k-ARRANGEMENTS, STATISTICS AND PATTERNS

SHISHUO FU, GUO-NIU HAN, AND ZHICONG LIN

Abstract. The k-arrangements are permutations whose fixed points are k-colored. We prove enumerative results related to statistics and patterns on k-arrangements, confirming several conjectures by Blitvić and Steingrímsson. In particular, one of their conjectures regarding the equdistribution of the number of descents over the derangement form and the permutation form of k-arrangements is strengthened in two interesting ways. Moreover, as one application of the so-called Decrease Value Theorem, we calculate the generating function for a symmetric pair of Eulerian statistics over permutations arising in our study. This generating function is expressed in terms of a newly introduced linear operator ρ on formal power series.

1. Introduction

In their paper [2] about interpreting moments of probability measures on the real line, Blitvić and Steingrímsson introduced the k-*arrangements*, which are permutations with k-colored fixed points. They posed several conjectures related to the equidistributions of statistics and enumeration of patterns on k-arrangements. The purpose of this note is to address these enumeration conjectures. Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, 2, \dots, n\}$. For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$, let $\mathrm{FIX}(\sigma) := \{i \in [n] : \sigma(i) = i\}$ be the set of *fixed points* of σ. For any nonnegative integer k, a k-*arrangement* of $[n]$ is a pair $\mathfrak{a} = (\pi, \phi)$ of a permutation $\pi \in \mathfrak{S}_n$ and an arbitrary function $\phi : \mathrm{FIX}(\pi) \to \{\bar{i} : 1 \le i \le k\}$, where $\bar{i} := -i$. Note that for $k = 0$, there is no function $\phi : \mathrm{FIX}(\pi) \to \emptyset$ unless $\mathrm{FIX}(\pi) = \emptyset$. We will refer to π as the *base permutation* of $\mathfrak{a}$. Let A_n^k denote the set of k-arrangements of $[n]$. For instance, the 0-arrangements and 1-arrangements can be identified with derangements and permutations, respectively. The 2-arrangements, also called *decorated permutations* by Postnikov [24, Def. 13.3], were investigated previously from different aspects [5, 19, 24, 28].

Blitvić and Steingrímsson [2] introduced two different representations of k-arrangements, called *permutation form* and *derangement form*. Define the *reduction* (resp. *positive reduction*) of a word w over integers, denoted by $\mathrm{red}(w)$ (resp. $\mathrm{red}^+(w)$), to be the word obtained from w by replacing all instances of the i-th smallest letter (resp. positive letter) of w with i, for all i. For example, we have $\mathrm{red}(55\bar{1}2\bar{1}\bar{2}) = 442321$ and $\mathrm{red}^+(55\bar{1}2\bar{1}\bar{2}) = 22\bar{1}1\bar{1}\bar{2}$. For a k-arrangement $\mathfrak{a} = (\pi, \phi)$ of $[n]$, the *derangement form* (resp. *permutation form*) of $\mathfrak{a}$, denoted $\mathrm{df}_k(\mathfrak{a})$ (resp. $\mathrm{pf}_k(\mathfrak{a})$), is the word obtained from π by changing $\pi(i)$ to $\phi(i)$ for

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each $i \in \text{FIX}(\pi)$ (resp. $i \in \text{FIX}(\pi)$ such that $\phi(i) \neq \bar{k}$) and then applying the positive reduction. For instance, let $\mathfrak{a}$ be the 3-arrangement (π, ϕ) with $\pi = 75\mathbf{34}1\mathbf{6}2$ and $\phi(3) = \bar{1}$, $\phi(4) = \bar{3}$ and $\phi(6) = \bar{3}$. Then $\mathfrak{a}$ has derangement form $43\bar{1}\bar{3}1\bar{3}2$ whose *derangement part* is $\mathrm{Der}(\mathfrak{a}) = 4312$, and permutation form $64\bar{1}3152$ whose *permutation part* is 643152. The set of permutation forms (resp. derangement forms) representing elements in A_n^k is denoted P_n^k (resp. D_n^k). Note that $\mathsf{P}_n^1 = \mathfrak{S}_n$. The above two representations of k-arrangements provide two bijections between these three sets:

$$\mathrm{pf}_k : \mathsf{A}_n^k \to \mathsf{P}_n^k \quad \text{and} \quad \mathrm{df}_k : \mathsf{A}_n^k \to \mathsf{D}_n^k.$$

For a word $w = w_1 w_2 \cdots w_n$ over $\mathbb{Z}$, an index $i \in [n-1]$ is called a *descent (position)* of w if $w_i > w_{i+1}$. Let $\text{DES}(w)$ (resp. $\mathsf{des}(w)$) be the set (resp. the number) of descents of a word w, and $\mathrm{Pos}(w)$ be the *positive subword* of w, i.e., the subword that consists of all the positive letters in w. It is well known (see [14]) that the *Eulerian polynomials* $A_n(t)$ can be defined by $A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathsf{des}(\pi)}$. We extend some classical statistics on permutations or words to k-arrangements $\mathfrak{a} = (\pi, \phi)$ by

$$\begin{aligned}
\mathsf{fix}(\mathfrak{a}) &:= |\text{FIX}(\pi)|, & \mathsf{fix}_i(\mathfrak{a}) &:= |\{j \in \text{FIX}(\pi) : \phi(j) = \bar{i}\}|, \\
\text{DES}(\mathfrak{a}) &:= \text{DES}(\mathrm{pf}_k(\mathfrak{a})), & \mathsf{des}(\mathfrak{a}) &:= \mathsf{des}(\mathrm{pf}_k(\mathfrak{a})), \\
\text{DEZ}(\mathfrak{a}) &:= \text{DES}(\mathrm{df}_k(\mathfrak{a})), & \mathsf{dez}(\mathfrak{a}) &:= \mathsf{des}(\mathrm{df}_k(\mathfrak{a})), \\
\mathrm{Der}(\mathfrak{a}) &:= \mathrm{Pos}(\mathrm{df}_k(\mathfrak{a})).
\end{aligned}$$

We split A_n^k into small subsets according to the multiplicity of the image of the funciton ϕ in the following way. Given an array of nonnegative integers $\mathbf{m} = (m_1, m_2, \cdots, m_k)$, we let

$$\mathsf{A}_n^k(\mathbf{m}) := \{\mathfrak{a} \in \mathsf{A}_n^k : \mathsf{fix}_i(\mathfrak{a}) = m_i \text{ for } 1 \leq i \leq k\}.$$

For the sake of simplicity, if $\mathrm{ST}_1, \mathrm{ST}_2, \ldots$ is a sequence of statistics on certain set of combinatorial objects $\mathcal{O}$, then we use the compact notation

$$(\mathrm{ST}_1, \mathrm{ST}_2, \ldots)\, \mathfrak{o} = (\mathrm{ST}_1(\mathfrak{o}), \mathrm{ST}_2(\mathfrak{o}), \ldots) \quad \text{for each } \mathfrak{o} \in \mathcal{O}$$

throughout this paper.

In the previous work of Foata and the second author [11, 12], derangement forms in D_n^1 were already studied under the term *shuffle class*. They also introduced the DEZ and dez statistics on permutations, and constructed the bijection $\Phi = \Phi_1$ to show the following equidistribution, in the case of $k = 1$.

Theorem 1.1. *Let $n, k \geq 1$ and $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ be an array of nonnegative integers. For any permutation $\tau \in \mathfrak{S}_k$, there exists a bijection $\Phi_k : \mathsf{A}_n^k(\mathbf{m}) \to \mathsf{A}_n^k(\mathbf{m}')$ such that for every $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m})$,*

$$(\text{DEZ}, \mathrm{Der})\ \mathfrak{a} = (\text{DES}, \mathrm{Der})\ \Phi_k(\mathfrak{a}), \tag{1.1}$$

where $\mathbf{m}' = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(k)})$.

The equidistribution of des over P_n^k and D_n^k, was first conjectured in [2, Conj. 1], which is generalized in two directions: a set-valued extension, as stated in Theorem 1.1, and a symmetrical generalization as stated in the next theorem. Notice that Theorem 1.2 is new even for $k = 1$ over $\mathfrak{S}_n$, since our triple equidistribution on $\mathfrak{S}_n$ can not be proven using the bijection in [11, Thm. 1.1].

Theorem 1.2. *Let $n, k \geq 1$ and $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ be an array of nonnegative integers. For any permutation $\tau \in \mathfrak{S}_{k-1}$, there exists a bijection $\Psi_k : \mathsf{A}_n^k(\mathbf{m}) \to \mathsf{A}_n^k(\mathbf{m}')$ such that for every $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m})$,*

$$(\mathsf{des}, \mathsf{dez}, \mathrm{Der})\ \mathfrak{a} = (\mathsf{dez}, \mathsf{des}, \mathrm{Der})\ \Psi_k(\mathfrak{a}). \tag{1.2}$$

where $\mathbf{m}' = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(k-1)}, m_k)$.

Taking the identity permutation τ, Theorems 1.1 and 1.2 imply that there exists two bijections $\mathfrak{a} \mapsto \mathfrak{a}'$ and $\mathfrak{a} \mapsto \mathfrak{a}''$ from A_n^k onto itself such that

$$(\mathrm{DEZ}, \mathrm{Der})\ \mathfrak{a} = (\mathrm{DES}, \mathrm{Der})\ \mathfrak{a}',$$
$$(\mathsf{des}, \mathsf{dez}, \mathrm{Der})\ \mathfrak{a} = (\mathsf{dez}, \mathsf{des}, \mathrm{Der})\ \mathfrak{a}''.$$

We will also investigate the enumerative aspect of pattern avoiding k-arrangements. We say a word $w = w_1 w_2 \cdots w_n \in \mathbb{Z}^n$ *avoids the pattern* $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in \mathfrak{S}_k$ $(k \leq n)$ if there does not exist $i_1 < i_2 < \cdots < i_k$ such that $\mathrm{red}(w_{i_1} w_{i_2} \cdots w_{i_k}) = \sigma$. For a set $\mathcal{W}$ of words, let $\mathcal{W}(\sigma)$ be the set of σ-avoiding words in $\mathcal{W}$. Two patterns σ and π are said to be *Wilf-equivalent* over $\mathcal{W}$ if $|\mathcal{W}(\sigma)| = |\mathcal{W}(\pi)|$. One of the most famous enumerative results in pattern avoiding permutations, attributed to MacMahon and Knuth (cf. [17,27]), is that $|\mathfrak{S}_n(\sigma)| = C(n)$ for each pattern $\sigma \in \mathfrak{S}_3$, where

$$C(n) := \frac{1}{n+1}\binom{2n}{n}$$

is the n-th *Catalan number*. The study of pattern avoiding derangements was initiated by Robertson, Saracino and Zeilberger [25] and further generalized by others in [4, 9, 10]. Since the k-arrangements in permutation form and derangement form can be considered as generalizations of permutations and derangements, we study k-arrangements of both forms avoiding a single pattern of length 3. In the case of permutation form, we verify all the enumerative conjectures (see Section 4) posed by Blitvić and Steingrímsson [2], while in the derangement form only one Wilf-equivalence is found and reported next.

Theorem 1.3. *For $n \geq 1$, we have $|\mathsf{D}_n^1(321)| = |\mathsf{D}_n^1(132)|$. In other words, the pattern 321 is Wilf-equivalent to 132 on 1-arrangements in derangement form. Moreover, we have the algebraic generating function for $|\mathsf{D}_n^1(321)|$:*

$$1 + \sum_{n \geq 1} |\mathsf{D}_n^1(321)| x^n = \frac{1 - 3x + 3x^2 + 2x^3 + (x^2 + x - 1)\sqrt{1-4x}}{2x^2(1-x)(2+x)}. \tag{1.3}$$

The rest of this paper is organized as follows. In Section 2, we provide explicit bijections Φ_k and Ψ_k for proving Theorems 1.1 and 1.2. In Section 3, using the so-called Decrease Value Theorem, we calculate the generating function for the symmetric pair of Eulerian statistics des and dez over permutations. The enumeration of pattern avoiding k-arrangements are carried out in Section 4, proving all the connections suspected by Blitvić and Steingrímsson, as well as Theorem 1.3.

