ON THE EXISTENCE OF PERMUTATIONS CONDITIONED BY CERTAIN RATIONAL FUNCTIONS

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Abstract. We prove several conjectures made by Z.-W. Sun on the existence of permutations conditioned by certain rational functions. Furthermore, we fully characterize all integer values of the "inverse difference" rational function. Our proofs consist of both investigation of the mathematical properties of the rational functions and brute-force attack by computer for finding special permutations.

1. Introduction

Permutations (see, for example, [1, 3]) are studied in almost every branch of mathematics and also in computer science. The number of permutations $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in \mathfrak{S}_n$ of $\{1, 2, \ldots, n\}$ is $n!$. In [4] Z.-W. Sun made several conjectures about the existence of permutations conditioned by certain rational functions. In the paper we confirm three of them by proving the following theorem.

Theorem 1. (i) *For any integer $n > 5$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k) - \pi(k+1)} = 0. \tag{1.1}$$

(ii) *For any integer $n > 7$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k) - \pi(k+1)} + \frac{1}{\pi(n) - \pi(1)} = 0. \tag{1.2}$$

(iii) *For any integer $n > 5$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k)\pi(k+1)} = 1. \tag{1.3}$$

Since $n!$ is a huge number for large n, the generation of all $n!$ permutations of n by computer is already a challenge [2]. For this reason (1.1)-(1.3) have only been verified for very small n. Our proof of Theorem 1 consists of both investigation of the mathematical properties of the rational functions and brute-force attack by computer for finding certain special permutations.

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Furthermore, we can fully characterize all integer values of the "inverse difference" rational function given in the left-hand side of (1.1). We define

$$V_n = \left\{ \sum_{k=1}^{n-1} \frac{1}{\pi(k) - \pi(k+1)} : \pi \in \mathfrak{S}_n \right\}. \tag{1.4}$$

Since $a \in V_n$ implies $-a \in V_n$ by the reverse of permutation, we only need to study the nonnegative integer values of V_n. For example, $n = 5$, the value set V_5 contains the following nonnegative rational numbers:

$$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, 1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{19}{12}, \frac{7}{4}, \frac{11}{6}, \frac{23}{12}, 2, \frac{13}{6}, \frac{11}{4}, 4.$$

We see that there are three integers $1, 2, 4$ in the above list.

Theorem 2. *We have $V_3 \cap \mathbb{N} = \{2\}$, $V_5 \cap \mathbb{N} = \{1, 2, 4\}$, and for $n \neq 3, 5$,*

$$V_n \cap \mathbb{N} = \{0 \le j \le n-1 \mid j \neq n-2\}. \tag{1.5}$$

The proofs of Theorems 1 and 2 will be given in Section 2. Notice that we are still not able to prove three other conjectures of Sun. Let us reproduce them below for interested readers.

Conjecture 3. *(i)* [4, Conj. 4.7(ii)] *For any integer $n > 6$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k) + \pi(k+1)} = 1. \tag{1.6}$$

Also, for any integer $n > 7$, there is a permutation $\pi \in \mathfrak{S}_n$ such that

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k) + \pi(k+1)} + \frac{1}{\pi(n) + \pi(1)} = 1. \tag{1.7}$$

(ii) [4, Conj. 4.8(ii)] *For any integer $n > 7$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k)^2 - \pi(k+1)^2} = 0. \tag{1.8}$$

Motivated by (1.8), we make the following conjecture.

Conjecture 4. *For any integer $n > 11$, there is a permutation $\pi \in \mathfrak{S}_n$ such that*

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k)^2 - \pi(k+1)^2} + \frac{1}{\pi(n)^2 - \pi(1)^2} = 0. \tag{1.9}$$

Conjecture 4 has been checked for $11 < n < 28$ by computer. We list below the permutations satisfying (1.9), which are found by our computer program in a highly non-trivial way.

