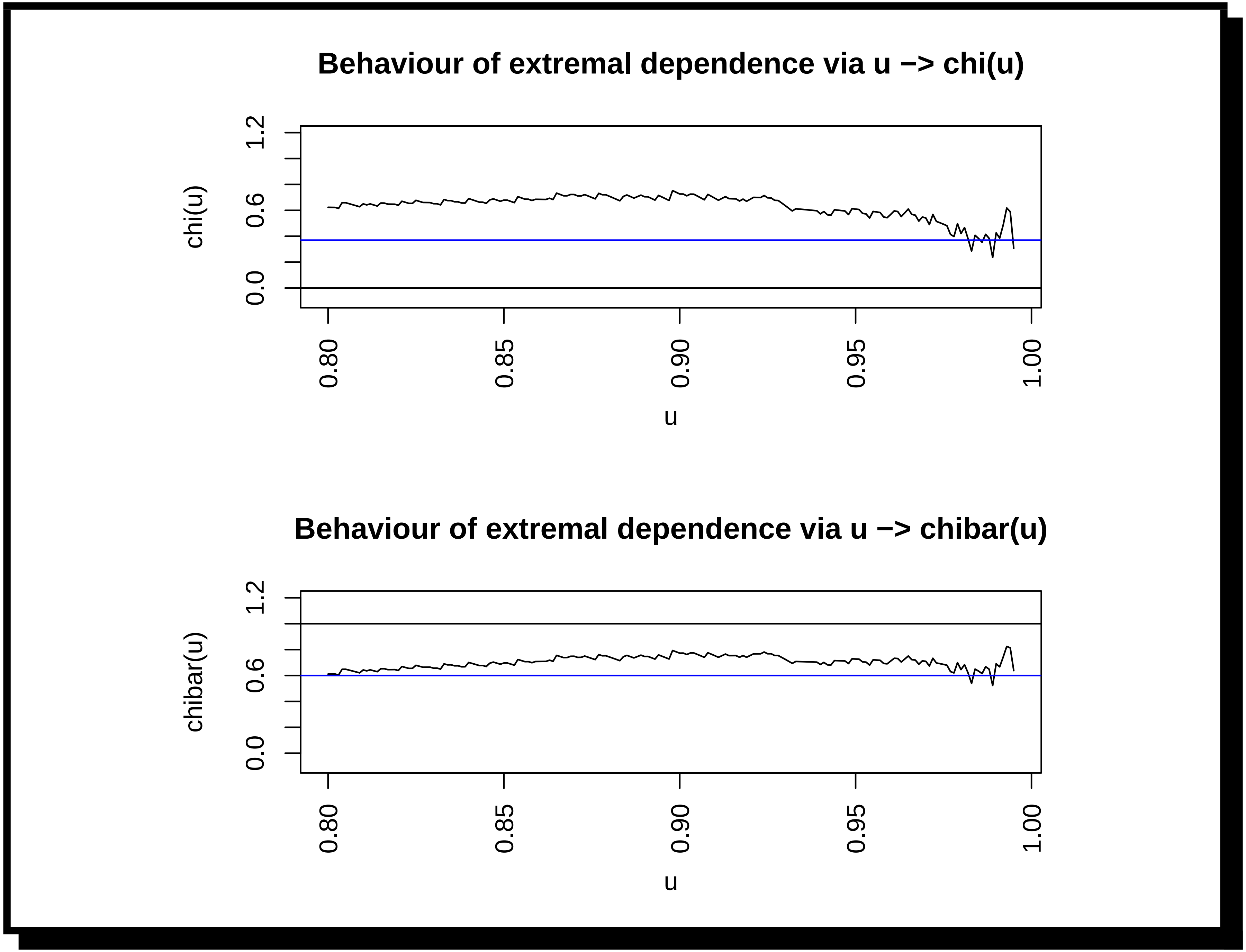
Reject independence pval$= 5.10^{-3}$.

When testing all the data, as well as when testing the highest 20% or 10% of the data.

– **Coles, Heffernan & Tawn (1999) measures** :

- $\lim_{u \to 1} \chi(u) := \chi = 0$ means Asymptotic Independence,

  and then $\lim_{u \to 1} \bar{\chi}(u) := \bar{\chi}$ is a second-order dependence parameter.
- $\chi > 0$ means Asymptotic Dependence, and then $\bar{\chi} = 1$.


