COMBINATORIAL PROOFS OF SOME PROPERTIES OF TANGENT AND GENOCCHI NUMBERS

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Abstract. The tangent number T_{2n+1} is equal to the number of increasing labelled complete binary trees with $2n+1$ vertices. This combinatorial interpretation immediately proves that T_{2n+1} is divisible by 2^n. However, a stronger divisibility property is known in the studies of Bernoulli and Genocchi numbers, namely, the divisibility of $(n+1)T_{2n+1}$ by 2^{2n}. The traditional proofs of this fact need significant calculations. In the present paper, we provide a combinatorial proof of the latter divisibility by using the hook length formula for trees. Furthermore, our method is extended to k-ary trees, leading to a new generalization of the Genocchi numbers.

1. Introduction

The *tangent numbers* [1] $(T_{2n+1})_{n\geq 0}$ appear in the Taylor expansion of $\tan(x)$:

$$\tan x = \sum_{n\geq 0} T_{2n+1} \frac{x^{2n+1}}{(2n+1)!}. \tag{1.1}$$

It is known that the tangent number T_{2n+1} is equal to the number of all *alternating permutations* of length $2n+1$ (see [1, 9, 13, 15]). Also, T_{2n+1} counts the number of *increasing labelled complete binary trees* with $2n+1$ vertices. This combinatorial interpretation immediately implies that T_{2n+1} is divisible by 2^n. However, a stronger divisibility property is known related to the study of Bernoulli and Genocchi numbers [4, 5, 16], as stated in the following theorem.

Theorem 1. *The number $(n+1)T_{2n+1}$ is divisible by 2^{2n}, and the quotient is an odd number.*

The quotient is called *Genocchi number* and denoted by

$$G_{2n+2} := (n+1)T_{2n+1}/2^{2n}. \tag{1.2}$$

Let

$$g(x) := \sum_{n\geq 0} G_{2n+2} \frac{x^{2n+2}}{(2n+2)!}$$

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[1] Some mathematical literature uses a slightly different notation where $\tan x$ is written $T_1 x + T_2 x^3/3! + T_3 x^5/5! + \cdots$ (See [13])

be the exponential generating function for the Genocchi numbers. Then, (1.2) is equivalent to

$$g(x) = x \tan \frac{x}{2}. \tag{1.3}$$

The initial values of the tangent and Genocchi numbers are listed below:

n	0	1	2	3	4	5	6
T_{2n+1}	1	2	16	272	7936	353792	22368256
G_{2n+2}	1	1	3	17	155	2073	38227

The fact that the Genocchi numbers are odd integers is traditionally proved by using the von Staudt-Clausen theorem on Bernoulli numbers and the little Fermat theorem [4, 5, 16]. Barsky [3, 10] gave a different proof by using the Laplace transform. To the best of the authors' knowledge, no simple combinatorial proof has been derived yet and it is the purpose of this paper to provide one. Our approach is based on the geometry of the so-called *leaf-labelled tree* and the fact that the hook length h_v of such a tree is always an odd integer (see Sections 2 and 3).

In Section 4 we consider the k-ary trees instead of the binary trees and obtain a new generalization of the Genocchi numbers. For each integer $k \geq 2$, let $L_{kn+1}^{(k)}$ be the number of increasing labelled complete k-ary trees with $kn+1$ vertices. Thus, $L_{kn+1}^{(k)}$ will appear to be a natural generalization of the tangent number. The general result is stated next.

Theorem 2. *(a) For each integer $k \geq 2$, the integer*

$$\frac{(k^2n - kn + k)!\, L_{kn+1}^{(k)}}{(kn+1)!}$$

is divisible by $(k!)^{kn+1}$.

(b) Moreover, the quotient

$$M_{k^2n-kn+k}^{(k)} := \frac{(k^2n - kn + k)!\, L_{kn+1}^{(k)}}{(k!)^{kn+1}(kn+1)!} \equiv \begin{cases} 1 \pmod{k}, & k = p, \\ 1 \pmod{p^2}, & k = p^t,\ t \geq 2, \\ 0 \pmod{k}, & \textit{otherwise}, \end{cases}$$

where $n \geq 1$ and p is a prime number.