$\pi_{12} = (1, 4, 3, 5, 7, 2, 12, 8, 10, 11, 9, 6)$,

$\pi_{13} = (1, 2, 12, 8, 9, 6, 11, 10, 7, 5, 13, 4, 3)$,

$\pi_{14} = (1, 2, 12, 9, 6, 4, 3, 13, 8, 7, 5, 10, 14, 11)$,

$\pi_{15} = (1, 9, 2, 3, 12, 10, 11, 5, 4, 14, 6, 15, 13, 8, 7)$,

$\pi_{16} = (1, 3, 2, 4, 5, 11, 16, 14, 10, 8, 6, 12, 9, 15, 13, 7)$,

$\pi_{17} = (1, 3, 2, 4, 5, 9, 15, 6, 12, 16, 11, 10, 14, 13, 8, 7, 17)$,

$\pi_{18} = (1, 3, 2, 4, 6, 5, 7, 13, 8, 14, 12, 16, 10, 18, 17, 9, 11, 15)$,
$\pi_{19} = (1, 3, 2, 4, 6, 5, 7, 8, 12, 18, 17, 13, 9, 15, 11, 10, 16, 19, 14)$,
$\pi_{20} = (1, 3, 2, 4, 6, 5, 7, 18, 8, 13, 12, 17, 9, 20, 16, 19, 10, 11, 15, 14)$,
$\pi_{21} = (1, 3, 2, 4, 6, 5, 7, 17, 8, 20, 16, 9, 12, 18, 15, 13, 19, 21, 11, 14, 10)$,
$\pi_{22} = (1, 3, 2, 4, 6, 5, 7, 8, 20, 13, 17, 22, 18, 12, 9, 15, 21, 19, 16, 11, 10, 14)$,
$\pi_{23} = (1, 3, 2, 4, 6, 14, 10, 18, 12, 8, 20, 7, 5, 21, 15, 11, 17, 13, 22, 23, 16, 19, 9)$,
$\pi_{24} = (1, 3, 2, 4, 6, 14, 10, 18, 12, 8, 5, 9, 21, 11, 24, 16, 20, 22, 17, 15, 13, 19, 23, 7)$,
$\pi_{25} = (1, 3, 2, 4, 6, 14, 10, 18, 12, 8, 5, 16, 24, 9, 21, 23, 7, 17, 15, 11, 13, 22, 20, 19, 25)$,
$\pi_{26} = (1, 3, 2, 4, 6, 14, 10, 18, 12, 8, 22, 13, 5, 23, 16, 20, 19, 21, 9, 7, 17, 11, 25, 15, 24, 26)$,
$\pi_{27} = (1, 3, 2, 4, 6, 14, 10, 18, 12, 8, 22, 13, 9, 5, 11, 21, 23, 16, 26, 19, 25, 27, 17, 15, 24, 20, 7)$.

2. Proofs

Let $\Phi_{\rm dif}(\pi)$, $\Phi_{\rm cycdif}(\pi)$, and $\Phi_{\rm prod}(\pi)$ denote the three rational functions expressed in the left-hand side of (1.1), (1.2), and (1.3), respectively. The following *Link lemma* is useful for our construction.

Lemma 5 (Link)**.** *Let $\sigma \in \mathfrak{S}_s$ and $\tau \in \mathfrak{S}_t$ be two permutations on $\{1, 2, \dots, s\}$ and $\{1, 2, \dots, t\}$, respectively, such that $\sigma(s) = s, \tau(1) = 1$ and $\Phi_{\rm dif}(\sigma) = \Phi_{\rm dif}(\tau) = 0$. We define the "link" of the two permutations $\rho \in \mathfrak{S}_{s+t-1}$ by*

$$\rho(k) = \begin{cases} \sigma(k), & \text{if } 1 \le k \le s \\ s - 1 + \tau(k - s + 1). & \text{if } s + 1 \le k \le s + t - 1 \end{cases}$$

Then, we have $\Phi_{\rm dif}(\rho) = 0$. Furthermore, if $\tau(t) = t$, we have $\rho(s+t-1) = s+t-1$.

