SKEW DOUBLED SHIFTED PLANE PARTITIONS: CALCULUS AND ASYMPTOTICS

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Abstract. In this paper, we establish a new summation formula for Schur processes, called the *complete summation formula*. As an application, we obtain the generating function and the asymptotic formula for the number of *doubled shifted plane partitions*, which can be viewed as plane partitions "shifted at the two sides". We prove that the order of the asymptotic formula depends only on the diagonal width of the doubled shifted plane partition, not on the profile (the skew zone) itself. By using the same methods, the generating function and the asymptotic formula for the number of *symmetric cylindric partitions* are also derived.

1. Introduction

An *ordinary plane partition* (resp. A *defective plane partition*) is a filling $\omega = (\omega_{i,j})$ of the quarter plane $\Lambda = \{(i, j) \mid i, j \geq 1\}$ (resp. of a connected area of the quarter plane Λ) with nonnegative integers such that rows and columns decrease weakly, and the size $|\omega| = \sum \omega_{i,j}$ is finite. The enumeration of various defective plane partitions have been widely studied (see [2, 10, 22, 23]). In particular, the generating functions for the following five types of defective plane partitions (see Fig. 1) have been obtained since MacMahon:

(A) the ordinary plane partitions (MacMahon [16], Stanley [22]);
(B) the skew plane partitions (Sagan [21]);
(C) the skew shifted plane partitions (Sagan [21]);
(D) the symmetric plane partitions (Andrews [2], Macdonald [15]);
(E) the cylindric partitions (Gessel and Krattenthaler [10], Borodin [4]).

In the literature there are several approaches to deriving the generating functions for various defective plane partitions, such as: (1) Determinant evaluation and nonintersecting lattice paths [1, 10]; (2) Hook lengths and combinatorial proofs [21, 23]; (3) Schur functions and Schur processes [3, 22]. (4) Lozenge tilings and Kuo condensation [6].

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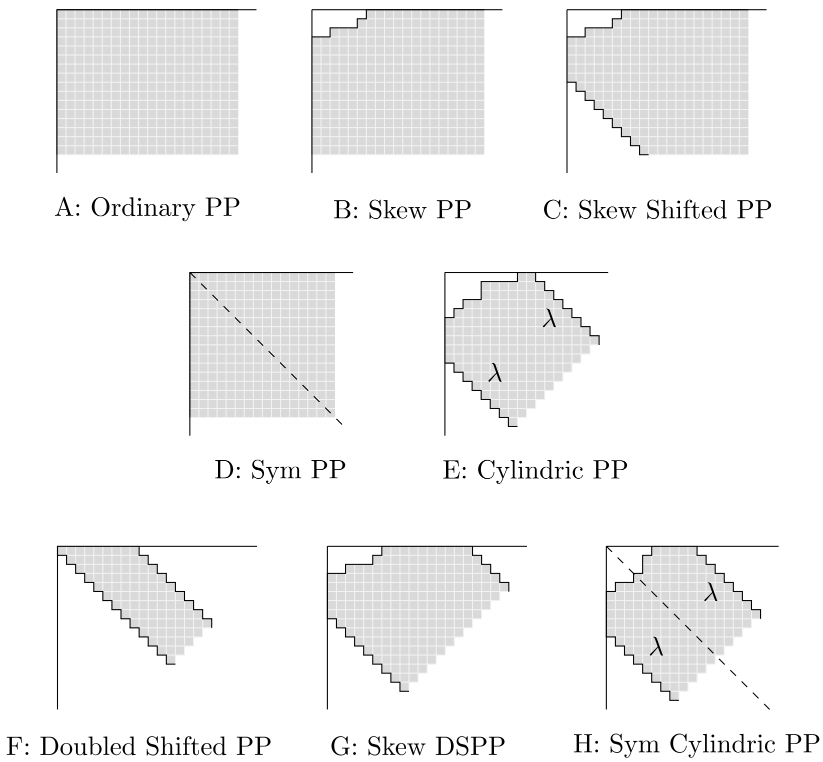


Fig. 1. Various kinds of defective plane partitions.

The Schur process was first introduced by Okounkov and Reshetikhin [18, 19] in 2001. Later, they used the Schur process to derive the cyclic symmetry of the topological vertex by considering a certain type of plane partitions [20]. This result was further developed by Iqbal et al., for providing a short proof of the Nekrasov-Okounkov formula [12, 17]. Borodin [4] used the Schur process to derive the generating function for cylindric partitions, introduced by Gessel and Krattenthaler [10]. The Macdonald process, which is a (q, t)-generalization of the Schur process, was first introduced by Vuletić [27], and further developed by Corteel, Savelief, Vuletić and Langer [7, 14, 26] in the study of weighted cylindric partitions and plane overpartitions. Finally, a survey of Macdonald processes was published by Borodin and Corwin [5].

The Schur process approach is shown to be a powerful tool in the study of various kinds of defective plane partitions. In fact, the above generating functions for defective plane partitions (A-E) are specializations of two general summation formulas for Schur processes, namely, the *open summation formula* (2.1) and the *cylindric summation formula* (2.3). Formulas (2.1) and (2.3) have been developed by Okounkov, Reshetikhin, Borodin, Corteel, Savelief, Vuletić and Langer [4, 7, 14, 18, 19, 26, 27]. For convenience, they are also reproduced in Theorem 2.1.

In the present paper we establish a new summation formula for Schur processes, called the *complete summation formula* (2.2). As an application, we obtain the generating functions for the skew doubled shifted plane partitions and the symmetric cylindric partitions (see Fig. 1 (F)/(G)/(H)).

Let us reproduce some classical formulas in this introduction (see, e.g., [21, 24]). The generating functions for ordinary plane partitions (PP), shifted plane partitions (ShiftPP) and symmetric plane partitions (SPP) are the following respectively:

$$\sum_{\omega\in\mathrm{PP}} z^{|\omega|} = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{1-z^{i+j-1}} = \prod_{k=1}^{\infty}(1-z^k)^{-k}; \tag{1.1}$$

$$\sum_{\omega\in\mathrm{ShiftPP}} z^{|\omega|} = \prod_{k=1}^{\infty}\frac{1}{1-z^k}\prod_{1\le i<j\le\infty}\frac{1}{1-z^{i+j}}; \tag{1.2}$$

$$\sum_{\omega\in\mathrm{SPP}} z^{|\omega|} = \prod_{k=1}^{\infty}\frac{1}{1-z^{2k-1}}\prod_{1\le i<j\le\infty}\frac{1}{1-z^{2(i+j-1)}}. \tag{1.3}$$

As a byproduct of our complete summation formula (2.2) we can further derive the following generating function for doubled shifted plane partitions of width m (see Fig. 1 (F) and Section 3 for the definition).

Theorem 1.1. *Let* DSPP_m *be the set of all doubled shifted plane partitions* ω *of width* m *(i.e., skew doubled shifted plane partitions with profile* $\delta = (-1)^{m-1}$*). Then*

$$\sum_{\omega\in\mathrm{DSPP}_m} z^{|\omega|} = \prod_{k=1}^{\infty}\frac{1}{1-z^k}\times\prod_{k=0}^{\infty}\prod_{1\le i<j\le m-1}\frac{1}{1-z^{2mk+i+j}}. \tag{1.4}$$

Inspired by the works of Dewar, Murty and Kotěšovec [8, 13], we establish some useful theorems for asymptotic formulas in [11] (see Theorem 4.2). Furthermore, the following asymptotic formula for the number of doubled shifted plane partitions can also be obtained.