2. Constructions of the bijections Φ_k and Ψ_k

In this section we describe explicit constructions of bijections Φ_k and Ψ_k mentioned in Section 1, and then prove Theorems 1.1 and 1.2. Foata and the second author have constructed a DES-preserving bijection Φ between D_n^1 and P_n^1 in a different form (see [11, Thm. 1.1]). The reader is referred to [11] for the definition and properties of Φ. Our general bijection Φ_k is constructed by using Φ composed with other simple transformations, including the following multiplicity changing bijection θ [16, Section 4].

Lemma 2.1. *Let $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ be an array of nonnegative integers and $\mathbf{n} = (n_1, n_2, \cdots, n_k)$ a rearrangement of $\mathbf{m}$. There exists a bijection $\theta : R(\mathbf{m}) \to R(\mathbf{n})$ such that for each $w \in R(\mathbf{m})$,*

$$\mathrm{DES}(w) = \mathrm{DES}(\theta(w)),$$

where $R(\mathbf{m})$ (resp. $R(\mathbf{n})$) denotes the set of all words on k linearly ordered letters $a_1 < a_2 < \cdots < a_k$, containing exactly m_i (resp. n_i) copies of the letter a_i for all $i = 1, 2, \ldots, k$.

Lemma 2.2. *Let $n, k \geq 1$ and $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ be an array of nonnegative integers. For any permutation $\tau \in \mathfrak{S}_k$, there exists a bijection $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m}) \mapsto \mathfrak{b} \in \mathsf{A}_n^k(\mathbf{m}')$ such that*

$$(\mathrm{DEZ}, \mathrm{Der})\ \mathfrak{a} = (\mathrm{DEZ}, \mathrm{Der})\ \mathfrak{b}, \tag{2.1}$$

where $\mathbf{m}' = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(k)})$.

Proof. (Step 1) In the derangement form $\mathrm{df}_k(\mathfrak{a})$ of $\mathfrak{a}$, hide the positive letters; (Step 2) Then, apply the appropriate θ in Lemma 2.1; (Step 3) Show the letters hidden in (Step 1). We get the derangement form $\mathrm{df}_k(\mathfrak{b})$ of $\mathfrak{b}$. There are three types of descent values in $\mathrm{df}_k(\mathfrak{a})$ and $\mathrm{df}_k(\mathfrak{b})$: positive letter to positive letter, negative letter to negative letter, positive letter to negative letter. Checking each type of descent values, we conclude that $\mathrm{DES}(\mathrm{df}_k(\mathfrak{a})) = \mathrm{DES}(\mathrm{df}_k(\mathfrak{b}))$. □

Lemma 2.3. *Let $n, k \geq 1$ and $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ be an array of nonnegative integers. For any permutation $\tau \in \mathfrak{S}_{k-1}$, there exists a bijection $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m}) \mapsto \mathfrak{b} \in \mathsf{A}_n^k(\mathbf{m}')$ such that*

$$(\mathrm{DES}, \mathrm{DEZ}, \mathrm{Der})\ \mathfrak{a} = (\mathrm{DES}, \mathrm{DEZ}, \mathrm{Der})\ \mathfrak{b}. \tag{2.2}$$

where $\mathbf{m}' = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(k-1)}, m_k)$.

Proof. Similar to the proof of Lemma 2.2, we construct $\mathfrak{b}$ and verify that $\mathrm{DES}(\mathrm{df}_k(\mathfrak{a})) = \mathrm{DES}(\mathrm{df}_k(\mathfrak{b}))$. Since all letters $\bar{k}$ are not changed in (Step 2), we also have $\mathrm{DES}(\mathrm{pf}_k(\mathfrak{a})) = \mathrm{DES}(\mathrm{pf}_k(\mathfrak{b}))$. □

Proof of Theorem 1.1. By Lemma 2.2, it suffices to prove the theorem for a special permutation $\tau \in \mathfrak{S}_k$. We then prove the theorem for $\tau = \tau(1)\tau(2)\cdots\tau(k-1)\tau(k) = 23\cdots k1 \in \mathfrak{S}_k$. The bijection $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m}) \mapsto \Phi_k(\mathfrak{a}) \in \mathsf{A}_n^k(\mathbf{m}')$ is constructed in the following way.

Step 1. Derive the derangement form $S_1 = \mathrm{df}_k(\mathfrak{a}) \in \mathsf{D}_n^k$ of $\mathfrak{a}$;
Step 2. From S_1, replace each $-j$ by $-j+1$; hide all negative letter;
Step 3. Apply the bijection Φ described in [11, Section 2] and obtain $S_3 = \Phi(S_2)$;
Step 4. From S_3, show the letters hidden in Step 2; replace 0 by $\bar{k}$. We get $S_4 \in \mathsf{D}_n^k$;
Step 5. Finally let $\Phi_k(\mathfrak{a}) = \mathrm{df}_k^{-1}(S_4)$. For convenience, we write $S_5 = \mathrm{pf}_k(\Phi_k(\mathfrak{a})) \in \mathsf{P}_n^k$.

The following example illustrates our construction by showing the result of each step.

$$
\begin{array}{lcccccccccccccccccl}
S_1 & = & 5 & \bar{1} & 1 & \bar{2} & 2 & \bar{3} & \bar{1} & \bar{3} & \bar{1} & 3 & 6 & \bar{2} & \bar{1} & 7 & 4 & \in \mathsf{D}_{15}^3 \\
S_2 & = & 5 & 0 & 1 & & 2 & & 0 & & 0 & 3 & 6 & & 0 & 7 & 4 & \\
S_3 & = & 5 & 1 & 0 & & 0 & & 2 & & 3 & 0 & 6 & & 0 & 7 & 4 & \\
S_4 & = & 5 & 1 & \bar{3} & \bar{1} & \bar{3} & \bar{2} & 2 & \bar{2} & 3 & \bar{3} & 6 & \bar{1} & \bar{3} & 7 & 4 & \in \mathsf{D}_{15}^3 \\
S_5 & = & 8 & 1 & 3 & \bar{1} & 4 & \bar{2} & 2 & \bar{2} & 5 & 7 & 10 & \bar{1} & 9 & 11 & 6 & \in \mathsf{P}_{15}^3
\end{array}
$$

With the bijection Φ_k constructed above, we verify easily

$$\mathrm{Der}\ (\mathfrak{a}) = \mathrm{Der}\ (\Phi_k(\mathfrak{a})),$$

and

$$(\mathsf{fix}_1, \mathsf{fix}_2, \mathsf{fix}_3, \ldots, \mathsf{fix}_k)\ \mathfrak{a} = (\mathsf{fix}_k, \mathsf{fix}_1, \mathsf{fix}_2, \ldots, \mathsf{fix}_{k-1})\ \Phi_k(\mathfrak{a}). \tag{2.3}$$

Moreover, by the construction and properties of Φ (see [11, Thm. 1.1]), we have

$$\mathrm{DEZ}(\mathfrak{a}) = \mathrm{DES}(S_1) = \mathrm{DES}(S_5) = \mathrm{DES}(\Phi_k(\mathfrak{a})).$$

In the above example, one can check that $\mathrm{DEZ}(\mathfrak{a}) = \{1, 3, 5, 7, 11, 14\} = \mathrm{DES}(\Phi_k(\mathfrak{a}))$. This proves Theorem 1.1. $\square$

We need some definitions to facilitate our construction of Ψ_k. Let us denote by $\mathrm{Der}_k(\mathfrak{a})$ the word obtained from $\mathrm{df}_k(\mathfrak{a})$ by removing all letters $\bar{k}$, called the *weak derangement part* of $\mathfrak{a}$, which is extremely important in our construction. Note that $\mathrm{Der}_k(\mathfrak{a})$ can be viewed as the derangement form of certain k-arrangement itself, hence $(\pi, \phi) = \mathrm{df}_k^{-1}(\mathrm{Der}_k(\mathfrak{a}))$ is well-defined with $\phi^{-1}(\bar{k}) = \emptyset$. For each permutation $\sigma = \sigma(1) \cdots \sigma(n) \in \mathfrak{S}_n$, we define its *excedance word* $\mathbf{e}(\sigma)$ to be the word made from two letters E and N, standing for *excedance* and *nonexcedance*, respectively. More precisely, we let

$$\mathbf{e}(\sigma) = e_1 e_2 \cdots e_n, \quad \text{where } e_i := \begin{cases} E & \text{if } \sigma(i) > i, \\ N & \text{if } \sigma(i) \le i, \end{cases} \quad \text{for } 1 \le i \le n.$$

The excedance word for $\mathrm{Der}_k(\mathfrak{a})$, is understood to be the excedance word for the base permutation of $\mathrm{df}_k^{-1}(\mathrm{Der}_k(\mathfrak{a}))$. A moment of reflection should reveal the following observation, which shows that inserting letters $\bar{k}$ back into $\mathrm{Der}_k(\mathfrak{a})$ does not change the excedance type of those letters contained in $\mathrm{Der}_k(\mathfrak{a})$.

Observation 2.4. *For all $n, k \ge 1$ and each $\mathfrak{a} = (\pi, \phi) \in \mathsf{A}_n^k$, suppose $\mathrm{Der}_k(\mathfrak{a}) := w = w_1 \cdots w_m \in \mathbb{Z}^m$ for some $m \le n$, and w_j is reduced from $\pi(i_j)$ for every $1 \le j \le m$. If $\mathbf{e}(\pi) = e_1 e_2 \cdots e_n$ and $\mathbf{e}(w) = e'_1 e'_2 \cdots e'_m$ are the excedance words for π and w respectively, then $e_{i_j} = e'_j$ for every $1 \le j \le m$.*

For the 3-arrangement $\mathfrak{a} = (\pi, \phi)$ given in the introduction, namely, $\pi = 75\mathbf{34}1\mathbf{6}2$ and $\phi(3) = \bar{1}$, $\phi(4) = \bar{3}$ and $\phi(6) = \bar{3}$, we see $\mathrm{Der}_3(\mathfrak{a}) = 43\bar{1}12$. Therefore $\mathbf{e}(\pi) = EENNNNN$ and $\mathbf{e}(\mathrm{Der}_3(\mathfrak{a})) = EENNN$, which agrees with the observation above.

Suppose $\mathfrak{a} = (\pi, \phi) \in \mathsf{A}_n^k$ with $\mathsf{fix}_k(\mathfrak{a}) = n - m$, $\mathrm{Der}_k(\mathfrak{a}) := w = w_1 \cdots w_m \in \mathbb{Z}^m$, and $\mathbf{e}(w) = e_1 \cdots e_m$ for some $1 \le m \le n$. Now $\mathrm{df}_k(\mathfrak{a})$ can be decomposed as

$$\mathrm{df}_k(\mathfrak{a}) = S_0\, w_1\, S_1\, w_2\, \cdots\, S_{m-1}\, w_m\, S_m,$$

where S_i, $0 \le i \le m$ is a (possibly empty) block of letters $\bar{k}$, referred to as the *i-th slot* of $\mathrm{df}_k(\mathfrak{a})$. Define the *slot length vector* of $\mathfrak{a}$ as $\mathbf{s}(\mathfrak{a}) := (s_0, s_1, \ldots, s_m)$, where $s_i = |S_i|$ for $0 \le i \le m$. Note that $n - m = \sum_{i=0}^{m} s_i$ and the pair $(\mathbf{s}(\mathfrak{a}), \mathrm{Der}_k(\mathfrak{a}))$ uniquely determines $\mathfrak{a}$ and vice versa.

Next, we set $w_0 = w_{m+1} = +\infty$, $e_0 = e_{m+1} = E$, and classify S_i into the following mutually exclusive types, according to the values of w_i, w_{i+1}, and the pair (e_i, e_{i+1}):

- type I: $w_i > w_{i+1}$ and $(e_i, e_{i+1}) = (E, N)$;
- type II: $w_i \le w_{i+1}$ and $(e_i, e_{i+1}) = (N, E)$;
- type III: $w_i \le w_{i+1}$ and $(e_i, e_{i+1}) \ne (N, E)$;
- type IV: $w_i > w_{i+1}$ and $(e_i, e_{i+1}) \ne (E, N)$.

