2008/05/09

An explicit expansion formula for the powers of the Euler Product in terms of partition hook lengths

Guo-Niu HAN

ABSTRACT. — We discover an explicit expansion formula for the powers s of the Euler Product (or Dedekind η-function) in terms of hook lengths of partitions, where the exponent s is *any* complex number. Several classical formulas have been derived for certain integers s by Euler, Jacobi, Klein, Fricke, Atkin, Winquist, Dyson and Macdonald. In particular, Macdonald obtained expansion formulas for the integer exponents s for which there exists a semi-simple Lie algebra of dimension s. For the type $A_l^{(a)}$ he has expressed the (t^2-1)-st power of the Euler Product as a sum of weighted integer vectors of length t for any integer t. Kostant has considered the general case for any positive integer s and obtained further properties.

The present paper proposes a new approach. We convert the weighted vectors of *length* t used by Macdonald in his identity for type $A_l^{(a)}$ to weighted partitions with *free parameter* t, so that a new identity on the latter combinatorial structures can be derived without any restrictions on t. The surprise is that the weighted partitions have a very simple form in terms of hook lengths of partitions. As applications of our formula, we find some new identities about hook lengths, including the "marked hook formula". We also improve a result due to Kostant. The proof of the Main Theorem is based on Macdonald's identity for $A_l^{(a)}$ and on the properties of a bijection between t-cores and integer vectors constructed by Garvan, Kim and Stanton.

Summary

Andrei Okounkov pointed out that the main identity already appeared in his joint paper arXiv:hep-th/0306238. The present paper will remain on arXiv verbatim and not be published anywhere else. Parts of the results, as well as several new ones are reproduced in a forthcoming paper arXiv:0805.1398v1 [math.CO].

1. Introduction

The powers of the Euler Product and the hook lengths of partitions are two mathematical objects widely studied in the Theory of Partitions, in Algebraic Combinatorics and Group Representation Theory. In the present paper we establish a new connection by giving an explicit expansion formula for all the powers s of the Euler Product in terms of partition hook lengths, where the exponent s is any complex number. Recall that the *Euler Product* is the infinite product $\prod_{m\geq 0}(1-x^m)$. A variation of the Euler Product, called the *Dedekind η-function*, is defined by $\eta(x) = x^{1/24}\prod_{m\geq 0}(1-x^m)$. The following two formulas [Eu83; An76, p.11, p.21] go back to Euler (the pentagonal theorem)

$$\prod_{m\geq 1}(1-x^m) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2} \tag{1.1}$$

and Jacobi (triple product identity)

$$\prod_{m\geq 1}(1-x^m)^3 = \sum_{m\geq 0}(-1)^m(2m+1)x^{m(m+1)/2}. \tag{1.2}$$

Further explicit formulas for the powers of the Euler Product

$$\prod_{m\geq 1}(1-x^m)^s = \sum_{k\geq 0} f_k(s)x^k \tag{1.3}$$

have been derived for certain integers

$$s = 1, 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots \tag{1.4}$$

by Klein and Fricke for $s = 8$, Atkin for $s = 14, 26$, Winquist for $s = 10$, and Dyson for $s = 24, \ldots$ [Wi69; Dy72]. The paper entitled "Affine root systems and Dedekind's η-function", written by Macdonald in 1972, is a milestone in the study of powers of Euler Product [Ma72]. The review of this paper for MathSciNet, written by Verma [Ve], contains seven pages! It has also inspired several followers, see [Ka74; Mo75; Ko76; Le78; Ko04; Mi85; AF02; CFP05; RS06]. The main achievement of Macdonald was to unify all the well-known formulas for the integers s listed in (1.4), except for $s = 26$. He obtained an expansion formula of

$$\prod_{m\geq 0} (1 - x^m)^{\dim \mathfrak{g}} \tag{1.5}$$

for every semi-simple Lie algebra $\mathfrak{g}$. In the case of type $A_l^{(a)}$, i.e., type A_l with l even [Ma72, p.134], he expressed the $(t^2 - 1)$-st power of the Euler Product as a sum of weighted integer vectors of length t for each odd positive integer t:

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0,\ldots,v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2+v_1^2+\cdots+v_{t-1}^2)/(2t)}.$$

Following this direction it seems difficult to obtain more expansion formulas for other exponents, because t is a vector length and has to be an integer. Kostant considered the general case for positive integer s and obtained further properties [Ko04].

The present paper proposes a new approach. The main difficulty is to find an appropriate "other object" and convert the weighted vector of *length* t in Macdonald's identity to a weighted "other object" with *free parameter* t. In fact, we find out that the "other object" is merely the classical partition of integer and that the weighted partition has a surprisingly simple form in terms of hook lengths.

Let us describe our Main Theorem. The basic notions needed here can be found in [Ma95, p.1; St99, p.287; La01, p.1; Kn98, p.59; An76, p.1]. A *partition* λ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The integers $(\lambda_i)_{i=1,2,\ldots,\ell}$ are called the *parts* of λ, the number ℓ of parts being the *length* of λ denoted by $\ell(\lambda)$. The sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is denoted by $|\lambda|$. Let n be an integer, a partition λ is said to be a partition of n if $|\lambda| = n$. We write $\lambda \vdash n$. The set of all partitions of n is denoted by $\mathcal{P}(n)$. The set of all partitions is denoted by $\mathcal{P}$, so that

$$\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}(n).$$

Each partition can be represented by its Ferrers diagram. For example, $\lambda = (6, 3, 3, 2)$ is a partition and its Ferrers diagram is reproduced in Fig. 1.1.

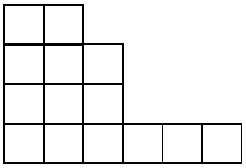


Fig. 1.1. Partition

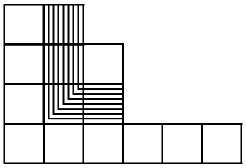
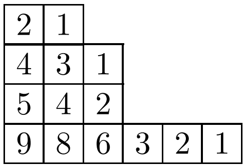


Fig. 1.2. Hook length



2	1				
4	3	1			
5	4	2			
9	8	6	3	2	1

Fig. 1.3. Hook lengths

For each box v in the Ferrers diagram of a partition λ, or for each box v in λ, for short, define the *hook length* of v, denoted by $h_v(\lambda)$ or h_v, to be the number of boxes u such that $u = v$, or u lies in the same column as v and above v, or in the same row as v and to the right of v (see Fig. 1.2). The *hook length multi-set* of λ is the multi-set of all hook lengths of λ. In Fig. 1.3 the hook lengths of all boxes for the partition $\lambda = (6, 3, 3, 2)$ have been written in each box. The hook length multi-set of λ is $\{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$.

Theorem 1.1 [Main]. *For any complex number β we have*

$$\prod_{m\geq 1} (1 - x^m)^{\beta-1} = \sum_{\lambda\in\mathcal{P}} \prod_{v\in\lambda} \Bigl(1 - \frac{\beta}{h_v^2}\Bigr) x. \tag{1.6}$$

Identity (1.6) will be called "Main Identity". Two numerical examples are given in §2 for verifying the Main Theorem. For convenience, the exponent s in the Euler Product has been replaced by $\beta - 1$. The proof of the Main Theorem is based on the Macdonald identities for $A_l^{(a)}$. We have also used the properties of the bijection between t-cores and N-codings constructed by Garvan, Kim and Stanton [GKS90] (see §5). From our Main Theorem we derive new formulas about hook lengths, including the "marked hook formula". We also improve a result due to Kostant (see §7). The five results we should like to single out are next stated. They will be further proved in Sections 2, 6, 7 and 9.

Corollary 1.2 [=2.4]. *For any positive integers n and k the following expression*

$$\sum_{\lambda\vdash n}\prod_{v\in\lambda}(1-\frac{k}{h_v^2}) \tag{1.7}$$

is an integer.

Theorem 1.3 [=6.10, marked hook formula]. *We have*

$$\sum_{\lambda\vdash n} f_\lambda^2 \sum_{v\in\lambda} h_v^2 = \frac{n(3n-1)}{2} n!, \tag{1.8}$$

where f_λ is the number of standard Young tableaux of shape λ (see §2.3 and §6.4).

Theorem 1.3 is to be compared with the following well-known formula

$$\sum_{\lambda\vdash n} f_\lambda^2 = n! \tag{1.9}$$

Theorem 1.4 [=6.9]. *We have*

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\Big(\sum_{v\in\lambda}\frac{1}{h_v^2}\Big)^2 = \prod_{m\geq 1}\frac{1}{1-x^m}\Big(\sum_{k\geq 1}\frac{x^k k^{-3}}{1-x^k}+\big(\sum_{k\geq 1}\frac{x^k k^{-1}}{1-x^k}\big)^2\Big). \tag{1.10}$$

Theorem 1.5 [=7.2]. *Let k be a positive integer and s be a real number such that $s \geq k^2-1$. Then $(-1)^k f_k(s) > 0$.*

Corollary 1.6 [=9.2]. *For any positive integer n the following expression*

$$\frac{1}{n+1}\sum_{\lambda\vdash n}\prod_{v\in\lambda}(1+\frac{n}{h_v^2}) \tag{1.11}$$

is a positive integer.

It would be interesting to find a direct proof of Corollaries 1.2, 1.6 and Theorems 1.3, 1.4. In particular, the “marked hook formula” suggests that a *marked* Robinson-Schensted-Knuth correspondence should be constructed between pairs of marked Young tableaux and marked permutations (see §6.4).

2. Basic consequences and specializations

2.1. Equivalent forms. With the notations recalled above for partitions the right-hand side of the Main Identity can be written as

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{v\in\lambda}(1-\frac{\beta}{h_v^2}) \;=\; \sum_{n\geq 0}x^n\sum_{\lambda\vdash n}\;\prod_{v\in\lambda}(1-\frac{\beta}{h_v^2}). \tag{2.1}$$

Using the identity (see [St99, p.316])

$$\prod_{m\geq 1}\frac{1}{1-x^m}=\exp\big(\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}\big), \tag{2.2}$$

the Main Identity can be written:

$$\prod_{m\geq 1}\frac{1}{1-x^m}\times\exp\Big(-\beta\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}\Big)=\sum_{n\geq 0}x^n\sum_{\lambda\vdash n}\;\prod_{v\in\lambda}(1-\frac{\beta}{h_v^2}) \tag{2.3}$$

or

$$\exp\Big((1-\beta)\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}\Big)=\sum_{n\geq 0}x^n\sum_{\lambda\vdash n}\;\prod_{v\in\lambda}(1-\frac{\beta}{h_v^2}). \tag{2.4}$$

The Main Theorem has also the following equivalent generating function form, which can be verified by comparing the coefficients of X^mY^n.

Theorem 2.1. *We have*

$$\sum_{k\geq 0}\frac{X^k}{1-Y\prod_{m\geq 1}(1-x^{mk})}=\sum_{k\geq 0}\sum_{\lambda\in\mathcal{P}}\frac{Y^k}{1-Xx^{|\lambda|}}\prod_{v\in\lambda}\Big(1-\frac{k+1}{h_v^2}\Big).$$

2.2. Corollaries. From the Main Theorem we immediately have the following results.

Corollary 2.2. *Let $F(\beta)$ be the function defined by*

$$F(\beta):=\sum_{\lambda\in\mathcal{P}}\;\prod_{v\in\lambda}(1-\frac{\beta+1}{h_v^2})x.$$

Then

$$F(\beta_1+\beta_2)=F(\beta_1)F(\beta_2).$$

In particular,

$$F(\beta)F(-\beta)=1.$$

Corollary 2.3. *For each positive integer n the following expression*

$$n!\sum_{\lambda\vdash n}\prod_{v\in\lambda}\left(1-\frac{\beta}{h_v^2}\right)$$

is a polynomial in β with integral coefficients.