We can realize that Theorem 2 is a direct generalization of Theorem 1, if we restate the problem in terms of generating functions. Let $\phi^{(k)}(x)$ and $\psi^{(k)}(x)$ denote the exponential generating functions for $L_{kn+1}^{(k)}$ and $M_{k^2n-kn+k}^{(k)}$, respectively, that is,

$$\phi^{(k)}(x) = \sum_{n\geq 0} L_{kn+1}^{(k)} \frac{x^{kn+1}}{(kn+1)!};$$

$$\psi^{(k)}(x) = \sum_{n\geq 0} M_{k^2n-kn+k}^{(k)} \frac{x^{k^2n-kn+k}}{(k^2n-kn+k)!}.$$

If k is clear from the context, the superscript (k) will be omitted. Thus, we will write $L_{kn+1} := L_{kn+1}^{(k)}$, $M_{k^2n-kn+k} := M_{k^2n-kn+k}^{(k)}$, $\phi(x) := \phi^{(k)}(x)$, $\psi(x) := \phi^{(k)}(x)$.

From Theorem 2 we have

$$\phi'(x) = 1 + \phi^k(x);$$

$$\psi(x) = x \cdot \phi\left(\frac{x^{k-1}}{k!}\right).$$

The last relation becomes the well-known formula (1.3) when $k = 2$.

Several generalizations of the Genocchi numbers have been studied in recent decades. They are based on the Gandhi polynomials [7, 5, 16], Seidel triangles [8, 18], continued fractions [17, 11], combinatorial models [11], etc. Our generalization seems to be the first extension dealing with the divisibility of $(n+1)T_{2n+1}$ by 2^{2n}. It also raises the following open problems.

Problem 1. Find a proof of Theorem 2 à la Carlitz, or à la Barsky.

Problem 2. Find the Gandhi polynomials, Seidel triangles, continued fractions and a combinatorial model for the new generalization of Genocchi numbers $M_{k^2n-kn+k}$ à la Dumont.

Problem 3. Evaluate $m_n := M_{k^2n-kn+k} \pmod k$ for $k = p^t$, where p is a prime number and $t \geq 3$. It seems that the sequence $(m_n)_{n\geq 0}$ is always periodic for any p and t. Computer calculation has provided the initial values:

$$(m_n)_{n\geq 0} = (1, 1, 5, 5, 1, 1, 5, 5, \cdots) \qquad \text{for } k = 2^3,$$
$$(m_n)_{n\geq 0} = (1, 1, 10, 1, 1, 10, 1, 1, 10 \cdots) \qquad \text{for } k = 3^3,$$
$$(m_n)_{n\geq 0} = (1, 1, 126, 376, 126, 1, 1, 126, 376, 126, \cdots) \qquad \text{for } k = 5^4,$$
$$(m_n)_{n\geq 0} = (1, 1, 13, 5, 9, 9, 5, 13, 1, 1, 13, 5, 9, 9, 5, 13, \cdots) \qquad \text{for } k = 2^4.$$

2. Increasing labelled binary trees

In this section we recall some basic notions on increasing labelled binary trees. Consider the set $\mathcal{T}(n)$ of all (unlabelled) binary trees with n vertices. For each $t \in \mathcal{T}(n)$ let $\mathcal{L}(t)$ denote the set of all *increasing labelled binary trees* of shape t, obtained from t by labeling its n vertices with $\{1, 2, \dots, n\}$ in such a way that the label of each vertex is less than that of its descendants. For each vertex v of t, the *hook length* of v, denoted by $h_v(t)$ or h_v, is the number of descendants of v (including v). The *hook length formula* ([12, §5.1.4. Ex. 20]) claims that the number of increasing labelled binary trees of shape t is equal to $n!$ divided by the product of the h_v's ($v \in t$)

$$\#\mathcal{L}(t) = \frac{n!}{\prod_{v\in t} h_v}. \tag{2.1}$$

Let $\mathcal{S}(2n+1)$ denote the set of all *complete binary trees* s with $2n+1$ vertices, which are defined to be the binary trees such that the two subtrees of each vertex are, either both empty, or both non-empty. For example, there are five complete binary trees with $2n+1 = 7$ vertices, labelled by their hook lengths in Fig. 1.

