Advances in Applied Mathematics, **14**(1993), 247–266

A STATISTICAL STUDY OF THE KOSTKA-FOULKES POLYNOMIALS

Guo-Niu Han

Département de mathématique
Université Louis Pasteur
7, rue René-Descartes, F-67084 Strasbourg

Abstract. — The sum of the series $\sum_{n\geq 0} z^n\, K_{\lambda\cup a^n,\mu\cup a^n}(q)$, where $K_{\nu,\theta}(q)$ denotes the Kostka-Foulkes polynomial associated with tableaux of shape ν and evaluation θ, is explicitly derived in the case $a=1$, $q=1$ and $\mu=11\ldots 1$. This sum is a rational function $P_\lambda(1-z)/(1-z)^{|\lambda|+1}$, where the numerator is the generating polynomial for the tableaux of shape λ by their first letter "pre". Another statistic "deu" is defined on the Young tableaux and the distribution of the pair (pre,deu) is symmetric over the set of the tableaux of the same shape. Finally an explicit calculation is made for the sum of the series $\sum_{n\geq 0} K_{t1^n}(q)z^n$ (arbitrary q) for the tableaux that are extensions $t1^n$ of a given tableau t.

Résumé. — Le calcul de la somme de la série $\displaystyle\sum_{n\geq 0} z^n\, K_{\lambda\cup a^n,\mu\cup a^n}(q)$, où $K_{\nu,\theta}(q)$ désigne le polynôme de Kostka-Foulkes associé aux tableaux de forme ν et d'évaluation θ est explicitement fait dans le cas $a=1$, $q=1$ et $\mu=11\ldots 1$. On trouve une fraction rationnelle $P_\lambda(1-z)/(1-z)^{|\lambda|+1}$, où le numérateur est le polynôme générateur des tableaux de forme λ par leur première lettre "pre". Une autre statistique "deu" est définie sur les tableaux de Young et la distribution du couple (pre,deu) est symétrique sur l'ensemble des tableaux de même forme. Enfin, un calcul explicite est fait pour la somme de la série $\sum_{n\geq 0} K_{t1^n}(q)z^n$ (q quelconque) pour les tableaux qui sont des extensions $t1^n$ d'un même tableau t.

1. Introduction

The Kostka-Foulkes polynomials $K_{\nu,\theta}(q)$ are defined to be the coefficients of the transition matrix from the basis of the Schur functions to the basis of the Hall-Littlewood functions in the algebra of the symmetric functions (*cf.* [Mac]). From the work of Lascoux and Schützengberger ([LS1, LS2, LS3]) it is known that those polynomials have non negative integral coefficients. In fact, those authors have proved that $K_{\nu,\theta}(q)$ was the generating polynomial for the set$\mathbb{T}_{\nu,\theta}$ of tableaux of shape ν and evaluation θ by a certain integral-valued statistic called the *charge* and denoted by "ch" in the sequel.

If $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_l)$ is a partition, the *length* of λ and its *weight* are respectively denoted by $l(\lambda)$ and $|\lambda|$. The union $\lambda\cup\mu$ of the two partitions λ, μ is defined to be the partition whose parts are those of

λ and of μ arranged in descending order [Mac]. If n is a positive integer, the notation $\lambda 1^n$ will also be used instead of $\lambda \cup 1^n$.

The purpose of this paper is to study the *growth* of the polynomials $K_{\nu,\theta}(a)$ in the following sense. In a previous paper we have proved the following property that was conjectured by GUPTA(BRYLINSKI) [Gup] :

THEOREM 1.1. — *For each nonnegative integer a, we have*

$$K_{\lambda,\mu}(q) \le K_{\lambda\cup a,\mu\cup a}(q).$$

Theorem 1.1 implies that for every integer n the following inequality

$$K_{\lambda,\mu}(q) \le K_{\lambda\cup a^n,\mu\cup a^n}(q),$$

holds. This led GUPTA(BRYLINSKI) to raise the problem of calculating the limit

$$\lim_{n\to\infty} K_{\lambda\cup a^n,\mu\cup a^n}(q).$$

In this general form, the problem remains too complex. Following a suggestion of LASCOUX, it seems more fruitful to try to find a simple expression for the sum of the series $\sum_{n\ge 0} z^n\, K_{\lambda\cup a^n,\mu\cup a^n}(q)$. It is conjectured that this sum is a rational fraction in q.

The conjecture is indeed true for $a = 1$ and $\mu = 11\ldots 1$, namely for the case of Young tableaux of shape λ. This result is the main result of the present paper. For $q = 1$ we prove that the forementioned sum is equal to $P_\lambda(1-z)/(1-z)^{|\lambda|+1}$, where the numerator $P_\lambda(z)$ is the generating function for the tableaux of shape λ by their first letters (*cf.* theorem 2.1).

We give two proofs of this theorem. In the first proof we are led to introduce a second statistic "deu" on the Young tableaux. The unexpected result is the fact that the joint distribution of the two statistics is *symmetric*. This is established by means of a simple algorithm (*cf.* theorem 3.6).

For an arbitrary q we obtain the following explicit result

$$\text{(1.1)} \qquad \sum_{n\ge 0} K_{t1^n}(q) z^n = \frac{q^{\operatorname{ch} t}}{(1-zq^c)(1-zq^{c+1})\ldots(1-zq^{c+m-p})},$$

where $K_{t1^n}(q)$ designates the generating polynomial of the charge over the set of tableaux s that reduce to t when all the elements not contained in the shape of t are deleted.

The paper is organized in eight sections. The next section contains the definition of the statistic "pre," the statement of theorem 2.1 and also a few consequences of the theorem. The definition of the statistic "deu" is given in section 3 together with the proof of the fact that the pair (pre, deu) has a symmetric distribution over the set of Young tableaux of the same shape. The first proof of theorem 2.1 is completed in section 4. In the next section we present a practical method for calculating the polynomial $P_\lambda(z)$. A second and direct proof of theorem 2.1 is derived in section 6. Finally, the last two sections are devoted to proving (1.1) for an arbitrary q.

2. The "pre" statistic

Let t be a tableau (or more generally a skew tableau). We define $\operatorname{pre} t$ ("pre" for "*première*") to be the first letter of the tableau t in the sense of Lascoux-Schützenberger (*cf.* [LS1]), i.e., the first letter of the word obtained by writing all the elements of t from top to bottom and left to right (see the example below). Denote by $P_\lambda(x)$ the generating polynomial of the statistic "pre" over the set $\mathbb{T}_\lambda$ of standard tableaux of shape λ, i.e.,

$$P_\lambda(x) = \sum_t x^{\operatorname{pre} t} \qquad (t \in \mathbb{T}_\lambda).$$

For example, for $\lambda = (4,2) = 42$, the elements of $\mathbb{T}_\lambda$ and the respective values of $\operatorname{pre} t$ are shown in the next table :

t	24 1356	25 1346	26 1345	34 1256	35 1246	36 1245	45 1236	46 1235	56 1234
$\operatorname{pre} t$	2			3			4		5

Accordingly $P_{42}(x) = 3x^2 + 3x^3 + 2x^4 + x^5$.