Corollary 2.4 [=1.2]. *For any positive integers n and k the following expression*

$$\sum_{\lambda\vdash n}\prod_{v\in\lambda}\left(1-\frac{k}{h_v^2}\right) \tag{2.5}$$

is an integer.

Note that unlike Corollary 2.3 there is no factor $n!$ in Corollary 2.4.

2.3. Specialization for $\beta=0$. Letting $\beta=0$ in the Main Theorem yields the well-known generating function for partitions (see [An76, p.3]).

Theorem 2.5. *Let $p(n)$ be the number of partitions of n. Then*

$$\prod_{m\geq 1}\frac{1}{1-x^m}=\sum_{n\geq 0}p(n)x^n. \tag{2.6}$$

Using (2.6) we can rewrite the Main Theorem as follows, which is probably the good form for finding combinatorial interpretations. It means that there exists a bijection between one weighted-partition (right-hand side of (2.7)) and sequences of $k+1$ partitions (left-hand side of (2.7)). When $\beta=-k$ ($k\in\mathbb{N}$), the left-hand side of the Main Identity may be rewritten $(\sum_{n\geq 0}p(n)x^n)^{k+1}$ because of (2.6). Hence the Main Theorem may be restated as the following corollary.

Corollary 2.6. *Let $k\in\mathbb{N}$. Then*

$$\sum p(n_1)p(n_2)\cdots p(n_{k+1}) \;=\; \sum_{\lambda\vdash n}\prod_{v\in\lambda}\left(1+\frac{k}{h_v^2}\right), \tag{2.7}$$

where the sum on the left-hand side ranges over all positive integer vectors $(n_1,n_2,\ldots,n_{k+1})$ such that $\sum n_i=n$.

2.4. Specialization for $\beta=1$. The case $\beta=1$ is really trivial. Every non-empty partition λ contains at least one box v of hook length $h_v=1$, so that

$$\prod_{v\in\lambda}\left(1-\frac{1}{h_v^2}\right)=0.$$

Hence

$$\sum_{n\geq 0} x^n \sum_{\lambda\vdash n} \prod_{v\in\lambda} (1-\frac{1}{h_v^2}) \;=\; 1.$$

2.5. Specialization for $\beta=\infty$. The hook length plays an important role in Algebraic Combinatorics thanks to the famous hook formula due to Frame, Robinson and Thrall [FRT54]

$$f_\lambda = \frac{n!}{\prod_{v\in\lambda} h_v(\lambda)}, \tag{2.8}$$

where f_λ is the number of standard Young tableaux of shape λ (see [St99, p.376; Kn98, p.59; Kr99, Ze84, GNW79, NPS97, RW83]). By using the Main Theorem we re-prove the following classical result, which is also a consequence of the Robinson-Schensted-Knuth correspondence (see, for example, [Kn98, p.49-59; St99, p.324]).

Theorem 2.7. *We have*

$$\sum_{\lambda\vdash n} f_\lambda^2 \;=\; n! \tag{2.9}$$

Proof. Put $\beta=-y/x$ in (2.4):

$$\sum_{\lambda\in\mathcal{P}} \prod_{v\in\lambda} \Bigl(1+\frac{y/x}{h_v^2}\Bigr)x \;=\; \exp\Bigl((1+\frac{y}{x})\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}\Bigr).$$

When $x\to 0$, we get

$$\sum_{\lambda\in\mathcal{P}} \prod_{v\in\lambda} \frac{y}{h_v^2} \;=\; \exp(y). \tag{2.10}$$

Comparing the coefficients of y^n yields

$$\sum_{\lambda\vdash n} \prod_{v\in\lambda} \frac{1}{h_v^2} \;=\; \frac{1}{n!}$$

or

$$\sum_{\lambda\vdash n} \Bigl(\frac{n!}{\prod_{v\in\lambda} h_v}\Bigr)^2 \;=\; n! \quad \square$$

2.6. Specialization for $\beta=-1$. Unlike the specializations $\beta=1,0,\infty$ as done previously, the case $\beta=-1$ does not relate to any classical result. However, we get a surprising formula for $pp(n)$, defined to be the number of ordered pairs π',π'' of partitions such that $|\pi'|+|\pi''|=n$ (see [BG06; CJW08]), as stated next.

Corollary 2.8. *We have*

$$pp(n) = \sum_{\lambda\vdash n}\prod_{v\in\lambda}\bigl(1+\frac{1}{h_v^2}\bigr). \tag{2.11}$$

Again, a direct proof of Corollary 2.8 would be welcome.

2.7. Specialization for $\beta = 2$. There is no direct specialization for $\beta = 2$. Nevertheless, by using the Euler pentagonal theorem (formula (1.1)) we obtain another expression for the alternate sum of the pentagonal powers, as stated in the next Proposition.

Proposition 2.9. *We have*

$$\sum_{\lambda\in\mathcal{P}}\prod_{v\in\lambda}\bigl(1-\frac{2}{h_v^2}\bigr)x = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2}. \tag{2.12}$$

Example 2.1. We see that the coefficient of x^4 in (2.12) is 0, i.e.:

$$\sum_{\lambda\vdash 4}\prod_{v\in\lambda}\bigl(1-\frac{2}{h_v^2}\bigr) = 0. \tag{2.13}$$

There are five partitions of 4 (see. Fig. 8.1) and their hook lengths are respectively $\{1,2,3,4\}, \{1,1,2,4\}, \{1,2,2,3\}, \{1,1,2,4\}$ and $\{1,2,3,4\}$. We verify that (2.13) is true by the following calculation.

$$2\bigl(1-\frac{2}{9}\bigr)\bigl(1-\frac{2}{16}\bigr)+2\bigl(1-\frac{2}{1}\bigr)\bigl(1-\frac{2}{16}\bigr)+\bigl(1-\frac{2}{4}\bigr)\bigl(1-\frac{2}{9}\bigr)=0.$$

This raises the question: is there a direct proof of Proposition 2.9?

2.8. Specialization for $\beta = 25$. Recall that the Ramanujan τ function is defined by (see [Se70, p.156]):

$$\begin{aligned} x\prod_{m\geq 1}(1-x^m)^{24} &= \sum_{n\geq 1}\tau(n)x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 + \cdots \end{aligned} \tag{2.14}$$

Putting $\beta = 25$ in the Main Theorem yields the next proposition.

Proposition 2.10. *We have*

$$\tau(n) = \sum_{\lambda}\prod_{v\in\lambda}\bigl(1-\frac{25}{h_v^2}\bigr) \tag{2.15}$$

where the sum ranges over all 5*-cores of* $n-1$.

Notice that there is cancellation between a box of hook length 3 and a box of hook length 4 in each 5-core, because

$$\big(1-\frac{25}{9}\big)\big(1-\frac{25}{16}\big)=1.$$

Example 2.2. Take $n=6$. There are two 5-cores of 5: $(3,2)$ and $(2,2,1)$. Those two 5-cores have their hook length multi-sets equal to $\{1,1,2,3,4\}$, so that

$$\tau(6)=2\big(1-\frac{25}{1}\big)\big(1-\frac{25}{1}\big)\big(1-\frac{25}{4}\big)=-6048.$$

3. Specialization for $\beta=4$

The $\beta=4$ case is very interesting. Unlike the case for $\beta=2$ in which Euler's pentagonal theorem is used, here the following well-known Jacobi triple product formula is re-proved! (see [An76, p.21; Kn98, p.20; JS89; FH99; FK99])

Theorem 3.1 [Jacobi]. *We have*

$$\prod_{m\geq 1}(1-x^m)^3=\sum_{m\geq 0}(-1)^m(2m+1)x^{m(m+1)/2}. \tag{3.1}$$

Proof. Put $\beta=4$ in The Main Identity:

$$\prod_{m\geq 1}(1-x^m)^3=\sum_{\lambda\in\mathcal{P}}\prod_{v\in\lambda}\big(1-\frac{4}{h_v^2}\big)x. \tag{3.2}$$

If a partition λ contains one box v whose hook length is $h_v=2$, then

$$\prod_{v\in\lambda}\big(1-\frac{4}{h_v^2}\big)x=0. \tag{3.3}$$

Otherwise λ must be a *staircase partition*

$$\Delta_m:=(m,m-1,\ldots,3,2,1).$$

We have (see Fig. 3.1 and 3.2 for an example):

$$\begin{aligned}\prod_{v\in\Delta_m}\big(1-\frac{4}{h_v^2}\big)&=\Big(\frac{(2m-1)^2-4}{(2m-1)^2}\Big)^1\cdots\Big(\frac{5}{9}\Big)^{m-1}\Big(\frac{-3}{1}\Big)^m\\&=(-1)^m(2m+1).\end{aligned}\tag{3.4}$$

As $|\Delta_m|=m(m+1)/2$, we conclude that

$$\prod_{m\geq 1}(1-x^m)^3=\sum_{m\geq 0}\prod_{v\in\Delta_m}\big(1-\frac{4}{h_v^2}\big)x=\sum_{m\geq 0}(-1)^m(2m+1)x^{m(m+1)/2}.\quad\square$$

1			
3	1		
5	3	1	
7	5	3	1

Fig. 3.1. Hook lengths h_v in Δ_4

-3			
$\frac{5}{9}$	-3		
$\frac{21}{25}$	$\frac{5}{9}$	-3	
$\frac{45}{49}$	$\frac{21}{25}$	$\frac{5}{9}$	-3

Fig. 3.2. $(1-4/h_v^2)$ in Δ_4

4. Specialization for $\beta = 9$

Recall that a partition λ is a *t-core*, if λ has no hook length equal to t. [GKS90; St99, p.468]. Hence, if λ is not a 3-core,

$$\prod_{v\in\lambda}(1-\frac{9}{h_v^2})x = 0. \tag{4.1}$$

By the Main Theorem

$$\prod_{m\geq 1}(1-x^m)^8 \;=\; \sum_{\lambda}\prod_{v\in\lambda}(1-\frac{9}{h_v^2})x \tag{4.2}$$

where the sum ranges over all 3-cores.