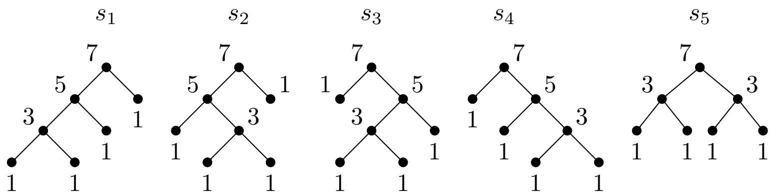

Fig. 1. Complete binary trees with 7 vertices

We now define an equivalence relation on $\mathcal{S}(2n+1)$, called *pivoting*. A *basic pivoting* is an exchange of the two subtrees of a non-leaf vertex v. For $s_1, s_2 \in \mathcal{S}(2n+1)$, if s_1 can be changed to s_2 by a finite sequence of basic pivotings, we write $s_1 \sim s_2$. It's routine to check that $\sim$ is an equivalence relation. Let $\bar{\mathcal{S}}(2n+1) = \mathcal{S}(2n+1)/\sim$. Since $s_1 \sim s_2$ implies that $\#\mathcal{L}(s_1) = \#\mathcal{L}(s_2)$, we define $\#\mathcal{L}(\bar{s}) = \#\mathcal{L}(s)$ for $s \in \bar{s}$. Then

$$T_{2n+1} = \sum_{\bar{s} \in \bar{\mathcal{S}}(2n+1)} T(\bar{s}), \tag{2.2}$$

where

$$T(\bar{s}) = \sum_{s \in \bar{s}} \#\mathcal{L}(s) = \#\bar{s} \times \#\mathcal{L}(\bar{s}). \tag{2.3}$$

For example, consider $\mathcal{S}(7)$ (see Fig. 1), we have

shape	s_1	s_2	s_3	s_4	s_5
$\prod_v h_v$	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7$	$3 \cdot 3 \cdot 7$
$n!/\prod_v h_v$	48	48	48	48	80

Trees s_1, s_2, s_3 and s_4 belong to the same equivalence class $\overline{s_1}$, while s_5 is in another equivalence class $\overline{s_5}$. Thus $T(\overline{s_1}) = 4 \times 48 = 192$, $T(\overline{s_5}) = 80$ and $T_7 = T(\overline{s_1}) + T(\overline{s_5}) = 272$.

The pivoting can also be viewed as an equivalence relation on the set $\cup_{s \in \bar{s}} \mathcal{L}(s)$, that is, all increasing labelled trees of shape s with $s \in \bar{s}$. Since the number of non-leaf vertices is n in s, there are exactly 2^n labelled trees in each equivalence class. Hence, $T(\bar{s})$ is divisible by 2^n. Take again the example above, $T(\overline{s_1})/2^3 = 24$, $T(\overline{s_5})/2^3 = 10$, and $T_7/2^3 = 24 + 10 = 34$.

This is not enough to derive that $2^{2n} \mid (n+1)T_{2n+1}$. However, the above process leads us to reconsider the question in each equivalence class. We can show that the divisibility actually holds in each $\bar{s}$, as stated below.

Proposition 3. *For each $\bar{s} \in \bar{\mathcal{S}}(2n+1)$, the integer $(n+1)T(\bar{s})$ is divisible by 2^{2n}.*

Let $G(\bar{s}) := (n+1)T(\bar{s})/2^{2n}$. Proposition 3 implies that $G(\bar{s})$ is an integer. By (1.2) and (2.2),

$$G_{2n+2} = \sum_{\bar{s} \in \bar{\mathcal{S}}(2n+1)} G(\bar{s}). \tag{2.4}$$

We give an example here and present the proof in the next section.

For $n = 4$, there are three equivalence classes.

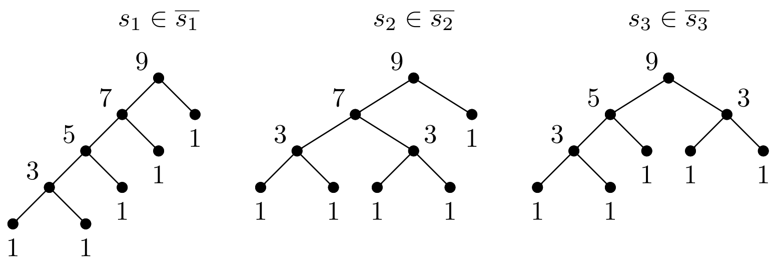

Fig. 2. Three equivalence classes for $n = 4$

In this case, Proposition 3 and relation (2.4) can be verified by the following table.

