ON THE RATIONAL APPROXIMATION TO THUE–MORSE RATIONAL NUMBERS

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ABSTRACT. Let $b \geq 2$ and $\ell \geq 1$ be integers. We establish that there is an absolute real number K such that all the partial quotients of the rational number

$$\prod_{h=0}^{\ell} (1 - b^{-2^h}),$$

of denominator $b^{2^{\ell+1}-1}$, do not exceed $\exp(K(\log b)^2 \sqrt{\ell}\, 2^{\ell/2})$.

1. INTRODUCTION

An easy covering argument which goes back to Cantelli shows that, for almost all real numbers ξ (with respect to the Lebesgue measure) and for every positive ε, the inequality

$$\left| \xi - \frac{p}{q} \right| > \frac{1}{q^{2+\varepsilon}}$$

holds for every sufficiently large q. However, it is often a very difficult problem to show that a given real number shares this property, unless its continued fraction expansion is explicitly determined. This is known to be the case for any irrational real algebraic number, by Roth's theorem, and for only a few other real numbers defined by their expansion in some integer base. Let $\mathbf{t} = t_0 t_1 t_2 \ldots$ denote the Thue–Morse word over $\{-1, 1\}$ defined by $t_0 = 1$, $t_{2k} = t_k$ and $t_{2k+1} = -t_k$ for $k \geq 0$. Then, the Thue–Morse generating series $\xi_{\mathbf{t}}(z)$ is given by

$$\begin{aligned} \xi_{\mathbf{t}}(z) &= \sum_{k \geq 0} t_k z^{-k} = 1 - z^{-1} - z^{-2} + z^{-3} - z^{-4} + z^{-5} + z^{-6} - \ldots \\ &= \prod_{h \geq 0} (1 - z^{-2^h}). \end{aligned}$$

By means of a non-vanishing result obtained in [1] for the Hankel determinants associated with the Thue–Morse sequence, Bugeaud [6] established

2010 *Mathematics Subject Classification.* 11J04, 11J70.

Key words and phrases. rational approximation, continued fraction.

that, for any given positive ε and any integer $b \geq 2$, the Thue–Morse–Mahler number

$$\xi_{\mathbf{t}}(b) = \sum_{k \geq 0} \frac{t_k}{b^k} = 1 - \frac{1}{b} - \frac{1}{b^2} + \frac{1}{b^3} - \frac{1}{b^4} + \frac{1}{b^5} + \frac{1}{b^6} - \frac{1}{b^7} - \frac{1}{b^8} + \dots$$

satisfies the inequality

$$\left|\xi_{\mathbf{t}}(b) - \frac{p}{q}\right| > \frac{1}{q^{2+\varepsilon}},$$

for every rational number p/q with q sufficiently large. Subsequently, his result has been considerably improved by Badziahin and Zorin [4, Th. 11], who showed that there exists a positive real number C such that the stronger inequality

$$\left|\xi_{\mathbf{t}}(b) - \frac{p}{q}\right| > \frac{1}{q^2 \exp(C \log b \sqrt{\log q \, \log\log(3q)})} \tag{1.1}$$

holds as soon as q is large enough. Thus, all the partial quotients of $\xi_{\mathbf{t}}(b)$ are rather small. Note also that, in view of [4, Th. 11], the number K occurring in [4, Th. 2] must depend on b.

Observe that the Thue–Morse power series $\xi_{\mathbf{t}}(z)$ is the limit of the sequence of rational functions

$$f_\ell(z) = \prod_{h=0}^{\ell} (1 - z^{-2^h}).$$

More precisely, we have

$$\xi_{\mathbf{t}}(z) = f_\ell(z) + O(z^{-2^{\ell+1}}), \quad \ell \geq 1,$$

and

$$|\xi_{\mathbf{t}}(x) - f_\ell(x)| \leq \frac{1}{(|x|-1)|x|^{2^{\ell+1}-1}}, \quad \ell \geq 1, x \in \mathbb{C}, |x| > 1. \tag{1.2}$$

Let $b \geq 2$ and $\ell \geq 1$ be integers. For a rational number p/q, we derive from (1.2) that

$$\left| \left|\xi_{\mathbf{t}}(b) - \frac{p}{q}\right| - \left|f_\ell(b) - \frac{p}{q}\right| \right| \leq \frac{1}{(b-1)b^{2^{\ell+1}-1}}. \tag{1.3}$$

Consequently, $\xi_{\mathbf{t}}(b)$ and $f_\ell(b)$ have the same first partial quotients. To see this, let p_n/q_n be the convergent to $\xi_{\mathbf{t}}(b)$ with $q_n \leq b^{2^\ell}$ and n maximal for this property. We assume that ℓ is sufficiently large to ensure that $n \geq 8$; then, a short calculation shows that

$$q_n \geq q_{n-1} + q_{n-2} \geq \cdots \geq 8q_{n-5}. \tag{1.4}$$

By a result of Borel [12, Ch. I, Th. 5B], there exists ε in $\{0,1,2\}$ such that

$$\Big|\xi_{\mathbf{t}}(b)-\frac{p_{n-5-\varepsilon}}{q_{n-5-\varepsilon}}\Big|\le\frac{1}{\sqrt{5}q_{n-5-\varepsilon}^2}.$$