The four types above clearly cover all the possibilities for the slot S_i, and by Observation 2.4, the type of S_i only depends on $\mathrm{Der}_k(w)$ and has nothing to do with s_i. We use $t_1(\mathfrak{a})$ (resp. $t_2(\mathfrak{a})$, $t_3(\mathfrak{a})$ and $t_4(\mathfrak{a})$) to denote the number of slots (possibly empty) of type I (resp. type II, type III and type IV) in $\mathrm{df}_k(\mathfrak{a})$, while the numbers of the non-empty ones are denoted as $t_1^+(\mathfrak{a})$, $t_2^+(\mathfrak{a})$, $t_3^+(\mathfrak{a})$ and $t_4^+(\mathfrak{a})$ respectively. We can easily verify that

$$\begin{aligned}
&t_1(\mathfrak{a}) + t_4(\mathfrak{a}) = \mathsf{des}(\mathrm{Der}_k(\mathfrak{a})) + 1,\\
&t_1(\mathfrak{a}) + t_2(\mathfrak{a}) + t_3(\mathfrak{a}) + t_4(\mathfrak{a}) = m + 1, \text{ and}\\
&t_1^+(\mathfrak{a}) + t_2^+(\mathfrak{a}) + t_3^+(\mathfrak{a}) + t_4^+(\mathfrak{a}) \le n - m.
\end{aligned}$$

The following lemma is the key to motivate our definition of Ψ_k.

Lemma 2.5. *For every $\mathfrak{a} \in \mathsf{A}_n^k$ with a non-empty $\mathrm{Der}_k(\mathfrak{a})$, we have*

1) $t_1(\mathfrak{a}) = t_2(\mathfrak{a})$ *and slots of type I and type II appear alternatingly in $\mathrm{df}_k(\mathfrak{a})$, starting with a block of type I.*
2) *the following relationships hold*

$$\mathsf{des}(\mathfrak{a}) = \mathsf{des}(\mathrm{Der}_k(\mathfrak{a})) + t_1^+(\mathfrak{a}) + t_3^+(\mathfrak{a}), \tag{2.4}$$

$$\mathsf{dez}(\mathfrak{a}) = \mathsf{des}(\mathrm{Der}_k(\mathfrak{a})) + t_2^+(\mathfrak{a}) + t_3^+(\mathfrak{a}). \tag{2.5}$$

Proof. Suppose $w = \mathrm{Der}_k(\mathfrak{a})$ as before. It is evident that

- A type I slot cannot precede another type I slot unless it precedes a type II slot first.
- A type II slot cannot precede another type II slot unless it precedes a type I slot first.
- A type IV slot S_i with $e_i = e_{i+1} = E$ cannot precede a type II slot unless it precedes a type I slot first.

Since we made the convention that $w_0 = w_{m+1} = +\infty$, and that the positive letters of w form a derangement word, we see S_m must be of type II, and S_0 is either of type IV with $w_0 > w_1 > 0$, $e_0 = e_1 = E$, or of type I when w_1 is negative. Hence by the discussion above, there exists at least one slot of type I among $S_0, S_1, \ldots, S_{m-1}$, and the claim in part 1) follows as well.

Next for part 2), when $\mathsf{fix}_k(\mathfrak{a}) = 0$ and $m = n$, we have $\mathrm{df}_k(\mathfrak{a}) = \mathrm{pf}_k(\mathfrak{a}) = \mathrm{Der}_k(\mathfrak{a})$, with all slots being empty, so $\mathsf{des}(\mathfrak{a}) = \mathsf{dez}(\mathfrak{a}) = \mathsf{des}(\mathrm{Der}_k(\mathfrak{a}))$. Otherwise, we can recover

$\mathrm{df}_k(\mathfrak{a})$ from $\mathrm{Der}_k(\mathfrak{a})$ by inserting $n-m$ copies of $\bar{k}$ into originally empty slots. Note that inserting j copies of $\bar{k}$ into certain slot S_i has the same effect on des (resp. dez) for each $1 \le j \le n-m$. Suppose S_i is an empty slot of type I, i.e., $w_i > w_{i+1}$ and $(e_i, e_{i+1}) = (E, N)$. Now if we insert j copies of $\bar{k}$ into it, transfering

$$w_1 \cdots w_i w_{i+1} \cdots w_m \quad \text{into} \quad w_1 \cdots w_i \underbrace{\bar{k}\bar{k}\cdots\bar{k}}_{j \text{ copies}} w_{i+1} \cdots w_m,$$

we see des increases by one while dez remains the same. This explains why we have the term $t_1^+(\mathfrak{a})$ in (2.4) but not in (2.5). Similar discussions of the other three types prove both (2.4) and (2.5). $\square$

Proof of Theorem 1.2. By Lemma 2.3, it suffices to prove the theorem for a special permutation $\tau \in \mathfrak{S}_{k-1}$. We then prove the theorem for the identity permutation $\tau = 12\cdots(k-1) \in \mathfrak{S}_{k-1}$. We proceed to construct a bijection $\mathfrak{a} \in \mathsf{A}_n^k(\mathbf{m}) \mapsto \Psi_k(\mathfrak{a}) \in \mathsf{A}_n^k(\mathbf{m})$ such that

$$(\mathsf{des}, \mathsf{dez}, \mathrm{Der}_k)\ \mathfrak{a} = (\mathsf{dez}, \mathsf{des}, \mathrm{Der}_k)\ \Psi_k(\mathfrak{a}), \tag{2.6}$$

which is more than we need, since $\mathrm{Der}(\mathfrak{a}) = \mathrm{Pos}(\mathrm{Der}_k(\mathfrak{a}))$. The only k-arrangement with empty weak derangement part is $\mathfrak{a} = (12\cdots n, \phi)$, where $\phi(i) = \bar{k}$ for all $1 \le i \le n$. In this case we let $\Psi_k(\mathfrak{a}) = \mathfrak{a}$ and see that (2.6) holds true. Otherwise, for a given k-arrangement $\mathfrak{a} \in \mathsf{A}_n^k$ with

$$\mathbf{s}(\mathfrak{a}) = (s_0, s_1, \ldots, s_m), \ \text{ and } \ \mathrm{Der}_k(\mathfrak{a}) \in \mathbb{Z}^m, \ m \ge 1.$$

The aforementioned map Ψ_k simply swaps the slots of types I and II in $\mathrm{df}_k(\mathfrak{a})$. Namely, by Lemma 2.5, we let $t := t_1(\mathfrak{a}) = t_2(\mathfrak{a})$, and let $S_{i_1}, S_{j_1}, \ldots, S_{i_t}, S_{j_t} = S_m$ be all of the types I and II blocks, appearing alternatingly. Then $\mathfrak{b} := \Psi_k(\mathfrak{a})$ is taken to be the unique k-arrangement corresponding to

$$\mathbf{s}(\mathfrak{b}) = (s_0', s_1', \ldots, s_m'), \ \text{ and } \ \mathrm{Der}_k(\mathfrak{b}) = \mathrm{Der}_k(\mathfrak{a}),$$

where

$$s_l' := \begin{cases} s_{j_r} & \text{if } l = i_r, \text{ for certain } 1 \le r \le t, \\ s_{i_r} & \text{if } l = j_r, \text{ for certain } 1 \le r \le t, \\ s_l & \text{otherwise.} \end{cases}$$

The swapping map Ψ_k defined in this way preserves the sum $\sum s_l = \sum s_l' = m_k$, hence it is an involution on $\mathsf{A}_n^k(\mathbf{m})$, and

$$(\mathsf{des}, \mathsf{dez})\ \mathfrak{a} = (\mathsf{dez}, \mathsf{des})\ \mathfrak{b}$$

follows from equations (2.4) and (2.5) immediately. The proof is now completed. $\square$

Example 2.6 (An example of Ψ_1)**.** For $k = 1$, $\mathsf{P}_n^1 = \mathfrak{S}_n$. Let

$$\pi = \mathbf{1}\ \mathbf{2}\ 5\ 3\ 9\ \mathbf{6}\ 4\ \mathbf{8}\ 16\ 11\ 7\ \mathbf{12}\ \mathbf{13}\ 10\ \mathbf{15}\ 14\ \mathbf{17} \in \mathfrak{S}_{17}$$

and $\mathfrak{a} := \mathrm{pf}_1^{-1}(\pi)$, then we see

$$\mathbf{s}(\mathfrak{a}) = (2,0,0,1,1,0,0,2,1,1) \quad \text{and} \quad \mathrm{Der}(\mathfrak{a}) = 3\ 1\ 5\ 2\ 9\ 7\ 4\ 6\ 8.$$

The type I (resp. type II) slots are S_1, S_3, S_6 (resp. S_2, S_4, S_9). So we have

$$\mathsf{s}(\Psi(\mathfrak{a})) = (2,0,0,1,1,0,1,2,1,0) \quad \text{and} \quad \mathrm{Der}(\Psi(\mathfrak{a})) = 3\ 1\ 5\ 2\ 9\ 7\ 4\ 6\ 8,$$

which gives us $\mathrm{df}_1(\Psi(\mathfrak{a})) = \bar{1}\ \bar{1}\ 3\ 1\ 5\ \bar{1}\ 2\ \bar{1}\ 9\ 7\ \bar{1}\ 4\ \bar{1}\ \bar{1}\ 6\ \bar{1}\ 8$. One can verify that indeed

$$(\mathsf{des}, \mathsf{dez}, \mathrm{Der})\ \mathfrak{a} = (7, 8, 315297468) = (\mathsf{dez}, \mathsf{des}, \mathrm{Der})\ \Psi(\mathfrak{a}).$$

Example 2.7 (An example of Ψ_2)**.** For $k = 2$, let $\mathfrak{b} = (\sigma, \phi)$ with $\sigma = 1\ 2\ 9\ 3\ 5\ 6\ 4\ 8\ 7$ and $\phi(1) = \phi(8) = \bar{1}, \phi(2) = \phi(5) = \phi(6) = \bar{2}$, then we see

$$\mathsf{s}(\mathfrak{b}) = (0,1,0,2,0,0,0,0) \quad \text{and} \quad \mathrm{Der}_2(\mathfrak{b}) = \bar{1}\ 4\ 1\ 2\ \bar{1}\ 3.$$

The type I (resp. type II) slots are S_0, S_2 (resp. S_1, S_6). So we have

$$\mathsf{s}(\Psi(\mathfrak{b})) = (1,0,0,2,0,0,0) \quad \text{and} \quad \mathrm{Der}_2(\Psi(\mathfrak{b})) = \bar{1}\ 4\ 1\ 2\ \bar{1}\ 3,$$

which gives us $\mathrm{df}_2(\Psi(\mathfrak{b})) = \bar{2}\ \bar{1}\ 4\ 1\ \bar{2}\ \bar{2}\ 2\ \bar{1}\ 3$. One checks to see

$$(\mathsf{des}, \mathsf{dez}, \mathrm{Der}_2)\ \mathfrak{b} = (3, 4, \bar{1}\ 4\ 1\ 2\ \bar{1}\ 3) = (\mathsf{dez}, \mathsf{des}, \mathrm{Der}_2)\ \Psi(\mathfrak{b}).$$

3. Bivariate joint generating function for des and dez

In section 2, we have established that the distribution of the two Eulerian statistics des and dez are symmetric over the permutation group. This section is devoted to the derivation of the bivariate joint generating function for those two statistics. Note that we will abuse the notation to write $\mathsf{dez}(\pi) := \mathsf{dez}(\mathrm{pf}_1^{-1}(\pi))$, for any permutation π. For instance, $\mathsf{dez}(41352) = \mathsf{des}(41\bar{1}52) = 3$.