Behaviour of extremal dependence via u −> chi(u)
chi(u)
0.0
0.6
1.2
0.80
0.85
0.90
0.95
1.00
u
Behaviour of extremal dependence via u −> chibar(u)
chibar(u)
0.0
0.6
1.2
0.80
0.85
0.90
0.95
1.00
u

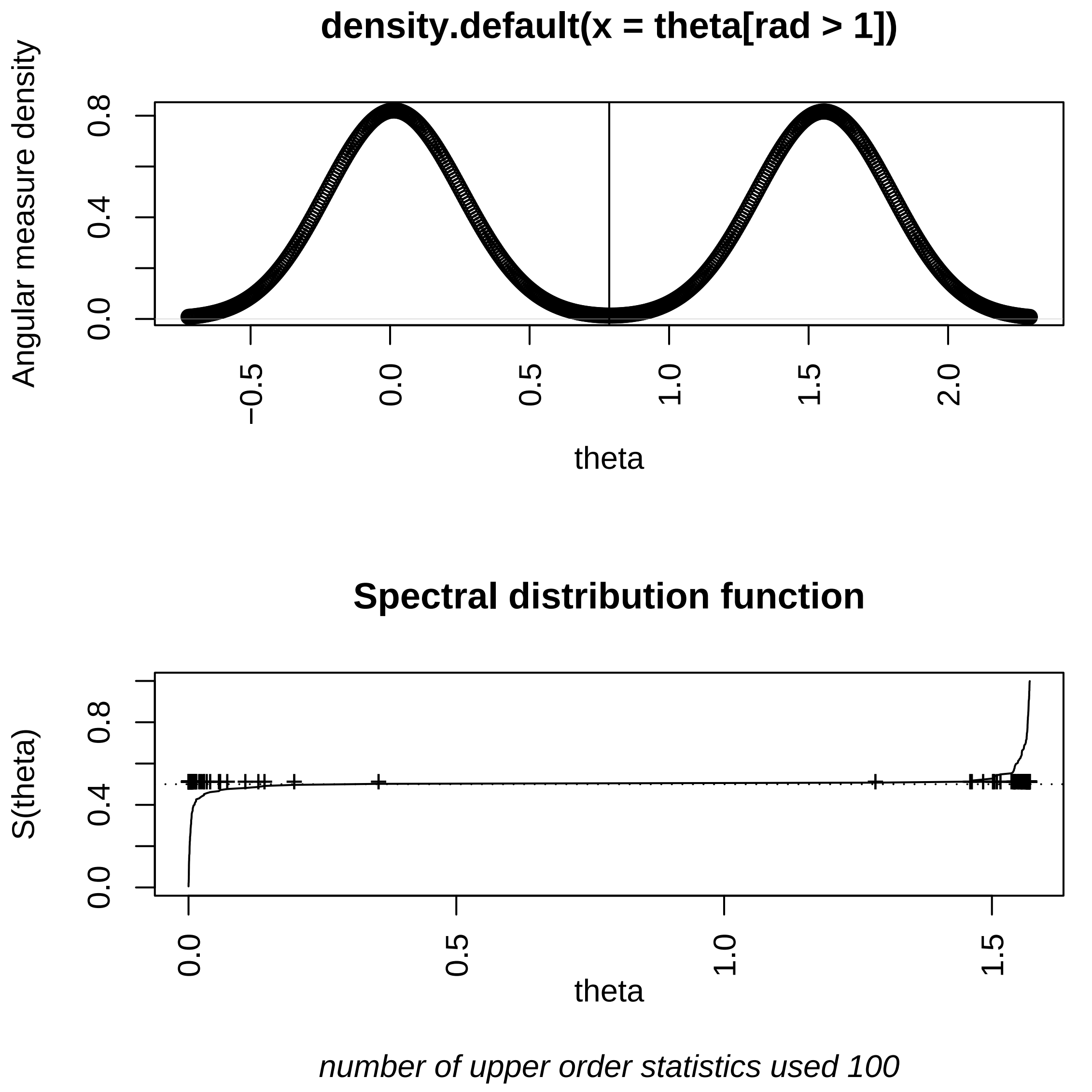
– **Estimating the spectral measure** :