Proof. Notice that in the definition of ρ, if we allow $k = s$ in the second case, the expression will give the same definition of $\rho(s)$ as in the first case, since $\sigma(s) = s = s - 1 + \tau(1)$. Hence,

$$\begin{aligned} \Phi_{\rm dif}(\rho) &= \sum_{k=1}^{s-1} \frac{1}{\rho(k) - \rho(k+1)} + \sum_{k=s}^{s+t-2} \frac{1}{\rho(k) - \rho(k+1)} \\ &= \sum_{k=1}^{s-1} \frac{1}{\sigma(k) - \sigma(k+1)} + \sum_{k=1}^{t-1} \frac{1}{\tau(k) - \tau(k+1)} \\ &= \Phi_{\rm dif}(\sigma) + \Phi_{\rm dif}(\tau) \\ &= 0. \end{aligned}$$

Furthermore, if $\tau(t) = t$, it is easy to see that $\rho(s + t - 1) = s + t - 1$. $\square$

Let us write the link ρ of σ and τ by $\langle \sigma, \tau \rangle$.

Example. Take $\sigma = (1, 4, 2, 5, 3, 6)$ and $\tau = (1, 3, 2, 4)$. We verify that

$$\begin{aligned} \Phi_{\rm dif}(\sigma) &= -\frac{1}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = 0, \\ \Phi_{\rm dif}(\tau) &= -\frac{1}{2} + \frac{1}{1} - \frac{1}{2} = 0. \end{aligned}$$

We have $\rho = \langle \sigma, \tau \rangle = (1, 4, 2, 5, 3, 6, 8, 7, 9)$ and

$$\Phi_{\rm dif}(\rho) = \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{3}\right) + \left(-\frac{1}{2} + \frac{1}{1} - \frac{1}{2}\right) = 0.$$

If $\tau(t) = t$, since the link $\rho = \langle \sigma, \tau \rangle$ also satisfies the conditions $\rho(s + t - 1) = s + t - 1$ and $\Phi(\rho) = 0$, we can "link" again and obtain $\langle \rho, \tau \rangle = \langle \langle \sigma, \tau \rangle, \tau \rangle$.

The following proposition is a slightly stronger version of Theorem 1(i) for the rational function $\Phi_{\rm dif}$.

Proposition 6. *For any integer $n > 5$, there is a permutation $\pi \in \mathfrak{S}_n$ such that $\pi(1) = 1, \pi(n) = n$, and $\Phi_{\rm dif}(\pi) = 0$.*

Proof. Let

$$\begin{aligned} \sigma_0 &= (1, 4, 2, 5, 3, 6), \\ \sigma_1 &= (1, 3, 2, 4), \\ \sigma_2 &= (1, 3, 6, 4, 7, 5, 2, 8), \end{aligned}$$

and $\tau = \sigma_1 = (1, 3, 2, 4)$. We have $\Phi_{\rm dif}(\sigma_j) = 0$ for $j = 0, 1, 2$, and $\tau(1) = 1, \tau(4) = 4$. By repeated application of the link algorithm, we obtain the following three families of permutations:

$$\begin{aligned} &(1, 4, 2, 5, 3, 6), \\ &(1, 4, 2, 5, 3, 6 \mid 8, 7, 9), \\ &(1, 4, 2, 5, 3, 6 \mid 8, 7, 9 \mid 11, 10, 12), \\ &\vdots \\ &(1, 3, 2, 4), \\ &(1, 3, 2, 4 \mid 6, 5, 7), \\ &(1, 3, 2, 4 \mid 6, 5, 7 \mid 9, 8, 10), \\ &\vdots \\ &(1, 3, 6, 4, 7, 5, 2, 8), \\ &(1, 3, 6, 4, 7, 5, 2, 8 \mid 10, 9, 11), \\ &(1, 3, 6, 4, 7, 5, 2, 8 \mid 10, 9, 11 \mid 13, 12, 14), \\ &\vdots \end{aligned}$$

of length

$$\begin{aligned} n &= 6, 9, 12, 15, \ldots \qquad (3k) \\ n &= 4, 7, 10, 13, \ldots \qquad (3k+1) \\ n &= 8, 11, 14, 17 \ldots \qquad (3k+2) \end{aligned}$$