A descent (position) i of a permutation π is called a *crossing descent*, if $\pi_i \geq i+1 \geq \pi_{i+1}$. Denote by $\mathsf{xdes}(\pi)$ the number of crossing descents of π. When restricted to the set of derangements, xdes is exactly the statistic t_1 that we introduce in section 2. Let

$$F(t, s; u) := 1 + \sum_{n\geq 2} \Big(\sum_{\pi \in \mathsf{D}_n} t^{\mathsf{des}(\pi)} s^{\mathsf{xdes}(\pi)} \Big) u^n, \tag{3.1}$$

$$G(x, y; u) := 1 + \sum_{n\geq 1} \Big(\sum_{\pi \in \mathfrak{S}_n} x^{\mathsf{des}(\pi)} y^{\mathsf{dez}(\pi)} \Big) u^n, \tag{3.2}$$

be the generating functions of $(\mathsf{des}, \mathsf{xdes})$ over the derangement set and $(\mathsf{des}, \mathsf{dez})$ over the permutation set respectively. The initial values of $F(t, s; u)$ and $G(x, y; u)$ are given below:

$$\begin{aligned} F(t, s; u) &= 1 + stu^2 + 2stu^3 + (st^3 + 2s^2t^2 + 2st^2 + 4st)u^4 + \cdots, \\ G(x, y; u) &= 1 + u + (xy+1)u^2 + (x^2y + xy^2 + 3xy + 1)u^3 \\ &\quad + (x^3y^3 + 7x^2y^2 + 4x^2y + 4xy^2 + 7xy + 1)u^4 + \cdots. \end{aligned}$$

Theorem 3.1. *For each nonnegative integer m the coefficient of t^m in $F(t, s; u)$ defined by* (3.1) *is a rational fraction in s and u.*

Theorem 3.1 is a consequence of Theorem 3.5 in view of (3.25).

Theorem 3.2. *We have*

$$G(x, y; u) = \frac{1}{1-u} \times F\Big(\frac{xy}{1-u+xyu}, \frac{(1-u+xu)(1-u+yu)}{1-u+xyu}; u(1 + \frac{xyu}{1-u}) \Big). \tag{3.3}$$

Remark 3.3. Depending on $F(t,s;u)$, Theorem 3.2 is not explicit. However, we can still see that $G(x,y;u)$ is a symmetric function in x and y from (3.3).

Proof. By the definition of $G(x,y;u)$ in (3.2), we have

$$G(x,y;u) = 1 + \sum_{n\geq 1}\Big(\sum_{\mathfrak{a}\in \mathsf{A}_n^1} x^{\mathsf{des}(\mathfrak{a})}y^{\mathsf{dez}(\mathfrak{a})}\Big)u^n = \frac{1}{1-u} + \sum_{n\geq 2}\Big(\sum_{\mathfrak{a}\in \tilde{\mathsf{A}}_n^1} x^{\mathsf{des}(\mathfrak{a})}y^{\mathsf{dez}(\mathfrak{a})}\Big)u^n, \tag{3.4}$$

where $\tilde{\mathsf{A}}_n^1 := \{\mathfrak{a} \in \mathsf{A}_n^1 : \mathrm{Der}(\mathfrak{a}) \neq \emptyset\}$. Recall from the last section that any 1-arrangement $\mathfrak{a} \in \tilde{\mathsf{A}}_n^1$ with weak derangement part $\pi = \mathrm{Der}(\mathfrak{a}) \in \mathsf{D}_m$ (for some $m \geq 2$) has the decomposition

$$\mathrm{df}_1(\mathfrak{a}) = B_0\pi_1B_1\pi_2\cdots B_{m-1}\pi_mB_m, \tag{3.5}$$

where each B_i (possibly empty) is a block with consecutive copies of $\bar{1}$. We also introduce four types of blocks for $\mathfrak{a}$, which are essentially four types of slots of the underlying derangement π. Note that the first slot of π must be of type IV and introduce the type generating function

$$\tilde{F}(x,a,b,y;u) := 1 + \sum_{n\geq 2}\Big(\sum_{\pi\in\mathsf{D}_n} x^{t_1(\pi)}a^{t_2(\pi)}b^{t_3(\pi)}y^{t_4(\pi)-1}\Big)u^n.$$

We aim to connect $\tilde{F}$ with both F and G, so as to establish (3.3). On the one hand, point 1) of Lemma 2.5 and the discussion preceding it give us

$$t_1(\pi) = t_2(\pi) \quad \text{and} \quad t_1(\pi)+t_2(\pi)+t_3(\pi)+t_4(\pi)-1 = m$$

for any $\pi \in \mathsf{D}_m$. Since $\mathsf{xdes}(\pi) = t_1(\pi)$ and $\mathsf{des}(\pi) = t_1(\pi)+t_4(\pi)-1$, we see

$$\tilde{F}(x,a,b,y;u) - 1 = \sum_{n\geq 2}(bu)^n \sum_{\pi\in\mathsf{D}_n}\Big(\frac{xa}{b^2}\Big)^{t_1(\pi)}\Big(\frac{y}{b}\Big)^{t_4(\pi)-1} = F\Big(\frac{y}{b},\frac{xa}{by};bu\Big) - 1. \tag{3.6}$$

On the other hand, invoking the interpretation (3.4) of $G(x,y;u)$, the decomposition (3.5) and relationships (2.4) and (2.5) give rise to the appropriate substitutions for variables x, a, b and y in F' to arrive at

$$G(x,y;u) = \frac{1}{1-u} \times \tilde{F}\Big(xy(1+\frac{xu}{1-u}), 1+\frac{yu}{1-u}, 1+\frac{xyu}{1-u}, \frac{xy}{1-u}; u\Big), \tag{3.7}$$

where the factor $1/(1-u)$ accounts for the contribution from inserting the block B_0. Now combining (3.6) and (3.7) completes the proof. $\square$

It remains to evaluate the generating function $F(t,s;u)$. As it turns out, the following trivariant generalization of $F(t,s;u)$ is more appropriate for calculation:

$$H(t,s,r;u) := \sum_{n\geq 0}\Big(\sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}(\pi)}s^{\mathsf{xdes}(\pi)}r^{\mathsf{fix}(\pi)}\Big)\frac{u^n}{(1-t)^{n+1}}. \tag{3.8}$$

The reduction to $F(t,s;u)$ is seen to be

$$F(t,s;u) = (1-t)H(t,s,0;(1-t)u).$$

To investigate $H(t,s,r;u)$, we consider the set $\mathbf{PS}(u;X,Y,Z)$ of the formal power series

$$\xi \in \mathbb{Z}[s,r][X_0,X_1,X_2,\ldots,Y_0,Y_1,Y_2,\ldots,Z_0,Z_1,Z_2,\ldots][[u]],$$

where $u, s, r, X_0, X_1, X_2, \ldots, Y_0, Y_1, Y_2, \ldots, Z_0, Z_1, Z_2, \ldots$ are commuting variables, such that the coefficient of u^n in ξ is a polynomial in $X_0, X_1, \ldots, Y_0, Y_1, \ldots, Z_0, Z_1, \ldots$ of total degree less than or equal to n, with coefficients in $\mathbb{Z}[s,r]$. We introduce a *linear operator* ρ on $\mathbf{PS}(u; X, Y, Z)$.

Definition 3.4 (The operator ρ)**.** For each monomial M in $X_0, X_1, \ldots, Y_0, Y_1, \ldots, Z_0, Z_1, \ldots$, the index i is said to be *effective* in M if

i) M contains Y_i or Z_i, and
ii) M contains certain X_k with $k > i$ such that
iii) M contains neither Y_j nor Z_j, for each $i < j < k$.

For example, both 0 and 1 are effective in $X_1X_2Y_0Y_3Z_0Z_1$, while only 1 is effective in $X_3X_4Y_1Z_0^2$. Let $\mathrm{eff}(M)$ denote the number of effective indices in M. Now we can define the operator $\rho : \mathbf{PS}(u; X, Y, Z) \to \mathbb{Z}[s,r][[u]]$ by setting

$$\rho(M) = s^{\mathrm{eff}(M)}, \tag{3.9}$$

and extending linearly to all formal power series in $\mathbf{PS}(u; X, Y, Z)$. For the previous examples, we have $\rho(X_1X_2Y_0Y_3Z_0Z_1) = s^2$ and $\rho(X_3X_4Y_1Z_0^2) = s$.

The following theorem can be viewed as the central result of this section.

Theorem 3.5. *We have*

$$H(t,s,r;u) = \sum_{m\geq 0} t^m \rho(S_m(u)), \tag{3.10}$$

where

$$S_m(u) = \frac{\dfrac{\prod_{1\leq j\leq m}(1-uX_j)}{\prod_{0\leq j\leq m}(1-ruZ_j)}}{1 - \displaystyle\sum_{1\leq l\leq m} \frac{uX_l\prod_{1\leq j\leq l-1}(1-uX_j)}{\prod_{0\leq j\leq l-1}(1-uY_j)}}. \tag{3.11}$$

Our strategy to prove Theorem 3.5 is as follows. First off, we utilize an updated version of the Gessel–Reutenauer standardization [12,13,15], denoted as Φ_{GR}, to map each word w from $[0,m]^n$ onto a pair (σ, c), where $\sigma \in \mathfrak{S}_n$ and $c = c_1c_2\cdots c_n$ is a word whose letters are nonnegative integers satisfying: $m - \mathsf{des}(\sigma) \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. This bijection entitles us to rewrite $H(t,s,r;u)$ as a weighted (each w weighted by $\psi(w)$, see Definition 3.6) generating function over all words w in $[0,m]^n$, after we make a key combinatorial observation (see Lemma 3.9) to connect the statistic xdes on a permutation to the statistic eff on the weight of the corresponding word. Secondly, Theorem 1.3 in [13] enables us to evaluate this weighted generating function, with the help of the operator ρ, to be the right-hand side of (3.10).

To begin the first step, we make some definitions and recall the Gessel–Reutenauer bijection. For $n, m \geq 0$, consider the set $\mathcal{W}_n(m) := [0,m]^n$ of all words of length n and alphabet being $[0,m] := \{0, 1, \ldots, m\}$. Denote the subset of ***non-increasing words*** as

$$\mathrm{NIW}_n(m) := \{c = c_1c_2\cdots c_n \in \mathcal{W}_n(m) : c_1 \geq c_2 \geq \cdots \geq c_n\}.$$

We use the lexicographic order "$>$" to compare words in $\mathcal{W}_n(r)$. This total order extends to words with different length (but same alphabet) naturally. Namely, let $u \in \mathcal{W}_n(m)$ and $v \in \mathcal{W}_l(m)$ be two nonempty primitive words (none of them can be expressed as w^b for some word w and integer $b \geq 2$), we write $u \succeq v$, if and only if $u^b \geq v^b$ when b is large enough. Here the multiplication is understood to be the concatenation of words.

Let $w = x_1x_2\cdots x_n$ be an arbitrary word over $\mathbb{Z}$ and set $x_{n+1} = +\infty$. For each $1 \leq i \leq n$, we say that i is a *decrease* (position) of w if

$$x_i = x_{i+1} = \cdots = x_j > x_{j+1}, \text{ for some } i \leq j \leq n.$$

So descent is the case of $i = j$. If on the contrary we have

$$x_i = x_{i+1} = \cdots = x_j < x_{j+1}, \text{ for some } i \leq j \leq n,$$

then we say that i is an *increase* (position) of w, and an *ascent* (position) of w if $i = j$. By our convention $x_{n+1} = +\infty$, thus n is always an ascent. Furthermore, a position i ($1 \leq i \leq n$) is said to be a *record* if

$$x_j \leq x_i, \text{ for all } 1 \leq j \leq i-1.$$

When the index i is a decrease (resp. increase, record) of w, the corresponding letter x_i is said to be a *decrease* (resp. *increase, record*) *value* of w. The set of all decreases (resp. increases, ascents, records) is denoted by $\textsc{dec}(w)$ (resp. $\textsc{inc}(w)$, $\textsc{asc}(w)$, $\textsc{rec}(w)$). In particular, a descent (resp. ascent) is always a decrease (resp. increase), thus $\textsc{des}(w) \subseteq \textsc{dec}(w)$ (resp. $\textsc{asc}(w) \subseteq \textsc{inc}(w)$). Now we can define the aforementioned weight ψ as was first introduced in [13].