For each partition ν, we denote by $K_\nu = \#\mathbb{T}_\nu$ the Kostka-Foulkes number, namely the number of tableaux of shape ν with standard evaluation. The first main result of this paper is the following.

Theorem 2.1. — *Let λ be a partition ; then*

$$\sum_{n\ge 0} z^n K_{\lambda 1^n} = \frac{P_\lambda(1-z)}{(1-z)^{|\lambda|+1}},$$

Notice that the above series is a rational function whose numerator is a polynomial with *nonnegative* integral coefficients in the variable $(1-z)$ and not in z.

Examine two particular cases of the previous identity. First, the Kostka-Foulkes number $K_{\lambda 1^n}$ can be calculated by means of the *hook formula.* When $\lambda = (a)$, the hook formula implies $K_{a1^n} = (a)_n/n!$, where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \ge 1$. On the other hand, there is only one tableau $12\cdots a$ of shape $\lambda = (a)$ and its first letter is 1. Therefore $P_\lambda(x) = x$. We then recover the binomial formula :

$$\sum_{n\ge 0} \frac{(a)_n}{n!} z^n = \frac{1}{(1-z)^a}.$$

For the partition $\lambda = 42$ we obtain the "binomial formula with *holes*" :

$$\sum_{n\ge 0} \frac{(n+1)(n+3)(n+4)(n+6)}{1\cdot 1\cdot 2\cdot 4} z^n = \frac{P_{42}(1-z)}{(1-z)^7}.$$

3. The “deu” statistic

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ $(\lambda_l \geq 1)$ is said to be *pointed*, if its last part λ_l is equal to 1 ; a tableau is said to be *pointed*, if its shape is a pointed partition. For example the tableau :

$$tA = \begin{array}{l} 8 \\ 4\;7\;9 \\ 1\;2\;3\;5\;6 \end{array}$$

is pointed.

Let t be a standard tableau of *shape* n, i.e., a tableau whose content is the set $[\,n\,] = \{1, 2, \ldots, n\}$. For each integer r $(1 \leq r \leq n)$ denote by $t|_r$ the standard subtableau of weight r obtained from t by removing all the terms greater than r. For example, $tA|_7 = {\scriptstyle \begin{array}{l} 4\;7 \\ 1\;2\;3\;5\;6 \end{array}}$.

The “deu” statistic is then defined by :

$$\operatorname{deu} t = \#\{r : t|_r \text{ is pointed}\}.$$

Keep the same example and write the letters r for which $t|_r$ is pointed in boldfaced type. We get :

$$tA = \begin{array}{l} \mathbf{8} \\ \mathbf{4}\;7\;\mathbf{9} \\ \mathbf{1}\;2\;3\;\mathbf{5}\;\mathbf{6} \end{array}\quad .$$

Hence $\operatorname{deu} tA = 6$.

The next two notions appear to be fundamental for the construction of our algorithm. Let t be a standard tableau of weight n. It is said to be *maximal*, if, either $\operatorname{pre} t = n$, or $t|_{(\operatorname{pre} t+1)}$ is not pointed. It is said to be *minimal*, if for every letter $r \neq 1$ belonging to the first column of t, the shape of the tableau $t|_r$ is of type $\mu 11$ with μ being a partition.

Suppose that t is of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and its coefficients $t_{i,j}$ are displayed as follows :

$$(\star) \qquad t = \begin{array}{|l} t_{l,1}\ t_{l,2}\ \ldots\ t_{l,\lambda_l} \\ \ldots\ldots\ldots\ldots\ldots \\ t_{2,1}\ t_{2,2}\ \ldots\ t_{2,\lambda_2} \\ t_{1,1}\ t_{1,2}\ \ldots\ t_{1,\lambda_1} \\ \hline \end{array}$$

Then t is *maximal*, if $t_{l,1} = n$, of if $t_{l,2} = t_{l,1} + 1$; the tableau is *minimal*, if $t_{j,1} < t_{j-1,2}$ for every j such that $t_{j,1}$ and $t_{j-1,2}$ exist.

The next lemma is easily verified by means of the latter characterization.

LEMMA 3.1. — *Let t be a tableau. The following properties hold :*

(i) *if t is pointed and maximal, then* $\operatorname{pre} t = n$ *;*

(ii) *if t is pointed and minimal, then* $\operatorname{deu} t = n$ *;*

(iii) *if t is non-pointed and maximal, then* $\operatorname{pre} t = t_{l,2} - 1$ *;*

(iv) *if t is non-pointed and minimal, then* $\operatorname{deu} t = t_{l,2} - 1$. ∎

For example, we can verify that the tableau

$$tB = \begin{array}{llll} 7 & 11 & & \\ 5 & 10 & 12 & \\ 2 & 6 & 8 & \\ 1 & 3 & 4 & 9 \end{array}$$

is minimal and non-maximal. In particular, $\operatorname{deu} t = t_{l,2} - 1 = 10$.

Let t be a standard tableau of weight n and $(p, p+1)$ $(1 \leq p \leq n-1)$ be a transposition. If $p+1$ is *lower* than p in the tableau t, the transposition of the coefficients p and $p+1$ in t also yields a tableau. We shall denote this new tableau by $u = (p, p+1) \circ t$.

LEMMA 3.2. — *Let t be a non-maximal standard tableau of order n. Let $p = \operatorname{pre} t$ and denote by u the tableau $u = (p, p+1) \circ t$. Then u is also a standard tableau. Moreover, by letting $s = u|_p$, we have :*

$$\operatorname{deu} u = \begin{cases} \operatorname{deu} t, & \text{if } s \text{ is pointed;} \\ \operatorname{deu} t - 1, & \text{if } s \text{ is non-pointed.} \end{cases}$$

Proof. — Keep the same notations as in $(\star)$. We see that $p = \operatorname{pre} t = t_{l,1} \neq n$ and also $t_{l,2} \neq t_{l,1} + 1 = p + 1$. The coefficient $(p+1)$ in tableau t is then located on a lower row than the row of ordinate l on which p is located. Thus, u is also a standard tableau.

If s is pointed, then the four tableaux $t|_{(p+1)}$, $t|_p$, $u|_{(p+1)}$ and $u|_p$ are pointed, so that $\operatorname{deu} u = \operatorname{deu} t$.