Hence we have constructed one permutation $\pi \in \mathfrak{S}_n$ for each $n > 5$ such that $\Phi_{\rm dif}(\pi) = 0$ and $\pi(1) = 1$ and $\pi(n) = n$. $\square$

The following proposition is another enhanced version of Theorem 1(i) for $\Phi_{\rm dif}$, which is crucial for proving Theorem 1(ii) for $\Phi_{\rm cycdif}$. The difference between Propositions 6 and 7 lies in the value of $\pi(n)$.

Proposition 7. *For any integer $n > 7$, there is a permutation $\pi \in \mathfrak{S}_n$ such that $\pi(1) = 1, \pi(n) = n - 1$ and $\Phi_{\mathrm{dif}}(\pi) = 0$.*

Proof. Let

$$\begin{aligned}
\alpha_8 &= (1, 2, 4, 8, 6, 5, 3, 7),\\
\alpha_9 &= (1, 4, 2, 5, 9, 3, 7, 6, 8),\\
\alpha_{10} &= (1, 2, 6, 3, 7, 8, 5, 4, 10, 9),\\
\alpha_{11} &= (1, 2, 3, 4, 6, 5, 9, 8, 7, 11, 10),\\
\alpha_{12} &= (1, 2, 3, 6, 4, 8, 12, 10, 9, 7, 5, 11).
\end{aligned}$$

We have $\alpha_j(1) = 1$, $\alpha_j(j) = j - 1$, and $\Phi_{\mathrm{dif}}(\alpha_j) = 0$ for $j = 8, 9, \ldots, 12$. The proposition is true for $j = 8, 9, \ldots, 12$. For $n \geq 13$ and $k = n - 7 \geq 6$, take the permutation $\sigma \in \mathfrak{S}_k$ obtained in Proposition 6, i.e., $\sigma(1) = 1, \sigma(k) = k, \Phi_{\mathrm{dif}}(\sigma) = 0$. Then, the link $\rho = \langle \sigma, \alpha_8 \rangle$ satisfies $\rho(1) = 1, \rho(n) = n - 1$ and $\Phi_{\mathrm{dif}}(\rho) = 0$. For example, for $n = 13$ and $k = 6$,

$$\rho = \langle (1, 4, 2, 5, 3, 6), (1, 2, 4, 8, 6, 5, 3, 7) \rangle = (1, 4, 2, 5, 3, 6, 7, 9, 13, 11, 10, 8, 12).$$

Hence, the proposition is true for any $n > 7$. $\square$

Now we are ready to prove part (ii) of Theorem 1 for the rational function Φ_{cycdif}.

Proof of Theorem 1(ii). If $n = 2k$ is even, we can easily check that the permutation

$$\pi = (1, 2, 3, \ldots, k - 1, k, 2k, 2k - 1, \ldots, k + 3, k + 2, k + 1),$$

which is obtained by the concatenation of the increasing permutation of $\mathfrak{S}_k$ and the decreasing permutation of $\{j \mid k + 1 \leq j \leq 2k\}$, satisfies

$$\Phi_{\mathrm{cycdif}}(\pi) = \left(-\frac{1}{1} - \frac{1}{1} - \cdots - \frac{1}{1} - \frac{1}{k} + \frac{1}{1} + \frac{1}{1} + \cdots + \frac{1}{1} \right) + \frac{1}{k} = 0.$$

The odd case is more complicated. First, we define

$$\begin{aligned}
\beta_9 &= (2, 1, 4, 5, 9, 3, 7, 6, 8),\\
\beta_{11} &= (1, 2, 11, 5, 4, 8, 7, 9, 3, 6, 10),\\
\beta_{13} &= (1, 2, 13, 3, 5, 4, 9, 8, 10, 6, 11, 7, 12).
\end{aligned}$$