Theorem 1.2. *Let* $\mathrm{DSPP}_m(n)$ *be the number of doubled shifted plane partitions* ω *of width* m *and size* n*. Then,*

$$\mathrm{DSPP}_m(n) \ \sim\ C(m)\times\frac{1}{n}\exp\Big(\pi\sqrt{\frac{(m^2+m+2)n}{6m}}\Big), \tag{1.5}$$

where $C(m)$ *is a constant independent of* n *given by the following expression:*

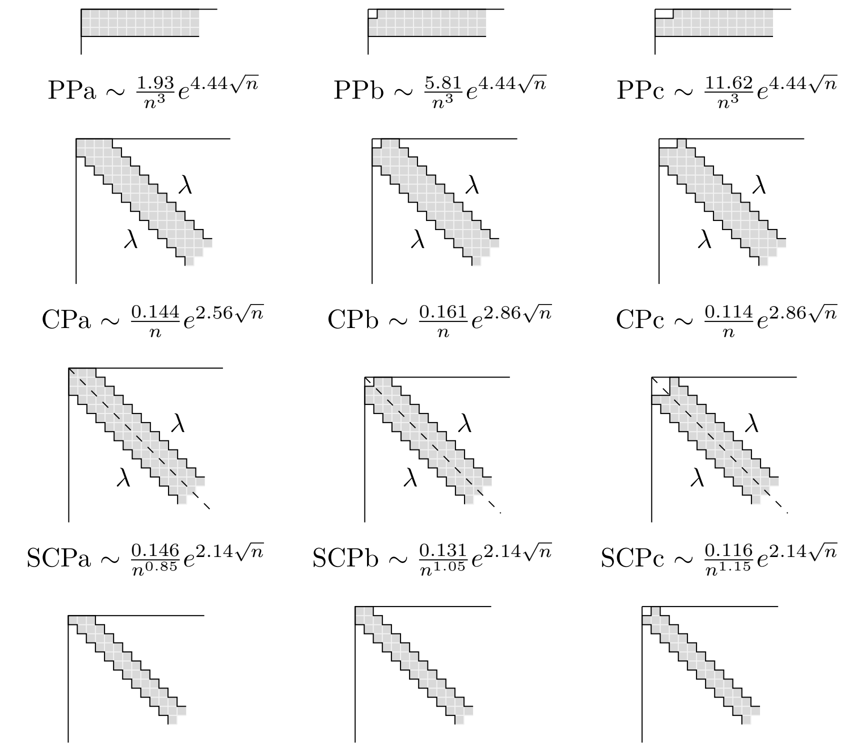
$$C(m) = \left(\prod_{i=1}^{m-2}\prod_{j=i+1}^{m-i-1}\sin\left(\frac{i+j}{2m}\pi\right)\right)^{-1}\frac{\sqrt{m^2+m+2}}{2^{(m^2-3m+14)/4}\sqrt{3m}}.$$

For example, the generating function and asymptotic formula for doubled shifted plane partitions of width $m = 3$ (see Fig. 2, case DSPPa) are

$$\sum_{\omega\in\mathrm{DSPP}_3} z^{|\omega|} \ = \ \prod_{k\ge1}\frac{1}{(1-z^k)(1-z^{6k-3})}; \tag{1.6}$$

$$\mathrm{DSPP}_3(n) \ \sim\ \frac{\sqrt{7}}{24}\,\frac{\exp(\pi\frac{\sqrt{7n}}{3})}{n}. \tag{1.7}$$

The proofs of Theorems 1.1 and 1.2 will be given in Section 4. In fact, Theorems 1.1 and 1.2 can be extended to *skew doubled shifted plane partitions* (see Sections 3 and 4). The asymptotic formulas for two other skew doubled shifted plane partitions, together with some ordinary plane partitions (PP), cylindric partitions (CP) and symmetric cylindric partitions (SCP) are also reproduced next. The proofs of those asymptotic formulas can be found in Sections 4 and 5 for DSPP and SCP respectively, and in [11] for PP and CP.



PPa $\sim \frac{1.93}{n^3} e^{4.44\sqrt{n}}$ PPb $\sim \frac{5.81}{n^3} e^{4.44\sqrt{n}}$ PPc $\sim \frac{11.62}{n^3} e^{4.44\sqrt{n}}$

CPa $\sim \frac{0.144}{n} e^{2.56\sqrt{n}}$ CPb $\sim \frac{0.161}{n} e^{2.86\sqrt{n}}$ CPc $\sim \frac{0.114}{n} e^{2.86\sqrt{n}}$

SCPa $\sim \frac{0.146}{n^{0.85}} e^{2.14\sqrt{n}}$ SCPb $\sim \frac{0.131}{n^{1.05}} e^{2.14\sqrt{n}}$ SCPc $\sim \frac{0.116}{n^{1.15}} e^{2.14\sqrt{n}}$

DSPPa $\sim \frac{0.110}{n} e^{2.77\sqrt{n}}$ DSPPb $\sim \frac{0.138}{n} e^{2.77\sqrt{n}}$ DSPPc $\sim \frac{0.174}{n} e^{2.77\sqrt{n}}$

Fig. 2. Asymptotic formulas for various kinds of defective plane partitions.

The skew doubled shifted plane partitions have some nice properties. (1) The order of the asymptotic formula depends only on the width of the doubled shifted plane partition, not on the profile (the skew zone) itself. The similar property holds for ordinary plane partitions. We may think that this is natural by intuition. However, the cylindric partitions (CP and SCP) show that this is not always the case. (2) We empirically observe that, the asymptotic formula for doubled shifted plane partitions gives already good approximative values for the numbers of DSPP, even for small integer n. While the asymptotic formula for PP needs a large integer n to produce an acceptable value, as shown in the following table.

n	5	10	15	20
#PPa	21	319	3032	22371
Asymptotic	$\sim$ 319	$\sim$ 2449	$\sim$ 17062	$\sim$ 103112
#CPa	7	42	176	627
Asymptotic	$\sim$ 8	$\sim$ 48	$\sim$ 198	$\sim$ 692
#SCPa	4	17	56	161
Asymptotic	$\sim$ 4	$\sim$ 18	$\sim$ 59	$\sim$ 169
#DSPPa	9	64	314	1244
Asymptotic	$\sim$ 10	$\sim$ 70	$\sim$ 336	$\sim$ 1325

It is amazing how the orders of the asymptotic formulas for CP and SCP differ. For the CP, the exponents of n in the denominator are always 1, but the exponents of e differ. While for the SCP, the exponents of n differ, but the exponents of n are constant. Let us summarize these observations in the following table.

	n^{Const}	$e^{\text{Const}\sqrt{n}}$	Fast Convergence
PP	Yes	Yes	No
CP	Yes	No	Yes
SCP	No	Yes	Yes
DSPP	Yes	Yes	Yes

The rest of the paper is arranged in the following way. In Section 2 we establish the complete summation formula for Schur processes. The basic notation and the trace generating functions for skew doubled shifted plane partitions can be found in Section 3. We derive the generating functions and the asymptotic formulas for the numbers of skew doubled shifted plane partitions and symmetric cylindric partitions, in Sections 4 and 5, respectively.