On the contrary, if s is not pointed, $t|_{(p+1)}$, $t|_p$ and $u|_{(p+1)}$ are pointed, but not $u|_p$, so that $\operatorname{deu} u = \operatorname{deu} t - 1$. ∎

Consider the following algorithm $\phi : t \mapsto t'$:

input : a non-maximal tableau t;
output : a non-minimal tableau t';

begin
$s := t' := t$;
repeat
$p := \operatorname{pre} s$;

$t' := (p, p+1) \circ t'$;
$s := t'|_p$;
until s is not pointed ;
end.

As the initial tableau t is supposed to be non-maximal, the three instructions following "repeat" apply at least once. Denote by $t_1 = t$, t_2, ... the sequence of the tableaux constructed successively. As the sequence of the parameters p attached to each of those tableaux t_i is decreasing, the sequence is *finite*, say, $t_1, t_2, \ldots, t_m = t'$ $(m \geq 2)$. Moreover,

$$1 + \operatorname{pre} t_1 = \operatorname{pre} t_2 = \cdots = \operatorname{pre} t_m$$
$$\operatorname{deu} t_1 = \operatorname{deu} t_2 = \cdots = \operatorname{deu} t_{m-1} = \operatorname{deu} t_m + 1.$$

Also note that the tableaux t_i have at least two rows $(l \geq 2)$ (otherwise t would be maximal).

LEMMA 3.3. — *Let t' be the non-minimal tableau derived from the non-maximal tableau t by applying the algorithm ϕ. Then*

$$1 + \operatorname{pre} t = \operatorname{pre} t' \qquad \textit{et} \qquad \operatorname{deu} t = \operatorname{deu} t' + 1. \quad \square$$

For example, apply ϕ to the tableau tB above (which is non-maximal). As $p = 7$, we first find :

$$t_2 = (7, 8) \circ t = \begin{array}{llll} 8 & 11 & & \\ 5 & 10 & 12 & \\ 2 & 6 & 7 & \\ 1 & 3 & 4 & 9 \end{array}$$

As $s = t_2|_7$ is pointed and $\operatorname{pre} s = 5$, we take $p = 5$ and obtain :

$$t_3 = (5, 6) \circ t_2 = \begin{array}{llll} 8 & 11 & & \\ 6 & 10 & 12 & \\ 2 & 5 & 7 & \\ 1 & 3 & 4 & 9 \end{array}$$

This time $s = t_3|_5$ is not pointed. The procedure ends. The final tableau is then $t' = \phi(t) = t_3$.

Conversely, designate by $c = (c_l > c_{l-1} > \cdots > c_1)$ $(c_1 = 1)$ the first column of the tableau t. Let m be the smallest integer such that the shapes of $t|_{c_l}, t|_{c_{l-1}}, \ldots, t|_{c_{m+1}}$ all contain at least two parts equal to 1 and $t|_{c_m}$ contains a single part equal to 1. Then $m = 1$, if and only if t is minimal. The integer $m = m(t)$ will be called the *minimality* of t. The following lemma is easy to verify.

LEMMA 3.4. — *If t' is a non minimal tableau (i.e., of minimality $m \geq 2$) and if $c' = (c'_l > c'_{l-1} > \cdots > c'_1)$ designates its first column, then*

$$t = (c'_1, c'_1 - 1) \circ (c'_2, c'_2 - 1) \circ \ldots \circ (c'_m, c'_m - 1) \circ t'$$

is a standard tableau such that $\phi(t) = t'$.

If $c = (c_l > c_{l-1} > \cdots > c_1)$ designates the first column of t, then

$$(L) \qquad m(t) \leq m(t') \qquad \text{and if } m(t) = m(t') = m, \text{ then } c_m < c'_m. \quad \square$$

Lemma 3.4 provides an explicit construction of the inverse algorithm ϕ^{-1}. Furthermore, property (L) says that when ϕ^{-1} is applied iteratively, we necessarily reach a *minimal* tableau after finitely many steps.

For example, by considering the elements of the first column $c' = (8, 6, 2, 1)$ of t', we see that the shape of $t'|_8$ contains two parts equal to 1, but not $t'|_6$. Accordingly,

$$t = (8, 7)(6, 5) \circ t'.$$

The minimality of t' is equal to 3 and that of t to 1. In this example t is then minimal.

The following lemma is also easily verified.

LEMMA 3.5. —

1) *There are as many maximal tableaux as minimal tableaux ;*

2) *If t is a tableau that is both minimal and maximal, then* $\operatorname{pre} t = \operatorname{deu} t$. $\square$

Start with a minimal tableau t and apply to t the algorithm ϕ, in an iterative manner, until a *maximal* tableau is reached. We then get a sequence of tableaux $(t_1 = t, t_2, \ldots, t_k)$ such that $t_i = \phi(t_{i-1})$ $(i = 2, \ldots, k)$ and that t_k is maximal. Let $\phi^*(t) = \{t_1, t_2, \ldots, t_k\}$.

It is worth noticing that the algorithm ϕ has no action on the element in position $(l, 2)$. Lemma 3.1 implies that

$$\operatorname{deu} t_1 = \operatorname{pre} t_k.$$

On the other hand, it follows from lemma 3.3 that :

$$1 + \operatorname{pre} t_1 = \operatorname{pre} t_2,\ 1 + \operatorname{pre} t_2 = \operatorname{pre} t_3,\ \ldots\ ,\ 1 + \operatorname{pre} t_{k-1} = \operatorname{pre} t_k\,;$$
$$\operatorname{deu} t_1 = 1 + \operatorname{deu} t_2,\ \operatorname{deu} t_2 = 1 + \operatorname{deu} t_3,\ \ldots\ ,\ \operatorname{deu} t_{k-1} = 1 + \operatorname{deu} t_k.$$

It then follows that the two pairs of statistics $(\operatorname{pre}, \operatorname{deu})$ and $(\operatorname{deu}, \operatorname{pre})$ are *equidistributed* on each orbit $\phi^*(t_1)$.

Finally, lemma 3.4 implies that the set of minimal tableaux $\mathbb{T}_\lambda^{\min}$ generates $\mathbb{T}_\lambda$ in the sense that every tableau in $\mathbb{T}_\lambda$ can be derived from a tableau in $\mathbb{T}_\lambda^{\min}$ by applying the algorithm ϕ, in an iterative manner. In other words,

$$\mathbb{T}_\lambda = \sum_t \phi^*(t) \qquad (t \in \mathbb{T}_\lambda^{\min}).$$

The following theorem has then be proved.

Theorem 3.6. — *The two pairs of statistics* (pre, deu) *et* (deu, pre) *are equidistributed on each set* $\mathbb{T}_\lambda$. *In other words,*

$$\sum_{t \in \mathbb{T}_\lambda} x^{\mathrm{pre}\, t} y^{\mathrm{deu}\, t}$$

is symmetric in the two variables x and y. ∎

4. Proof of theorem 2.1

Let λ be a partition. Denote by $\lambda\backslash 1$ the set of all partitions of weight $|\lambda| - 1$ the diagrams of which are contained in the diagram of λ. For example, if $\lambda = 4331$, we have $\lambda\backslash 1 = \{433, 4321, 3321\}$.