It then follows from (1.3) and (1.4) that

$$\Big|f_\ell(b)-\frac{p_{n-5-\varepsilon}}{q_{n-5-\varepsilon}}\Big|\le\frac{1}{\sqrt{5}q_{n-5-\varepsilon}^2}+\frac{2}{q_n^2}\le\Big(\frac{1}{\sqrt{5}}+\frac{1}{32}\Big)\frac{1}{q_{n-5-\varepsilon}^2}<\frac{1}{2q_{n-5-\varepsilon}^2},$$

which, by a classical theorem of Legendre [12, Ch. I, Th. 5C], implies that $p_{n-5-\varepsilon}/q_{n-5-\varepsilon}$ is a convergent of $f_\ell(b)$. Consequently, $\xi_{\mathbf{t}}(b)$ and $f_\ell(b)$ have the same $n-7$ first partial quotients. By (1.1), these partial quotients are rather small. However, (1.1) gives no information on the remaining partial quotients of $f_\ell(b)$, thus, in particular, on the rate with which the rational number $f_\ell(b)$ of denominator $b^{2^{\ell+1}-1}$ is approximated by rational numbers p/q of denominator q greater than b^{2^ℓ}. In the present note, we address this question and show that an inequality like (1.1) remains true for every convergent of $f_\ell(b)$.

Theorem 1.1. *There exists a positive real number K such that, for every integer $b\ge 2$ and every integer $\ell\ge 2$, the inequality*

$$\Big|\prod_{h=0}^{\ell}(1-b^{-2^h})-\frac{p}{q}\Big|>\frac{1}{q^2\exp(K\log b\,\sqrt{\log q\,\log\log(3q)})},$$

holds for every rational number p/q different from $f_\ell(b)$. Write

$$f_\ell(b)=\prod_{h=0}^{\ell}(1-b^{-2^h})=[0;a_1^{(\ell)},a_2^{(\ell)},\dots,a_{\mathcal{L}(\ell)}^{(\ell)}],$$

with $a_{\mathcal{L}(\ell)}^{(\ell)}\ge 2$. The partial quotients $a_j^{(\ell)}$ of $f_\ell(b)$ are all at most equal to $b^{2K\sqrt{\ell}\sqrt{\log b\,\log\log 3b}\,2^{\ell/2}}$. There exists a positive real number C, depending only on b, such that the length $\mathcal{L}(\ell)$ of the continued fraction of $f_\ell(b)$ exceeds $C2^{\ell/2}/\sqrt{\ell}$.

The second assertion of Theorem 1.1 immediately follows from the first one and from the classical theory of continued fractions. The last assertion already follows from (1.1).

Theorem 1.1 is mainly motivated by the very few known results on continued fraction expansions of sequences of rational numbers. Pourchet [10] (see also [5, 11]) proved that, for all coprime integers a and b with $1<b<a$ and for every positive ε, there exists a positive C, depending only on ε and on the prime divisors of a and b, such that all the partial quotients of $(a/b)^n$ are less than $Cb^{\varepsilon n}$. This was subsequently extended to quotients of power sums by Corvaja and Zannier [7], with a similar conclusion. Consequently, the length of the continued fraction expansion of $(a/b)^n$ (resp., of

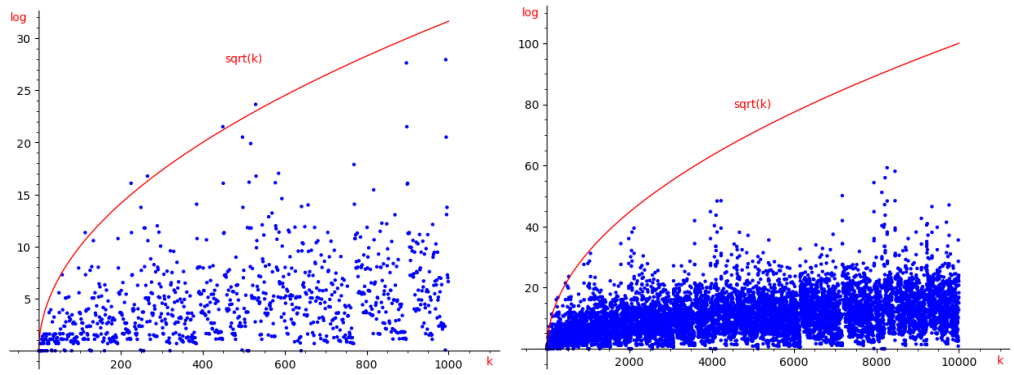

FIGURE 1. Logarithm of the absolute values of the coefficients of the k-th convergent

$(a^n-1)/(b^n-1)$) tends to infinity with n. We stress that the conclusion of Theorem 1.1 is much stronger.