Definition 3.6. Take six sequences of commuting variables $(X_i), (Y_i), (Z_i), (T_i), (Y_i')$ and (T_i') $(i = 0, 1, 2, \ldots)$, and for each word $w \in \mathcal{W}_n(m)$ define the weight $\psi(w)$ of $w = x_1x_2\cdots x_n$ to be

$$\begin{aligned}\psi(w) := {} & \prod_{i \in \mathrm{DES}} X_{x_i} \prod_{i \in \mathrm{ASC} \setminus \mathrm{REC}} Y_{x_i} \prod_{i \in \mathrm{DEC} \setminus \mathrm{DES}} Z_{x_i} \\ & \times \prod_{i \in (\mathrm{INC} \setminus \mathrm{ASC}) \setminus \mathrm{REC}} T_{x_i} \prod_{i \in \mathrm{ASC} \cap \mathrm{REC}} Y'_{x_i} \prod_{i \in (\mathrm{INC} \setminus \mathrm{ASC}) \cap \mathrm{REC}} T'_{x_i},\end{aligned} \tag{3.12}$$

where the argument "(w)" has been suppressed for typographic reasons. For example, if $w = 1\ 2\ \underline{8}\ 0\ \underline{8}\ 2\ 10\ \underline{13}\ 4\ 8\ \underline{13}\ \underline{11}\ \underline{11}\ 2\ 5\ 5\ \underline{11}\ \underline{6}\ \underline{3}\ 0$ with decrease values underlined, then

$$\psi(w) = Y_1'Y_2'X_8Y_0X_8Y_2Y_{10}'X_{13}Y_4Y_8X_{13}Z_{11}X_{11}Y_2T_5Y_5X_{11}X_6X_3Y_0. \tag{3.13}$$

For the sake of convenience, we review the Lyndon factorization of words.

Definition 3.7 (Lyndon factorization)**.** A word $l = x_1x_2\cdots x_n \in \mathcal{W}_n(m)$ is said to be a *Lyndon word* [12, 21], if either $n = 1$, or if $n \geq 2$ and $x_1x_2\cdots x_n > x_ix_{i+1}\cdots x_nx_1\cdots x_{i-1}$ holds for every i such that $2 \leq i \leq n$. As shown for instance in [21, Theorem 5.1.5], each nonempty word w composed of nonnegative integers, can be written uniquely as a product $w = l_1l_2\cdots l_k$, where each l_i is a Lyndon word and $l_1 \preceq l_2 \preceq \cdots \preceq l_k$. This word factorization is called *Lyndon factorization*. For instance, we have the Lyndon factorization

$$w = 1\,2\,1\,0\,0\,2\,2\,4\,5\,3\,1\,0\,2\,1\,2\,5 = 1\,|\,2\,1\,0\,0\,|\,2\,|\,2\,|\,4\,|\,5\,3\,1\,0\,2\,1\,2\,|\,5,$$

where factors are separated by vertical bars.

Finally, we recall the construction of the inverse $\Phi_{\mathrm{GR}}^{-1} : (\sigma, c) \mapsto w$ by means of one example. A description of this correspondence with more details can be found in Foata and the second author's previous paper [12].

$$
\begin{array}{rcllllllllllllllllllll}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\to \sigma & = & \underline{3} & \underline{13} & \underline{5} & \underline{10} & \underline{16} & \mathbf{6} & 2 & \underline{15} & \underline{20} & \underline{14} & 4 & 11 & 7 & \underline{19} & 8 & 12 & \mathbf{17} & \mathbf{18} & 1 & 9 \\
z & = & 8 & 8 & 7 & 7 & 7 & 6 & 5 & 5 & 5 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\to c & = & 5 & 5 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\bar{c} & = & 13 & 13 & 11 & 11 & 11 & 10 & 8 & 8 & 8 & 6 & 5 & 5 & 4 & 3 & 2 & 2 & 2 & 1 & 0 & 0 \\
\sigma & = & (\mathbf{18}) & (\mathbf{17}) & (9 & 20) & (8 & 15) & (\mathbf{6}) & (2 & 13 & 7) & (1 & 3 & 5 & 16 & 12 & 11 & 4 & 10 & 14 & 19) \\
\check{\sigma} & = & \mathbf{18} & \mathbf{17} & 9 & 20 & 8 & 15 & \mathbf{6} & 2 & 13 & 7 & 1 & 3 & 5 & 16 & 12 & 11 & 4 & 10 & 14 & 19 \\
\mapsto w & = & 1\,| & 2\,| & \underline{8} & 0\,| & \underline{8} & 2\,| & 10\,| & \underline{13} & 4 & 8\,| & \underline{13} & \underline{11} & \underline{11} & 2 & 5 & 5 & \underline{11} & \underline{6} & \underline{3} & 0
\end{array}
$$

In above example $n = 20$. The second row contains the values $\sigma(i)$ $(i = 1, 2, \ldots, n)$ of the *starting* permutation σ. The fixed points in σ are written in boldface, while the *excedances* $\sigma(i) > i$ are underlined. The third row is the vector $z = z_1 z_2 \cdots z_n$ defined as

$$z_i := |\{j : i \le j \le n-1,\ \sigma(j) > \sigma(j+1)\}|, \text{ for } 1 \le i \le n, \tag{3.14}$$

so that $z_1 = \mathsf{des}(\sigma)$. The fourth row is the *starting* nonincreasing word $c = c_1 c_2 \cdots c_n$. The fifth row $\bar{c} = \bar{c}_1 \bar{c}_2 \cdots \bar{c}_n$ is the word defined by

$$\bar{c}_i := z_i + c_i, \text{ for } 1 \le i \le n.$$

The sixth row is again the permutation σ but now in its *cycle notation*, with the minima leading each cycle and cycles listed with their first letters decreasing. When removing the parentheses in the sixth row we arrive at the seventh row denoted as $\check{\sigma} = \check{\sigma}(1)\check{\sigma}(2)\cdots\check{\sigma}(n)$. The bottom row is the word $w = x_1 x_2 \cdots x_n$ corresponding to the pair (σ, c) defined by

$$x_i := \bar{c}_{\check{\sigma}(i)}, \text{ for } 1 \le i \le n. \tag{3.15}$$

The underlined letters in w are decrease values of w. Finally, the vertical bars inserted into w indicate its Lyndon factorization.

It is known (cf. [12, 13]) that all the above steps are reversible and $\Phi_{\mathrm{GR}} : w \mapsto (\sigma, c)$ is indeed a bijection, essentially due to Gessel and Reutenauer [15], from $\mathcal{W}_n(m)$ onto the set of pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\mathsf{des}(\sigma) \le m$ and $c \in \mathrm{NIW}_n(m - \mathsf{des}(\sigma))$. The following observation was made in [13].

Observation 3.8. *Suppose $\Phi_{\mathrm{GR}}(w) = (\sigma, c)$, then we have*

(i) *$i \in \mathrm{DEC}(w)$ if and only if $\check{\sigma}(i) < \check{\sigma}(i+1)$, i.e., $\check{\sigma}(i)$ is an excedance of σ.*
(ii) *$i \in \mathrm{INC}(w) \cap \mathrm{REC}(w)$ if and only if $\check{\sigma}(i) \in \mathrm{FIX}(\sigma)$.*

Guided by Observation 3.8, we let γ be the homomorphism defined by the following substitutions of variables:

$$\gamma := \{X_j \leftarrow uX_j,\ Z_j \leftarrow uX_j,\ Y_j \leftarrow uY_j,\ T_j \leftarrow uY_j,\ Y'_j \leftarrow ruZ_j,\ T'_j \leftarrow ruZ_j\}. \tag{3.16}$$

The following feature of Φ_{GR} regarding crossing descents is key to our calculation.

Lemma 3.9. *Suppose $\Phi_{\mathrm{GR}}(w) = (\sigma, c)$ and $i \in \mathrm{DES}(\sigma)$, then i is a crossing descent of σ, if and only if the index $\bar{c}_{i+1}$ is effective in $\gamma\psi(w)$.*

Take w as in the running example of Φ_{GR}^{-1} above. By the calculation in (3.13), we have

$$\gamma\psi(w) = r^3u^{20}X_3X_6X_8^2X_{11}^3X_{13}^2Y_0^2Y_2^2Y_4Y_5^2Y_8Z_1Z_2Z_{10},$$

and so the effective indices in $\gamma\psi(w)$ are 2, 5 and 10, which correspond respectively to the crossing descents 14, 10 and 5 of the permutation σ.

Proof of Lemma 3.9. First we show the "only if" part. Suppose i is a crossing descent of σ, i.e., $\sigma(i) \geq i+1 \geq \sigma(i+1)$. Note that i is a descent of σ so $z_i = z_{i+1}+1$ hence $\bar{c}_i > \bar{c}_{i+1}$. In view of Observation 3.8 (i), we have

- $i+1 \geq \sigma(i+1)$ means that $i+1 = \check{\sigma}(\check{\sigma}^{-1}(i+1))$ is not an excedance of σ, which implies that either $Y_{\bar{c}_{i+1}}$ or $Z_{\bar{c}_{i+1}}$ appears in $\gamma\psi(w)$, and
- $\sigma(i) > i$ means that $i = \check{\sigma}(\check{\sigma}^{-1}(i))$ is an excedance of σ, which indicates that $X_{\bar{c}_i}$ appears in $\gamma\psi(w)$.

By definition this means that $\bar{c}_{i+1}$ is effective in $\gamma\psi(w)$.

It remains to show the "if" part. Conversely, suppose that certain $a = \bar{c}_{i+1}$, with $i \in \mathrm{DES}(\sigma)$, is effective in $\gamma\psi(w)$. Then, we can find indices, $j \geq 1$ and $k \geq 0$ such that $b = \bar{c}_i > a = \bar{c}_{i+1} = \bar{c}_{i+2} = \cdots = \bar{c}_{i+j} > \bar{c}_{i+j+1}$ and $\bar{c}_{i-k-1} > \bar{c}_{i-k} = \bar{c}_{i-k+1} = \cdots = \bar{c}_i = b$. We aim to show that i is a crossing descent of σ. Since a is effective in $\gamma\psi(w)$, at least one of Y_a and Z_a appears in $\gamma\psi(w)$, which implies that one of $i+\ell$, $1 \leq \ell \leq j$, must be a non-excedance of σ. This forces $i+1$ to be a non-excedance, as $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+\ell)$. On the other hand, we must have X_b appear in $\gamma\psi(w)$, which implies one of $i-\ell'$, $0 \leq \ell' \leq k$, must be an excedance of σ. This forces i to be an excedance of σ, as $\sigma(i-\ell') < \sigma(i-\ell'+1) < \cdots < \sigma(i)$. In conclusion, i is a crossing descent of σ, as desired. □

We will also make use of the following version of the so-called "Decrease Value Theorem".

Theorem 3.10 (Theorem 1.3 in [13])**.** *We have:*

$$\sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(m)}\psi(w) = \frac{\dfrac{\prod\limits_{1\leq j\leq m}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\leq j\leq m}\dfrac{1-T'_j}{1-T'_j+Y'_j}}}{1-\sum\limits_{1\leq l\leq m}\dfrac{\prod\limits_{1\leq j\leq l-1}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\leq j\leq l-1}\dfrac{1-T_j}{1-T_j+Y_j}}\dfrac{X_l}{1-Z_l+X_l}}. \tag{3.17}$$

We are in a position to prove Theorem 3.5.