Theorem 4.1. *We have*

$$\prod_{k\geq 1}(1-q^k)^8 = \sum_{k,m\geq 0}\Big(\frac{1}{2}(3k+1)(3m+1)(3k+3m+2)q^{k^2+k+m^2+m+km} \\ -\frac{1}{2}(3k+2)(3m+2)(3k+3m+4)q^{k^2+k+m^2+m+(k+1)(m+1)}\Big).$$

Proof. We need characterize all 3-cores. In fact, a partition is a 3-core if and only if it has one of the forms described in Fig. 4.1 (type A) and Fig. 4.2 (type B). Let $\Delta_k = (k,\ldots,3,2,1)$ be a staircase partition. Define

$$\Delta_k^2 = (k,k,\ldots,3,3,2,2,1,1)$$

and ${\Delta_k^2}'$ be the transposition of Δ_k^2. Then type A (resp. type B) is made of a partition of form Δ_k^2, a partition of form ${\Delta_m^2}'$ and a rectangle of form

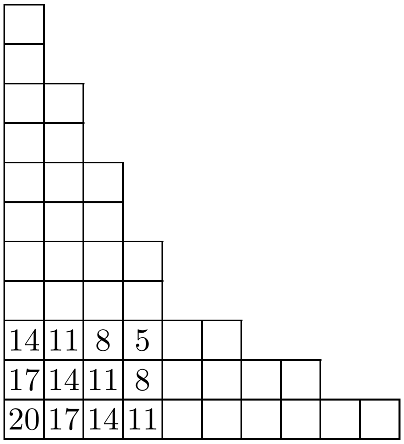


Fig. 4.1. Type A 3-core

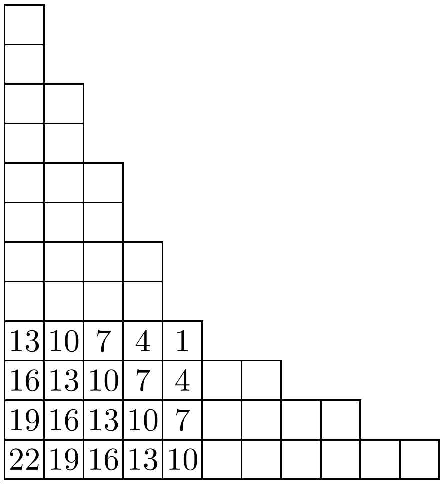
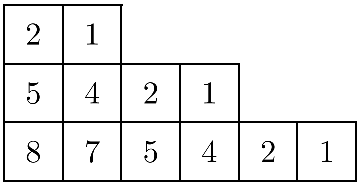
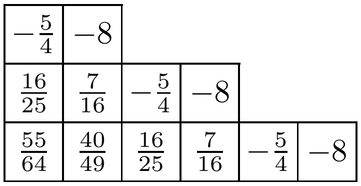


Fig. 4.2. Type B 3-core



2	1				
5	4	2	1		
8	7	5	4	2	1

Fig. 4.3. Hook lengths h_v in $\Delta_3^{2'}$



$-\frac{5}{4}$	-8				
$\frac{16}{25}$	$\frac{7}{16}$	$-\frac{5}{4}$	-8		
$\frac{55}{64}$	$\frac{40}{49}$	$\frac{16}{25}$	$\frac{7}{16}$	$-\frac{5}{4}$	-8

Fig. 4.4. $(1-9/h_v^2)$ in $\Delta_3^{2'}$

$k \times m$ (resp. of form $(k+1)\times(m+1)$). We have (see Fig. 4.3 and Fig. 4.4)

$$\prod_{v\in\Delta_k^2}\Big(1-\frac{9}{h_v^2}\Big) = (-8)^k\Big(-\frac{5}{4}\Big)^k\Big(\frac{7}{16}\Big)^{k-1}\Big(\frac{16}{25}\Big)^{k-1}\cdots$$

$$\times\Big(\frac{(3k-2)^2-9}{(3k-2)^2}\Big)^1\Big(\frac{(3k-1)^2-9}{(3k-1)^2}\Big)^1$$

$$= (3k+1)(3k+2)/2. \tag{4.3}$$

The product of $1-9/h_v^2$ for all boxes v in the rectangle of type A (Fig. 4.1) is:

$$\prod_{v\in A(k,m)}\Big(1-\frac{9}{h_v^2}\Big) = \prod_{j=1}^{m}\prod_{i=1}^{k}\frac{(3i+3j-1)^2-9}{(3i+3j-1)^2}$$

$$= \prod_{j=1}^{m}\frac{(3j-1)(3j+3k+2)}{(3j+2)(3j+3k-1)}$$

$$= \frac{2(3k+3m+2)}{(3k+2)(3m+2)}. \tag{4.4}$$

The product of $1-9/h_v^2$ for all boxes v in the rectangle of type B (Fig. 4.2) is:

$$\prod_{v\in B(k,m)}\Bigl(1-\frac{9}{h_v^2}\Bigr)=\prod_{j=1}^{m+1}\prod_{i=1}^{k+1}\frac{(3i+3j-5)^2-9}{(3i+3j-5)^2}$$

$$= \frac{-2(3k+3m+4)}{(3k+1)(3m+1)}. \tag{4.5}$$

Combining equations (4.3), (4.4) and (4.5) yields Theorem 4.1. □

5. Proof of the Main Theorem

In this section we always suppose that $t=2t'+1$ is an odd positive integer.

5.1. Fundamental properties of t-cores and V-codings. Recall that a partition λ is a t-core if the *hook length multi-set* of λ does not contain the integer t. It is known that the hook length multi-set of each t-core does not contain any *multiple* of t [Kn98. p.69, p.612; St99, p.468].

Definition 5.1. Each vector of integers $(v_0,v_1,\ldots,v_{t-1})\in\mathbb{Z}^t$ is called *V-coding* if the following conditions hold:

(i) $v_i\equiv i(\bmod t)$ for $0\le i\le t-1$;

(ii) $v_0+v_1+\cdots+v_{t-1}=0$.

The V-coding can be identified with the *set* $\{v_0,v_1,\ldots,v_{t-1}\}$ thanks to condition (i). In this section we present a bijection between t-cores and V-codings, that constitutes the crucial step in the proof of the Main Theorem.

Theorem 5.1. *There is a bijection $\phi_V:\lambda\mapsto(v_0,v_1,\ldots,v_{t-1})$ which maps each t-core onto a V-coding such that*

$$|\lambda|=\frac{1}{2t}(v_0^2+v_1^2+\cdots+v_{t-1}^2)-\frac{t^2-1}{24} \tag{5.1}$$

and

$$\prod_{v\in\lambda}\Bigl(1-\frac{t^2}{h_v^2}\Bigr)=\frac{(-1)^{t'}}{1!\cdot2!\cdot3!\cdots(t-1)!}\prod_{0\le i<j\le t-1}(v_i-v_j). \tag{5.2}$$

We will describe the bijection ϕ_V and prove the two equalities (5.1) and (5.2) in §5.2, §5.3 and §5.4 respectively. An example is given after the construction of the bijection ϕ_V.

5.2. The bijection ϕ_V and an example. Each finite set of integers $A = \{a_1, a_2, \ldots, a_n\}$ is said to be *t-compact* if the following conditions hold:

(i) $-1, -2, \ldots, -t \in A$;

(ii) for each $a \in A$ such that $a \neq -1, -2, \ldots, -t$, we have $a \geq 1$ and $a \not\equiv 0 \bmod t$;

(iii) let $b > a \geq 1$ be two integers such that $a \equiv b \bmod t$. If $b \in A$, then $a \in A$.

Let A be a t-compact set. An element $a \in A$ is said to be *t-maximal* if $b \notin A$ for every $b > a$ such that $a \equiv b \bmod t$. The set of t-maximal letters of A is denoted by $\max_t(A)$. Let λ be a t-core. The *H-set* of the t-core λ is defined to be

$$H(\lambda) = \{h_v \mid v \text{ is a box in the leftmost column of } \lambda\} \cup \{-1, -2, \ldots - t\}.$$

Lemma 5.2. *For each t-core λ its H-set $H(\lambda)$ is a t-compact set.*

Proof. Let $c = tk + r$ $(k \geq 1, 0 \leq r \leq t-1)$ be an element in $H(\lambda)$ and a be the maximal letter in $H(\lambda)$ such that $a < t(k-1) + r$. We must show that $t(k-1) + r$ is also in $H(\lambda)$. If it were not the case, let $z > t(k-1) + r, y_1, y_2, \ldots, y_d$ be the hook lengths as shown in Fig. 5.1, where only the relevant horizontal section of the partition diagram has been represented. We have $y_1 = c - a - 1 \geq tk + r - t(k-1) - r = t$ and $y_d = c - z + 1 \leq tk + r - t(k-1) - r = t$; so that there is one hook $y_i = t$. This is a contradiction since λ is supposed to be a t-core. $\square$

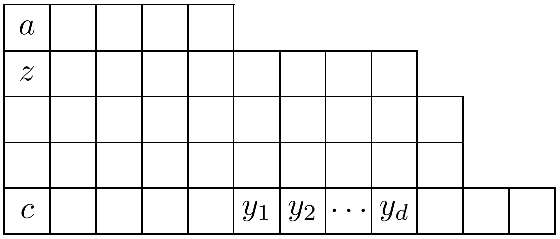


Fig. 5.1. Hook length and t-compact set

Construction of ϕ_V. Let λ be a t-core and $H(\lambda)$ be its H-set. The *U-coding* of λ is defined to be the set $U := \max_t(H(\lambda))$, which can be identified with the vector $(u_0, u_1, \ldots, u_{t-1})$ such that $u_0 = -t$, $u_i > -t$ and $u_i \equiv i \bmod t$ for $1 \leq i \leq t-1$. In general,

$$S := u_0 + u_1 + \cdots + u_{t-1} \neq 0.$$

The integer S is a multiple of t because

$$S = \sum u_i = \sum (tk_i + i) = t \sum k_i + t(t-1)/2$$

(remember that $t = 2t' + 1$ is an odd integer). The V-coding $\phi_V(\lambda)$ is the set V obtained from U by the following *normalization*:

$$\phi_V(\lambda) = V := \{u - S/t \ : \ u \in U\}. \tag{5.3}$$

In fact, we can prove that $S/t = \ell(\lambda) - t' - 1$ (see (5.8)). The set V can be identified with a vector V-coding because

$$\sum v_i = \sum (u_i - S/t) = \sum u_i - S = 0.$$

Example 5.1. Consider the 5-core

$$\lambda = (14, 10, 6, 6, 4, 4, 4, 2, 2, 2).$$

The H-set of λ (see Fig. 5.2)

$$H(\lambda) = \{23, 18, 13, 12, 9, 8, 7, 4, 3, 2, -1, -2, -3, -4, -5\}$$

is 5-compact. The U-coding of λ is $U = \max_5(H(\lambda)) = \{23, 12, 9, -4, -5\}$, or in vector form

$$(u_0, u_1, u_2, u_3, u_4) = (-5, -4, 12, 23, 9).$$

As $S = \sum u_i = 35$, the V-coding is given by

$$V = \{-5 - 7, -4 - 7, 12 - 7, 23 - 7, 9 - 7\} = \{-12, -11, 5, 16, 2\},$$

or in vector form

$$\phi_V(\lambda) = (v_0, v_1, v_2, v_3, v_4) = (5, 16, 2, -12, -11).$$

We have

$$\begin{aligned} |\lambda| &= \frac{1}{2t}(v_0^2 + v_1^2 + \cdots + v_{t-1}^2) - \frac{t^2 - 1}{24} \\ &= \frac{1}{2 \cdot 5}(5^2 + 16^2 + 2^2 + (-12)^2 + (-11)^2) - \frac{5^2 - 1}{24} = 54. \end{aligned}$$

and

$$\begin{aligned} \prod_{v \in \lambda}\Bigl(1 - \frac{5^2}{h_v^2}\Bigr) &= \frac{1}{1! \cdot 2! \cdot 3! \cdots (t-1)!} \prod_{0 \le i < j \le t-1} (v_i - v_j) \\ &= (-11)(3)(17)(16) \cdot (14)(28)(27) \cdot (14)(13) \cdot (-1)/288 \\ &= 60035976. \end{aligned}$$

Notice that, as expected, the above two numbers are positive integers.