2. Summation formulas for skew Schur functions

For the definitions and basic properties of skew Schur functions we refer to the books [15, 24]. Let

$$\Psi(X,Y) = \prod_{i,j}(1-x_iy_j)^{-1},$$
$$\Phi(X) = \prod_{i}(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1},$$

where $X=\{x_1,x_2,\ldots\}$ and $Y=\{y_1,y_2,\ldots\}$ are two alphabets. Each ±1-sequence $\delta=(\delta_i)_{1\le i\le h}$ of length $h\ge 1$ is called a *profile*. Let $|\delta|_1$ (resp. $|\delta|_{-1}$) be the number of letters 1 (resp. -1) in δ. Therefore, $h=|\delta|_1+|\delta|_{-1}$. The following theorem contains three fundamental summation formulas for skew Schur functions, namely, the *open summation formula* (2.1), the *cylindric summation formula* (2.3) and the *complete summation formula* (2.2). The open and cylindric formulas have already been derived by Okounkov, Reshetikhin, Borodin, Corteel, Savelief, Vuletić and Langer [4, 7, 14, 18, 19, 26, 27]. For convenience, they are also reproduced next. Our main contribution is the complete summation formula (2.2).

Theorem 2.1. *Let h be a positive integer, $\delta=(\delta_i)_{1\le i\le h}$ be a profile of length h, and $Z^1,\ldots,Z^h$ be a sequence of alphabets. Write $\mathbf{Z} := \mathbf{Z}_-^\delta + \mathbf{Z}_+^\delta = \sum_{1\le i\le h} Z^i$ as the union of $Z^1,Z^2,\ldots,Z^h$, $\mathbf{Z}_-^\delta := \sum_{i:\,\delta_i=-1} Z^i$, and $\mathbf{Z}_+^\delta := \sum_{i:\,\delta_i=1} Z^i$. For a sequence of partitions $\lambda^0,\lambda^1,\lambda^2,\ldots,\lambda^h$, let s_i^δ denote the skew Schur function $s_{\lambda^i/\lambda^{i-1}}$ if $\delta_i=1$ and $s_{\lambda^{i-1}/\lambda^i}$ if $\delta_i=-1$. We have*

$$\sum_{\lambda^1,\ldots,\lambda^{h-1}}\prod_{i=1}^h s_i^\delta(Z^i) = \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}}\Psi(Z^i,Z^j)\times\sum_\gamma s_{\lambda^0/\gamma}(\mathbf{Z}_-^\delta)s_{\lambda^h/\gamma}(\mathbf{Z}_+^\delta); \tag{2.1}$$

$$\sum_{\lambda^0,\ldots,\lambda^h} z^{|\lambda^h|}\prod_{i=1}^h s_i^\delta(Z^i) = \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}}\Psi(Z^i,Z^j)\times\Phi(\mathbf{Z}_-^\delta)\prod_{k\ge1}\frac{\Phi(z^k\mathbf{Z})}{1-z^k}; \tag{2.2}$$

$$\sum_{\substack{\lambda^0,\ldots,\lambda^h\\ \lambda^0=\lambda^h}} z^{|\lambda^h|}\prod_{i=1}^{h} s_i^{\delta}(Z^i) = \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}} \Psi(Z^i,Z^j)\times\prod_{k\ge1}\frac{\Psi(z^k\mathbf{Z}_-^{\delta},\mathbf{Z}_+^{\delta})}{1-z^k}. \tag{2.3}$$

Actually, Theorem 2.1 is equivalent to the following Theorem 2.2, since with $X^{i-1}=\emptyset$ and $Y^{i-1}=Z^i$ if $\delta_i=1$; $Y^{i-1}=\emptyset$ and $X^{i-1}=Z^i$ if $\delta_i=-1$, we recover Theorem 2.1. Therefore we just need to prove Theorem 2.2.

Theorem 2.2. *Suppose that $X^0,X^1,\ldots,X^{h-1}$ and $Y^0,Y^1,\ldots,Y^{h-1}$ are $2h$ alphabets. Let $\mathbf{X}=\sum_{i=0}^{h-1}X^i$ and $\mathbf{Y}=\sum_{i=0}^{h-1}Y^i$ be the union of $X^0,X^1,\ldots,X^{h-1}$ and $Y^0,Y^1,\ldots,Y^{h-1}$ respectively. Then we have*

$$\begin{aligned}\sum_{\lambda^1,\ldots,\lambda^{h-1}}\ \sum_{\mu^0,\ldots,\mu^{h-1}}\ \prod_{i=0}^{h-1} s_{\lambda^i/\mu^i}(X^i)s_{\lambda^{i+1}/\mu^i}(Y^i)\\ =\prod_{0\le i<j\le h-1}\Psi(Y^i,X^j)\sum_{\gamma}s_{\lambda^0/\gamma}(\mathbf{X})s_{\lambda^h/\gamma}(\mathbf{Y}).\end{aligned} \tag{2.4}$$

$$\begin{aligned}\sum_{\lambda^0,\ldots,\lambda^h}\ \sum_{\mu^0,\ldots,\mu^{h-1}} z^{|\lambda^h|}\prod_{i=0}^{h-1} s_{\lambda^i/\mu^i}(X^i)s_{\lambda^{i+1}/\mu^i}(Y^i)\\ =\prod_{0\le i<j\le h-1}\Psi(Y^i,X^j)\times\Phi(\mathbf{X})\prod_{k\ge1}\frac{\Phi(z^k(\mathbf{X}+\mathbf{Y}))}{1-z^k}.\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sum_{\substack{\lambda^0,\ldots,\lambda^h\\ \lambda^0=\lambda^h}}\ \sum_{\mu^0,\ldots,\mu^{h-1}} z^{|\lambda^h|}\prod_{i=0}^{h-1} s_{\lambda^i/\mu^i}(X^i)s_{\lambda^{i+1}/\mu^i}(Y^i)\\ =\prod_{0\le i<j\le h-1}\Psi(Y^i,X^j)\times\prod_{k\ge1}\frac{\Psi(z^k\mathbf{X},\mathbf{Y})}{1-z^k}.\end{aligned} \tag{2.6}$$

To give the proof of Theorem 2.2, let us recall the following two formulas stated in Macdonald's book [15] (see p. 93, ex. 26(1) and ex. 27(3)).

$$\sum_{\rho}s_{\rho/\lambda}(X)s_{\rho/\mu}(Y)=\Psi(X,Y)\sum_{\rho}s_{\lambda/\rho}(Y)s_{\mu/\rho}(X), \tag{2.7}$$

$$\sum_{\rho}s_{\rho/\nu}=\Phi(X)\sum_{\rho}s_{\nu/\rho}. \tag{2.8}$$

First we use (2.8) to prove some lemmas.

Lemma 2.3. *We have*

$$\sum_{\mu,\tau}z^{|\mu|}s_{\mu/\tau}(X)=\prod_{k\ge1}\frac{\Phi(z^kX)}{1-z^k}. \tag{2.9}$$

Proof. Let $F(X)$ be the left-hand side of (2.9). Then, by (2.8) we have

$$\begin{aligned}F(X)&=\sum_{\mu,\tau}z^{|\mu|}s_{\mu/\tau}(X)=\sum_{\tau}z^{|\tau|}\sum_{\mu}s_{\mu/\tau}(zX)\\ &=\Phi(zX)\sum_{\tau}z^{|\tau|}\sum_{\rho}s_{\tau/\rho}(zX)=\Phi(zX)F(zX).\end{aligned}$$

Hence, we obtain

$$\sum_{\mu,\tau} z^{|\mu|} s_{\mu/\tau}(X) = \prod_{k\geq 1} \Phi(z^k X) \times F(\emptyset).$$

Since

$$F(\emptyset) = \sum_{\mu,\tau} z^{|\mu|} s_{\mu/\tau}(\emptyset) = \sum_{\mu} z^{|\mu|} = \prod_{k\geq 1} \frac{1}{1-z^k},$$

then (2.9) is proved. □

Lemma 2.4. *We have*

$$\sum_{\lambda,\mu,\gamma} z^{|\mu|} s_{\mu/\gamma}(X) s_{\lambda/\gamma}(Y) = \Phi(Y) \prod_{k\geq 1} \frac{\Phi(z^k(X+Y))}{1-z^k}. \tag{2.10}$$