Proof of Theorem 3.5. For each word $w = x_1x_2\cdots x_n \in \mathcal{W}_n(m)$, we have:

$$\gamma\psi(w) = u^n\prod_{i\in\mathrm{DEC}}X_{x_i}\prod_{i\in\mathrm{INC}\setminus\mathrm{REC}}Y_{x_i}\prod_{i\in\mathrm{INC}\cap\mathrm{REC}}rZ_{x_i}. \tag{3.18}$$

Applying γ to (3.17) we get:

$$\sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(m)} u^n \prod_{i\in\mathrm{DEC}} X_{x_i} \prod_{i\in\mathrm{INC}\setminus\mathrm{REC}} Y_{x_i} \prod_{i\in\mathrm{INC}\cap\mathrm{REC}} rZ_{x_i} = S_m(u). \tag{3.19}$$

By $\Psi_{\mathrm{GR}}: w\mapsto(\sigma,c)$ and Observation 3.8, the left-hand side of (3.19) is equal to

$$\sum_{n\geq 0} u^n \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathsf{des}(\sigma)\leq m}} \sum_{c\in\mathrm{NIW}_n(m-\mathsf{des}(\sigma))} r^{\mathsf{fix}(\sigma)} W_{(\sigma,c)}, \tag{3.20}$$

where

$$W_{(\sigma,c)} = \prod_{j<\sigma(j)} X_{c_j+z_j} \prod_{j>\sigma(j)} Y_{c_j+z_j} \prod_{j=\sigma(j)} Z_{c_j+z_j}.$$

With

$$W(\sigma;t) := \sum_{k\geq 0} t^k \sum_{c\in\mathrm{NIW}_n(k)} W_{(\sigma,c)},$$

the graded form of (3.20) reads

$$\sum_{n\geq 0} u^n \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{des}(\sigma)} r^{\mathsf{fix}(\sigma)} W(\sigma;t) = \sum_{m\geq 0} t^m S_m(u). \tag{3.21}$$

Now Lemma 3.9 says that i is a crossing descent of σ, if and only if $\bar{c}_{i+1}$ is effective in $W_{(\sigma,c)}$, therefore we have

$$\rho(W_{(\sigma,c)}) = s^{\mathrm{eff}(W_{(\sigma,c)})} = s^{\mathsf{xdes}(\sigma)}. \tag{3.22}$$

Consequently,

$$\rho(W(\sigma;t)) = s^{\mathsf{xdes}(\sigma)} \sum_{k\geq 0} t^k \times |\mathrm{NIW}_n(k)| = \frac{s^{\mathsf{xdes}(\sigma)}}{(1-t)^{n+1}}.$$

Applying ρ to both sides of (3.21) yields (3.10). $\square$

Although Theorem 3.5 is somewhat complicated, it allows us to derive some formulae for special cases, with the help of a computer algebra system. A trick to evaluate the fraction at the right-hand side of (3.10) by the operator ρ is that we can replace $x^k u^k$, for $x = X_j$, Y_j or Z_j, by xu^k for $k\geq 1$. That is

$$\rho\Big(\frac{1}{a+bxu}\Big) = \rho\Big(\frac{1}{a}(1-\frac{bxu}{a+bu})\Big). \tag{3.23}$$

Using this trick, we can derive the formulae for $m = 1, 2$ as follows.

(I) Special case $m = 1$. The term $\rho(S_1(u))$ is equal to

$$\begin{aligned}
&\rho\left(\frac{1-uX_1}{(1-ruZ_0)(1-ruZ_1)}\Big/(1-\frac{uX_1}{1-uY_0})\right)\\
&=\rho\left(\frac{(1-uX_1)(1-uY_0)}{(1-ruZ_0)(1-ruZ_1)(1-uX_1-uY_0)}\right)\\
&=\frac{1}{(1-ru)^2}\left(1+\rho(\frac{u^2X_1Y_0}{1-uX_1-uY_0})\right) = \frac{1}{(1-ru)^2}\Big(1+\frac{su^2}{1-2u}\Big).
\end{aligned}$$

We now give a combinatorial proof of the formula above. When we extract the coefficient of t^1 from

$$H(t,s,r;u)=\sum_{n\geq 0}\Big(\sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{des}(\sigma)}s^{\mathsf{xdes}(\sigma)}r^{\mathsf{fix}(\sigma)}\Big)\frac{u^n}{(1-t)^{n+1}},$$

there are two cases to be considered:

(1) The factor $1/(1-t)^{n+1}$ contributes $(n+1)t^1$ while the permutation satisfies $\mathsf{des}(\sigma)=0$. Then the permutation must be the identity and contributes $r^n u^n$. So we sum up over all $n\geq 0$ to get the term $1/(1-ru)^2$.

(2) The factor $1/(1-t)^{n+1}$ contributes $t^0=1$ while the permutation satisfies $\mathsf{des}(\sigma)=1$. Now note that any permutation $\sigma\in\mathfrak{S}_n$ that has exactly one descent (the so-called "Grassmannian permutation"), can be uniquely decomposed as

$$\sigma=12\cdots i\ \sigma_{i+1}\sigma_{i+2}\cdots\sigma_{i+j}\ (i+j+1)(i+j+2)\cdots n,$$

where $0\leq i\leq n-2$, $2\leq j\leq n-i$, and $\sigma_{i+1}\cdots\sigma_{i+j}$ contains no fixed points and exactly one descent at $i+j=\sigma_k>\sigma_{k+1}=i+1$ for certain k $(i+1\leq k<i+j)$, hence this descent is a crossing descent. All the other letters $i+2,i+3,\ldots,i+j-1$ can appear either to the left of $i+j$, or to the right of $i+1$, making 2^{j-2} choices in total. So we see the contributions from all such σ's are

$$\frac{s}{(1-ru)^2}\cdot\Big(\sum_{j\geq 2}2^{j-2}u^j\Big)=\frac{su^2}{(1-ru)^2(1-2u)},$$

which is precisely the remaining term in the formula above.

(II) Special case $m=2$. With the help of a computer algebra program, we obtain the term $\rho(S_2(u))$, which is equal to

$$\frac{1}{(1-ru)^3}\Big(1+\frac{((3-r)u^2-7u+3)u^2\cdot s}{(u^2-3u+1)(1-2u)}+\frac{(-3ru+2u^2+r-4u+2)u^4\cdot s^2}{(u^2-3u+1)(1-3u)(1-2u)}\Big).$$

Let $\mathfrak{S}_n^{(m)}=\{\pi\in\mathfrak{S}_n:\mathsf{des}(\pi)=m\}$ and

$$P_m(u)=\sum_{n\geq 0}\Big(\sum_{\pi\in\mathfrak{S}_n^{(m)}} s^{\mathsf{xdes}(\pi)}r^{\mathsf{fix}(\pi)}\Big)u^n \tag{3.24}$$

be the generating function for the xdes and fix statistics over the set of all permutations with exactly m descents. Theorem 3.5 implies that

$$\sum_{n\geq 0}\Big(\sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}(\pi)}s^{\mathsf{xdes}(\pi)}r^{\mathsf{fix}(\pi)}\Big)u^n=(1-t)\sum_{m\geq 0}t^m\rho(S_m((1-t)u)). \tag{3.25}$$

Comparing the coefficients of t^0, t^1, t^2 in (3.25) with the explicit values of $\rho(S_1(u))$ and $\rho(S_2(u))$ previously obtained, we derive

$$\begin{aligned}
P_0(u) &= \frac{1}{1-ru};\\
P_1(u) &= \frac{u^2 s}{(1-ru)^2(1-2u)};\\
P_2(u) &= \frac{u^3}{(1-ru)^3(u^2-3u+1)(1-2u)}\times\\
&\quad \left(\frac{(3ru-r-2u)(u-1)}{1-2u}s+\frac{(-3ru+2u^2+r-4u+2)u}{1-3u}s^2\right).
\end{aligned}$$

To end this section, we connect our results with two kinds of Genocchi numbers, and the new statistic xdes with earlier work of Ehrenborg and Steingrímsson [8].

• For $m = 1, 2, \ldots$, the coefficient of $[s^m u^{2m}]$ in $\rho(S_m(u))|_{r=0}$ are

$$1, 2, 8, 56, 608, \ldots$$

which are the *Genocchi numbers of the second kind* (registered as A005439 in [22]), or *Genocchi medians*. In fact, by Theorem 3.5 this coefficient is the the coefficient of $[t^m s^m u^{2m}]$ in $H(t, s, 0; u)$, i.e., the coefficient of $[t^m s^m]$ in

$$\Big(\sum_{\pi\in D_{2m}} t^{\mathsf{des}(\pi)} s^{\mathsf{xdes}(\pi)}\Big)\frac{1}{(1-t)^{2m+1}},$$

which is equal to the number of derangements π in D_{2m} such that $\mathsf{des}(\pi) = \mathsf{xdes}(\pi) = m$. By [7] we know that the Genocchi number of second kind is the number of derangements σ on $\{1, 2, \cdots, 2m\}$ such that $\sigma(i) > i$ iff i is odd, which is equivalent to the condition $\mathsf{des}(\sigma) = \mathsf{xdes}(\sigma) = m$.

• For $m = 1, 2, \ldots$, the coefficient of $[s^m u^{2m}]$ in $\rho(S_m(u))|_{r=1}$ are

$$1, 3, 17, 155, 2073, \ldots$$

which are the *Genocchi numbers of the first kind* (registered as A110501 in [22]). In fact, by Theorem 3.5 this coefficient is the the coefficient of $[t^m s^m u^{2m}]$ in $H(t, s, 1; u)$, i.e., the coefficient of $[t^m s^m]$ in

$$\Big(\sum_{\pi\in \mathfrak{S}_{2m}} t^{\mathsf{des}(\pi)} s^{\mathsf{xdes}(\pi)}\Big)\frac{1}{(1-t)^{2m+1}},$$

which is equal to the number of permutations π in $\mathfrak{S}_{2m}$ such that $\mathsf{des}(\pi) = \mathsf{xdes}(\pi) = m$. By [6, 7] we know that the Genocchi number of first kind is the number of permutations σ on $\{1, 2, \cdots, 2m\}$ such that for $\sigma(i) > i$ iff i is odd, which is equivalent to the condition $\mathsf{des}(\sigma) = \mathsf{xdes}(\sigma) = m$.

• Although the definition of xdes might seem a little peculiar, it has disguisedly showed up in the literature. In [8, Def. 4.1], Ehrenborg and Steingrímsson introduced the notion of "excedance run" on *ab*-words (certain equivalence classes on permutations determined by their excedance sets), which is essentially the same as our xdes, defined on permutations.

More precisely, for any permutation π, the number of crossing descents of π equals the number of (excedance) runs of the ab-word of π.

4. Patterns on k-arrangements

In this section, we denote $\max(w)$ and $\min(w)$ the maximal and the minimal letters of a word w over integers, respectively.

4.1. **Pattern avoiding 3-arrangements in permutation form.**

Recall that $C(n)$ is the n-th Catalan number. The following enumeration result was conjectured in [2, Conj. 2].

Theorem 4.1. *The number of 3-arrangements of $[n]$ whose permutation form avoids any single pattern of length 3 is $C(n+2)-2^n$.*

Proof. First of all, it was shown by Savage and Wilf [26, Thm. 3] that the number of permutations (or rearrangements) of a given multiset that avoid a pattern of length 3 is independent of the pattern. The same holds true for the permutation form of k-arrangements for any $k \geq 1$, since P_n^k is a union of rearrangement classes [2, Prop. 3]. Thus, it is sufficient to prove

$$C^{(3)}(x) := 1 + \sum_{n\geq 1} |\mathsf{P}_n^3(312)| x^n = \sum_{n\geq 0} (C(n+2)-2^n)x^n = \frac{C(x)-1-x}{x^2} - \frac{1}{1-2x}, \tag{4.1}$$

where $C(x)$ is the generating function for Catalan numbers

$$C(x) := \sum_{n\geq 0} C(n)x^n = \frac{1-\sqrt{1-4x}}{2x}. \tag{4.2}$$

On the other hand, Blitvić and Steingrímsson [2, Prop. 6] showed that

$$C^{(2)}(x) := 1 + \sum_{n\geq 1} |\mathsf{P}_n^2(312)| x^n = \frac{C(x)-1}{x}. \tag{4.3}$$

We view $\mathsf{P}_n^3(312)$ as the disjoint union of $\mathsf{P}_n^2(312)$ and $\bar{\mathsf{P}}_n^3(312) := \mathsf{P}_n^3(312) \setminus \mathsf{P}_n^2(312)$. Any $w = w_1 \dots w_n \in \bar{\mathsf{P}}_n^3(312)$ with w_j being the rightmost letter $\bar{2}$ can be written as $w = \alpha\,\bar{2}\,\beta$ with $\max(\alpha) \leq \min(\beta)$. This decomposition can be fully characterized according to the following two cases:

- if β contains the letter $\bar{1}$, then we have $\mathrm{red}(\beta) \in \mathsf{P}_{n-j}^2(312) \setminus \mathsf{P}_{n-j}^1(312)$ and $\alpha \in \{\bar{1},\bar{2}\}^{j-1}$;
- otherwise β has purely positive letters, and $\mathrm{red}(\beta) \in \mathsf{P}_{n-j}^1(312)$, $\alpha \in \mathsf{P}_{n-j}^3(312)$.