The construction of the tableaux implies the following recurrence relation :

$$\begin{cases} K_{\lambda 1^n} = \displaystyle\sum_{\mu \in \lambda\backslash 1} K_{\mu 1^n}, & \text{if } \lambda \text{ is pointed, or if } n = 0\,; \\ K_{\lambda 1^n} = \displaystyle\sum_{\mu \in \lambda\backslash 1} K_{\mu 1^n} + K_{\lambda 1^{n-1}}, & \text{if } \lambda \text{ is not pointed and if } n \geq 1\ . \end{cases}$$

Let $G_\lambda(z) = \sum_{n \geq 0} z^n K_{\lambda 1^n}$; then

$$\begin{cases} G_\lambda(z) = \displaystyle\sum_{\mu \in \lambda\backslash 1} G_\mu(z), & \text{if } \lambda \text{ is pointed}\,; \\ G_\lambda(z) = \displaystyle\sum_{\mu \in \lambda\backslash 1} G_\mu(z) + zG_\lambda(z), & \text{if } \lambda \text{ is not pointed.} \end{cases}$$

This can also be rewritten as :

$$G_\lambda(z) = c_\lambda \sum_{\mu \in \lambda\backslash 1} G_\mu(z),$$

with

$$c_\lambda = \begin{cases} 1, & \text{if } \lambda \text{ is pointed}\,; \\ \dfrac{1}{1-z}, & \text{if } \lambda \text{ is not pointed.} \end{cases}$$

By applying this procedure in an iterative manner we get

$$(A) \qquad G_\lambda(z) = \sum_{(\mu_0,\mu_1,\cdots,\mu_{|\lambda|})} c_{\mu_0} c_{\mu_1} \cdots c_{\mu_{|\lambda|-1}} G_\emptyset(z),$$

where the sum is over all sequences of partitions $(\mu_0, \mu_1, \cdots, \mu_{|\lambda|})$ such that $\mu_0 = \lambda$, $\mu_{|\lambda|} = \emptyset$ and $\mu_i \in \mu_{i-1} \backslash 1$ for $1 \le i \le |\lambda|$.

Note that

$$G_\emptyset(z) = \sum_{n \ge 0} z^n K_{1^n} = \frac{1}{1-z}$$

and

$$(B) \qquad \sum_{(\mu_0,\mu_1,\cdots,\mu_{|\lambda|})} c_{\mu_0} c_{\mu_1} \cdots c_{\mu_{|\lambda|-1}} = \sum_{t \in \mathbb{T}_\lambda} c(t),$$

where

$$\begin{aligned} c(t) &= \prod_{1 \le r \le n} \left(\frac{1}{1-z}\right)^{\chi(t|_r \text{ is not pointed})} \\ &= \frac{1}{(1-z)^{|\lambda|}} \prod_{1 \le r \le n} (1-z)^{\chi(t|_r \text{ is pointed})} \\ &= \frac{(1-z)^{\operatorname{deu} t}}{(1-z)^{|\lambda|}}. \end{aligned}$$

From relations (A) and (B) we conclude that

$$G_\lambda(z) = \frac{1}{(1-z)^{|\lambda|+1}} \sum_{t \in \mathbb{T}_\lambda} (1-z)^{\operatorname{deu} t} = \frac{U_\lambda(1-z)}{(1-z)^{|\lambda|+1}},$$

where $U_\lambda(x)$ is the generating polynomial of the statistic "deu" over the set $\mathbb{T}_\lambda$. But from theorem 3.6 we know that $U_\lambda(x) = P_\lambda(x)$. This completes the proof of theorem 2.1. $\square$

5. A practical method for the calculation of $P_\lambda(x)$.

First define a linear operator D_r on the linear space of polynomials of degree less than or equal to $r-1$ by :

$$D_r(x^p) = x^{p+1} + x^{p+2} + \cdots + x^r \qquad (p \le r-1).$$

With each tableau t whose first column is $c = (c_l > c_{l-1} > \cdots > c_1 = 1)$ associate the polynomial

$$D(t) = D_{c_l} D_{c_{l-1}} \cdots D_{c_1} * 1 \ .$$

Using the notation $(\star)$ of section 3 we say that a tableau t is *compact*, if $t|_r$ is maximal for every r. Denote by $\overline{\mathbb{T}}_\lambda$ the set of all compact tableaux.

For example, there are only two compact tableaux of shape 3211 :

$$\overline{\mathbb{T}}_{3211} = \left\{ \begin{smallmatrix} 7 & & \\ 6 & & \\ 3 & 4 & \\ 1 & 2 & 5 \end{smallmatrix} \quad , \quad \begin{smallmatrix} 7 & & \\ 6 & & \\ 4 & 5 & \\ 1 & 2 & 3 \end{smallmatrix} \right\} .$$

THEOREM 5.1. — *We have*

$$P_\lambda(x) = \sum_t D(t) \qquad (t \in \overline{\mathbb{T}}_\lambda).$$

Proof. — The result is proved by induction on the weight. There is only one compact tableau of weight 1 and it is true that $P_1(x) = D_1 * 1 = x$. Assume that the result holds for partitions of weight less than or equal to $(n-1)$. Two cases are to be considered depending on whether the partition is pointed or not.

(α) *The partition $\lambda = \mu 1$ is pointed of weight n.* If t is an injective tableau, namely a tableau whose coefficients are all distinct, define its *reduced form* $\Omega(t)$ as being the tableau obtained from t by replacing the ith smallest integer by i for all i. For example,

$$\Omega\left(\begin{smallmatrix} 5 & 9 & \\ 2 & 7 & 8 \end{smallmatrix} \right) = \begin{smallmatrix} 2 & 5 & \\ 1 & 3 & 4 \end{smallmatrix} .$$

Now let t be a tableau of shape μ and weight$(n-1)$ whose first letter is p. For all r such that $p+1 \le r \le n$ there is exactly one tableau t^r that can be written in the form $t^r = r.t'$ (*cf.* [LS1]), where t' is a tableau satisfying $\Omega(t') = t$.