Proof. By (2.8) and Lemma 2.3 we have

$$\begin{aligned}
&\sum_{\lambda,\mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X) s_{\lambda/\gamma}(Y)\\
=&\sum_{\mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X) \sum_{\lambda} s_{\lambda/\gamma}(Y)\\
=&\Phi(Y) \sum_{\mu} z^{|\mu|} \sum_{\gamma} s_{\mu/\gamma}(X) \sum_{\tau} s_{\gamma/\tau}(Y)\\
=&\Phi(Y) \sum_{\mu} z^{|\mu|} \sum_{\tau} s_{\mu/\tau}(X+Y)\\
=&\Phi(Y) \prod_{k\geq 1} \frac{\Phi(z^k(X+Y))}{1-z^k}.
\end{aligned}$$

□

Remark. Formula (2.10) is similar to the following formula stated in Macdonald's book [15] (see p. 94, ex. 28(a)), which has two free partitions λ and γ:

$$\sum_{\lambda,\gamma} z^{|\lambda|} s_{\lambda/\gamma}(X) s_{\lambda/\gamma}(Y) = \prod_{k\geq 1} \frac{\Psi(z^k X, Y)}{1-z^k}. \tag{2.11}$$

Now we are ready to give the proof of Theorem 2.2.

Proof of the Theorem 2.2. Let $F(X^0, X^1, \ldots, X^{h-1}, Y^0, Y^1, \ldots, Y^{h-1})$ be the left-hand side of (2.4). By (2.7) we have

$$\begin{aligned}
&F(X^0, X^1, \ldots, X^{h-1}, Y^0, Y^1, \ldots, Y^{h-1})\\
&= \sum_{\lambda^1,\ldots,\lambda^{h-2}} \sum_{\mu^0,\ldots,\mu^{h-1}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i) s_{\lambda^{i+1}/\mu^i}(Y^i)\\
&\quad \times s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2}) s_{\lambda^h/\mu^{h-1}}(Y^{h-1}) \sum_{\lambda^{h-1}} s_{\lambda^{h-1}/\mu^{h-2}}(Y^{h-2}) s_{\lambda^{h-1}/\mu^{h-1}}(X^{h-1})\\
&= \Psi(Y^{h-2}, X^{h-1}) \sum_{\lambda^1,\ldots,\lambda^{h-2}} \sum_{\mu^0,\ldots,\mu^{h-1}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i) s_{\lambda^{i+1}/\mu^i}(Y^i)\\
&\quad \times s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2}) s_{\lambda^h/\mu^{h-1}}(Y^{h-1}) \sum_{\lambda^{h-1}} s_{\mu^{h-2}/\lambda^{h-1}}(X^{h-1}) s_{\mu^{h-1}/\lambda^{h-1}}(Y^{h-2})
\end{aligned}$$

$$= \Psi(Y^{h-2}, X^{h-1}) \sum_{\lambda^1,\ldots,\lambda^{h-1}} \sum_{\mu^0,\ldots,\mu^{h-3}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i) s_{\lambda^{i+1}/\mu^i}(Y^i)$$
$$\times \sum_{\mu^{h-2}} s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2}) s_{\mu^{h-2}/\lambda^{h-1}}(X^{h-1}) \sum_{\mu^{h-1}} s_{\lambda^h/\mu^{h-1}}(Y^{h-1}) s_{\mu^{h-1}/\lambda^{h-1}}(Y^{h-2})$$
$$= \Psi(Y^{h-2}, X^{h-1}) \sum_{\lambda^1,\ldots,\lambda^{h-1}} \sum_{\mu^0,\ldots,\mu^{h-3}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i) s_{\lambda^{i+1}/\mu^i}(Y^i)$$
$$\times s_{\lambda^{h-2}/\lambda^{h-1}}(X^{h-2} + X^{h-1}) s_{\lambda^h/\lambda^{h-1}}(Y^{h-2} + Y^{h-1})$$
$$= \Psi(Y^{h-2}, X^{h-1}) \sum_{\lambda^1,\ldots,\lambda^{h-2}} \sum_{\mu^0,\ldots,\mu^{h-2}} \prod_{i=0}^{h-3} s_{\lambda^i/\mu^i}(X^i) s_{\lambda^{i+1}/\mu^i}(Y^i)$$
$$\times s_{\lambda^{h-2}/\mu^{h-2}}(X^{h-2} + X^{h-1}) s_{\lambda^h/\mu^{h-2}}(Y^{h-2} + Y^{h-1})$$
$$= \Psi(Y^{h-2}, X^{h-1}) F(X^0, \ldots, X^{h-3}, X^{h-2} + X^{h-1}, Y^0, \ldots, Y^{h-3}, Y^{h-2} + Y^{h-1})$$
$$= \cdots$$
$$= \Big(\prod_{0 \le i < j \le h-1} \Psi(Y^i, X^j)\Big) F(X^0 + \ldots + X^{h-1}, Y^0 + \ldots + Y^{h-1})$$
$$= \Big(\prod_{0 \le i < j \le h-1} \Psi(Y^i, X^j)\Big) \sum_{\gamma} s_{\lambda^0/\gamma}(X^0 + \ldots + X^{h-1}) s_{\lambda^h/\gamma}(Y^0 + \ldots + Y^{h-1}).$$

Therefore, identity (2.4) is true. Then identities (2.5) and (2.6) hold by (2.4), (2.11) and Lemma 2.4. □

3. Definitions for skew doubled shifted plane partitions

In this section we give the definition and the trace generating function of skew doubled shifted plane partitions. Each profile δ is associated with a connected area $\Delta := \Delta(\delta)$ of the quarter plane Λ in a unique manner. For a given profile

$$\delta = 1^{a_0}(-1)^{b_1} 1^{a_1}(-1)^{b_2} \ldots 1^{a_{r-1}}(-1)^{b_r}$$

with $a_0, b_r \ge 0$, $a_i, b_i \ge 1$ for $1 \le i \le r-1$, let

$$\Delta_1 = \bigcup_{i=1}^{r-1} \{(c,d) \in \Lambda : \sum_{j=1}^{r-i-1} a_{r-j} \le c \le \sum_{j=1}^{r-i} a_{r-j}, 1 \le d \le \sum_{j=1}^{i} b_i\},$$
$$\Delta_2 = \{(c,d) \in \Lambda : c - d > \sum_{i=0}^{r-1} a_i\},$$
$$\Delta_3 = \{(c,d) \in \Lambda : d - c > \sum_{i=1}^{r} b_i\}.$$

The connected area Δ is defined to be $\Delta := \Lambda \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. For example, with the profile $\delta = (1, -1, -1, 1, -1, 1, -1, 1)$, the four areas $\Delta_1, \Delta_2, \Delta_3, \Delta$ are illustrated in Fig. 3.

Let λ and μ be two integer partitions. We write $\lambda \succ \mu$ or $\mu \prec \lambda$ if λ/μ is a horizontal strip (see [14, 15, 19, 24]).

Definition 3.1. Let $\delta = (\delta_i)_{1\le i\le h}$ be a profile. A *skew doubled shifted plane partition* (DSPP) with profile δ is a filling $\omega = (\omega_{i,j})$ of $\Delta(\delta)$ with nonnegative integers such that the size $|\omega| = \sum_{(i,j)} \omega_{i,j}$ is finite, and the rows and columns are weakly decreasing, i.e.,

$$\omega_{i,j} \ge \omega_{i,j+1}, \quad \omega_{i,j} \ge \omega_{i+1,j}$$

whenever these numbers are well-defined.