$\bar{s}$	$\#\bar{s}$	$\prod h_v$	$\#\mathcal{L}(\bar{s})$	$T(\bar{s})$	$G(\bar{s})$
$\overline{s_1}$	8	$3 \cdot 5 \cdot 7 \cdot 9$	384	3072	60
$\overline{s_2}$	2	$3 \cdot 3 \cdot 7 \cdot 9$	640	1280	25
$\overline{s_3}$	4	$3 \cdot 3 \cdot 5 \cdot 9$	896	3584	70
sum	14			7936	155

3. Combinatorial proof of Theorem 1

Let n be a nonnegative integer and $\bar{s} \in \bar{\mathcal{S}}(2n+1)$ be an equivalence class in the set of increasing labelled complete binary trees. The key of the proof is the fact that the hook length h_v is always an odd integer. For each complete binary tree s, we denote the product of all hook lengths by $H(s) = \prod_{v \in s} h_v$. Also, let $H(\bar{s}) = H(s)$ for $s \in \bar{s}$, since all trees in the equivalence class $\bar{s}$ share the same product of all hook lengths.

Lemma 4. *For each complete binary tree s, the product of all hook lengths $H(s)$ is an odd integer.*

By Lemma 4, Proposition 3 has the following equivalent form.

Proposition 5. *For each $\bar{s} \in \bar{\mathcal{S}}(2n+1)$, the integer $(2n+2)H(\bar{s})T(\bar{s})$ is divisible by 2^{2n+1}.*

Proof. By identities (2.3) and (2.1) we have

$$\begin{aligned}(2n+2)H(\bar{s})T(\bar{s}) &= (2n+2)H(\bar{s}) \times \#\bar{s} \times \#\mathcal{L}(\bar{s}) \\ &= (2n+2) \times \#\bar{s} \times (2n+1)! \\ &= (2n+2)! \times \#\bar{s}. \end{aligned} \tag{3.1}$$

Suppose that s is a complete binary tree with $2n+1$ vertices, then s has $n+1$ leaves. Let s^+ be the complete binary tree with $4n+3$ vertices obtained from s by replacing each leaf of s by the complete binary tree with 3 vertices. So s^+ has $2n+2$ leaves. Let $\mathcal{L}^+(s^+)$ be the set of all leaf-labelled trees of shape s^+, obtained from s^+ by labeling its $2n+2$ leaves with $\{1, 2, \ldots, 2n+2\}$. It is clear that $\#\mathcal{L}^+(s^+) = (2n+2)!$. By (3.1) we have the following combinatorial interpretation:

For each $\bar{s} \in \bar{\mathcal{S}}(2n+1)$, the number of all leaf-labelled trees of shape s^+ such that $s \in \bar{s}$ is equal to $(2n+2)H(\bar{s})T(\bar{s})$.

This time we take the pivoting for an equivalence equation on the set of leaf-labelled trees $\cup_{s \in \bar{s}} \mathcal{L}^+(s^+)$. Since a leaf-labelled tree s^+ has $2n+1$ non-leaf vertices, and each non-trivial sequence of pivotings will make a difference on the labels of leaves, every equivalence class contains 2^{2n+1} elements. Hence, we can conclude that $(2n+2)H(\bar{s})T(\bar{s})$ is divisible by 2^{2n+1}. $\square$

For example, in Fig. 3, we reproduce a labelled tree with 9 vertices and a leaf-labelled tree with 19 vertices. There are 4 non-leaf vertices in the labelled tree and the 9 non-leaf vertices in the leaf-labelled tree, as indicated by the fat dot symbol "$\bullet$". Comparing with the traditional combinatorial model, our method increases the number of non-leaf vertices. Consequently, we establish a stronger divisibility property.

Labeled tree
$n = 4$ non-leaf vertices

Leaf-labelled tree
$2n+1 = 9$ non-leaf vertices

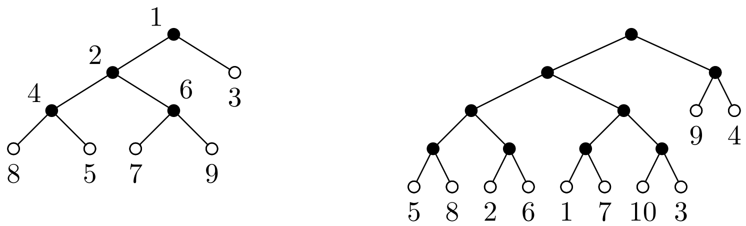

Fig. 3. Trees, non-leaf vertices and divisibilities

For proving Theorem 1, it remains to show that $G_{2n+2} = \sum G(\bar{s})$ is an odd number. Since $H(\bar{s})$ is odd, we need only to prove that the *weighted Genocchi number*

$$f(n) = \sum_{\bar{s} \in \bar{S}(2n+1)} H(\bar{s})G(\bar{s}) \tag{3.2}$$

is odd. For example, in Fig. 2., $G_{10} = G(\overline{s_1}) + G(\overline{s_2}) + G(\overline{s_3}) = 60 + 25 + 70 = 155$, and

$$\begin{aligned} f(4) &= H(\overline{s_1})G(\overline{s_1}) + H(\overline{s_2})G(\overline{s_2}) + H(\overline{s_3})G(\overline{s_3}) \\ &= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 60 + 3 \cdot 3 \cdot 7 \cdot 9 \cdot 25 + 3 \cdot 3 \cdot 5 \cdot 9 \cdot 70 \\ &= (3 \cdot 5 \cdot 7)^2 \cdot 9. \end{aligned}$$