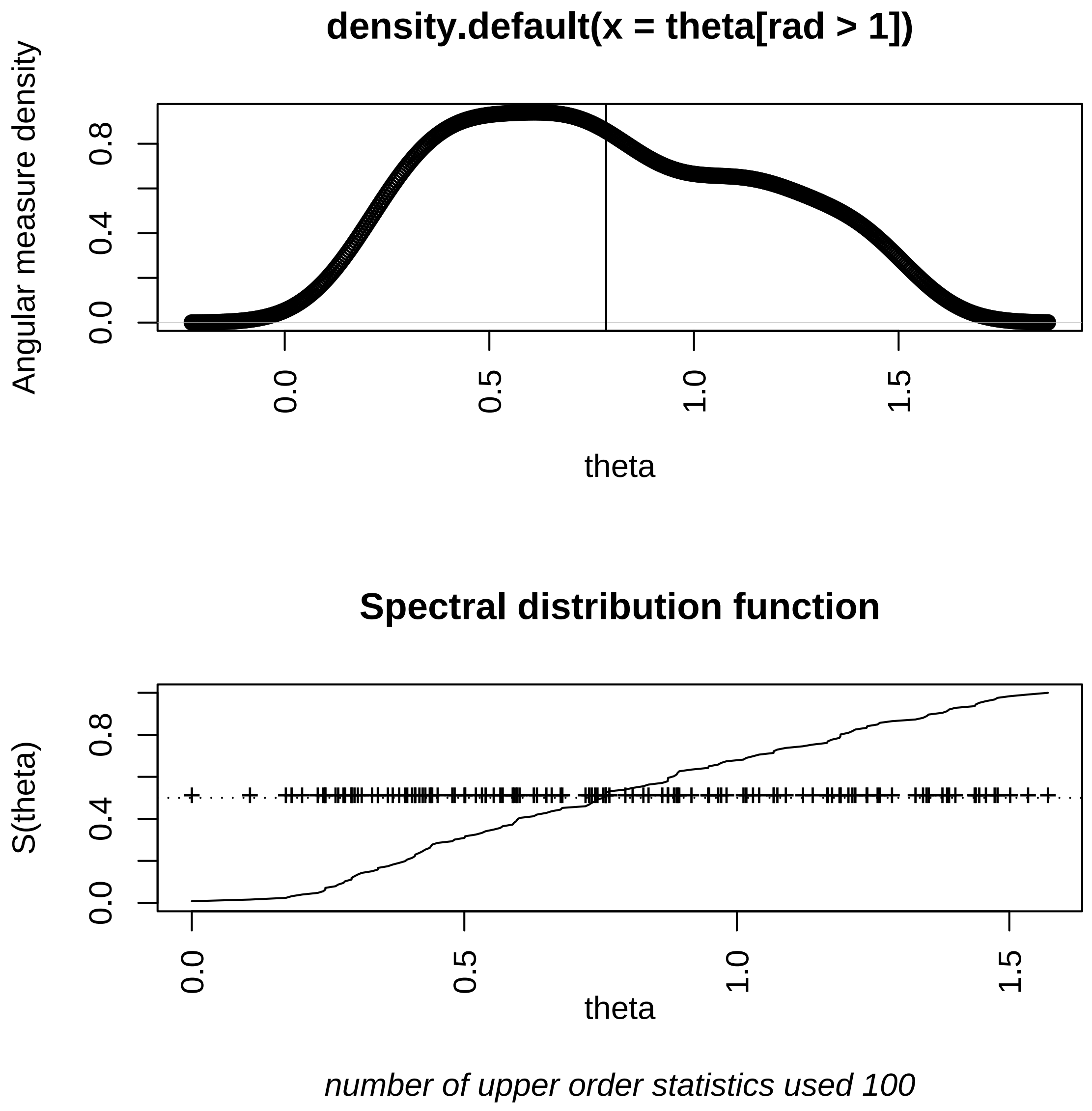
Estimation of the associated distribution function and the density function (assuming it exists).

In case of **Asymptotic Independence** :

S is concentrated on the axes ; via another parametrization (using L^2-norm and usual polar decomposition), we should see a distribution **concentrated on 0 and** $\pi/2$.