This decomposition is reversible and in terms of generating function gives

$$C^{(3)}(x) - C^{(2)}(x) = x(C^{(2)}(x) - C(x))(1-2x)^{-1} + xC^{(3)}(x)C(x).$$

Combining this with (4.3) and the explicit expression of $C(x)$ in (4.2) yields (4.1) after simplification using Maple. $\square$

4.2. **The statistic des on pattern avoiding 2-arrangements in permutation form.** This subsection is devoted to the classification of the des-Wilf equivalences for patterns of length 3 for permutation form of 2-arrangements. These des-Wilf equivalences were stated as Conjectures 3 and 4 in [2, Sec. 3.6].

Theorem 4.2. *The distribution of* des *on* 2*-arrangements of* $[n-1]$ *whose permutation form avoids any single one of the patterns* $213, 312, 231$ *or* 132*, is given by the triangle sequence A108838 in [22], which counts, among other things, parallelogram polyominoes of semiperimeter* $n+1$ *having* k *corners, and has formula* $\frac{2}{n+1}\binom{n+1}{k+2}\binom{n-2}{k}$.

Proof. It is known (cf. [23, Sec. 2.3]) that the size generating function

$$N = N(t,x) := 1 + \sum_{n\geq 1} x^n \sum_{\pi\in\mathfrak{S}_n(\sigma)} t^{\mathsf{des}(\pi)},$$

where σ is one of the patterns $213, 312, 231$ or 132, satisfies the functional equation

$$txN^2 - (1-x+tx)N + 1 = 0. \tag{4.4}$$

First we consider the pattern 312. Any $w = w_1\cdots w_n \in \mathsf{P}_n^2(312)\setminus \mathsf{P}_n^1(312)$ with w_j being the rightmost letter $\bar{1}$ can be written as $w = \alpha\,\bar{1}\,\beta$ with $\max(\alpha)\leq \min(\beta) > 0$ such that $\alpha\,\bar{1}\in\tilde{\mathsf{P}}_j^2(312)$ and $\mathrm{red}(\beta)\in\mathsf{P}_{n-j}^1(312) = \mathfrak{S}_{n-j}(312)$, where $\tilde{\mathsf{P}}_j^2(312)$ denotes the set of words $w\in\mathsf{P}_j^2(312)$ whose last letter is $\bar{1}$. Moreover, we have

$$\mathsf{des}(w) = \mathsf{des}(\alpha\,\bar{1}) + \mathsf{des}(\beta) \quad\text{and}\quad \mathsf{des}(\alpha\,\bar{1}) = \mathsf{des}(\alpha) + \chi(w_{j-1}\neq\bar{1}),$$

where $\chi(\mathsf{S})$ equals 1, if the statement S is true and 0 otherwise. Let us introduce

$$F(t,x) := 1 + \sum_{n\geq 1} x^n \sum_{w\in\mathsf{P}_n^2(312)} t^{\mathsf{des}(w)} \quad\text{and}\quad G(t,x) := \sum_{n\geq 1} x^n \sum_{w\in\tilde{\mathsf{P}}_n^2(312)} t^{\mathsf{des}(w)}.$$

The above decomposition then gives the system of equations

$$\begin{cases} F = N + GN, \\ G = tx(F-1-G) + x(1+G). \end{cases}$$

Solving this system of equations yields $N = \frac{(1-x+tx)F}{1+txF}$. Substituting this into (4.4) results in

$$t^2x^2F^2 - (t^2x^2 - 2tx^2 + x^2 - 2x + 1)F + 1 = 0. \tag{4.5}$$

Comparing with the generating function for sequence A108838 in [22] proves the desired result for pattern 312. The proof for the pattern 213 is identical and will be omitted.

Next we consider the pattern 231. Let $\bar{\mathsf{P}}_n^2(231)$ denote the set of $w\in\mathsf{P}_n^2(231)$ with $\max(w)\neq\bar{1}$. Any $w = w_1\cdots w_n\in\bar{\mathsf{P}}_n^2(231)$ with the largest letter $w_j = m > 0$ can be written as $\alpha\,m\,\beta$, where $\max(\alpha)\leq\min(\beta)$. We have two cases:

- if $\max(\alpha) = \bar{1}$ (or α is empty), i.e., α is a word with all letters being $\bar{1}$, then $\mathrm{red}(\beta)\in\mathsf{P}_{n-j}^2(231)$ (possibly empty);
- otherwise $\max(\alpha)\geq 1$, then $\alpha\in\bar{\mathsf{P}}_{j-1}^2(231)$ and $\mathrm{red}(\beta)\in\mathsf{P}_{n-j}^1(231)$ (possibly empty).

By this decomposition, if we define

$$H = H(t,x) := 1 + \sum_{n\geq 1} x^n \sum_{w\in \mathsf{P}_n^2(231)} t^{\mathsf{des}(w)},$$

then

$$H = \frac{1}{1-x} + \frac{x(1+t(H-1))}{1-x} + \left(H - \frac{1}{1-x}\right)x(1+t(N-1)).$$

Thus, we have $N = \frac{H(tx^2-x^2+2x-1)+1}{tx(Hx-H+1)}$. Substituting this into (4.4) results in

$$t^2x^2H^2 - (t^2x^2 - 2tx^2 + x^2 - 2x + 1)H + 1 = 0,$$

which proves the statement for the pattern 231 after comparing with (4.5). The proof for the pattern 132 is the same as for 231 and thus is omitted. The proof of the theorem is now completed. □

Finally, we deal with the patterns 321 and 123, thus completing the classification of all six patterns of length 3, in terms of their des-Wilf equivalences on P_n^2. Our proof of the following connection is algebraic. A bijective proof would be interesting.

Theorem 4.3. *The distribution of* des *on* 2*-arrangements of* $[n-1]$ *whose permutation form avoids the pattern* 321*, is given by the triangle sequence A236406 in [22], which counts* 321*-avoiding permutations of* $[n]$ *with* k *peaks.*

In order to prove Theorem 4.3, we need to compute the joint distribution of the number of descents and the position of the leftmost descent on 321-avoiding permutations. We will apply Krattenthaler's classical bijection [18] (see also [9]) from Dyck paths to 321-avoiding permutations.

A *Dyck path* of semilength n is a lattice path in $\mathbb{N}^2$ from $(0,0)$ to (n,n) using only *east steps* $(1,0)$ and *north steps* $(0,1)$, which does not pass above the line $y=x$. The *height of an east step* in a Dyck path is the number of north steps before this east step. For the sake of convenience, we represent a Dyck path as $d_1d_2\cdots d_n$, where d_i is the height of its i-th east step. See Fig. 1 for the Dyck path 012224566. Denote by $\mathcal{D}_n$ the set of all Dyck paths of semilength n. In particular, we denote by

$$\mathrm{ID}_n = 01\cdots n-1 \text{ and } \mathrm{id}_n = 12\cdots n$$

the zigzag Dyck path of semilength n and the identity permutation of length n, with ID_0 and id_0 being the empty path and the empty permutation, respectively. We will use the description of Krattenthaler's bijection $\psi : \mathcal{D}_n \to \mathfrak{S}_n(321)$ in [20]. Given a Dyck path $D = d_1d_2\cdots d_n \in \mathcal{D}_n$, define $\psi(D) = \pi = \pi_1\pi_2\cdots\pi_n$, where

- $\pi_i = d_i + 1$ if $d_i \neq d_{i+1}$ or $i = n$; otherwise
- if i is the j-th smallest integer in $\{k \in [n-1] : d_k = d_{k+1}\}$, then π_i is the j-th smallest integer in $[n] \setminus \{d_1+1, d_2+1, \ldots, d_n+1\}$.

See Fig. 1 for a visualization of this bijection for the Dyck path 012224566.

Let us introduce the following three statistics for $D \in \mathcal{D}_n$:

- $\mathsf{hill}(D)$, the number of hills of D, where a *hill* of a Dyck path is an east step touching the diagonal $y = x$ and followed immediately by a north step.

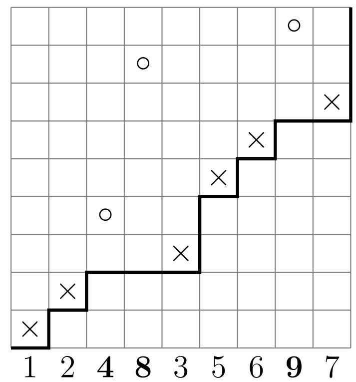

FIGURE 1. Krattenthaler's bijection $\psi : \mathcal{D}_n \to \mathfrak{S}_n(321)$.

- $\mathsf{seg}(D)$, the number of **seg**ments of D, where a *segment* is a maximal string of at least two consecutive east steps of the same height;
- $\mathsf{lseg}(D) = i$, if the i-th east step is the last step of the **l**eftmost **seg**ment of D. Otherwise, $D = \mathrm{ID}_n$ has no segments, then we let $\mathsf{lseg}(D) = n + 1$. In particular, $\mathsf{lseg}(\mathrm{ID}_0) = 1$.

Continuing with our Dyck path in Fig. 1, we have $\mathsf{hill}(D) = \mathsf{seg}(D) = 2$ and $\mathsf{lseg}(D) = 5$. For a permutation $\pi \in \mathfrak{S}_n$, let

$$\mathsf{ldes}(\pi) := \min\{i : \pi_i > \pi_{i+1} \text{ or } i = n\} \tag{4.6}$$

be the the position of the ***l**eftmost **des**cent* of a permutation π. In particular, $\mathsf{ldes}(\mathrm{id}_0) = 0$. The following property is clear from the above description of ψ.

Lemma 4.4. *For each $n \geq 0$, the bijection $\psi : \mathcal{D}_n \to \mathfrak{S}_n(321)$ transforms $(\mathsf{hill}, \mathsf{seg}, \mathsf{lseg})D$ to $(\mathsf{fix}, \mathsf{des}, \mathsf{ldes} + 1)\psi(D)$.*

We continue to compute the generating function

$$\begin{aligned} C(t,p) = C(t,p;x) &:= p + \sum_{n\geq 1} x^n \sum_{D\in\mathcal{D}_n} t^{\mathsf{seg}(D)} p^{\mathsf{lseg}(D)} = p + p^2x + (p+t)p^2x^2 + \cdots \\ &= \frac{p}{1-px} + \sum_{n\geq 1} x^n \sum_{\substack{\pi\in\mathfrak{S}_n(321)\\ \pi\neq \mathrm{id}_n}} t^{\mathsf{des}(\pi)} p^{\mathsf{ldes}(\pi)+1} \end{aligned}$$

using the *classical decomposition* of Dyck paths.