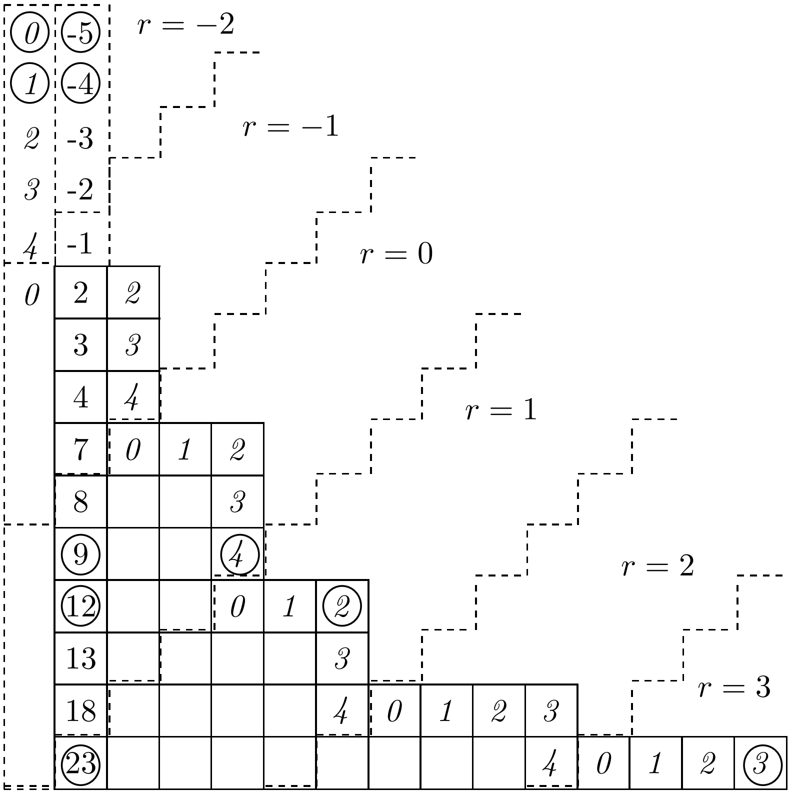


Fig. 5.2. U-coding and N-coding of t-core

5.3. Proof of (5.1). A vector of integers $(n_0, n_1, \ldots, n_{t-1}) \in \mathbb{Z}^t$ is said to be an *N-coding* if $n_0 + n_1 + \cdots + n_{t-1} = 0$. Garvan, Kim and Stanton have defined a bijection ϕ_N between N-codings and t-cores. We now recall its definition using their own words [GKS90,p.3] (see also [BG06]).

Let λ be a t-core. Define the vector $(n_0, \ldots, n_{t-1}) = \phi_N(\lambda)$ in the following way. Label a box in the i-th row and j-column of λ by $j - i \bmod t$. We also label the boxes in column 0 (in dotted lines in Fig. 5.2) in the same way, and call the resulting diagram the *extended t-residue diagram*. A box is called *exposed* if it is at the end of a row of the extended t-residue diagram. The set of boxes (i, j) satisfying $t(r-1) \le j - i < tr$ of the extended t-residue diagram of λ is called *region* and numbered r. In Fig. 5.2 the regions have been bordered by dotted lines. We now define n_i to be the maximum region r which contains an exposed box labeled i.

In Fig. 5.2 the labels of all boxes lying on the maximal border strip (but the leftmost one) have been written in italic. This includes all the exposed boxes: 3,3,3,2,4,3,2,4,3,2,4,3,2,1,0, when reading from bottom to top. We have $(n_0, n_1, n_2, n_3, n_4) = (-2, -2, 1, 3, 0)$.

Theorem 5.3 [Garvan-Kim-Stanton]. *The bijection*

$$\phi_N : \lambda \mapsto (n_0, n_1, \ldots, n_{t-1})$$

has the following property:

$$|\lambda| = \frac{t}{2}\sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} i n_i. \tag{5.4}$$

Let $t' = (t-1)/2$ and let

$$\phi_V^N : (n_0, n_1, \ldots, n_{t-1}) \mapsto (v_0, v_1, \ldots, v_{t-1})$$

be the bijection that maps each N-coding onto the V-coding defined by

$$v_i = \begin{cases} t n_{i+t'} + i & \text{if } 0 \le i \le t'; \\ t n_{i-t'-1} + i - t & \text{if } t'+1 \le i \le t-1 \end{cases} \tag{5.5}$$

or in set form

$$\{v_i \mid 0 \le i \le t-1\} = \{t n_i + i - t' \mid 0 \le i \le t-1\}. \tag{5.6}$$

The bijective property is easy to verify. More essentially, the bijection ϕ_V defined in §5.2 is the composition product of the two previous bijections as is now shown.

Lemma 5.4. *We have* $\phi_V = \phi_V^N \circ \phi_N$.

Proof. Let $(v_0, \ldots, v_{t-1}) = \phi_V(\lambda)$, $(n_0, \ldots, n_{t-1}) = \phi_N(\lambda)$ and

$$(v'_0, \ldots, v'_{t-1}) = \phi_V^N(n_0, \ldots, n_{t-1}).$$

We need prove that $v_i = v'_i$. The number n_i in the N-coding is defined to be the maximum region r which contains an exposed box labelled i. This exposed box is called *critical italic box.* In Fig. 5.2, a circle is drawn around the label of each critical italic box. On the other hand, the U-coding is defined to be the set $\max_t(H(\lambda))$, where $H(\lambda)$ is the H-set of λ. A box in the leftmost column whose hook length is an element of the U-coding is called *critical roman box.* In Fig. 5.2, a circle is drawn around the hook length number of each critical roman box. Let us write the labels of all the exposed boxes (the vector $L = (L_i)$) with its region numbers (the vector $R = (R_i)$) and the H-set of λ (the vector $H = (H_i) = H(\lambda)$), read from bottom to top.

$$\begin{array}{rccccccccccccccc}
L = & \textcircled{\textit{3}} & \mathit{3} & \mathit{3} & \textcircled{\textit{2}} & \textcircled{\textit{4}} & \mathit{3} & \mathit{2} & \mathit{4} & \mathit{3} & \mathit{2} & \mathit{4} & \mathit{3} & \mathit{2} & \textcircled{\textit{1}} & \textcircled{\textit{0}} \\
R = & \textcircled{3} & 2 & 1 & \textcircled{1} & \textcircled{0} & 0 & 0 & -1 & -1 & -1 & -2 & -2 & -2 & \textcircled{-2} & \textcircled{-2} \\
H = & \textcircled{23} & 18 & 13 & \textcircled{12} & \textcircled{9} & 8 & 7 & 4 & 3 & 2 & -1 & -2 & -3 & \textcircled{-4} & \textcircled{-5}
\end{array}$$

It is easy to see that $L_j = (H_j - \ell(\lambda)) \bmod t$ and $R_j = \lfloor (H_j - \ell(\lambda))/t \rfloor + 1$. This means that L_i has a circle symbol if and only if H_i has a circle symbol. We then have a natural bijection

$$f : u_i \mapsto \lfloor (u_i - \ell(\lambda))/t \rfloor + 1 = n_{(u_i - \ell) \bmod t} \tag{5.7}$$

between the set $\{u_0, \ldots, u_{t-1}\}$ and $\{n_0, \ldots, n_{t-1}\}$. By (5.6) and (5.7) we have

$$\begin{aligned}\{v'_i\} &= \{tn_i + i - t'\} \\ &= \{tn_{(u_i-\ell) \bmod t} + (u_i - \ell) \bmod t - t'\} \\ &= \{t(\lfloor (u_i - \ell)/t \rfloor + 1) + (u_i - \ell) \bmod t - t'\} \\ &= \{u_i - \ell + t' + 1\}.\end{aligned}$$

On the other hand, (v'_i) is a V-coding, because $v'_i \equiv i \bmod t$ and $\sum v'_i = t\sum n_i + \sum i - t(t-1)/2 = 0$; so that

$$(\sum_i u_i)/t = \ell - t' - 1. \tag{5.8}$$

Hence

$$\{v'_i\} = \{u_i - \ell + t' + 1\} = \{u_i - (\sum_i u_i)/t\} = \{v_i\}. \quad \square$$

Take again the same partition as in Example 5.1; the N-coding is

$$(n_0, n_1, n_2, n_3, n_4) = (-2, -2, 1, 3, 0).$$

We verify that

$$\begin{aligned}&(v'_0, v'_1, v'_2, v'_3, v'_4) \\ &\quad = (1 \times 5 + 0,\ 3 \times 5 + 1,\ 0 \times 5 + 2,\ -2 \times 5 - 2,\ -2 \times 5 - 1). \\ &\quad = (5, 16, 2, -12, -11) = (v_0, v_1, v_2, v_3, v_4).\end{aligned}$$

Proof of (5.1) in Theorem 5.1. From (5.6) we have

$$\begin{aligned}\sum v_i^2 &= \sum (tn_i + i - t')^2 \\ &= \sum \big((tn_i)^2 + 2tin_i - 2tt'n_i + i^2 + t'^2 - 2it'\big) \\ &= t^2 \sum n_i^2 + 2t \sum in_i + \frac{(t-1)t(2t-1)}{6} + tt'^2 - t't(t-1) \\ &= t^2 \sum n_i^2 + 2t \sum in_i + \frac{t(t^2-1)}{12}.\end{aligned}$$

Hence

$$\frac{1}{2t}\sum v_i^2 = \frac{t}{2}\sum n_i^2 + \sum in_i + \frac{t^2-1}{24} = |\lambda| + \frac{t^2-1}{24}. \quad \square$$

5.4. Proof of (5.2). We first etablish the following two lemmas.

Lemma 5.5. *For any t-compact set A we have*

$$\prod_{a\in A, a>0}\Bigl(1-\frac{t^2}{a^2}\Bigr)=\prod_{a\in \max_t(A), a\neq -t}\frac{a+t}{a}. \tag{5.9}$$

Example 5.2. Take $t=5$. Then the set

$$A=\{-5,-4,-3,-2,-1,2,3,4,7,8,9,12,13,18,23\}$$

is 5-compact. We have $\max_t(A)=\{-5,-4,9,12,23\}$. Hence

$$\prod_{a\in A, a>0}\Bigl(1-\frac{25}{a^2}\Bigr)=\frac{1\cdot 14\cdot 17\cdot 28}{(-4)\cdot 9\cdot 12\cdot 23}. \tag{5.10}$$

Proof. Write

$$\prod_{a\in A, a>0}\Bigl(1-\frac{t^2}{a^2}\Bigr)=\prod_{a\in A, a>0}\frac{(a-t)\cdot(a+t)}{a\cdot a},$$

then delete the common factors in numerator and denominator, as illustrated by means of Example 5.2.