Next page : Estimation of the density function and the distribution function of the angular measure of a gaussian sample first, and of the wave data second.


density.default(x = theta[rad > 1])
Angular measure density
0.0
0.4
0.8
−0.5
0.0
0.5
1.0
1.5
2.0
theta
Spectral distribution function
S(theta)
0.0
0.4
0.8
0.0
0.5
1.0
1.5
theta
number of upper order statistics used 100



density.default(x = theta[rad > 1])
Angular measure density
0.0
0.4
0.8
0.0
0.5
1.0
1.5
theta
Spectral distribution function
S(theta)
0.0
0.4
0.8
0.0
0.5
1.0
1.5
theta
number of upper order statistics used 100

2. Modeling under asymptotic independence

- **Bivariate case ?** **Sibuya** (1960) :

(X_1, X_2) has *asymptotically independent* components iff

(AI) : $$\lim_{u \to 1} P\left\{X_2 > F_{X_2}^{-1}(u) \,|\, X_1 > F_{X_1}^{-1}(u)\right\} = \chi = 0.$$

- **Multivariate case ?**

Theorem : (**Berman,** 1961) Let $\{\mathbf{X}_n, n \geq 1\}$ be i.i.d. from F, with common univariate margins F_1 s.t. $F_1^n(a_n\, x + b_n) \longrightarrow G_1(x)$. The following assertions are equivalent :

(i) $F^n(a_n\mathbf{x} + b_n\mathbf{1}) = P\left(\bigvee_{i=1}^{n} \mathbf{X}_i \leq a_n\mathbf{x} + b_n\mathbf{1}\right) \longrightarrow \prod_{j=1}^{d} G_1(x_j).$

(ii) For all $1 \leq k < \ell \leq d$,

$$P\left(\bigvee_{i=1}^{n} X_{i,k} \leq a_n x_k + b_n \; , \; \bigvee_{i=1}^{n} X_{i,\ell} \leq a_n x_\ell + b_n\right) \longrightarrow G_1(x_k)G_1(x_\ell).$$

(iii) For all $1 \leq k < \ell \leq d$,

$$\lim_{t \to x^\star} P(\, X_{1,k} > t \mid X_{1,\ell} > t\,) = 0.$$

2.1 Alternative models for joint tails

Ledford & Tawn (1996, 1997)

Main model (for unit Fréchet margins) :

$$P(Z_1 > z_1 \, , \, Z_2 > z_2) \sim \frac{\mathcal{L}(z_1, z_2)}{z_1^{c_1} \, z_2^{c_2}}, \qquad (2)$$

when $z_1, z_2 \to \infty$, where $c_1, c_2 > 0$ and $c_1 + c_2 \geq 1$, and $\mathcal{L} : \mathbf{x} \in \mathbb{R}^2 \mapsto \mathcal{L}(\mathbf{x}) > 0$ is a bivariate slowly varying function, ie :

$$\lim_{t \to \infty} \frac{\mathcal{L}(t\mathbf{x})}{\mathcal{L}(t\mathbf{1})} \;=\; \lambda(\mathbf{x}),$$

for some positive function λ satisfying $\lambda(t\mathbf{x}) = \lambda(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$, $t > 0$. (See **Bingham, Goldie & Teugels,** 1989).

Under this model

$$P(Z_1 > r\ ,\ Z_2 > r) \sim \frac{\mathcal{L}(r,r)}{r^{c_1+c_2}}$$

Coefficient of tail dependence : $\eta = \dfrac{1}{c_1 + c_2} \in (0,1]$

Asymptotic dependence $\leftrightsquigarrow$ $\eta = 1$ and $\mathcal{L}(r,r) \to 0$

Asymptotic independence $\leftrightsquigarrow$ $\eta < 1$.

Remark : $\bar{\chi} = 2\eta - 1$

More formalization : See De Haan & Ferreira (2006)

Suppose (X, Y) has distribution function F, with continuous marginal distribution functions denoted by F_1 and F_2.

Second-order model : Assume the existence and positivity of

$$H(x, y) = \lim_{t \to 0} \frac{P\{1 - F_1(X) < tx, 1 - F_2(Y) < ty\}}{P\{1 - F_1(X) < t, 1 - F_2(Y) < t\}}.$$

Then $P\{1 - F_1(X) < t, 1 - F_2(Y) < t\}$ is a regularly varying function with index $1/\eta$:

$$H(tx, ty) = t^{1/\eta} H(x, y).$$

Estimation of η and inference in submodels of (2) :

Ledford & Tawn (1996, 1997) − **Bruun & Tawn** (1998)

de Haan & de Ronde (1998) − **Peng** (1999)

Draisma et al. (2004) **Beirlant et al.** (2004).

Related models : **Hidden regular variation**

Resnick (2002), **Maulik & Resnick** (2002, 2004),
Heffernan & Resnick (2005)...

2.2 Alternative models for conditional excesses

Modeling and estimation procedures :

Heffernan & Tawn (05) - **Heffernan & Resnick** (06)

Balkema & Embrechts (07) - **Fougères & Soulier** (08)