The set of all DSPP with profile δ is denoted by DSPP_δ. Recall that the Schur process for plane partitions was first introduced by Okounkov and Reshetikhin [19] (see also [14]); the main idea was to read the plane partitions along the diagonals. When reading the DSPP ω with profile δ along the diagonals from left to right, we obtain a sequence of integer partitions $(\lambda^0, \lambda^1, \dots, \lambda^h)$ such that $\lambda^{i-1} \prec \lambda^i$ (resp. $\lambda^{i-1} \succ \lambda^i$) if $\delta_i = 1$ (resp. $\delta_i = -1$), and $|\omega| = \sum_{i=0}^{h} |\lambda^i|$. For simplicity, we identify the skew doubled shifted plane partition ω and the sequence of integer partitions by writing

$$\omega = (\lambda^0, \lambda^1, \dots, \lambda^h).$$

The *(diagonal) width* of the skew doubled shifted plane partition ω is defined to be $h + 1$.

For example, with the DSPP ω given in Fig. 3, we obtain a sequence of partitions: $(4,1) \prec (5,4) \succ (5,2) \succ (3) \prec (4,1) \succ (2) \prec (2,2) \succ (2,1) \prec (5,2,1)$. Hence, $\omega = \big((4,1),(5,4),(5,2),(3),(4,1),(2),(2,2),(2,1),(5,2,1)\big)$ is a DSPP of width 9 with profile $\delta = (1,-1,-1,1,-1,1,-1,1)$.

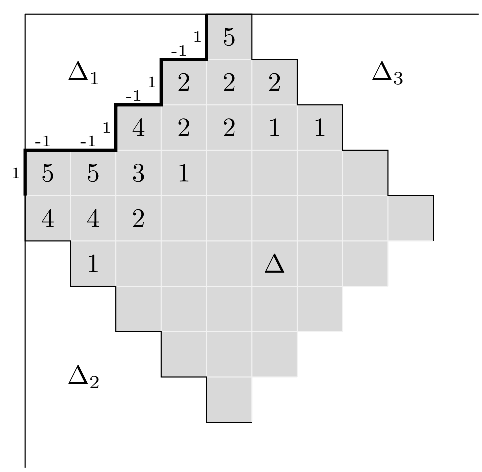


Fig. 3. A skew doubled shifted plane partition.

For a sequence of parameters u_i $(i \ge 0)$, write $U_j = u_0 u_1 \cdots u_{j-1}$ $(j \ge 0)$. Let $\mathrm{DSPP}_\delta(\lambda^0, \lambda^h)$ denote the set of the skew doubled shifted plane partitions $\omega = (\lambda^0, \lambda^1, \dots, \lambda^h)$ starting from λ^0 and ending at λ^h with profile δ. Letting $Z^i = \{U_i^{-\delta_i}\}$ in Theorem 2.1, we obtain the following trace generating functions for skew doubled shifted plane partitions.

Theorem 3.1. *Let $\delta = (\delta_i)_{1\le i\le h}$ be a profile. We have*

$$\sum_{\omega\in \mathrm{DSPP}_\delta(\lambda^0,\lambda^h)} \prod_{i=0}^{h} u_i^{|\lambda^i|} = U_{h+1}^{|\lambda^h|} \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}} \frac{1}{1-U_i^{-1}U_j} \times \sum_\gamma s_{\lambda^0/\gamma}(\{U_i : \delta_i=-1\}) s_{\lambda^h/\gamma}(\{U_i^{-1} : \delta_i = 1\}); \tag{3.1}$$

$$\sum_{\omega\in \mathrm{DSPP}_\delta} \prod_{i=0}^{h} u_i^{|\lambda^i|} = \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}} \frac{1}{1-U_i^{-1}U_j} \times \Phi(\{U_i : \delta_i=-1\}) \prod_{k\ge 1} \frac{\Phi(\{U_i^{-\delta_i}U_{h+1}^k : 1\le i\le h\})}{1-U_{h+1}^k}; \tag{3.2}$$

$$\sum_{\omega\in \mathrm{DSPP}_\delta(\lambda^0=\lambda^h)} \prod_{i=0}^{h} u_i^{|\lambda^i|} = \prod_{\substack{1\le i<j\le h\\ \delta_i>\delta_j}} \frac{1}{1-U_i^{-1}U_j} \times \prod_{k\ge 1} \frac{\Psi(\{U_iU_{h+1}^k : \delta_i=-1\}, \{U_j^{-1} : \delta_j = 1\})}{1-U_{h+1}^k}. \tag{3.3}$$

The above theorem implies many classical results on various defective plane partitions, including the trace generating function of ordinary plane partitions (PP) (Stanley [23]), and the generating functions of symmetric plane partitions (SPP) (Andrews [2], Macdonald [15]), skew plane partitions (Sagan [21]) and skew shifted plane partitions (Sagan [21]).

4. Formulas for skew doubled shifted plane partitions

Let $u_i = z$ for $i\ge 0$ in (3.2). We then derive the generating function for DSPP_δ:

$$\sum_{\omega\in \mathrm{DSPP}_\delta} z^{|\omega|} = \prod_{\substack{1\le j<i\le h\\ \delta_i<\delta_j}} \frac{1}{1-z^{i-j}} \times \Phi(\{z^i : \delta_i=-1\}) \times \prod_{k\ge 1} \frac{\Phi(\{z^{(h+1)k+i} : \delta_i=-1\} + \{z^{(h+1)k-j} : \delta_j = 1\})}{1-z^{(h+1)k}}. \tag{4.1}$$

The right-hand side of the above identity can be further simplified. For each profile $\delta = (\delta_i)_{1\le i\le m-1}$, we define the following multisets as

$$\begin{aligned} W_1(\delta) &:= \{m\} \cup \{i \mid \delta_i = -1\} \cup \{m-i \mid \delta_i = 1\};\\ W_2(\delta) &:= \{\quad i+j \quad \mid 1\le i<j\le m-1,\ \delta_i=\delta_j=-1\}\\ &\quad\cup \{2m-i-j \mid 1\le i<j\le m-1,\ \delta_i=\delta_j=1\}\\ &\quad\cup \{2m+i-j \mid 1\le i<j\le m-1,\ \delta_i<\delta_j\}\\ &\quad\cup \{\quad j-i \quad \mid 1\le i<j\le m-1,\ \delta_i>\delta_j\}. \end{aligned}$$

Therefore, Eq. (4.1) implies the following theorem.

Theorem 4.1. *The generating function for the skew doubled shifted plane partitions with profile* $\delta = (\delta_i)_{1 \le i \le m-1}$ *is*

$$\sum_{\omega \in \mathrm{DSPP}_\delta} z^{|\omega|} = \prod_{k \ge 0} \left(\prod_{t \in W_1(\delta)} \frac{1}{1 - z^{mk+t}} \right) \left(\prod_{t \in W_2(\delta)} \frac{1}{1 - z^{2mk+t}} \right).$$

Define

$$\psi_n(v, r, b; p) := v \sqrt{\frac{p(1-p)}{2\pi}} \frac{r^{b+(1-p)/2}}{n^{b+1-p/2}} \exp(n^p r^{1-p}) \tag{4.2}$$

for $n \in \mathbb{N}$, $v, b \in \mathbb{R}$, $r > 0$, $0 < p < 1$.

Inspired by the works of Dewar, Murty and Kotěšovec [8, 13], we have established the following useful result for asymptotic formulas in [11].