$$\begin{array}{lllllll}
\frac{1}{-4} & & & & & & (a\equiv 1 \bmod 5)\\
\frac{2}{-3} & \frac{-3}{2}\times\frac{7}{2} & \frac{2}{7}\times\frac{12}{7} & \frac{7}{12}\times\frac{17}{12} & & & (a\equiv 2 \bmod 5)\\
\frac{3}{-2} & \frac{-2}{3}\times\frac{8}{3} & \frac{3}{8}\times\frac{13}{8} & \frac{8}{13}\times\frac{18}{13} & \frac{13}{18}\times\frac{23}{18} & \frac{18}{23}\times\frac{28}{23} & (a\equiv 3 \bmod 5)\\
\frac{4}{-1} & \frac{-1}{4}\times\frac{9}{4} & \frac{4}{9}\times\frac{14}{9} & & & & (a\equiv 4 \bmod 5)
\end{array}$$

The product $(a-5)(a+5)/a^2$ for $a>0$ is reproduced in the row determined by $a \bmod 5$ in the above table, except for the leftmost column. But the product of the factors in the leftmost column is equal to 1 because t is an odd integer; so that the left-hand side of (5.10) is the product of the factors in the above table. After deleting the common factors, it remains the rightmost fraction in each row. $\square$

Lemma 5.6. *Let λ be a t-core and $(u_0,u_1,\ldots,u_{t-1})$ be its U-coding (defined in the body of the construction of ϕ_V). Let λ' be the t-core obtained from λ by erasing the leftmost column of λ and $(u'_0,u'_1,\ldots,u'_{t-1})$ be its U-coding. Then*

$$\prod_{0\le i<j\le t-1}\frac{u_i-u_j}{u'_i-u'_j}=\prod_{j=1}^{t-1}\frac{u_j+t}{u_j}.$$

Example 5.3. Take the 5-core λ given in Example 5.1. The U-coding of λ is $(u_0, u_1, u_2, u_3, u_4) = (-5, -4, 12, 23, 9)$. We have

$$\lambda' = (13, 9, 5, 5, 3, 3, 3, 1, 1, 1).$$

The U-coding of λ' is $(u'_0, u'_1, u'_2, u'_3, u'_4) = (-5, 11, 22, 8, -1)$. Now, consider the cyclic rearrangement

$$(u''_0, u''_1, u''_2, u''_3, u''_4) = (-1, -5, 11, 22, 8)$$

of $(u'_0, u'_1, u'_2, u'_3, u'_4)$. We have $\prod(u'_i - u'_j) = \prod(u''_i - u''_j)$ because t is an odd integer. Moreover $u''_i = u_i - 1$ for all $1 \le i \le 4$. Hence

$$\begin{aligned} \prod_{0\le i<j\le t-1} \frac{u_i - u_j}{u''_i - u''_j} &= \prod_{j=1}^{t-1} \frac{u_0 - u_j}{u''_0 - u''_j} \\ &= \frac{(-5+4)(-5-12)(-5-23)(-5-9)}{(-1+5)(-1-11)(-1-22)(-1-8)} \\ &= \frac{(-4+5)(12+5)(23+5)(9+5)}{(-4)(12)(23)(9)}. \end{aligned}$$

Proof. We suppose that λ contains δ parts equal to 1. Its H-set $H(\lambda)$ (viewed as a vector in decreasing order if necessary) can be split into six segments $H(\lambda) = A_1A_2A_3A_4A_5A_6$ defined by (see Fig. 5.3)
(i) $a \ge \delta + 2$ for each $a \in A_1$;
(ii) $A_2 = (\delta, \delta - 1, \ldots, 3, 2, 1)$;
(iii) $A_3 = (-1, -2, -3, \ldots, \delta + 2 - t)$;
(iv) $A_4 = (\delta + 1 - t)$;
(v) $A_5 = (\delta - t, \delta - 1 - t, \ldots, 1 - t)$;
(vi) $A_6 = (-t)$.

On the other hand the H-set $H(\lambda')$ of λ' is split into five segments $H(\lambda') = A'_1A'_2A'_3A'_4A'_5$ defined by
(i') $A'_1 = \{a - \delta - 1 \ :\ a \in A_1\}$;
(ii') $A'_2 = \{a - \delta - 1 \ :\ a \in A_2\} = (-1, -2, \ldots, -\delta)$;
(iii') $A'_3 = (-\delta - 1)$;
(iv') $A'_4 = \{a - \delta - 1 \ :\ a \in A_3\} = (-\delta - 2, -\delta - 3, \ldots, -t + 1)$;
(v) $A'_5 = (-t)$.

Notice that some segments A_i and A'_i may be empty. More precisely,

$$\begin{cases} A_2 = A_5 = A'_2 = \emptyset, & \text{if } \delta = 0; \\ A_3 = A'_4 = \emptyset, & \text{if } \delta = t - 2; \\ A_3 = A_4 = A'_3 = A'_4 = \emptyset, & \text{if } \delta = t - 1. \end{cases}$$

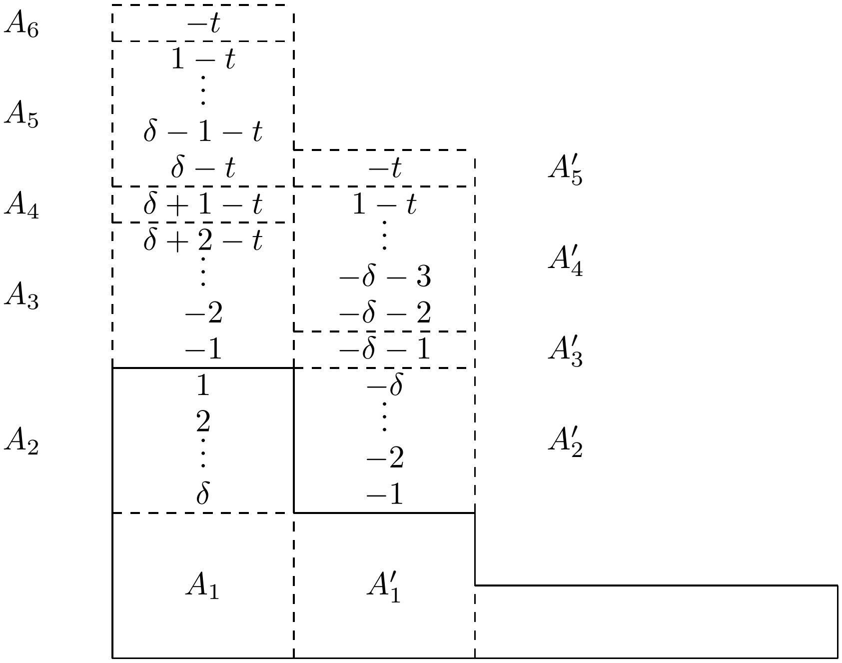


Fig. 5.3. Comparison the hook lengths of λ and λ'

The basic facts are:

(i) $a \notin \max_t(H(\lambda))$ for every $a \in A_5$; because $\{a \bmod t \ : \ a \in A_5\} = \{a \bmod t \ : \ a \in A_3\}$. In other words the set A_5 is *masked* by A_3.

(ii) $\delta + 1 - t \in \max_t(H(\lambda))$; because $a \not\equiv 0 \bmod t$ for every $a \in A_1'$ so that $a \not\equiv \delta+1 \bmod t$ for every $a \in A_1$. It is easy to see that $a \not\equiv \delta+1 \bmod t$ for every $a \in A_2 \cup A_3$.

(iii) $-\delta - 1 \in \max_t(H(\lambda'))$; because $a \not\equiv 0 \bmod t$ for every $a \in A_1 \cup A_2$ so that $a \not\equiv -\delta - 1 \bmod t$ for every $a \in A_1' \cup A_2'$.

(iv) Since that $a \mapsto a - \delta - 1$ is a bijection between $A_1 \cup A_2 \cup A_3$ and $A_1' \cup A_2' \cup A_4'$, it is also a bijection between $\max_t(H(\lambda)) \setminus \{-t, \delta - t + 1\}$ and $\max_t(H(\lambda')) \setminus \{-t, -\delta - 1\}$.

The above facts enable us to derive the U-coding of λ' from the U-coding of λ as follows. Let

$$(u_i) = (u_0 = -t, u_1, u_2, \ldots, u_{k-1}, \delta + 1 - t, u_{k+1}, u_{k+1}, \ldots, u_{t-1})$$

be the U-coding of λ and define

$$(u_i'') = (u_0'' = -\delta - 1, u_1'', u_2'', \ldots, u_{k-1}'', -t, u_{k+1}'', u_{k+1}'', \ldots, u_{t-1}'')$$

where $u_i'' = u_i - \delta - 1$ for $i \geq 1$. Then, the U-coding of λ' is simply

$$(u_i') = (u_0' = -t, u_{k+1}'', u_{k+1}'', \ldots, u_{t-1}'', -\delta - 1, u_1'', u_2'', \ldots, u_{k-1}'').$$

We have $\prod(u'_i - u'_j) = \prod(u''_i - u''_j)$ because t is an odd integer. On the other hand, $u''_i - u''_j = u_i - u_j$ for all $1 \le i < j \le t-1$. Hence

$$\prod_{0\le i<j\le t-1} \frac{u_i - u_j}{u'_i - u'_j} = \prod_{0\le i<j\le t-1} \frac{u_i - u_j}{u''_i - u''_j} = \prod_{j=1}^{t-1} \frac{u_0 - u_j}{u''_0 - u''_j}$$
$$= \prod_{j=1}^{t-1} \frac{-t - u_j}{-\delta - 1 - u''_j} = \prod_{j=1}^{t-1} \frac{u_j + t}{u_j}. \qquad \square$$

Proof of (5.2) in Theorem 5.1. Because the U-coding and V-coding of λ only differ by the normalization given in (5.3) and t is an odd integer, we have $\prod(v_i - v_j) = \prod(u_i - u_j)$. By Lemmas 5.6 and 5.5 we have

$$\prod_{0\le i<j\le t-1} (u_i - u_j) = \prod_{j=1}^{t-1} \frac{u_j + t}{u_j} \times \prod_{0\le i<j\le t-1} (u'_i - u'_j)$$
$$= \prod_{a\in H(\lambda), a>0} \Bigl(1 - \frac{t^2}{a^2}\Bigr) \times \prod_{0\le i<j\le t-1} (u'_i - u'_j)$$
$$= \cdots = K \times \prod_{v\in\lambda} \Bigl(1 - \frac{t^2}{h_v^2}\Bigr).$$

Taking λ as the empty t-core, the U-coding of λ is $(-t, -t+1, -t+2, \ldots, -3, -2, -1)$. We then obtain $K = (-1)^{t'} 1! \cdot 2! \cdot 3! \cdots (t-1)!$ $\square$

5.5. Enf of the proof of the Main Theorem. Recall that the Dedekind η-function is defined by

$$\eta(x) = x^{1/24} \prod_{m\ge 1} (1 - x^m). \tag{5.11}$$

Let $t = 2t' + 1$ be an odd integer. Macdonald obtained the following result [Ma72] (see comments in §1).

Theorem 5.7 [Macdonald]. *We have*

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0,\ldots,v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2+v_1^2+\cdots+v_{t-1}^2)/(2t)}, \tag{5.12}$$

where the sum ranges over all V-codings $(v_0, v_1, \ldots, v_{t-1})$ (see Definition 5.1) and c_0 is a numerical constant.