The weighted Genocchi number $f(n)$ is more convenient for us to study, since it has an explicit simple expression.

Theorem 6. *Let $f(n)$ be the weighted Genocchi number defined in* (3.2). *Then,*

$$f(n) = (1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1))^2 \cdot (2n+1) = (2n-1)!! \cdot (2n+1)!!. \tag{3.3}$$

Proof. We successively have

$$\begin{aligned} f(n) &= \sum_{\bar{s}} H(\bar{s})G(\bar{s}) \\ &= \sum_{\bar{s}} \frac{H(\bar{s})(n+1)T(\bar{s})}{2^{2n}} \\ &= \sum_{\bar{s}} \frac{(2n+2)! \times \#\bar{s}}{2^{2n+1}} \\ &= \frac{(2n+2)!}{2^{2n+1}} \sum_{\bar{s}} \#\bar{s} \end{aligned}$$

$$= \frac{(2n+2)!}{2^{2n+1}} \cdot \#\mathcal{S}(2n+1).$$

While $\#\mathcal{S}(2n+1)$ equals to the Catalan number C_n, we can calculate that

$$\begin{aligned} f(n) &= \frac{(2n+2)!}{2^{2n+1}} \cdot C_n \\ &= \frac{(2n+2)!}{2^{2n+1}} \cdot \frac{1}{n+1}\binom{2n}{n} \\ &= (2n-1)!! \cdot (2n+1)!!. \end{aligned}$$ □

From Theorem 6, the weighted Genocchi number $f(n)$ is an odd number. Therefore, the normal Genocchi number G_{2n+2} is also odd. This achieves the proof of Theorem 1.

4. Generalizations to k-ary trees

In this section we assume that $k \geq 2$ is an integer.

Recall the *hook length formula* for binary trees described in Section 2. For general rooted trees t (see [12, §5.1.4, Ex. 20]), we also have

$$\#\mathcal{L}(t) = \frac{n!}{\prod_{v\in t} h_v}, \tag{4.1}$$

where $\mathcal{L}(t)$ denote the set of all *increasing labelled trees* of shape t.

Let L_{kn+1} be the number of increasing labelled complete k-ary trees with $kn+1$ vertices. Then,

$$L_{kn+1} = \sum_{n_1+\cdots+n_k=n-1} \binom{kn}{kn_1+1, \cdots, kn_k+1} L_{kn_1+1}\cdots L_{kn_k+1}. \tag{4.2}$$

Equivalently, the exponential generating function $\phi(x)$ for L_{kn+1}

$$\phi(x) = \sum_{n\geq 0} L_{kn+1}\frac{x^{kn+1}}{(kn+1)!}$$

is the solution of the differential equation

$$\phi'(x) = 1 + \phi^k(x) \tag{4.3}$$

such that $\phi(0) = 0$.

Let $\psi(x)$ be the exponential generating function for $M_{k^2n-kn+k}$ which is defined in Theorem 2,

$$\psi(x) := \sum_{n\geq 0} M_{k^2n-kn+k}\frac{x^{k^2n-kn+k}}{(k^2n-kn+k)!}.$$

Then

$$\psi(x) = x\cdot\phi\left(\frac{x^{k-1}}{k!}\right). \tag{4.4}$$

From identities (4.3) and (4.4), Theorem 2 can be restated in the form of power series and differential equations:

Corollary 7. *Let $\psi(x)$ be a power series satisfying the following differential equation*

$$x\psi'(x) - \psi(x) = \frac{k-1}{k!}\Big(x^k + \psi^k(x)\Big),$$

with $\psi(0) = 0$. Then, for each $n \geq 1$, the coefficient of $\dfrac{x^{k^2n-kn+k}}{(k^2n-kn+k)!}$ in $\psi(x)$ is an integer. Moreover, it is congruent to
(i) $1 \pmod{k}$, if $k = p$;
(ii) $1 \pmod{p^2}$, if $k = p^t$ with $t \geq 2$;
(iii) $0 \pmod{k}$, otherwise.