For example, if we take $tC = \begin{smallmatrix} 4 & 5 & 7 & \\ 1 & 2 & 3 & 6 \end{smallmatrix}$ of weight $n-1 = 7$, we get

$$(tC)^5 = \begin{smallmatrix} 5 & & & \\ 4 & 6 & 8 & \\ 1 & 2 & 3 & 7 \end{smallmatrix} , \quad (tC)^6 = \begin{smallmatrix} 6 & & & \\ 4 & 5 & 8 & \\ 1 & 2 & 3 & 7 \end{smallmatrix} , \quad (tC)^7 = \begin{smallmatrix} 7 & & & \\ 4 & 5 & 8 & \\ 1 & 2 & 3 & 6 \end{smallmatrix} , \quad (tC)^8 = \begin{smallmatrix} 8 & & & \\ 4 & 5 & 7 & \\ 1 & 2 & 3 & 6 \end{smallmatrix} .$$

Let $\mathbb{T}(t) = \{t^r \mid p+1 \le r \le n\}$; then $\mathbb{T}_\lambda = \sum_t \mathbb{T}(t)$ where the sum is over all $t \in \mathbb{T}_\mu$. Therefore

$$\begin{aligned} P_\lambda(x) &= \sum_{t \in \mathbb{T}_\mu} (x^{\mathrm{pre}\, t+1} + x^{\mathrm{pre}\, t+2} + \cdots + x^n) \\ &= D_n \sum_{t \in \mathbb{T}_\mu} x^{\mathrm{pre}\, t} \\ &= D_n P_\mu(x) \\ &= D_n \sum_{t \in \mathbb{T}_\mu} D(t) \quad \text{(by induction).} \end{aligned}$$

As $t \mapsto n.t$ is a bijection of $\overline{\mathbb{T}}_\mu$ onto $\overline{\mathbb{T}}_\lambda$ and as $D(nt) = D_n D(t)$, we conclude that

$$P_\lambda(x) = \sum_{t \in \mathbb{T}_\mu} D(nt) = \sum_{t \in \mathbb{T}_\lambda} D(t)\ .$$

(β) *The partition λ is not pointed of weight n.* Let $\mathbb{P}(\lambda)$ be the set of pointed partitions $\mu \subset \lambda$ such that $l(\mu) = l(\lambda)$ and such that μ has a single part equal to 1.

Let $\mu \in \mathbb{P}(\lambda)$; denote by $\mathbb{T}_{\lambda\backslash\mu}$ the set of all skew tableaux of shape $\lambda\backslash\mu$, of content $\{|\mu|+1, |\mu|+2, \ldots, n\}$ whose first letter is $|\mu|+1$. Let $x = t_{l,2}$; then the bijection $t \mapsto (t|_{x-1}, t|_{[x,n]})$ implies the following relations :

$$\mathbb{T}_\lambda = \sum_{\mu \in \mathbb{P}(\lambda)} \mathbb{T}_\mu \otimes \mathbb{T}_{\lambda\backslash\mu}\ ; \tag{C}$$

$$\overline{\mathbb{T}}_\lambda = \sum_{\mu \in \mathbb{P}(\lambda)} \overline{\mathbb{T}}_\mu \otimes \mathbb{T}_\lambda\ ; \tag{D}$$

Since $\operatorname{pre}(t \otimes s) = \operatorname{pre} t$ and $D(t \otimes s) = D(t)$ for all $t \in \mathbb{T}_\mu$ and $s \in \mathbb{T}_{\lambda\backslash\mu}$, we have

$$\begin{aligned}
P_\lambda(x) &= \sum_{\mu \in \mathbb{P}(\lambda)} P_\mu(x)(\#\mathbb{T}_{\lambda\backslash\mu}) && \text{(by } (C)\text{)}\\
&= \sum_{\mu \in \mathbb{P}(\lambda)} \sum_{t \in \overline{\mathbb{T}}_\mu} D(t)(\#\mathbb{T}_{\lambda\backslash\mu}) && \text{(by induction)}\\
&= \sum_{\mu \in \mathbb{P}(\lambda)} \sum_{t \otimes s \in \overline{\mathbb{T}}_\mu \otimes \mathbb{T}_{\lambda\backslash\mu}} D(t \otimes s) &&\\
&= \sum_{t \in \overline{\mathbb{T}}_\lambda} D(t) \quad . \quad \square && \text{(by } (D)\text{)}
\end{aligned}$$

For example, the following diagram shows how to calculate $P_{3211}(x)$.

$$\begin{array}{cccccccccccccc}
 & x^1 & & x^2 & & x^3 & & x^4 & & x^5 & & x^6 & & x^7 \\
D_1 & 1 \\
 & & \searrow \\
D_3{+}D_4 & & & 1{+}1 & \to & 1{+}1 & \to & 1 \\
 & & & & \searrow & & \searrow & & \searrow \\
D_6 & & & & & 2 & \to & 4 & \to & 5 & \to & 5 \\
 & & & & & & \searrow & & \searrow & & \searrow & & \searrow \\
D_7 & & & & & & & 2 & \to & 6 & \to & 11 & \to & 16
\end{array}$$

Therefore $P_{3211}(x) = D_7D_6D_3D_1 * 1 + D_7D_6D_4D_1 * 1 = 2x^4 + 6x^5 + 11x^6 + 16x^7$.

6. A combinatorial proof of theorem 2.1

Let s be a tableau of shape μ and $\lambda \subset \mu$ be a partition contained in μ. Denote by $s|_\lambda$ the standard tableau of shape λ deduced from s by first removing all letters contained in the skew form $\mu \backslash \lambda$ and then applying the reduction Ω defined in the preceding section. Also let

$$\mathbb{T}_{t1^n} = \big\{ s \in \mathbb{T}_{\lambda 1^n} \ : \ s|_\lambda = t \big\},$$

where λ is the shape of t.

On the other hand, let $1 \le p \le m$ and denote by $\mathbb{S}(p,m)$ the set of all sequences $\alpha = (\alpha_p, \alpha_{p+1}, \ldots, \alpha_m)$ of length $(m-p+1)$ where the α_i's are nonnegative. Also for each $n \ge 0$ let $\mathbb{S}(p,m;n)$ be the subset of all sequences $\alpha = (\alpha_p, \alpha_{p+1}, \ldots, \alpha_m)$ of *weight* $|\alpha| = \alpha_p + \alpha_{p+1} + \cdots + \alpha_m$ equal to n.

LEMMA 6.1. — *Let* $\operatorname{pre} t = p$ *and* $|\lambda| = m$. *Then there is a bijection* ψ *between the two sets :*

$$\begin{array}{ccc} \mathbb{S}(p,m;n) & \xrightarrow{\ \ \psi\ \ } & \mathbb{T}_{t1^n} \\ \alpha = (\alpha_p, \alpha_{p+1}, \cdots, \alpha_m) & \longmapsto & s \end{array}$$

CONSTRUCTION OF ψ . — Consider the word $w = m+n, m+n-1, \cdots, p+2, p+1$ and with the element $\alpha \in \mathbb{S}(p,m;n)$ associate the factorization of w defined by

$$w = w_m x_m w_{m-1} x_{m-1} \ldots x_{p+1} w_p,$$

where each x_i is a letter and where each w_i is a word of length $|w_i| = \alpha_i$ $(i = m, m-1, \cdots, p)$.