Consider the term of lowest degree in the above power series. We immediately get

$$c_0 = \frac{(-1)^{t'}}{1! \cdot 2! \cdot 3! \cdots (t-1)!}. \tag{5.13}$$

Proof of the Main Theorem. Let $n \geq 0$ be a positive integer. The coefficient $C_n(\beta)$ of x^n on the left-hand side of the Main Identity is a polynomial in β of degree n. The coefficient $D_n(\beta)$ of x^n on the right-hand side of the Main Identity is also a polynomial in β of degree n thanks to (2.3). For proving $C_n(\beta) = D_n(\beta)$, it suffices to find $n+1$ explicit numerical values $\beta_0, \beta_1, \ldots, \beta_n$ such that $C_n(\beta_i) = D_n(\beta_i)$ for $0 \leq i \leq n$ by using the Lagrange interpolation formula. The basic fact is that

$$\prod_{v\in\lambda}\Bigl(1-\frac{t^2}{h_v^2}\Bigr) = 0$$

for every partition λ which is not a t-core. By comparing Theorems 5.1 and 5.7 we see that the Main Identity is true when $\beta = t^2$ for every odd integer t, i.e.,

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{v\in\lambda}\Bigl(1-\frac{t^2}{h_v^2}\Bigr) \ = \ \prod_{m\geq 1}(1-x^m)^{t^2-1},$$

so that $C_n(\beta) = D_n(\beta)$ for every complex number β. ∎

Note that Kostant already observed that $C_n(\beta)$ is a polynomial in β, but did not mention any explicit expression [Ko04].

6. New formulas about hook lengths

In Sections 2-4 we have taken special numerical values for β. In this section we compare the coefficients of β^k to derive further identities.

6.1. Comparing the coefficients of β.

Proposition 6.1. *We have*

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda}\frac{1}{h_v^2} = \prod_{m\geq 1}\frac{1}{1-x^m}\times\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}. \tag{6.1}$$

Proof. Identity (6.1) follows from (2.3) by comparing the coefficients of β^1 on both sides. ∎

6.2. The Stanley-Elder-Bessenrodt-Bacher-Manivel Theorem. We also have a second proof of Proposition 6.1 that is direct and provides a more general result about the power sum of the hook lengths.

Theorem 6.2. *We have*

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda} h_v^{\alpha} = \prod_{m\geq 1}\frac{1}{1-x^m}\times\sum_{k\geq 1}\frac{x^k k^{\alpha+1}}{1-x^k}. \tag{6.2}$$

Recall that $\sigma_\alpha(n)=\sum_{d|n} d^\alpha$ is the α-th power sum of all positive divisors of n (see [Se70, p.149]) whose generating function is classically given by

$$\sum_{k\geq 1} \frac{x^k k^\alpha}{1-x^k} = \sum_{n\geq 1} \sigma_\alpha(n) x^n. \tag{6.3}$$

Using (6.3) identity (6.2) can be rewritten as

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n} \sum_{v\in\lambda} h_v^\alpha = \prod_{m\geq 1} \frac{1}{1-x^m} \times \sum_{n\geq 1} \sigma_{\alpha+1}(n) x^n. \tag{6.4}$$

The proof of Theorem 6.2 is based on an elegant result about the multi-set of hook lengths and the multi-set of parts of all partitions of n. Many studies have been done along those lines [Be98, BM02, Ho86, St04, KS82, We1, We2]. Each hook length h_v can be split into $h_v=a_v+l_v+1$ where a_v is the *arm length* and l_v is the *leg length* (see [St99, p.457]). The ordered pair (a_v, l_v) is called a *hook type*.

Theorem 6.3 [Stanley-Elder-Bessenrodt-Bacher-Manivel]. *Let $n\geq k\geq 1$ be two integers. Then for every positive $j<k$ the total number of occurrences of the part k among all partitions of n is equal to the number of boxes whose hook type is $(j, k-j-1)$.*

We now state a weaker form of the SEBBM Theorem, much easier to figure out. Let A be a multi-set of positive integers. Define $\dot{A}$ to be the multi-set derived from A by replacing each element a of A by a copies of a. For instance, with $A=\{1,1,2,5\}$ we obtain $\dot{A}=\{1,1,2,2,5,5,5,5,5\}=\{1^1,1^1,2^2,5^5\}$.

Proposition 6.4. *Let $H(n)$ (resp. $G(n)$) be the multi-set of all hook lenghts (resp. the parts) of all partitions of n. Then*

$$H(n)=\dot{G}(n). \tag{6.5}$$

For example, the set of all partitions of 4 with their hook length multi-sets is reproduced in Fig. 6.1. We see that $H(4)=\{1^7,2^6,3^3,4^4\}$. On the other hand, $G(4)=\{1,1,1,1,2,1,1,2,2,3,1,4\}$. We have $\dot{G}(4)=\{1^7,2^6,3^3,4^4\}=H(4)$. Notice that $\dot{G}(4)$ can be represented as in Fig. 6.2.

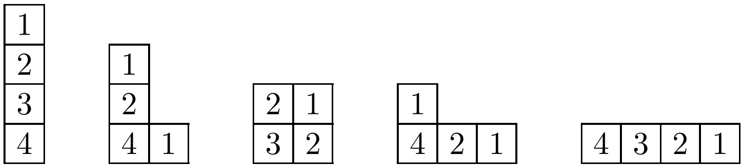


Fig. 6.1. The multi-set $H(4)$ of hook lengths

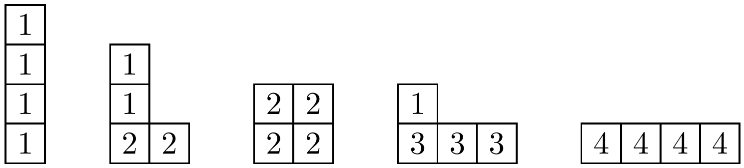


Fig. 6.2. The multi-set $\dot{G}(4)$ of parts with duplications

Theorem 6.3 can be used to evaluate the power sum of the hook lengths. We obtain immediately the following Theorem, which means that the r-th power sum of the hook lengths is equal to the $(r+1)$-st power sum of the parts.

Corollary 6.5. *For each positive integer n and each complex number α we have*

$$\sum_{\lambda\vdash n}\sum_{v\in\lambda} h_v^{\alpha} = \sum_{\lambda\vdash n}\sum_{i} \lambda_i^{\alpha+1}. \tag{6.6}$$

By Corollary 6.5, we see that Theorem 6.2 is equivalent to the next Theorem.

Theorem 6.6. *We have*

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{i} \lambda_i^{\alpha} = \prod_{m\geq 1}\frac{1}{1-x^m} \times \sum_{k\geq 1}\frac{x^k k^{\alpha}}{1-x^k}. \tag{6.7}$$

Proof. Let

$$F(k) := \sum_{n_k, n_{k+1},\ldots\geq 0} x^{kn_k+(k+1)n_{k+1}+\cdots}(n_k k^{\alpha} + n_{k+1}(k+1)^{\alpha}+\cdots). \tag{6.8}$$

We have

$$\begin{aligned}
F(k) &= \sum_{n_k} x^{kn_k} n_k k^{\alpha} \times \sum_{n_{k+1},\ldots} x^{(k+1)n_{k+1}+\cdots} + \sum_{n_k} x^{kn_k} F(k+1)\\
&= \frac{k^{\alpha}}{(1-x^{k+1})(1-x^{k+2})\cdots}\sum_{n} x^{kn} n + \frac{1}{1-x^k}F(k+1)\\
&= \frac{1}{(1-x^k)(1-x^{k+1})\cdots}\frac{k^{\alpha}x^k}{1-x^k} + \frac{1}{1-x^k}F(k+1). \qquad (6.9)
\end{aligned}$$

Let

$$F'(k) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})}F(k). \tag{6.10}$$

Then identity (6.9) becomes

$$F'(k) = \frac{1}{(1-x)(1-x^2)\cdots} \frac{k^\alpha x^k}{1-x^k} + F'(k+1). \tag{6.11}$$

By iteration

$$F'(1) = \prod_{m\ge 1} \frac{1}{1-x^m} \times \sum_{k\ge 1} \frac{x^k k^\alpha}{1-x^k}. \tag{6.12}$$

Thus, the left-hand side of (6.7) is equal to $F(1) = F'(1)$. □

Putting $\alpha = 0$ and $\alpha = -1$ we obtain the following specializations.

Corollary 6.7. *We have*

$$\begin{aligned}\sum_{n\ge 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda} 1 &= \prod_{m\ge 1} \frac{1}{1-x^m} \times \sum_{k\ge 1} \frac{x^k k}{1-x^k} \\ &= x\frac{d}{dx} \prod_{m\ge 1} \frac{1}{1-x^m}\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}\sum_{n\ge 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda} \frac{1}{h_v} &= \prod_{m\ge 1} \frac{1}{1-x^m} \times \sum_{k\ge 1} \frac{x^k}{1-x^k} \\ &= \sum_{m\ge 1} \frac{m x^m}{(1-x)(1-x^2)\cdots(1-x^m)}.\end{aligned} \tag{6.14}$$

For the second equality of (6.14), see [Slo, A006128].

Historical Remarks about the SEBBM theorem. In the present paper we do not give the proof of the SEBBM theorem. We only want to make the following historical remarks. Stanley proved the case $j = 0$ in 1972. Independent discoveries and proofs were given by Kirdar and Skyrme (1982), Paul Elder (1984) and Hoare (1986) (see [We1, We2, St04, KS82, Ho86]). This result is called Elder's Theorem. Bessenrodt [Be98] proved the general case of Theorem 6.3 in 1998. The final version of this result was given by Bacher and Manivel [BM02] in 2002. In fact, when we re-discovered Theorem 6.3, as will be further explained, we noticed that Elder's Theorem, stated in this hook length language, was just the particular case $j = 0$ of Theorem 6.3.

When preparing the present paper we rediscovered Theorem 6.3 in the following manner. First, we obtained Proposition 6.1, as mentioned earlier by comparing the coefficients of β in the Main Theorem. We then expanded the right-hand side of (6.1) and calculated the first terms:

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda}\frac{1}{h_v^2} = x + 5\frac{x^2}{2!} + 29\frac{x^3}{3!} + 218\frac{x^4}{4!} + 1814\frac{x^5}{5!} + \cdots \tag{6.15}$$

When searching for the sequence $1, 5, 29, 218, 1814, \ldots$ in *The On-Line Encyclopedia of Integer Sequences* [Slo] we got the sequence A057623, that referred to "`n! * (sum of reciprocals of all parts in unrestricted partitions of n).`" Next we calculated

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda}\frac{1}{h_v} = x + 3x^2 + 6x^3 + 12x^4 + 20x^5 + 35x^6 + \cdots$$

by enumerating all partitions. Going back to the The On-Line Encyclopedia the sequence $(1, 3, 6, 12, 20, 35, \ldots)$ referred to the sequence A006128 with the following description: "`Total number of parts in all partitions of n.`" Those facts led us to discover equality (6.6), and then Theorem 6.3.

6.3. Comparing the coefficients of β^2. By selecting the coefficients of β^2 in our Main Identity we obtain the following equality about hook lengths. Unlike Theorem 6.2 the following results can not be derived from the SEBBM theorem.