Theorem 4.2 ([11])**.** *Let* m *be a positive integer. Suppose that* x_i *and* y_i $(1 \le i \le m)$ *are positive integers such that* $\gcd(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = 1$. *Then, the coefficients* d_n *in the following infinite product*

$$\prod_{i=1}^{m} \prod_{k \ge 0} \frac{1}{1 - q^{x_i k + y_i}} = \sum_{n=0}^{\infty} d_n q^n$$

have the following asymptotic formula

$$d_n \sim v \frac{1}{2\sqrt{2\pi}} \frac{r^{b+1/4}}{n^{b+3/4}} \exp(\sqrt{nr}), \tag{4.3}$$

where

$$v = \prod_{i=1}^{m} \frac{\Gamma(y_i/x_i)}{\sqrt{x_i \pi}} (\frac{x_i}{2})^{y_i/x_i}, \qquad r = \sum_{i=1}^{m} \frac{2\pi^2}{3x_i}, \qquad b = \sum_{i=1}^{m} (\frac{y_i}{2x_i} - \frac{1}{4}).$$

By applying the above two results, our next theorem gives asymptotic formulas for the number of skew doubled shifted plane partitions. For simplicity, write

$$\epsilon(\delta) := -\sum_i \delta_i \, i = \sum_{\delta_i = -1} i - \sum_{\delta_j = 1} j.$$

Then we obtain the following result.

Theorem 4.3. *Let* $m \ge 2$ *be a positive integer and* $\delta = (\delta_j)_{1 \le j \le m-1}$ *be a profile of length* $m - 1$. *We denote by* $\mathrm{DSPP}_\delta(n)$ *the number of skew doubled shifted plane partitions* ω *with profile* δ *and size* n. *Then,*

$$\mathrm{DSPP}_\delta(n) \sim C_1(\delta) C_2(m) \times \frac{1}{n} \exp\Big(\pi \sqrt{\frac{(m^2 + m + 2)n}{6m}}\Big),$$

where $C_1(\delta)$ *and* $C_2(m)$ *are two constants with respect to* n*:*

$$C_1(\delta) = 2^{-\frac{\epsilon(\delta)}{m} - |\delta|_1} \times \prod_{t \in W_1(\delta)} \Gamma(\frac{t}{m}) \prod_{t \in W_2(\delta)} \Gamma(\frac{t}{2m}),$$

$$C_2(m) = \Big(2^{m^2 - 3m + 14} \pi^{m^2 - m}\Big)^{-\frac{1}{4}} \times \sqrt{\frac{m^2 + m + 2}{3}}.$$

Proof. By the definitions of $W_1(\delta)$ and $W_2(\delta)$ we have

$$\begin{aligned}
&\#W_1(\delta) = m;\\
&\#W_2(\delta) = \binom{m-1}{2};\\
&\sum_{t\in W_1(\delta)} t = m(|\delta|_1+1)+\epsilon(\delta);\\
&\sum_{t\in W_2(\delta)} t = (m-2)\epsilon(\delta) + 2m\binom{|\delta|_1}{2} + 2m \sum_{\substack{1\le i<j\le m-1\\ \delta_i<\delta_j}} 1.
\end{aligned}$$

Hence,

$$\sum_{t\in W_1(\delta)} \frac{1}{m} + \sum_{t\in W_2(\delta)} \frac{1}{2m} = \frac{m}{m} + \frac{1}{2m}\binom{m-1}{2} = \frac{m^2+m+2}{4m}. \tag{4.4}$$

Furthermore,

$$\sum_{t\in W_1(\delta)} \frac{t}{m} + \sum_{t\in W_2(\delta)} \frac{t}{2m} = 1 + |\delta|_1 + \binom{|\delta|_1}{2} + \sum_{\substack{1\le i<j\le m-1\\ \delta_i<\delta_j}} 1 + \frac{\epsilon(\delta)}{2}.$$

If we exchange any two adjacent letters in δ, the sum of the last two terms doesn't change, therefore it is equal to $\frac12\binom{m}{2} - \binom{|\delta|_1+1}{2}$. Then we obtain

$$\sum_{t\in W_1(\delta)} \frac{t}{m} + \sum_{t\in W_2(\delta)} \frac{t}{2m} = \frac{m^2-m+4}{4}$$

and

$$\sum_{t\in W_1(\delta)} (\frac{t}{2m} - \frac14) + \sum_{t\in W_2(\delta)} (\frac{t}{4m} - \frac14) = \frac14. \tag{4.5}$$

By Theorems 4.2 and 4.1 the number of DSPP with profile δ and size n is asymptotic to

$$v\frac{1}{2\sqrt{2\pi}}\frac{r^{b+1/4}}{n^{b+3/4}}\exp(\sqrt{nr}),$$

where

$$\begin{aligned}
v &= \prod_{t\in W_1(\delta)} \left(\frac{\Gamma(t/m)}{\sqrt{m\pi}}(\frac{m}{2})^{t/m}\right) \prod_{t\in W_2(\delta)} \left(\frac{\Gamma(t/(2m))}{\sqrt{2m\pi}}(m)^{t/(2m)}\right)\\
&= 2^{-\frac{\epsilon(\delta)}{m}-|\delta|_1-1-\frac12\binom{m-1}{2}} m^{1/2}\pi^{(-m^2+m-2)/4}\\
&\qquad \times \prod_{t\in W_1(\delta)} \Gamma(\frac{t}{m}) \prod_{t\in W_2(\delta)} \Gamma(\frac{t}{2m}),\\
r &= \sum_{t\in W_1(\delta)} \frac{2\pi^2}{3m} + \sum_{t\in W_2(\delta)} \frac{2\pi^2}{6m} = \frac{(m^2+m+2)\pi^2}{6m},\\
b &= \sum_{t\in W_1(\delta)} \left(\frac{t}{2m} - \frac14\right) + \sum_{t\in W_2(\delta)} \left(\frac{t}{4m} - \frac14\right) = \frac14.
\end{aligned}$$

This achieves the proof. □

For example, consider the three skew doubled shifted plane partitions (DSPPa)-(DSPPc) given in Fig. 2. Their profiles, generating functions and asymptotic formulas are respectively:

(a) Fig. 2, case DSPPa. $\delta = (1,1)$, $W_1(\delta) = \{3,2,1\}$, $W_2(\delta) = \{3\}$,

$$\begin{aligned} \sum_{\omega \in \mathrm{DSPPa}} z^{|\omega|} &= \prod_{k\geq 0} \frac{1}{(1-z^{k+1})(1-z^{6k+3})}, \\ \mathrm{DSPPa}(n) &\sim \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n}. \end{aligned}$$

(b) Fig. 2, case DSPPb. $\delta = (1,-1)$, $W_1(\delta) = \{3,2,2\}$, $W_2(\delta) = \{1\}$,

$$\begin{aligned} \sum_{\omega \in \mathrm{DSPPb}} z^{|\omega|} &= \prod_{k\geq 0} \frac{1}{(1-z^{3k+3})(1-z^{3k+2})^2(1-z^{6k+1})}, \\ \mathrm{DSPPb}(n) &\sim \sqrt{2}\alpha \times \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n}, \end{aligned}$$

where

$$\alpha = 2^{-\frac{11}{6}}\sqrt{3}\pi^{-\frac{3}{2}}\Gamma(\frac{2}{3})^2\Gamma(\frac{1}{6}) = 0.8908\cdots$$

(c) Fig. 2, case DSPPc. $\delta = (-1,1)$, $W_1(\delta) = \{3,1,1\}$, $W_2(\delta) = \{5\}$,

$$\begin{aligned} \sum_{\omega \in \mathrm{DSPPc}} z^{|\omega|} &= \prod_{k\geq 0} \frac{1}{(1-z^{3k+3})(1-z^{3k+1})^2(1-z^{6k+5})}, \\ \mathrm{DSPPc}(n) &\sim \sqrt{2}\alpha^{-1} \times \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n}. \end{aligned}$$

Theorems 1.1 and 1.2 for doubled shifted plane partitions are specializations of Theorems 4.1 and 4.3 for skew doubled shifted plane partitions when taking the profile $\delta = (-1)^{m-1}$.