Then define

$$s = w_m w_{m-1} \ldots w_p . s' \ ,$$

where s' is a tableau such that $\Omega(s') = s|_\lambda = t$. $\square$

For example, if we take $t = \begin{smallmatrix} 3 & 4 & & \\ 1 & 2 & 5 & 6 \end{smallmatrix}$, we get $p = 3$ et $m = 6$. We can take

$$\alpha = (\alpha_3, \alpha_4, \alpha_5, \alpha_6) = (0, 2, 0, 3) \ ;$$

so that $n = 5, m + n = 11$. The factorization

$$(11, 10, 9, 8, 7, 6, 5, 4) = (11, 10, 9) 8 () 7 (6, 5) 4 ()$$

yields the tableau $s = 11.10.9.6.5.s'$. We then write

$$s = \begin{array}{llll} 11 & & & \\ 10 & & & \\ 9 & & & \\ 6 & & & \\ 5 & & & \\ 3 & 4 & & \\ 1 & 2 & 7 & 8 \end{array}$$

By using the preceding lemma the generating function

$$G_t(z) = \sum_{n \geq 0} \big(\#\mathbb{T}_{t1^n}\big) z^n$$

is calculated as follows :

$$\begin{aligned} G_t(z) &= \sum_{n \geq 0} \big(\#\mathbb{S}(p, m; n)\big) z^n \\ &= \sum z^{\alpha_p + \alpha_{p+1} + \cdots + \alpha_m} \qquad (\alpha_p, \alpha_{p+1}, \cdots, \alpha_m \geq 0) \\ &= \frac{1}{(1-z)^{m-p+1}} = \frac{(1-z)^p}{(1-z)^{m+1}} \ ; \end{aligned}$$

therefore

$$G_\lambda(z) = \sum_{t \in \mathbb{T}_\lambda} G_t(z) = \frac{P_\lambda(1-z)}{(1-z)^{m+1}} \cdot$$

We have then completed the second proof of theorem 2.1.

7. The q-case

Recall that the *charge* (see [LS2]) of a tableau t of shape λ and of weight m is defined as follows : first, write t as a single row in a canonical manner (using the notation of $(\star)$ of section 3) :

$$(\star\star) \qquad t = x_1 x_2 \cdots x_m \quad (= t_{l,1} \cdots t_{1,\lambda_1}) \ ;$$

then, let 0 be the charge ch(1) of the letter 1 and for $i \geq 1$ let the charge $\mathrm{ch}(i+1)$ of $(i+1)$ be equal to $\mathrm{ch}(i)$, if $(i+1)$ is to the left of i, and to $\mathrm{ch}(i)+1$, if $(i+1)$ is to the right of i. The charge $\mathrm{ch}(t)$ of t is then defined as the sum of all $\mathrm{ch}(i)$:

$$\mathrm{ch}(t) = \sum_i \mathrm{ch}(i).$$

For example the tableau $t = {\scriptstyle\begin{smallmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{smallmatrix}}$, as a single row, is written : $t = 246135$. The charges of its letters are : $\mathrm{ch}(1) = \mathrm{ch}(2) = 0$; $\mathrm{ch}(3) = \mathrm{ch}(4) = 1$; $\mathrm{ch}(5) = \mathrm{ch}(6) = 2$. Therefore $\mathrm{ch}(t) = 6$.

The Kostka-Foulkes polynomial for the standard tableaux is then (see [LS2])

$$K_\lambda(q) = \sum_t q^{\operatorname{ch} t} \quad (t \in \mathbb{T}_\lambda)\ .$$

The purpose of this section is to calculate the generating function $\sum_{n\ge 0} K_{\lambda 1^n}(q) z^n$. As we will see below, the expression we obtain depends not only on λ, but also on all the tableaux t in the set $\mathbb{T}_\lambda$. As

$$K_{\lambda 1^n}(q) = \sum_t K_{t1^n}(q) \quad (t \in \mathbb{T}_\lambda),$$

with

$$K_{t1^n}(q) = \sum_s q^{\operatorname{ch} s} \quad (s \in \mathbb{T}_{t1^n}),$$

we then have :

$$(\star\star\star) \qquad \sum_{n\ge 0} K_{\lambda 1^n}(q) z^n = \sum_{t\in\mathbb{T}_\lambda} \sum_{n\ge 0} K_{t1^n}(q) z^n.$$

As usual, let

$$(u;q)_n = \begin{cases} 1, & \text{if } n = 0\,; \\ (1-u)(1-uq)\dots(1-uq^{n-1}), & \text{if } n \ge 1\,; \end{cases}$$

denote the q-ascending factorial. The next theorem gives a closed expression for the generating function $\sum_{n\ge 0} K_{t1^n}(q) z^n$ for each tableau t.

Theorem 7.1. — *Let λ be a partition of the integer m and t be a standard tableau of shape λ. Furthermore, let p be the first letter* $\operatorname{pre} t$ *of t and c be the charge* $\operatorname{ch}(p)$ *of p in t. Then the following identity holds :*

$$\begin{aligned} \sum_{n\ge 0} K_{t1^n}(q) z^n &= \frac{q^{\operatorname{ch} t}}{(zq^c;q)_{m-p+1}} \\ &= \frac{q^{\operatorname{ch} t}}{(1-zq^c)(1-zq^{c+1})\cdots(1-zq^{m+c-p})}. \end{aligned}$$

For example, let $\lambda = 33$ and then $m = 6$. The tableaux t in $\mathbb{T}_\lambda$ and their statistics pre, c and ch are shown in the next table.

t	$\begin{smallmatrix}2&4&6\\1&3&5\end{smallmatrix}$	$\begin{smallmatrix}2&5&6\\1&3&4\end{smallmatrix}$	$\begin{smallmatrix}3&5&6\\1&2&4\end{smallmatrix}$	$\begin{smallmatrix}3&4&6\\1&2&5\end{smallmatrix}$	$\begin{smallmatrix}4&5&6\\1&2&3\end{smallmatrix}$
p	2	2	3	3	4
c	0	0	1	1	2
$\operatorname{ch}(t)$	6	8	9	10	12

Using identity $(\star\star\star)$ we then have

$$\sum_{n\geq 0} K_{331^n}(q)z^n = \frac{q^{12}}{(zq^2;q)_3} + \frac{q^9+q^{10}}{(zq;q)_4} + \frac{q^6+q^8}{(z;q)_5}.$$