When $k = 2$, L_{2n+1} is just the tangent number T_{2n+1} and M_{2n+2} is the Genocchi number G_{2n+2}. For $k = 3$ and 4, the initial values of L_{kn+1} and $M_{k^2n-kn+k}$ are reproduced below:

n	L_{3n+1}	M_{6n+3}
0	1	1
1	6	70
2	540	500500
3	184680	43001959000
4	157600080	21100495466050000
5	270419925600	39781831724228093500000

Table for $k = 3$

n	L_{4n+1}	M_{12n+4}
0	1	1
1	24	525525
2	32256	10258577044340625
3	285272064	42645955937142729593062265625
4	8967114326016	6992644904557760596067178252404694486328125

Table for $k = 4$

Now we define an equivalence relation (*k-pivoting*) on the set of all (unlabelled) complete k-ary trees $\mathcal{R}(kn+1)$. A *basic k-pivoting* is a rearrangement of the k subtrees of a non-leaf vertex v. Let r_1, r_2 be two complete k-ary trees, if r_1 can be changed to r_2 by a finite sequence of basic k-pivotings, we write $r_1 \sim r_2$. Hence the set of all complete k-ary trees can be partitioned into several equivalence classes. Let $\bar{\mathcal{R}}(kn+1) = \mathcal{R}(kn+1)/\sim$, define $\#\mathcal{L}(\bar{r}) = \#\mathcal{L}(r)$ for $r \in \bar{r}$, then we have

$$\sum_{\bar{r}\in\bar{\mathcal{R}}(kn+1)} L(\bar{r}) = L_{kn+1}, \tag{4.5}$$

where

$$L(\bar{r}) = \sum_{r\in\bar{r}} \#\mathcal{L}(r) = \#\bar{r} \times \#\mathcal{L}(\bar{r}). \tag{4.6}$$

Similar to the case of the tangent numbers, this equivalence relation implies that $L(\bar{r})$ is divisible by $(k!)^n$. There is still a stronger divisibility, stated as below:

Lemma 8. *For each* $\bar{r} \in \bar{\mathcal{R}}(kn+1)$*, the number* $(k^2n - kn + k)!L(\bar{r})/(kn+1)!$ *is divisible by* $(k!)^{kn+1}$.

Proof. First, we show that the coefficient $(k^2n - kn + k)!/(kn+1)!$ is divisible by $(k-1)!^{kn+1}$. In fact,

$$\frac{(k^2n - kn + k)!}{(kn+1)! \cdot (k-1)!^{kn+1}} = (k^2n - kn + k) \cdot \prod_{i=1}^{kn+1} \binom{i(k-1)-1}{k-2}. \tag{4.7}$$

It remains to prove

$$k^{kn+1} \mid \frac{(k^2n - kn + k)!\ L(\bar{r})}{(kn+1)! \cdot (k-1)!^{kn+1}}. \tag{4.8}$$

For each vertex v in a complete k-ary tree r, we observe that the hook length h_v satisfies $h_v \equiv 1 \pmod{k}$. Thus,

$$H(\bar{r}) = \prod_{v \in r} h_v \equiv 1 \pmod{k}.$$

Consequently, relation (4.8) is equivalent to

$$k^{kn+1} \mid \frac{(k^2n - kn + k)!\ L(\bar{r})H(\bar{r})}{(kn+1)! \cdot (k-1)!^{kn+1}},$$

which can be rewritten as

$$(k!)^{kn+1} \mid (k^2n - kn + k)! \times \frac{L(\bar{r})H(\bar{r})}{(kn+1)!}. \tag{4.9}$$

We will prove this divisibility using the following combinatorial model. Let r be a complete k-ary tree with $kn+1$ vertices. It is easy to show that r has $(k-1)n+1$ leaves. Replacing all leaves of r by the complete k-ary tree with $k+1$ vertices, we get a new tree with $k^2n - kn + k$ leaves, denoted by r^+. Let $\mathcal{L}^+(r^+)$ be the set of all leaf-labelled tree of shape r^+, obtained from r^+ by labeling all the leaves with $1, 2, \ldots, k^2n - kn + k$. It is clear that $\#\mathcal{L}^+(r^+) = (k^2n - kn + k)!$. On the other hand, by the hook length formula we have

$$\frac{L(\bar{r})H(\bar{r})}{(kn+1)!} = \frac{H(\bar{r}) \times \#\bar{r} \times \#\mathcal{L}(r)}{(kn+1)!} = \#\bar{r}.$$

Thus, the right-hand side of (4.9) is equal to $(k^2n - kn + k)! \times \#\bar{r}$, that is, the number of all leaf-labelled trees of shape r^+ such that $r \in \bar{r}$.