Proof of Theorems 1.1 and 1.2. Take $\delta = (-1)^{m-1}$. Then we have

$$W_1(\delta) = \{1,2,3,\ldots,m\}, \quad W_2(\delta) = \{i+j : 1 \leq i < j \leq m-1\}.$$

Therefore Theorem 4.1 implies Theorem 1.1. Since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}$ and $\prod_{j=1}^{m-1}\sin(\frac{j\pi}{m}) = \frac{m}{2^{m-1}}$, we have

$$\prod_{t\in W_1(\delta)} \Gamma(\frac{t}{m}) = \sqrt{\frac{\pi^{m-1}}{\prod_{j=1}^{m-1}\sin(\frac{j\pi}{m})}} = \sqrt{\frac{(2\pi)^{m-1}}{m}}$$

and

$$\prod_{t\in W_2(\delta)} \Gamma(\frac{t}{2m}) = \pi^{(m-1)(m-2)/4} \left(\prod_{i=1}^{m-2} \prod_{j=i+1}^{m-i-1} \sin\left(\frac{i+j}{2m}\pi\right)\right)^{-1}.$$

Since $\delta = (-1)^{m-1}$, by Theorem 4.3 we verify that

$$C_1(\delta) = 2^{-\frac{\epsilon(\delta)}{m} - |\delta|_1} \times \prod_{t\in W_1(\delta)} \Gamma(\frac{t}{m}) \prod_{t\in W_2(\delta)} \Gamma(\frac{t}{2m})$$

$$= \frac{\pi^{(m^2-m)/4}}{\sqrt{m}} \left(\prod_{i=1}^{m-2} \prod_{j=i+1}^{m-i-1} \sin\left(\frac{i+j}{2m}\pi\right) \right)^{-1},$$

and that $C_1(\delta)C_2(m)$ is equal to $C(m)$ given in Theorem 1.2. □

For example, consider the three skew doubled shifted plane partitions (DSPPa)-(DSPPc) given in Fig. 2. Their profiles, generating functions and asymptotic formulas are respectively:

(a) Fig. 2, case DSPPa. $\delta = (1,1)$, $W_1(\delta) = \{3,2,1\}$, $W_2(\delta) = \{3\}$,

$$\sum_{\omega \in \text{DSPPa}} z^{|\omega|} = \prod_{k \ge 0} \frac{1}{(1-z^{k+1})(1-z^{6k+3})},$$

$$\text{DSPPa}(n) \sim \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n}.$$

(b) Fig. 2, case DSPPb. $\delta = (1,-1)$, $W_1(\delta) = \{3,2,2\}$, $W_2(\delta) = \{1\}$,

$$\sum_{\omega \in \text{DSPPb}} z^{|\omega|} = \prod_{k \ge 0} \frac{1}{(1-z^{3k+3})(1-z^{3k+2})^2(1-z^{6k+1})},$$

$$\text{DSPPb}(n) \sim \sqrt{2}\alpha \times \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n},$$

where

$$\alpha = 2^{-\frac{11}{6}} \sqrt{3} \pi^{-\frac{3}{2}} \Gamma(\frac{2}{3})^2 \Gamma(\frac{1}{6}) = 0.8908 \cdots$$

(c) Fig. 2, case DSPPc. $\delta = (-1,1)$, $W_1(\delta) = \{3,1,1\}$, $W_2(\delta) = \{5\}$,

$$\sum_{\omega \in \text{DSPPc}} z^{|\omega|} = \prod_{k \ge 0} \frac{1}{(1-z^{3k+3})(1-z^{3k+1})^2(1-z^{6k+5})},$$

$$\text{DSPPc}(n) \sim \sqrt{2}\alpha^{-1} \times \frac{\sqrt{7}}{24} \frac{\exp(\pi \frac{\sqrt{7n}}{3})}{n}.$$

Remark 4.1. In Theorem 1.2, we derive an asymptotic formula for bounded m and large n. We would like to mention that, a recent paper of Fang, Hwang and Kang [9] generalized our Theorem 1.2 and obtained stronger asymptotic formulas for all possible values of m, n with $m \le n$.

5. Formulas for symmetric cylindric partitions

Cylindric partitions were introduced by Gessel and Krattenthaler [10]. They obtained the generating function for cylindric partitions of some given shape that satisfy certain row bounds as some summation of determinants related to q-binomial coefficients. Later, Borodin gave an equivalent definition [4] and obtained the generating function for cylindric partitions. A *cylindric partition* (CP) with profile δ is a DSPP

$$\omega = (\lambda^0, \lambda^1, \dots, \lambda^{h-1}, \lambda^h)$$

with profile δ such that $\lambda^0 = \lambda^h$. The *size* of such partition is defined by $|\omega| = \sum_{i=0}^{h-1} |\lambda^i|$ (notice that λ^h is not counted here). The following generating function for cylindric partitions, first proved by Borodin [4], later by Tingley [25] and Langer [14], can be obtained by letting $u_i = z$ $(0 \le i \le h-1)$ and $u_h = 1$ in (3.3).

Theorem 5.1 (Borodin[4])**.** *Let $\delta = (\delta_i)_{1\le i\le h}$ be a profile. Then the generating function for the cylindric partitions with profile δ is*

$$\sum_{\omega\in \mathrm{CP}_\delta} z^{|\omega|} = \prod_{k\ge 0}\prod_{t\in W_3(\delta)} \frac{1}{1-z^{hk+t}},$$

where $W_3(\delta)$ is the following multiset

$$W_3(\delta) := \{h\} \cup \{j-i : i<j,\ \delta_i > \delta_j\} \cup \{h+i-j : i<j,\ \delta_i<\delta_j\}.$$

A *symmetric cylindric partition* (SCP) with profile $\delta = (\delta_1, \delta_2, \dots, \delta_h)$ is a DSPP

$$\omega = (\lambda^h, \lambda^{h-1}, \dots, \lambda^1, \lambda^0, \lambda^1, \dots, \lambda^{h-1}, \lambda^h)$$

with profile $(-\delta_h, -\delta_{h-1}, \dots, -\delta_2, -\delta_1, \delta_1, \delta_2, \dots, \delta_{h-1}, \delta_h)$. Notice that a symmetric cylindric partition is always a cylindric partition, and when $\lambda^h = \emptyset$, the SCP ω becomes an SyPP. The size of the symmetric cylindric partition ω is defined by

$$|\omega| = |\lambda^0| + 2\sum_{i=1}^{h} |\lambda^h|.$$

Theorem 5.2. *The generating function for symmetric cylindric partitions with profile δ is*

$$\sum_{\omega\in \mathrm{SCP}_\delta} z^{|\omega|} = \prod_{\substack{i<j\\ \delta_i>\delta_j}} \frac{1}{1-z^{2(j-i)}}\Phi(\{z^{2i-1} : \delta_i = -1\}) \times \prod_{k\ge 1} \frac{\Phi(z^{(2h+1)k}(\{z^{-2i+1} : \delta_i = 1\} + \{z^{2i-1} : \delta_i = -1\}))}{1-z^{(2h+1)k}}. \tag{5.1}$$