Again consider the set $\mathbb{S}(p,m)$ introduced in section 6 (where it was proved that each subset $\mathbb{S}(p,m;n)$ was in one-to-one correspondence with $\mathbb{T}_{t1^n}$ when $\operatorname{pre} t = p$ and $|\lambda| = m$). For each sequence $\alpha = (\alpha_p, \alpha_{p+1}, \ldots, \alpha_m)$ in $\mathbb{S}(p,m)$ define the evaluation :

$$\mathrm{Ev}(\alpha) = c\alpha_p + (c+1)\alpha_{p+1} + \cdots + (c+m-p)\alpha_m.$$

Clearly

$$\sum_{\alpha\in\mathbb{S}(p,m)} z^{|\alpha|}q^{\mathrm{Ev}(\alpha)} = \sum_{n\geq 0} z^n \sum_{\alpha\in\mathbb{S}(p,m;n)} q^{\mathrm{Ev}(\alpha)} = \frac{1}{(zq^c;q)_{m-p+1}},$$

which is the denominator of the right-hand side of the identity of Theorem 7.1. To prove the theorem it then suffices to find a bijection $\alpha \mapsto s$ of $\mathbb{S}(p,m;n)$ onto $\mathbb{T}_{t1^n}$ for each $n \geq 0$ such that $\operatorname{ch} t + \mathrm{Ev}(\alpha) = \operatorname{ch} s$. Although the bijection $\psi : \alpha \mapsto s$ constructed in the preceding section looks to be very natural, it does not satisfy the latter property. However we can find a bijection $\theta : \alpha \mapsto \beta$ of $\mathbb{S}(p,m;n)$ onto itself such that

$$(R) \qquad \operatorname{ch} t + \mathrm{Ev}(\beta) = \operatorname{ch} \psi(\alpha).$$

More generally, let c, p, m (with $p \leq m$) and a subset A of the interval $[p,m] = \{p, p+1, \ldots, m-1, m\}$ be given. We first define a *c-charge* $\operatorname{ch}^c A$ of A. Furthermore, for each sequence α in $\mathbb{S}(p,m;n)$ we also define a (c,α)-*charge* $\operatorname{ch}^{c,\alpha} A$ of A and give the construction of a bijection $\theta : \alpha \mapsto \beta$ of $\mathbb{S}(p,m;n)$ onto itself having the property :

$$(S) \qquad \operatorname{ch}^c A + \mathrm{Ev}(\beta) = \operatorname{ch}^{c,\alpha} A.$$

The *c-charge* $\operatorname{ch}^c A$ of A is defined as follows. First, let $\operatorname{ch}^c p = c$ and for $p \leq i \leq m-1$ let

$$\operatorname{ch}^c(i+1) = \begin{cases} \operatorname{ch}^c i, & \text{if } i \in A; \\ \operatorname{ch}^c i + 1 & \text{if } i \notin A. \end{cases}$$

Then define

$$\operatorname{ch}^c A = \operatorname{ch}^c p + \operatorname{ch}^c(p+1) + \cdots + \operatorname{ch}^c m\ .$$

For example, for $c = 0$, $p = 1$, $m = 8$ and with the elements of A written in boldfaced type we get :

$$\begin{array}{llcccccccc} A & = & \mathbf{1} & \mathbf{2} & \mathbf{3} & 4 & 5 & \mathbf{6} & \mathbf{7} & 8 \\ \operatorname{ch}^c i & = & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \end{array}$$

so that $\operatorname{ch}^c(A) = (4,1,3) * {}^t(0,1,2) = 7$.

Next let $\alpha = (\alpha_p, \alpha_{p+1}, \cdots, \alpha_m)$ belong to $\mathbb{S}(p,m;n)$. The (c,α)-*charge* $\operatorname{ch}^{c,\alpha} A$ of A is defined in an equivalent manner as follows. Let

1) $\mathrm{ch}^{c,\alpha}\, p = c$;

2) for $p \le i \le m-1$ let

$$\mathrm{ch}^{c,\alpha}\, i+1 = \begin{cases} \mathrm{ch}^{c,\alpha}\, i, & \text{if } i \in A \textit{ and } \alpha_i = 0\,; \\ \mathrm{ch}^{c,\alpha}\, i+1, & \text{otherwise.} \end{cases}$$

Then define

$$\mathrm{ch}^{c,\alpha} A = (1+\alpha_p)\,\mathrm{ch}^{c,\alpha}\, p + (1+\alpha_{p+1})\,\mathrm{ch}^{c,\alpha}(p+1) + \cdots + (1+\alpha_m)\,\mathrm{ch}^{c,\alpha}\, m.$$

For example, for $c = 0$ and with the same A we obtain :

$$\begin{array}{lcccccccc} A & = \mathbf{1} & \mathbf{2} & \mathbf{3} & 4 & 5 & \mathbf{6} & \mathbf{7} & 8 \\ \alpha & = 2 & 0 & 0 & 3 & 0 & 2 & 1 & 2 \\ \mathrm{ch}^{c,\alpha}\, i & = 0 & 1 & 1 & 1 & 2 & 3 & 4 & 5 \end{array}$$

Accordingly $\mathrm{ch}^{c,\alpha} A = (3,6,1,3,2,3) * {}^t(0,1,2,3,4,5) = 40$.

Construction of θ . — To get β from α we form a chain of sequences $(\beta^m, \beta^{m-1}, \ldots, \beta^{p+1}, \beta^p)$ defined as follows :

1) the length $l(\beta^i)$ of β^i is equal to $m+1-i$ $(m \le i \le p)$;

2) β^m is the sequence of length 1 equal to (α_m); also, $\beta = \beta^p$;

3) if $\beta^{i+1} = (x_{i+1}, x_{i+2}, \ldots, x_{m-1}, x_m)$, let

$$\beta^i = \begin{cases} (\alpha_i - 1, x_{i+1}, x_{i+2}, \ldots, x_{m-1}, x_m + 1), & \text{if } i \in A \text{ and } \alpha_i \ge 1\,; \\ (\beta^{i+1}, 0), & \text{if } i \in A \text{ and } \alpha_i = 0\,; \\ (\alpha_i, \beta^{i+1}), & \text{if } i \notin A\ . \end{cases}$$

We can verify by induction on the weight of α, that we do construct a bijection satisfying relation (S).