We have $\Phi_{\mathrm{cycdif}}(\beta_j) = 0$ for $j = 9, 11, 13$. Next, for $n = 2k + 1 \geq 15$, i.e., $k \geq 7$ and $m = k + 1 \geq 8$, by Propositions 6 and 7, there exist two permutations $\sigma \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_m$ such that

(i) $\sigma(1) = 1$, $\sigma(k) = k$, and $\Phi_{\mathrm{dif}}(\sigma) = 0$;
(ii) $\tau(1) = 1$, $\tau(m) = m - 1$, and $\Phi_{\mathrm{dif}}(\tau) = 0$.

Let τ' be the permutation of $\{j \mid k + 1 \leq j \leq 2k + 1\}$ obtained by adding k in the reverse of τ:

$$\tau' = (k + \tau(m), k + \tau(m - 1), k + \ldots, k + \tau(3), k + \tau(2), k + \tau(1)).$$

Notice that the first and last elements of τ' are $k + \tau(m) = 2k$ and $k + \tau(1) = k + 1$, respectively. Let ρ be the concatenation of σ and τ'. We can verity that $\Phi_{\mathrm{dif}}(\tau') = -\Phi_{\mathrm{dif}}(\tau) = 0$ and

$$\Phi_{\mathrm{cycdif}}(\rho) = \Phi_{\mathrm{dif}}(\sigma) + \frac{1}{k - 2k} + \Phi_{\mathrm{dif}}(\tau') + \frac{1}{(k + 1) - 1} = 0.$$

For example, for $n=15$, $k=7$ and $m=8$, we have $\sigma=(1,3,2,4,6,5,7)$ and $\tau=(1,2,4,8,6,5,3,7)$, so that $\tau'=(14,10,12,13,15,11,9,8)$. Our final permutation ρ is the concatenation of σ and τ':

$$\rho=(1,3,2,4,6,5,7,\ 14,10,12,13,15,11,9,8).$$

We can check that $\Phi_{\text{cycdif}}(\rho)=0$. $\square$

To prove part (iii) of Theorem 1 concerning the rational function Φ_{prod}, we need the following lemma. Also, it is much more convenient to describe the construction in the increasing binary trees model (see, for example, [3, p. 51], [1, p. 143]).

Lemma 8 (Insertion)**.** *Let $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n-1)\in\mathfrak{S}_{n-1}$ be a permutation and $\tau\in\mathfrak{S}_n$ be the permutation obtained by insertion of the letter n into σ:*

$$\tau=\sigma(1)\cdots\sigma(j)\,n\,\sigma(j+1)\cdots\sigma(n-1). \qquad (j=1,2,\ldots,n-2)$$

Then, $\Phi_{\text{prod}}(\sigma)=\Phi_{\text{prod}}(\tau)$ if and only if $\sigma(j)+\sigma(j+1)=n$.

Proof. We have

$$\begin{aligned}\Phi_{\text{prod}}(\sigma)&=\sum_{k=1}^{n-2}\frac{1}{\sigma(k)\sigma(k+1)}=\cdots+\frac{1}{\sigma(j)\sigma(j+1)}+\cdots\\ \Phi_{\text{prod}}(\tau)&=\sum_{k=1}^{n-1}\frac{1}{\tau(k)\tau(k+1)}=\cdots+\frac{1}{\sigma(j)\,n}+\frac{1}{n\,\sigma(j+1)}+\cdots\\ &=\cdots+\frac{\sigma(j)+\sigma(j+1)}{n\,\sigma(j)\sigma(j+1)}+\cdots\end{aligned}$$

Hence, $\Phi_{\text{prod}}(\sigma)=\Phi_{\text{prod}}(\tau)$ if and only if

$$\frac{1}{\sigma(j)\sigma(j+1)}=\frac{\sigma(j)+\sigma(j+1)}{n\,\sigma(j)\sigma(j+1)},$$

i.e., $\sigma(j)+\sigma(j+1)=n$. $\square$

Now we use the insertion lemma to prove part (iii) of our main theorem.