Translate the k-pivoting to the set of all leaf-labelled trees of shape r^+ such that $r \in \bar{r}$. It is easy to check that the k-pivoting is still an equivalence relation. Since a leaf-labelled tree has $kn+1$ non-leaf vertices, there are $(k!)^{kn+1}$ leaf-labelled trees in each equivalence class, which implies that the right-hand side of (4.9) is divisible by $(k!)^{kn+1}$. □

The following two lemmas will be used for proving Theorem 2.

Lemma 9 (Legendre's formula)**.** *Suppose that* p *is prime number. For each positive integer* k*, let* $\alpha(k)$ *be the highest power of* p *dividing* $k!$ *and* $\beta(k)$ *be the sum of all digits of* k *in base* p*. Then,*

$$\alpha(k) = \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor = \frac{k - \beta(k)}{p-1}. \tag{4.10}$$

For the proof of Lemma 9, see [6, p. 263].

Lemma 10. *Let $p \geq 3$ be a prime number, then*

$$(pk+1)(pk+2)\cdots(pk+p-1) \equiv (p-1)! \pmod{p^2}. \tag{4.11}$$

Proof. The left-hand side of (4.11) is equal to

$$(pk)^{p-1}e_0 + \cdots + (pk)^2 e_{p-3} + (pk)e_{p-2} + e_{p-1} \equiv (pk)e_{p-2} + (p-1)! \pmod{p^2},$$

where $e_j := e_j(1, 2, \cdots, p-1)$ are the elementary symmetric functions. See [14]. Since

$$e_{p-2} = (p-1)! \sum_i i^{-1} \equiv (p-1)! \sum_i i \equiv (p-1)! \frac{p(p-1)}{2} \equiv 0 \pmod{p},$$

equality (4.11) is true. □

We are ready to prove Theorem 2.

Proof of Theorem 2. The first part (a) is an immediate consequence of Lemma 8 and (4.5). Let $n \geq 1$, we construct the following weighted function

$$f(n) = \sum_{\bar{r} \in \bar{\mathcal{R}}(kn+1)} H(\bar{r}) M(\bar{r}),$$

where

$$M(\bar{r}) = \frac{(k^2 n - kn + k)!\, L(\bar{r})}{(k!)^{kn+1}\,(kn+1)!}.$$

Since $H(\bar{r}) \equiv 1 \pmod{k}$, we have

$$f(n) \equiv \sum_{\bar{r} \in \bar{\mathcal{R}}(kn+1)} M(\bar{r}) = M_{k^2 n - kn + k} \pmod{k}. \tag{4.12}$$

Thus, we only need to calculate $f(n)$.

$$\begin{aligned} f(n) &= \sum_{\bar{r}} H(\bar{r}) M(\bar{r}) \\ &= \sum_{\bar{r}} \frac{H(\bar{r}) \times (k^2 n - kn + k)!\, L(\bar{r})}{(k!)^{kn+1}\,(kn+1)!} \\ &= \sum_{\bar{r}} \frac{(k^2 n - kn + k)! \times \#\bar{r}}{(k!)^{kn+1}} \\ &= \frac{(k^2 n - kn + k)!}{(k!)^{kn+1}} C_k(n), \end{aligned}$$

where $C_k(n)$ is the number of all (unlabelled) complete k-ary trees, that is equal to the Fuss-Catalan number [2]

$$C_k(n) = \frac{(kn)!}{n!(kn-n+1)!}.$$

Consequently,

$$f(n) = \frac{(k^2 n - kn + k)!}{(k!)^{kn-n+1}(kn-n+1)!} \cdot \frac{(kn)!}{(k!)^n n!} \tag{4.13}$$

$$= \prod_{i=0}^{kn-n} \binom{ik+k-1}{k-1} \times \prod_{j=0}^{n-1} \binom{jk+k-1}{k-1}. \tag{4.14}$$

For proving the second part (b), there are three cases to be considered depending on the value of k.

(b1) $k = p$ is a prime integer. We have

$$\binom{ip+p-1}{p-1} = \frac{(ip+1)(ip+2)\cdots(ip+p-1)}{1\times 2\times\cdots\times(p-1)} \equiv 1 \pmod{p}.$$

Thus $f(n) \equiv 1 \pmod{p}$ by identity (4.14).