For example,

$$\begin{array}{lcccccccc} A & = \mathbf{1} & \mathbf{2} & \mathbf{3} & 4 & 5 & \mathbf{6} & \mathbf{7} & 8 \\ \alpha & = 2 & 0 & 0 & 3 & 0 & 2 & 1 & 2 \\ \hline \beta^8 & = & & & & & & & 2 \\ \beta^7 & = & & & & & & 0 & 3 \\ \beta^6 & = & & & & & 1 & 0 & 4 \\ \beta^5 & = & & & & 0 & 1 & 0 & 4 \\ \beta^4 & = & & & 3 & 0 & 1 & 0 & 4 \\ \beta^3 & = & & 3 & 0 & 1 & 0 & 4 & 0 \\ \beta^2 & = & 3 & 0 & 1 & 0 & 4 & 0 & 0 \\ \beta^1 & = 1 & 3 & 0 & 1 & 0 & 4 & 0 & 1 \\ \hline \beta & = 1 & 3 & 0 & 1 & 0 & 4 & 0 & 1 \end{array}$$

Hence

$$\sum_i (i-1)\beta_i = (1,3,0,1,0,4,0,1) * {}^t(0,1,2,3,4,5,6,7) = 33$$
$$= \mathrm{ch}^{c,\alpha} A - \mathrm{ch}^c A\ .$$

We now have all the ingredients to complete the proof of Theorem 7.1. Let t be a tableau of weight m such that $\mathrm{pre}\, t = p$ et $\mathrm{ch}\, p = c$. Form the subset $A = A(t)$ of $[p, m]$ consisting of all the i's such that $(i+1)$ is to the left of i when the entries of t are written in a single row as in $(\star\star)$. Clearly

$$\mathrm{ch}(t) = \sum_{i \le p-1} \mathrm{ch}(i) + \mathrm{ch}^c A(t).$$

Moreover, whenever α is in $\mathbb{S}(p, m; n)$ and $s = \psi(\alpha)$, we have :

$$\mathrm{ch}(s) = \sum_{i \le p-1} \mathrm{ch}(i) + \mathrm{ch}^{c,\alpha} A(t).$$

Reporting the values of $\mathrm{ch}^c A(t)$ and $\mathrm{ch}^{c,\alpha} A(t)$ in identity (S) above yields identity (R). This achieves the proof of theorem 7.1.

8. The Gupta limit

Write Theorem 7.1 in the form

$$\sum_{n\ge 0} K_{t1^n}(q) z^n = \frac{q^{\mathrm{ch}\, t}}{(zq^c; q)_{m-p+1}} = q^{\mathrm{ch}\, t} \frac{(zaq^c; q)_\infty}{(zq^c; q)_\infty},$$

where $a = q^{m-p+1}$.

From the q-binomial theorem it follows that

$$K_{t1^n}(q) = q^{\mathrm{ch}\, t} q^{cn} \frac{(a; q)_n}{(q; q)_n}\cdot$$

Notice that if t is a tableau such that $c = \mathrm{ch}\, p \ge 1$, we obtain 0 for the value of $\lim_{n\to\infty} K_{t1^n}(q)$. Consider the set $\mathbb{T}^0_\lambda$ of all tableaux t of shape λ such that $c = \mathrm{ch}\, p = 0$ $\big($or tableaux whose first column is $(l(\lambda), l(\lambda)-1, \cdots, 3, 2, 1)$ $\big)$. If t is such a tableau, we have

$$\lim_{n\to\infty} K_{t1^n}(q) = \lim_{n\to\infty} q^{\mathrm{ch}\, t} \frac{(a; q)_n}{(q; q)_n} = \frac{q^{\mathrm{ch}\, t}}{(q; q)_{m-p}}\cdot$$

Hence

$$\lim_{n\to\infty} K_{t1^n}(q) = \frac{1}{(q; q)_{|\lambda| - l(\lambda)}} \sum_{t \in \mathbb{T}^0_\lambda} q^{\mathrm{ch}\, t}.$$

We have then proved the following corollary. (*)

(*) J. Zeng has another proof of this result by means of the q-hook formula.

COROLLARY 8.1. — *We have*

$$\lim_{n\to\infty} K_{\lambda 1^n}(q) = \frac{q^{|\lambda^-|}}{(q;q)_{|\lambda^-|}} K_{\lambda^-}(q)\ .$$

where the partition λ^- is obtained by deleting the first column of λ. In other words : $\lambda^- = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_l - 1)$. $\square$

For example, for $\lambda = 33$, then $\lambda^- = 22$ and accordingly,

$$\begin{aligned}\lim_{n\to\infty} K_{\lambda 1^n}(q) &= \frac{q^4}{(q;q)_4} K_{22}(q) = \frac{q^4}{(q;q)_4}(q^2+q^4)\\ &= q^6 + q^7 + 3q^8 + 4q^9 + 7q^{10} + 9q^{11} + 14q^{12} + \cdots\ .\end{aligned}$$

Here are listed the first polynomials of $K_{\lambda 1^n}(q)$ that satisfy that limit

$$\begin{aligned}
K_{33}(q) &= q^6 + q^8 + q^9 + q^{10} + q^{12},\\
K_{331}(q) &= q^6 + q^7 + 2q^8 + 2q^9 + 3q^{10} + 3q^{11} + 3q^{12} + \cdots,\\
K_{331^2}(q) &= q^6 + q^7 + 3q^8 + 3q^9 + 5q^{10} + 5q^{11} + 7q^{12} + \cdots,\\
K_{331^3}(q) &= q^6 + q^7 + 3q^8 + 4q^9 + 6q^{10} + 7q^{11} + 10q^{12} + \cdots,\\
K_{331^4}(q) &= q^6 + q^7 + 3q^8 + 4q^9 + 7q^{10} + 8q^{11} + 12q^{12} + \cdots,\\
K_{331^5}(q) &= q^6 + q^7 + 3q^8 + 4q^9 + 7q^{10} + 9q^{11} + 13q^{12} + \cdots,\\
K_{331^6}(q) &= q^6 + q^7 + 3q^8 + 4q^9 + 7q^{10} + 9q^{11} + 14q^{12} + \cdots.
\end{aligned}$$

REFERENCES

[Gup] G. GUPTA. — Problem No. 9, in *Combinatorics and Algebra*, C. GREENE, éd., *Contemporary Mathematics*, vol. 34, 1984, pp. 310.

[Han] G.-N. HAN. — Croissance des polynômes de Kostka, *C. R. Acad. Sc. Paris*, **311,** série I (1990), pp. 269-272.

[Knu] D.-E. KNUTH . — *The Art of Computer Programming, vol.3, Sorting and Searching*, Don Mills, Ontario, Addison-Wesley, 1972.

[LS1] A. LASCOUX ET M.-P. SCHÜTZENBERGER. — "Le monoïde plaxique," *Non-commutative structures in Algebra and Geometric Combinatorics*, A. DE LUCA, éd., Napoli, 1978, pp. 129-156.

[LS2] A. LASCOUX ET M.-P. SCHÜTZENBERGER. — Croissance des polynômes de Foulkes-Green, *C. R. Acad. Sc. Paris*, **288,** série A (1979), pp. 95-98.

[LS3] M.-P. SCHÜTZENBERGER. — "Propriétés nouvelles des tableaux de Young," *Séminaire Delange-Pisot-Poitou*, 19e année, n°26, 1977/1978.

[Mac] I.-G. MACDONALD. — *Symmetric functions and Hall polynomials*, Oxford, Clarendon Press, 1979.