Proof of Theorem 1(iii). Let

$$\begin{aligned}\delta_6&=(2,1,3,4,5,6),\\ \delta_7&=(2,1,3,7,4,5,6),\\ \delta_8&=(6,4,1,2,7,5,3,8).\end{aligned}$$

We verify that $\Phi_{\text{prod}}(\delta_j)=1$ for $j=6,7,8$. By insertion lemma, we define σ_9 by inserting the letter 9 between 2 and 7 in σ_8:

$$\sigma_9=(6,4,1,2,\mathbf{9},7,5,3,8).$$

Next, we insert 10 in σ_9 between 6 and 4:

$$\sigma_{10}=(6,\mathbf{10},4,1,2,9,7,5,3,8).$$

For the insertion of 11, we have two possible positions, namely, between $(2,9)$ and $(3,8)$. We choose the position $(2,9)$ and define

$$\sigma_{11}=(6,10,4,1,2,\mathbf{11},9,7,5,3,8).$$

The crucial idea is to show that this kind of insertion can be repeatedly applied as many times as we want, starting from σ_8. Thus, we obtain the desired permutation

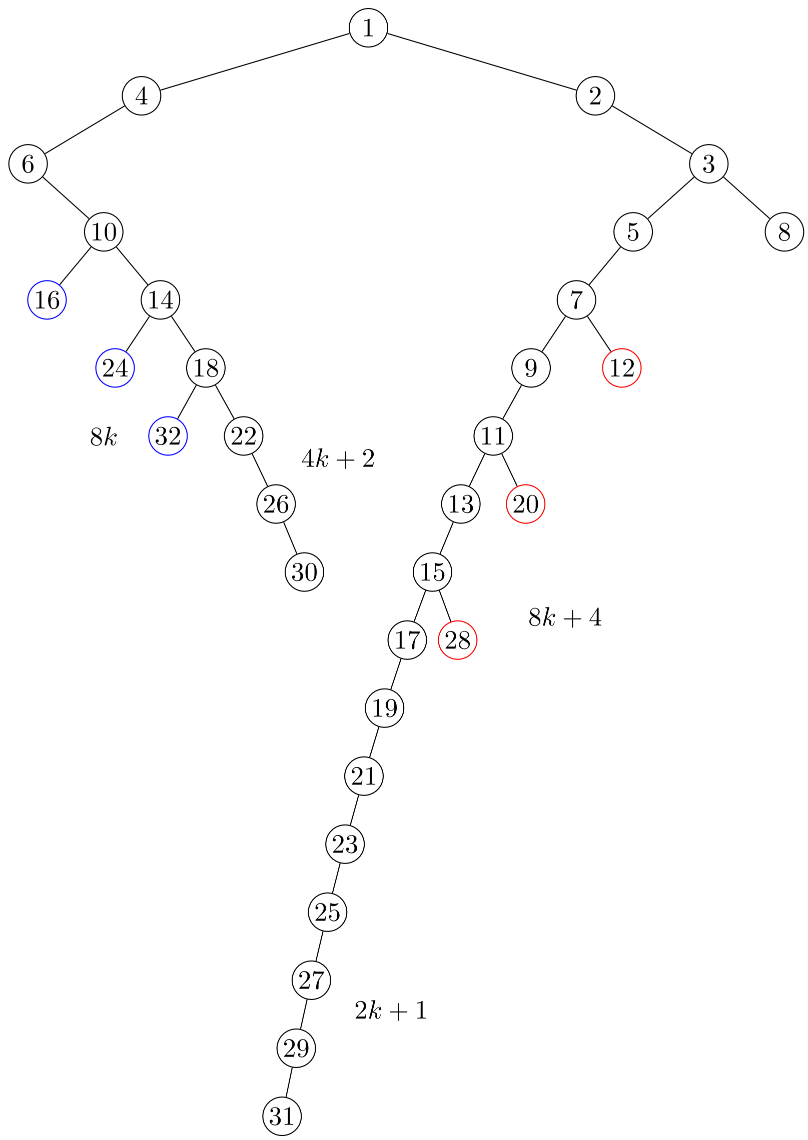

FIGURE 1. The increasing binary tree for δ_{32}

$\sigma_n \in \mathfrak{S}_n$ for each $n \geq 8$. To understand the general pattern, we take a rather big example with $n = 32$. Our permutation δ_{32} is as follows:

$$\delta_{32} = (6, 16, 10, 24, 14, 32, 18, 22, 26, 30, 4, 1, 2,\\ 31, 29, 27, 25, 23, 21, 19, 17, 15, 28, 13, 11, 20, 9, 7, 12, 5, 3, 8),$$

which can be represented by the increasing binary tree (see, for example, [3, p. 51], [1, p. 143]) in Figure 1. We see that the nodes of the form $2k+1, 4k+2, 8k, 8k+4$ all appear in the tree structure. Hence, the insertion can be repeatedly applied to reach each δ_n for $n \geq 8$. $\square$

As proved in Theorem 1, for any integer $n > 5$, there is a permutation $\pi \in \mathfrak{S}_n$ such that $\Phi_{\mathrm{dif}}(\pi) = 0$. For a permutation π, the value of $\Phi_{\mathrm{dif}}(\pi)$ is, a priori, a rational number. Theorem 2 provides a full characterization of the integer values of the rational function Φ_{dif}.