(b2) $k = p^t$ $(t \geq 2)$ where p is a prime integer. If $p \geq 3$, by Lemma 10, we have

$$\begin{aligned}
\binom{ip^t+p^t-1}{p^t-1} &= \prod_{s=0}^{p^{t-1}-1} \frac{(ip^t+sp+1)\cdots(ip^t+sp+p-1)}{(sp+1)\cdots(sp+p-1)} \cdot \prod_{s=1}^{p^{t-1}-1} \frac{ip^t+sp}{sp} \\
&\equiv \left[\frac{(p-1)!}{(p-1)!}\right]^{p^{t-1}} \cdot \binom{ip^{t-1}+p^{t-1}-1}{p^{t-1}-1} \pmod{p^2} \\
&\equiv \binom{ip^{t-1}+p^{t-1}-1}{p^{t-1}-1} \pmod{p^2} \\
&\equiv \cdots \\
&\equiv \binom{ip+p-1}{p-1} \pmod{p^2} \\
&= \frac{(ip+1)(ip+2)\cdots(ip+p-1)}{1\times 2\times\cdots\times(p-1)} \\
&\equiv 1 \pmod{p^2}.
\end{aligned}$$

Thus $f(n) \equiv 1 \pmod{p^2}$ for $k = p^t$ with $p \geq 3$ and $t \geq 2$.

Now suppose $p = 2$ and $k = 2^t$ $(t \geq 2)$. We have

$$\begin{aligned}
\binom{i2^t+2^t-1}{2^t-1} &= \prod_{s=0}^{2^{t-1}-1} \frac{i\cdot 2^t+2s+1}{2s+1} \cdot \prod_{s=1}^{2^{t-1}-1} \frac{i\cdot 2^t+2s}{2s} \\
&= \prod_{s=0}^{2^{t-2}-1} \frac{(i\cdot 2^t+4s+1)(i\cdot 2^t+4s+3)}{(4s+1)(4s+3)} \cdot \prod_{s=1}^{2^{t-1}-1} \frac{i\cdot 2^{t-1}+s}{s} \\
&\equiv \left(\frac{-1}{-1}\right)^{2^{t-2}} \cdot \binom{i\cdot 2^{t-1}+2^{t-1}-1}{2^{t-1}-1} \pmod{4} \\
&\equiv \binom{i\cdot 2^{t-1}+2^{t-1}-1}{2^{t-1}-1} \pmod{4} \\
&\equiv \cdots \\
&\equiv \binom{i\cdot 2+2-1}{2-1} \pmod{4} \\
&= 2i+1.
\end{aligned}$$

Therefore, by identity (4.14), we can check that

$$f(n) \equiv \prod_{i=0}^{(2^t-1)n} (2i+1) \times \prod_{j=0}^{n-1} (2j+1) \equiv 1 \pmod 4.$$

(b3) Suppose that k has more than one prime factors. We want to prove $f(n) \equiv 0 \pmod k$. Let p be a prime factor of k, and write $k = bp^m$ with $b \geq 2$ and $p \nmid b$. Notice that $f(n) \mid f(n+1)$ by identity (4.13). Thus, it suffices to show that

$$f(1) = \frac{(k^2)!}{(k!)^{k+1}} \equiv 0 \pmod{p^m}, \tag{4.15}$$

which is equivalent to

$$\alpha(b^2p^{2m}) - (bp^m + 1)\,\alpha(bp^m) \geq m. \tag{4.16}$$

By Legendre's formula (4.10), the left-hand side of (4.16) is equal to

$$\begin{aligned} \Delta &= \frac{1}{p-1}\Big(b^2p^{2m} - \beta(b^2) - (bp^m+1)(bp^m - \beta(b))\Big) \\ &= \frac{1}{p-1}\Big(\beta(b) - \beta(b^2) + bp^m\beta(b) - bp^m\Big). \end{aligned}$$

Since $\beta(b^2) \leq b\beta(b)$ and $\beta(b) \geq 2$, $b \geq 2$, we have

$$\begin{aligned} \Delta &\geq \frac{1}{p-1}\Big((bp^m - b + 1)\beta(b) - bp^m\Big) \\ &\geq \frac{1}{p-1}\Big(b(p^m-2) + 2\Big) \\ &\geq \frac{1}{p-1}\Big(2p^m - 2\Big) \\ &\geq m. \end{aligned} \tag{4.17}$$

This completes the proof. ☐

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