Lemma 4.5. *The generating function $C(t,p;x)$ satisfies the algebraic functional equation*

$$tp^2x^2C^2(t,1) + px(C - pxC - p)C(t,1) - (C - pxC - p) = 0. \tag{4.7}$$

Proof. Let $\mathcal{B}_n$ be the set of Dyck paths in $\mathcal{D}_n$ that begin with an east step followed immediately by a north step. If we introduce

$$B(t,p;x) := \sum_{n\geq 1} x^n \sum_{D\in\mathcal{B}_n} t^{\mathsf{seg}(D)} p^{\mathsf{lseg}(D)},$$

then clearly

$$B(t,p;x) = pxC(t,p;x). \tag{4.8}$$

For $n \geq 2$, a Dyck path $D = d_1 \cdots d_n \in \mathcal{D}_n \setminus \mathcal{B}_n$ with $\min\{i \geq 2 : d_{i+1} = i \text{ or } i = n\} = j$ can be decomposed uniquely into a pair (D_1, D_2) of Dyck paths, where $D_1 = d_2 d_3 \cdots d_j \in \mathcal{D}_{j-1}$ and $D_2 = (d_{j+1}-j)(d_{j+2}-j)\cdots(d_n-j) \in \mathcal{D}_{n-j}$ (possibly empty). This decomposition is reversible and satisfies the following properties:

$$\mathsf{lseg}(D) = \begin{cases} 2, & \text{if } D_1 \in \mathcal{B}_{j-1} \\ 1 + \mathsf{lseg}(D_1), & \text{otherwise} \end{cases}$$

and

$$\mathsf{seg}(D) = \mathsf{seg}(D_1) + \mathsf{seg}(D_2) + \chi(D_1 \in \mathcal{B}_{j-1}).$$

Turning this decomposition into generating functions yields

$$C - B - p = tp^2 x B(t,1)C(t,1) + px(C - B - p)C(t,1). \tag{4.9}$$

Substituting (4.8) into (4.9) gives (4.7) after simplification. □

We are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Setting $p = 1$ in (4.7) and solving for $C(t,1)$ gives

$$C(t,1) = \frac{1 - \sqrt{-4tx^2 + 4x^2 - 4x + 1}}{2x(tx - x + 1)}. \tag{4.10}$$

Substituting this into (4.7) and solving for $C = C(t,p;x)$ yields

$$C - \frac{p}{1-px} = \frac{tp^2(2tx^2 - 2x^2 + 2x - 1 + \sqrt{-4tx^2 + 4x^2 - 4x + 1})}{(tx - x + 1)(px - 1)(2tx - 2x + 2 - p + p\sqrt{-4tx^2 + 4x^2 - 4x + 1})}. \tag{4.11}$$

Let $\tilde{C}(p) = (C - \frac{p}{1-px})/p^2$. Then $\tilde{C}(p)$ is the size generating function for 321-avoiding permutations with at least one descent counted by the pair $(\mathsf{des}, \mathsf{ldes} - 1)$. For $w \in \mathsf{P}_n^2(321)$, let

$$\mathsf{plat}(w) := |\{i \in [n-1] : w_i = w_{i+1}\}|$$

be the number of *plateaux* of w. Since each permutation form $w \in \mathsf{P}_n^2(321)$, whose permutation part π is a 321-avoiding permutation, can be obtained from π by inserting some copies of $\bar{1}$ into the spaces not after the leftmost descent slot of π, we have

$$f(t,q;x) := 1 + \sum_{n\geq 1} x^n \sum_{w \in \mathsf{P}_n^2(321)} t^{\mathsf{des}(w)} q^{\mathsf{plat}(w)} = \Big(1 + \frac{x}{1-qx}\Big)^2 \tilde{C}(p) + \Big(1 + \frac{x}{1-qx}\Big)\frac{1}{1-px},$$

where we set $p = 1 + \frac{tx}{1-qx}$, and the case with $\pi = \mathrm{id}_n, n \geq 0$ is dealt with separately to form the second product. Combining this relationship with (4.11) results in

$$f(t,q;x) = \frac{(1 - 2tx^2 - qx + 2x^2 - x + (qx - x - 1)S)(1 + x - qx)}{2x^2(tx - x + 1)(q^2x - 2qx + tx - q + 2)}, \tag{4.12}$$

Pattern p	First values of $\lvert\mathsf{D}_n^1(p)\rvert$:	counted?	in OEIS?
321	$1,2,5,15,48,159,538,1850,6446,22712,\ldots$	Algebraic g.f.	A289589?
132	Wilf-equivalent to pattern 321	Algebraic g.f.	A289589?
231	$1,2,5,14,42,131,420,1376,4595,15573,\ldots$	open	new
123	$1,2,6,19,61,202,688,2367,8316,29356,\ldots$	open	new
312	$1,2,4,10,27,78,235,736,2366,7772,\ldots$	open	new
213	$1,2,6,19,63,210,716,2462,8604,30296,\ldots$	open	new

TABLE 1. Length-3 patterns for 1-arrangements in derangement form

where $S := \sqrt{1+4x(x-tx-1)}$. Setting $q=1$ in (4.12) yields

$$\frac{1-2tx^2+2x^2-2x-\sqrt{1+4x(x-tx-1)}}{2x^2(tx-x+1)^2},$$

which proves the theorem after comparing with the size generating function for 321-avoiding permutations by the number of peaks derived recently by Bukata et al. in [3, Thm. 3]. □

Remark 4.6. The expression (4.10) was first proved by Barnabei et al. [1]. Our expression (4.11) is a generalization of (4.10). See also [20] for a different generalization of (4.10).

For $w = w_1\cdots w_n \in \mathsf{P}_n^2(321)$, let $w^r = w_n\cdots w_1 \in \mathsf{P}_n^2(123)$ be the *reversal* of w. Clearly, we have

$$\mathsf{des}(w^r)+1 = n-\mathsf{des}(w)-\mathsf{plat}(w).$$

Thus, making the substitution $q \leftarrow t^{-1}$, $x \leftarrow tx$ and $t \leftarrow t^{-1}$ in (4.12) gives the following generating function formula for counting 123-avoiding 2-arrangements in permutation form by $\mathsf{des}+1$.

Theorem 4.7. *We have the generating function formula*

$$1+\sum_{n\geq 1}x^n\sum_{w\in\mathsf{P}_n^2(123)}t^{\mathsf{des}(w)+1} = \frac{1-x-tx-2tx^2+2t^2x^2-(tx-x+1)T}{2tx^2(tx-x-1)(1-\frac{2t}{tx-x+1})}, \tag{4.13}$$

where $T := \sqrt{1+4tx(tx-x-1)}$.

The refinement $\sum_{w\in\mathsf{P}_n^2(123)}t^{\mathsf{des}(w)}$ of Catalan numbers appears to be new and the first few polynomials are

$$1,\quad 2,\quad 3+2t,\quad 2+10t+2t^2,\quad 2+12t+26t^2+2t^3,\quad 2+12t+56t^2+60t^3+2t^4.$$

4.3. **Length-3 patterns for 1-arrangements in derangement form.** In general, the enumeration of pattern avoiding 1-arrangements in derangement form is harder than that in permutation form. Our computer program indicates that only one Wilf-equivalence exists for length-3 patterns on 1-arrangements in derangement form (see Theorem 1.3). The enumerative sequences for the number of the other four Wilf-equivalence classes turn out to be new in OEIS (see Table 1).

The rest of this section is devoted to the proof of Theorem 1.3. We begin with a refinement of an intriguing result due to Robertson, Saracino and Zeilberger [25] which asserts that fix has the same distribution over $\mathfrak{S}_n(321)$ and $\mathfrak{S}_n(132)$. Let $\pi \in \mathfrak{S}_n$ be a permutation. Recall from (4.6) that $\mathsf{ldes}(\pi)$ is the position of the leftmost descent of π. Similarly, let

$$\mathsf{rdes}(\pi) := n - \max(\{i : \pi_i > \pi_{i+1}\} \cup \{0\})$$

be the complement of the position of the ***r**ightmost **des**cent* of π. Let $\mathsf{exc}(\pi) := |\{i : \pi_i > i\}|$ and $\mathsf{aexc}(\pi) := |\{i : \pi_i < i\}|$ be the number of *excedances* and *anti-excedances* of π, respectively. Also, define

$$\begin{aligned} \mathsf{lmax}(\pi) &= |\{i : \pi_j < \pi_i \text{ for all } j < i\}| \quad \text{and} \\ \mathsf{rmin}(\pi) &= |\{i : \pi_j > \pi_i \text{ for all } j > i\}|, \end{aligned}$$

the number of ***l**eft-to-right **max**ima* and ***r**ight-to-left **min**ima* of π, respectively. It is clear that

$$(\mathsf{exc}, \mathsf{lmax})\pi = (\mathsf{aexc}, \mathsf{rmin})\pi^{-1}. \tag{4.14}$$

Lemma 4.8. *There exists a bijection $\mathcal{K} : \mathfrak{S}_n(321) \to \mathfrak{S}_n(132)$ such that for each $\pi \in \mathfrak{S}_n$,*

$$(\mathsf{fix}, \mathsf{exc}, \mathsf{ldes})\pi = (\mathsf{fix}, \mathsf{aexc}, \mathsf{rdes})\mathcal{K}(\pi).$$

Proof. The bijection $\mathcal{K}$ is a composition of a bijection due to Knuth and the inverse of permutations. For $\pi \in \mathfrak{S}_n(321)$, let π' be the 132-avoiding permutation under Knuth's bijection (see the description in [4, Sec. 3.1] or [10]). Then noting that a permutation $\sigma \in \mathfrak{S}_n(132)$ if and only if $\sigma^{-1} \in \mathfrak{S}_n(132)$, we define $\mathcal{K}(\pi) = (\pi')^{-1}$. It has been shown by Elizalde and Pak [10] that $(\mathsf{fix}, \mathsf{exc})\pi = (\mathsf{fix}, \mathsf{exc})\pi'$ and by Claesson and Kitaev [4, Sec. 6.4] that $\mathsf{ldes}(\pi) = \mathsf{lmax}(\pi')$. Thus, in view of (4.14), it remains to show that $\mathsf{rmin}(\sigma) = \mathsf{rdes}(\sigma)$ for each $\sigma \in \mathfrak{S}_n(132)$. This follows from the observation that the suffix of σ starting from the letter 1 is monotonously increasing. □

The next observation is obvious, but useful.

Observation 4.9. *Let $w \in \mathsf{D}_n^1$ be a word with weak derangement part* $\mathrm{Der}(w) = \pi$*. Then, w is* 321*-avoiding (resp.* 132*-avoiding) if and only if*

(1) π is a 321*-avoiding (resp.* 132*-avoiding) derangement, and*
(2) all copies of $\bar{1}$ appear not after the leftmost (resp. not before the rightmost) descent slot of π.

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The first statement that $\mathsf{D}_n^1(321)$ and $\mathsf{D}_n^1(132)$ have the same cardinality follows directly from Lemma 4.8 and Observation 4.9.

Let $\mathcal{D}'_n$ be the set of Dyck paths $D \in \mathcal{D}_n$ such that D has no hills. In order to compute (1.3), we need to calculate the generating function

$$D(p, x) := p + \sum_{n \geq 1} x^n \sum_{\pi \in \mathsf{D}_n(321)} p^{\mathsf{ldes}(\pi)+1} = p + \sum_{n \geq 1} x^n \sum_{D \in \mathcal{D}'_n} p^{\mathsf{lseg}(D)},$$

where the second equality follows from Lemma 4.4. Using the classical decomposition of Dyck paths as in the proof of Lemma 4.5 we obtain

$$D(p,x)=\frac{2p(1+2(1-p)x+\sqrt{1-4x})}{(1+2x+\sqrt{1-4x})(2-p+p\sqrt{1-4x})}. \tag{4.15}$$

By Observation 4.9 we have

$$1+\sum_{n\geq 1}|\mathsf{D}_n^1(321)|x^n=D\big(\frac{1}{1-x},x\big).$$

Combining this with (4.15) we get (1.3), completing the proof. □

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(Shishuo Fu) College of Mathematics and Statistics, Chongqing University, Huxi campus, Chongqing 401331, P.R. China
Email address: `fsshuo@cqu.edu.cn`

(Guo-Niu Han) I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, F-67084 Strasbourg, France
Email address: `guoniu.han@unistra.fr`

(Zhicong Lin) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao 266237, P.R. China
Email address: `linz@sdu.edu.cn`