Proof of Theorem 2. In fact, we need to prove a stronger statement by replacing each V_n in the theorem by

$$V'_n = \left\{ \sum_{k=1}^{n-1} \frac{1}{\pi(k) - \pi(k+1)} : \pi \in \mathfrak{S}_n, \pi(n) = n \right\}. \tag{2.1}$$

It is easy to see that $n - 1 \in V'_n$ by taking the identity permutation $(1, 2, \ldots n)$. Also, the maximal value of $\Phi_{\mathrm{dif}}(\pi)$ is $n - 1$, so that $m \notin V_n$ for $m \geq n$. We can check by computer for all $n \leq 6$. If $n \geq 7$, we prove by induction on n. First, $0 \in V'_n$ by Proposition 6. Next, for $1 \leq m \leq n - 3$, by the induction hypothesis, there is a permutation $\tau \in \mathfrak{S}_{n-1}$ such that $\tau(n-1) = n - 1$ and $\Phi_{\mathrm{dif}}(\tau) = m - 1$. We define

$$\pi = (\tau(1), \tau(2), \ldots, \tau(n-1), n).$$

We verify that $\Phi_{\mathrm{dif}}(\pi) = 1 + \Phi_{\mathrm{dif}}(\tau) = m$. Finally, for each $\sigma \in \mathfrak{S}_n$ which is not the identity permutation, we see that there exists at least one negative term in the summation (1.1). Hence $\Phi_{\mathrm{dif}}(\sigma) < n - 2$, and $n - 2 \notin V_n$. $\square$

For $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n-1$, let $\eta(n, k)$ be the number of permutations $\pi \in \mathfrak{S}_n$ satisfying

$$\sum_{k=1}^{n-1} \frac{1}{\pi(k) - \pi(k+1)} = k.$$

We list the first values of $\eta(n, k)$ in the following table.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	0	1										
3	0	0	1									
4	2	2	0	1								
5	0	3	4	0	1							
6	12	16	3	6	0	1						
7	18	44	38	8	8	0	1					
8	348	339	136	73	11	10	0	1				
9	906	1284	802	264	112	18	12	0	1			
10	9740	8112	4081	1974	448	181	27	14	0	1		
11	40992	44462	28028	10695	3754	842	212	14	16	0	1	
12	376976	340032	178236	81037	25266	6548	1272	288	24	18	0	1

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