Proposition 6.8. *We have*

$$\sum_{n\geq 2} x^n \sum_{\lambda\vdash n}\sum_{\{u,v\}}\frac{1}{h_u^2 h_v^2} = \frac{1}{2}\prod_{m\geq 1}\frac{1}{1-x^m}\times\Bigl(\sum_{k\geq 1}\frac{x^k}{k(1-x^k)}\Bigr)^2, \tag{6.16}$$

where the third sum ranges over all unordered pairs $\{u, v\}$ *such that* $u, v \in \lambda$ *and* $u \neq v$.

By Theorem 6.7 and Proposition 6.1, we have

$$\sum_{n\geq 2} x^n \sum_{\lambda\vdash n}\sum_{(u,v)}\frac{1}{h_u^2 h_v^2} = \prod_{m\geq 1}(1-x^m)\times\Bigl(\sum_{n\geq 1} x^n \sum_{\lambda\vdash n}\sum_{v\in\lambda}\frac{1}{h_v^2}\Bigr)^2, \tag{6.17}$$

where the third sum ranges over all ordered pairs (u, v) such that $u, v \in \lambda$ and $u \neq v$.

Theorem 6.9 [=1.4]. *We have*

$$\sum_{n\geq 1} x^n \sum_{\lambda\vdash n} \Big(\sum_{v\in\lambda} \frac{1}{h_v^2}\Big)^2 = \prod_{m\geq 1} \frac{1}{1-x^m} \Big(\sum_{k\geq 1} \frac{x^k k^{-3}}{1-x^k} + \Big(\sum_{k\geq 1} \frac{x^k k^{-1}}{1-x^k}\Big)^2\Big). \tag{6.18}$$

Proof. For each partition λ we have

$$\Big(\sum_{v\in\lambda} \frac{1}{h_v^2}\Big)^2 = \sum_{v\in\lambda} \frac{1}{h_v^4} + 2 \sum_{\{u,v\}} \frac{1}{h_u^2 h_v^2}$$

and conclude in view of Theorem 6.2 and Proposition 6.8. ∎

6.4. Comparing the coefficients of $\beta^n x^n$ *and* $\beta^{n-1}x^n$. Recall that f_λ is the number of standard Young tableaux of shape λ. Comparing the coefficients of $(-\beta)^n x^n$ on both sides of the Main Theorem (see, for example (2.3)), we get

$$\sum_{\lambda\vdash n} f_\lambda^2 = n! \tag{6.19}$$

Theorem 6.10 [=1.3, marked hook formula]. *We have*

$$\sum_{\lambda\vdash n} f_\lambda^2 \sum_{v\in\lambda} h_v^2 = \frac{n(3n-1)}{2} n! \tag{6.20}$$

Proof. Selecting the coefficients of $(-\beta)^{n-1}x^n$ on the right-hand side of equation (2.3) we obtain

$$\begin{aligned}
&[(-\beta)^{n-1}x^n] \prod_{m\geq 1} \frac{1}{1-x^m} \times \exp\Big(-\beta \sum_{k\geq 1} \frac{x^k}{k(1-x^k)}\Big)\\
=&[x^n]\ \frac{1}{(n-1)!} \prod_{m\geq 1} \frac{1}{1-x^m} \times \Big(\sum_{k\geq 1} \frac{x^k}{k(1-x^k)}\Big)^{n-1}\\
=&[x^1]\ \frac{1}{(n-1)!} \frac{1}{1-x} \times \Big(\frac{1}{1-x} + \frac{x}{2(1-x^2)}\Big)^{n-1}\\
=&\frac{n(3n-1)}{2n!}. \qquad (6.21)
\end{aligned}$$

Selecting the coefficients of $(-\beta)^{n-1}x^n$ on the left-hand side of equation (2.3) we get

$$\sum_{\lambda\vdash n} \sum_{u\in\lambda} \prod_{v\neq u} \frac{1}{h_v^2} = \sum_{\lambda\vdash n} \prod_{u\in\lambda} \frac{1}{h_u^2} \sum_{v\in\lambda} h_v^2 = \sum_{\lambda\vdash n} \frac{f_\lambda^2}{n!^2} \sum_{v\in\lambda} h_v^2. \tag{6.22}$$ ∎

Remark. Is there a combinatorial proof of the marked hook formula (6.20), analogous to the Robinson-Schensted-Knuth correspondence for proving (6.19) ? Let T be a standard Young tableau of shape λ (see [Kn98, p.47]), u be a box in λ and m an integer such that $1 \le m \le h_u(\lambda)$. The triplet (T, u, m) is called a *marked Young tableau* of shape (λ, u). The number of marked Young tableaux of shape (λ, u) is then $f_\lambda h_u$. On the other hand, call *marked permutation* each triplet (σ, j, k) where $\sigma \in \mathfrak{S}_n$, $1 \le j \le n$ and $1 \le k \le n + j - 1$. We say that the letter j within the permutation σ is marked k. The total number of marked permutations of order n is

$$\sum_{j=1}^{n} (n + j - 1) n! = \frac{n(3n-1)}{2} n!$$

Example. The sequence 6 4 9 5_k 7 1 2 8 3 with $1 \le k \le 13$ is a marked permutation, whose letter 5 is marked k. The following two diagrams are two marked Young tableaux of the same shape, where $1 \le i, j \le 3$.

$$\begin{array}{|c|c|c} \cline{1-2} 5 & 9 & \\ \cline{1-2} 2_i & 8 & \\ \hline 1 & 4 & \multicolumn{1}{c|}{7} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c} \cline{1-2} 8 & 9 & \\ \cline{1-2} 4_j & 5 & \\ \hline 1 & 3 & \multicolumn{1}{c|}{7} \\ \hline \end{array}$$

For proving the marked hook formula we need find a *marked* Robinson-Schensted-Knuth correspondence between pairs of marked Young tableaux and marked permutations.

6.5. Comparing the coefficients of $\beta^{n-2}x^n$ and $\beta^{n-3}x^n$. In the same manner as in the proof of the marked hook formula we obtain the following results by selecting the coefficients of $(-\beta)^{n-2}x^n$ and $(-\beta)^{n-3}x^n$ in (2.3).

Proposition 6.11. *We have*

$$\sum_{\lambda \vdash n} f_\lambda^2 \sum_{\{u,v\}} h_u^2 h_v^2 = \frac{n(n-1)(27n^2 - 67n + 74)}{24} n!,$$

where the second sum ranges over all unordered pairs $\{u, v\}$ such that $u, v \in \lambda$ and $u \neq v$.

Proposition 6.12. *We have*

$$\sum_{\lambda \vdash n} f_\lambda^2 \sum_{\{u,v,w\}} h_u^2 h_v^2 h_w^2 = \frac{n(n-1)(n-2)(27n^3 - 174n^2 + 511n - 600)}{48} n!,$$

where the second sum ranges over all unordered distinct triplets $\{u, v, w\}$ of boxes of the partition λ.

7. Improvement of a result due to Kostant

Let

$$\prod_{n\ge 1}(1-x^n)^s=\sum_{k\ge 0} f_k(s)x^k. \tag{7.1}$$

Kostant proved the following result [Ko04, Th. 4.28].

Theorem 7.1 [Kostant]. *Let k and m be two positive integers such that $m\ge \max(k,4)$. Then $f_k(m^2-1)\ne 0$.*

The condition $m>1$ in the original Theorem of Kostant should be replaced by $m\ge 4$, as, for example, we have $f_3(8)=0$ (see Theorem 7.3).

Theorem 7.2 [=1.5]. *Let k be a positive integer and s be a real number such that $s\ge k^2-1$. Then $(-1)^k f_k(s)>0$.*

Remarks. We extend Kostant's result in two directions: first, we claim that $(-1)^k f_k(s)>0$ instead of $f_k(s)\ne 0$; second, s is any real number instead of an integer of the form m^2-1.

Proof. By the Main Theorem we may write

$$(-1)^k f_k(s)=\sum_{\lambda\vdash k} W(\lambda), \tag{7.2}$$

where

$$W(\lambda)=\prod_{v\in\lambda}\Bigl(\frac{s+1}{h_v^2}-1\Bigr)=\prod_{v\in\lambda}\Bigl(\frac{s+1-h_v^2}{h_v^2}\Bigr). \tag{7.3}$$

For each $\lambda\vdash k$ and $v\in\lambda$ we have $h_v(\lambda)\le k$, so that $W(\lambda)\ge 0$. This means that there is *no cancellation* in the sum (7.2). If $s>k^2-1$ then $W(\lambda)>0$. If $s=k^2-1\ge 15$ we have $k\ge 4$. In that case there is at least one partition λ, whose hook lengths are strictly less than k. Hence $W(\lambda)>0$. ∎

Here is another result of Kostant [Ko04, Th.4.27].

Theorem 7.3 [Kostant]. *We have*

$$\begin{aligned} f_4(s)&=1/4!\ s(s-1)(s-3)(s-14);\\ -f_3(s)&=1/3!\ s(s-1)(s-8);\\ f_2(s)&=1/2!\ s(s-3). \end{aligned}$$

Even though we do not see how to factorize each $f_k(s)$, the occurrences of some factors in the above formulas have some relevance in terms of hook lengths. Every partition contains one hook length $h_v=1$, so that $f_k(s)$ has the factor $s+1-h_v^2=s$ (see (7.3)). Every partition of 3 contains a hook length $h_v=3$, so that $f_3(s)$ has the factor $s-8$. Every partition of 2 or 4 has a hook length $h_v=2$, so that $s-3$ is a factor of $f_2(s)$ and $f_4(s)$.

8. The magic partition formula

Let s_λ be the Schur functions corresponding to the partition λ (see [Ma95, p.40; St99, p.308; La01, p.8]). Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be two alphabets. The famous Cauchy formula is stated as follows (see [Ma95, p.63; St99, p.322; La01, p.13; Kn70]):

Theorem 8.1 [Cauchy].

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(X) s_\lambda(Y). \tag{8.1}$$

Let d be a positive integer. Taking

$$X = \{x, x^2, x^3, \ldots\} \quad \text{and} \quad Y = \{1, 1, \ldots, 1\} = \{1^d\},$$

we get the following specialization:

$$\prod_{m \geq 1} \Bigl(\frac{1}{1 - x^m}\Bigr)^d = \sum_{\lambda \in \mathcal{P}} s_\lambda(x, x^2, \ldots) s_\lambda(1^d). \tag{8.2}$$

Also recall the classical hook-content formula [Ro58; St99, p.374]:

Theorem 8.2. *For any partition λ and positive integer n we have*

$$s_\lambda(1, x, x^2, \ldots, x^{n-1}) = x^{b(\lambda)} \prod_{v \in \lambda} \frac{1 - x^{n + c(v)}}{1 - x^{h_v}} \tag{8.3}$$

where $b(\lambda) = \sum_i (i-1)\lambda_i$ and $c(v) = j - i$ if $v = (i, j) \in \lambda$.