Proof. By (3.1), we have

$$\begin{aligned}
\sum_{\omega\in \mathrm{SCP}_\delta} z^{|\omega|} &= \sum_{\lambda^0,\lambda^h} z^{-|\lambda^0|} \sum_{\omega'\in \mathrm{DSPP}_\delta(\lambda^0,\lambda^h)} z^{2|\omega'|}\\
&= \prod_{\substack{i<j\\ \delta_i>\delta_j}} \frac{1}{1-z^{2(j-i)}}\\
&\times \sum_{\lambda,\mu} z^{2(h+1)|\mu|-|\lambda|} \sum_{\gamma\subset\lambda,\mu} s_{\lambda/\gamma}(\{z^{2i} : \delta_i=-1\}) s_{\mu/\gamma}(\{z^{-2i} : \delta_i = 1\})\\
&= \prod_{\substack{i<j\\ \delta_i>\delta_j}} \frac{1}{1-z^{2(j-i)}}\\
&\times \sum_{\lambda,\mu} z^{(2h+1)|\mu|} \sum_{\gamma\subset\lambda,\mu} s_{\lambda/\gamma}(\{z^{2i-1} : \delta_i=-1\}) s_{\mu/\gamma}(\{z^{-2i+1} : \delta_i = 1\}).
\end{aligned}$$

By Lemma 2.4, this is equal to the right hand side of (5.1). □

The right-hand side of the above identity can be further simplified. For each profile $\delta = (\delta_i)_{1\le i\le m-1}$, let $W_4(\delta)$ and $W_5(\delta)$ be the following multisets:

$$\begin{aligned}
W_4(\delta) &= \{2m-1\} \cup \{2i-1 \mid \delta_i = -1\} \cup \{2m-2i \mid \delta_i = 1\};\\
W_5(\delta) &= \{2i+2j-2 \quad \mid 1\le i<j\le m-1,\ \delta_i=\delta_j=-1\}\\
&\quad \cup \{4m-2i-2j \ \mid 1\le i<j\le m-1,\ \delta_i=\delta_j=1\}
\end{aligned}$$

$$\cup \{2(2m-1)+2i-2j \mid 1 \le i < j \le m-1,\ \delta_i < \delta_j\}$$
$$\cup \{2j-2i \mid 1 \le i < j \le m-1,\ \delta_i > \delta_j\}.$$

Then we obtain the following result.

Theorem 5.3. *The generating function for symmetric cylindric partitions with profile $\delta = (\delta_i)_{1\le i\le m-1}$ is*

$$\sum_{\omega\in \mathrm{SCP}_\delta} z^{|\omega|} = \prod_{k\ge 0}\left(\prod_{t\in W_4(\delta)} \frac{1}{1-z^{(2m-1)k+t}}\right)\left(\prod_{t\in W_5(\delta)} \frac{1}{1-z^{2(2m-1)k+t}}\right).$$

By the definitions of W_4 and W_5, it is easy to verify that $|W_4(\delta)| = m$ and $|W_5(\delta)| = \binom{m-1}{2}$. Hence,

$$\sum_{t\in W_4(\delta)} \frac{1}{2m-1} + \sum_{t\in W_5(\delta)} \frac{1}{2(2m-1)} = \frac{m^2+m+2}{4(2m-1)}. \tag{5.2}$$

By Theorems 4.2 and 5.3 we obtain the following asymptotic formula for the number of SCP with size n.

Theorem 5.4. *Let $m \ge 2$ be a positive integer and $\delta = (\delta_j)_{1\le j\le m-1}$ be a profile of length $m-1$. Let $\mathrm{SCP}_\delta(n)$ denote the number of symmetric cylindric partitions with profile δ and size n. Then*

$$\mathrm{SCP}_\delta(n) \sim v\sqrt{\frac{1}{8\pi}\frac{r^{b+1/4}}{n^{b+3/4}}}\exp(\sqrt{nr}),$$

where r, b, v are given below:

$$r = \frac{(m^2+m+2)\pi^2}{6(2m-1)},$$
$$b = \sum_{t\in W_4(\delta)}\left(\frac{t}{2(2m-1)} - \frac{1}{4}\right) + \sum_{t\in W_5(\delta)}\left(\frac{t}{4(2m-1)} - \frac{1}{4}\right),$$
$$v = 2^{-\frac{1}{2m-1}\sum_{t\in W_4(\delta)} t - \frac14(m^2-3m+2)}\pi^{-\frac14(m^2-m+2)}(2m-1)^{2b}$$
$$\times \prod_{t\in W_4(\delta)} \Gamma\left(\frac{t}{2m-1}\right)\prod_{t\in W_5(\delta)}\Gamma\left(\frac{t}{2(2m-1)}\right).$$

Remark 5.1. The term r depends only on the width of the symmetric cylindric partitions, not on the profile itself, while the term b depends on the profile. It is interesting to compare these phenomena with the asymptotic formula for cylindric partitions [11]. There, b depends only on the width, and r depends on the profile.

For example, consider the three symmetric cylindric partitions (SCPa)-(SCPc) given in Fig. 2. Their profiles, generating functions and asymptotic formulas are respectively:

(a) Fig. 2, case SCPa. $\delta = (-1,-1)$. $W_4(\delta) = \{1,3,5\}$ and $W_5(\delta) = \{4\}$.

$$\sum_{\omega\in \mathrm{SCPa}} z^{|\omega|} = \prod_{k\ge 0}\frac{1}{(1-z^{5k+1})(1-z^{5k+3})(1-z^{5k+5})(1-z^{10k+4})},$$

$$\mathrm{SCPa}(n) \sim \Gamma(\frac15)\Gamma(\frac25)\Gamma(\frac35)2^{-\frac{19}{5}}5^{-\frac{3}{20}}\pi^{-\frac95}(\frac73)^{\frac{7}{20}} \times \frac{1}{n^{17/20}}\exp\left(\pi\sqrt{\frac{7n}{15}}\right).$$

(b) Fig. 2, case SCPb. $\delta = (1, -1)$. $W_4(\delta) = \{3, 4, 5\}$ and $W_5(\delta) = \{2\}$.

$$\sum_{\omega \in \text{SCPb}} z^{|\omega|} = \prod_{k \geq 0} \frac{1}{(1 - z^{5k+3})(1 - z^{5k+4})(1 - z^{5k+5})(1 - z^{10k+2})},$$

$$\text{SCPb}(n) \sim \Gamma(\frac{1}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5}) 2^{-\frac{22}{5}} 5^{\frac{1}{20}} \pi^{-\frac{7}{5}} (\frac{7}{3})^{\frac{11}{20}} \times \frac{1}{n^{21/20}} \exp\Big(\pi \sqrt{\frac{7n}{15}}\Big).$$

(c) Fig. 2, case SCPc. $\delta = (1, 1)$. $W_4(\delta) = \{2, 4, 5\}$ and $W_5(\delta) = \{6\}$.

$$\sum_{\omega \in \text{SCPc}} z^{|\omega|} = \prod_{k \geq 0} \frac{1}{(1 - z^{5k+2})(1 - z^{5k+4})(1 - z^{5k+5})(1 - z^{10k+6})},$$

$$\text{SCPc}(n) \sim \Gamma(\frac{2}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5}) 2^{-\frac{21}{5}} 5^{\frac{3}{20}} \pi^{-\frac{6}{5}} (\frac{7}{3})^{\frac{13}{20}} \times \frac{1}{n^{23/20}} \exp\Big(\pi \sqrt{\frac{7n}{15}}\Big).$$

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