By Theorem 8.2 we then have

$$s_\lambda(x, x^2, x^3, \ldots) = x^{|\lambda| + b(\lambda)} \prod_{v \in \lambda} \frac{1}{1 - x^{h_v}}, \tag{8.4}$$

$$s_\lambda(1^d) = \prod_{v \in \lambda} \frac{d + c(v)}{h_v}. \tag{8.5}$$

Theorem 8.3. *For any complex number β we have*

$$\prod_{m \geq 1} (1 - x^m)^\beta = \sum_{\lambda \in \mathcal{P}} x^{|\lambda| + b(\lambda)} \prod_{v \in \lambda} \frac{c(v) - \beta}{h_v (1 - x^{h_v})}. \tag{8.6}$$

Proof. From (8.2), (8.4) and (8.5) we see that (8.6) is true for any negative integer β. Thus (8.6) is true for any complex number β (see the explanation given in the proof of the Main Theorem, §5.5). ∎

Remark 8.1. Theorem 8.3 appears to be another formula for all the powers of the Euler Product. Although its form is analogous with the Main Theorem, it has fewer applications. As the variable x occurs in the denominator on the right-hand side of (8.6), it becomes cumbersome to select specific coefficients of x^n. There are apparently no specialization leading to Macdonald identities.

Remark 8.2. When β is given the value 1 in (8.6) we recover the following identity due to Euler [An76, p.11].

Corollary 8.4 [Euler]. *We have*

$$\prod_{m\ge 1}(1-x^m)=\sum_{n\ge 0}\frac{(-1)^n x^{n(n+1)/2}}{(1-x)(1-x^2)\cdots(1-x^n)}.$$

Combining Thorem 8.3 and the Main Theorem we get the following result, called *magic partition formula* because the sum and the product on both sides range over the same sets $\lambda\in\mathcal{P}$ and $v\in\lambda$.

Theorem 8.5 [Magic partition formula]. *For any complex number β we have*

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|+b(\lambda)}\prod_{v\in\lambda}\frac{c(v)+1-\beta}{h_v(1-x^{h_v})} \;=\; \sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{v\in\lambda}\frac{h_v^2-\beta}{h_v^2}. \tag{8.7}$$

9. Reversion of the Euler Product

Let $y(x)$ be a formal power series satisfying the following relation

$$\begin{aligned} x &= y(1-y)(1-y^2)(1-y^3)\cdots \\ &= y-y^2-y^3+y^6+y^8-y^{13}-y^{16}+\cdots \end{aligned} \tag{9.1}$$

The first coeficients of the reversion series in (9.1) are the following

$$y(x)=x+x^2+3\,x^3+10\,x^4+38\,x^5+153\,x^6+646\,x^7+\cdots \tag{9.2}$$

They are referred to as the first values of the sequence A109085 in The On-Line Encyclopedia of Integer Sequences [Slo].

Theorem 9.1. *We have the following explicit formula for the reversion of (9.1) in terms of hook lengths:*

$$y(x)=\sum_{n\ge 1}\frac{x^n}{n}\sum_{\lambda\vdash n-1}\prod_{v\in\lambda}\big(1+\frac{n-1}{h_v^2}\big). \tag{9.3}$$

Proof. Rewrite (9.1) as $y = x\phi(y)$ where $\phi(y) = \prod_{m\geq 1}(1-y^m)^{-1}$. By the Lagrange inversion formula and the Main Theorem we have

$$\begin{aligned}
[x^n]\ y &= \frac{1}{n}[x^{n-1}]\ \phi(x)^n \\
&= \frac{1}{n}[x^{n-1}]\ \prod_{m\geq 1}(1-y^m)^{-n} \\
&= \frac{1}{n}[x^{n-1}]\ \sum_{\lambda\in\mathcal{P}}\prod_{v\in\lambda}(1+\frac{n-1}{h_v^2})x \\
&= \frac{1}{n}\sum_{\lambda\vdash n-1}\ \prod_{v\in\lambda}(1+\frac{n-1}{h_v^2}). \quad \square
\end{aligned}$$

As the coefficients of $y(x)$ are all positive integers we have the following result.

Corollary 9.2 [=1.6]. *For any positive integer n the following expression*

$$\frac{1}{n+1}\sum_{\lambda\vdash n}\prod_{v\in\lambda}(1+\frac{n}{h_v^2})$$

is a positive integer.

Acknowledgements. The author wishes to thank Dominique Foata for helpful discussions during the preparation of this paper. He also thank Alain Lascoux, Richard Stanley and Kathy Ji for comments on a previous version.

References

[AF02] Adin, Ron M.; Frumkin, Avital, Rim Hook Tableaux and Kostant's η-Function Coefficients, *arXiv: math.CO/0201003.*

[An76] Andrews, George E., *The Theory of Partitions*, Addison-Wesley, Reading, 1976 (*Encyclopedia of Math. and Its Appl.*, vol. **2**).

[Be98] Bessenrodt, Christine, On hooks of Young diagrams, *Ann. of Comb.*, **2** (1998), pp. 103–110.

[BM02] Bacher, Roland; Manivel, Laurent, Hooks and Powers of Parts in Partitions, *Sém. Lothar. Combin.*, vol. **47**, article B47d, 2001, 11 pages.

[BG06] Berkovich, Alexander; Garvan, Frank G., The BG-rank of a partition and its applications, *arXiv: math/0602362.*

[CFP05] Cellini, Paola; Frajria, Pierluigi M.; Papi, Paolo, The $\hat{W}$-orbit of ρ, Kostant's formula for powers of the Euler product and affine Weyl groups as permutations of $\mathbb{Z}$, *arXiv: math.RT/0507610.*

[CJW08] Chen, William Y.C.; Ji, Kathy Q.; Wilf, Herbert S., BG-ranks and 2-cores, *arXiv: math/0605474v2*, 2008.

[Dy72] Dyson, Freeman J., Missed opportunities, *Bull. Amer. Math. Soc.*, **78** (1972), pp. 635–652.

[Eu83] Euler, Leonhard, The expansion of the infinite product $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. into a single series, *English translation from the Latin by Jordan Bell* on *arXiv:math.HO/0411454*.

[FK99] Farkas, Hershel M.; Kra, Irwin, On the Quintuple Product Identity, *Proc. Amer. Math. Soc.*, **27** (1999), pp. 771–778.

[FH99] Foata, Dominique; Han, Guo-Niu, The triple, quintuple and septuple product identities revisited. *Sem. Lothar. Combin.* Art. B42o, 12 pp.

[FRT54] Frame, J. Sutherland; Robinson, Gilbert de Beauregard; Thrall, Robert M., The hook graphs of the symmetric groups, *Canadian J. Math.*, **6** (1954), pp. 316–324.

[GKS90] Garvan, Frank; Kim, Dongsu; Stanton, Dennis, Cranks and t-cores, *Invent. Math.*, **101** (1990), pp. 1–17.

[GNW79] Greene, Curtis; Nijenhuis, Albert; Wilf, Herbert S., A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. in Math.*, **31** (1979), pp. 104–109.

[Ho86] Hoare, A. Howard M., An Involution of Blocks in the Partitions of n, *Amer. Math. Monthly*, **93** (1986), pp. 475–476.

[JS89] Joichi, James T.; Stanton, Dennis, An involution for Jacobi's identity, *Discrete Math.*, **73** (1989), pp. 261–271.

[Ka74] Kac, Victor G., Infinite-dimensional Lie algebras and Dedekind's η-function, *Functional Anal. Appl.*, **8** (1974), pp. 68–70.

[Kn70] Knuth, Donald E., Permutations, matrices, and generalized Young tableaux, *Pacific J. Math.*, **34** (1970), pp. 709-727.

[Kn98] Knuth, Donald E., *The Art of Computer Programming,* **vol. 3**, *Sorting and Searching, 2nd ed.*, Addison Wesley Longman, 1998.

[Ko76] Kostant, Bertram, On Macdonald's η-function formula, the Laplacian and generalized exponents, *Adv. in Math.*, **20** (1976), pp. 179–212.

[Ko04] Kostant, Bertram, Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra, *Invent. Math.*, **158** (2004), pp. 181–226.

[Kr99] Krattenthaler, Christian, Another involution principle-free bijective proof of Stanley's hook-content formula, *J. Combin. Theory Ser. A*, **88** (1999), pp. 66–92.

[KS82] Kirdar, M. S.; Skyrme, Tony H. R., On an Identity Related to Partitions and Repetitions of Parts, *Canad. J. Math.*, **34** (1982), pp. 194-195.

[La01] Lascoux, Alain, *Symmetric Functions and Combinatorial Operators on Polynomials*, CBMS Regional Conference Series in Mathematics, Number 99, 2001.

[Le78] Lepowsky, James, Macdonald-type identities, *Advances in Math.*, **27** (1978), pp. 230–234.

[Ma72] Macdonald, Ian G., Affine root systems and Dedekind's η-function, *Invent. Math.*, **15** (1972), pp. 91–143.

[Ma95] Macdonald, Ian G., *Symmetric Functions and Hall Polynomials*, Second Edition, Clarendon Press, Oxford, 1995.

[Mi85] Milne, Stephen C., An elementary proof of the Macdonald identities for $A_l^{(1)}$, *Adv. in Math.*, **57** (1985), pp. 34–70.

[Mo75] Moody, Robert V., Macdonald identities and Euclidean Lie algebras, *Proc. Amer. Math. Soc.*, **48** (1975), pp. 43–52.

[NPS97] Novelli, Jean-Christophe; Pak, Igor; Stoyanovskii, Alexander V., A direct bijective proof of the hook-length formula, *Discrete Math. Theor. Comput. Sci.*, **1** (1997), pp. 53–67.

[Ro58] Robinson, G. de B., A remark by Philip Hall, *Canad. Math. Bull.*, **1** (1958), pp. 21–23.

[RS06] Rosengren, Hjalmar; Schlosser, Michael, Elliptic determinant evaluations and the Macdonald identities for affine root systems, *Compositio Math.*, **142** (2006), pp. 937-961.

[RW83] Remmel, Jeffrey B.; Whitney, Roger A, bijective proof of the hook formula for the number of column strict tableaux with bounded entries, *European J. Combin.*, **4** (1983), pp. 45–63.

[Se70] Serre, Jean-Pierre, *Cours d'arithmétique*, Collection SUP: "Le Mathématicien", 2 Presses Universitaires de France, Paris 1970.

[Slo] Sloane, Neil; *al.*, The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences/`.

[St99] Stanley, Richard P., *Enumerative Combinatorics, vol. 2*, Cambridge university press, 1999.

[St04] Stanley, Richard P., *Errata and Addenda to Enumerative Combinatorics Volume 1, Second Printing*, Rev. Feb. 13, 2004. `http://www-math.mit.edu/~rstan/ec/newerr.ps`.

[Ve] Verma, Daya-Nand, Review of the paper "Affine root systems and Dedekind's η-function" written by Macdonald, I. G., MR0357528(50#9996), MathSciNet, 7 pages.

[We1] Weisstein, Eric W., Elder's Theorem, from MathWorld – A Wolfram Web Resource.

[We2] Weisstein, Eric W., Stanley's Theorem, from MathWorld – A Wolfram Web Resource.

[Wi69] Winquist, Lasse, An elementary proof of $p(11m+6) \equiv 0 \,(\mathrm{mod} 11)$, *J. Combinatorial Theory*, **6** (1969), pp. 56–59.

[Ze84] Zeilberger, Doron, A short hook-lengths bijection inspired by the Greene-Nijenhuis-Wilf proof, *Discrete Math.*, **51** (1984), pp. 101–108.

I.R.M.A. UMR 7501
Université Louis Pasteur et CNRS,
7, rue René-Descartes
F-67084 Strasbourg, France
`guoniu@math.u-strasbg.fr`