The function $q \mapsto \exp(\sqrt{\log q \log\log(3q)})$ occurring in (1.1) is a consequence of the bound of order $(c_1 k)^{c_2 k}$ obtained in [4] for the absolute values of the coefficients of the numerator and denominator of the k-th convergent to $\xi_{\mathbf{t}}(z)$. However, numerical experiments suggest that a better bound of the shape $c_3^{\sqrt{k}}$ should hold (here, c_1, c_2 and c_3 are absolute, positive real numbers); such a result seems difficult to establish, see Figure 1.

To prove (1.1), Badziahin and Zorin [4] used that all the partial quotients of the continued fraction expansion of $\xi_{\mathbf{t}}(z)$ are linear, a result established by Badziahin [2]. Here, we first show that all the partial quotients of the rational functions $f_\ell(z)$, $\ell \geq 1$, are linear. This is the main novelty of the present note and the object of Section 2. Then, in Section 3, we prove Theorem 1.1 by adapting to our purpose the argument of [4]. Finally, in the last section, we discuss another example.

2. The partial quotients of the rational functions $f_\ell(z)$

For a non-zero rational number x, we let $\nu_2(x)$ denote its 2-adic valuation, that is, the exponent of 2 in its decomposition in product of prime factors. We put $\nu_2(0) = +\infty$.

Proposition 2.1. *For every non-negative integers ℓ and $j \leq 2^{\ell+1}-1$, let $v_j^{(\ell)}$ be the family of rational constants defined by*

$$v_0^{(0)} = 1, \quad v_1^{(0)} = 1,$$

and, for $\ell \geq 1$,

$$v_0^{(\ell)} = 1,$$

$$v_1^{(\ell)} = 2,$$
$$v_{2j}^{(\ell)} = -\frac{v_j^{(\ell-1)}}{v_{2j-1}^{(\ell)}}, \quad 1 \le j \le 2^\ell - 1,$$
$$v_{2j+1}^{(\ell)} = 1 + (-1)^j - v_{2j}^{(\ell)}, \quad 1 \le j \le 2^\ell - 1.$$

Then, for all $\ell \ge 0$ and $0 \le j \le 2^{\ell+1} - 1$, we have

$$\nu_2(v_j^{(\ell)}) = \begin{cases} 1, & \text{for } \ell \ge 1 \text{ and } j = 1; \\ -1, & \text{for } \ell \ge 1 \text{ and } j = 2^\ell \text{ or } j = 2^\ell + 1; \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, we have $v_j^{(\ell)} \neq 0$ for all $\ell \ge 0$ and $0 \le j \le 2^{\ell+1} - 1$.

The first values of $v_j^{(\ell)}$ are given in the following table:

$\ell \setminus j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1														
1	1	2	$-\frac{1}{2}$	$\frac{1}{2}$												
2	1	2	-1	1	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$								
3	1	2	-1	1	1	1	-1	1	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{3}{5}$	$\frac{3}{5}$	$\frac{5}{9}$	$\frac{13}{9}$	$-\frac{3}{13}$	$\frac{3}{13}$

Consequently, the first values of $\nu_2(v_j^{(\ell)})$ are given in the following table:

$\ell \setminus j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0														
1	0	1	-1	-1												
2	0	1	0	0	-1	-1	0	0								
3	0	1	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0

Proof. We proceed by induction on ℓ. Recall that, for any non-zero rational numbers x, y, we have $\nu_2(x/y) = \nu_2(x) - \nu_2(y)$ and $\nu_2(x+y) \ge \min\{\nu_2(x), \nu_2(y)\}$, with equality if $\nu_2(x) \neq \nu_2(y)$. The tables above show that the proposition holds for $0 \le \ell \le 3$. Let $\ell \ge 4$ be an integer such that the proposition holds for $\ell - 1$. By definition, we have

$$\nu_2(v_0^{(\ell)}) = 0, \quad \nu_2(v_1^{(\ell)}) = 1, \quad \nu_2(v_2^{(\ell)}) = \nu_2(-1) = 0.$$

Since $\nu_2(v_{2j+1}^{(\ell)}) = \nu_2(v_{2j}^{(\ell)})$ for every $j = 1, \dots, 2^\ell - 1$ such that $\nu_2(v_{2j}^{(\ell)}) \neq 1$, we derive that $\nu_2(v_3^{(\ell)}) = 0$, thus, $v_3^{(\ell)}$ is nonzero and $\nu_2(v_4^{(\ell)}) = 0$. Reiterating the argument, we get that $v_5^{(\ell)}, \dots, v_{2^\ell-1}^{(\ell)}$ are all nonzero and

$$\nu_2(v_5^{(\ell)}) = \dots = \nu_2(v_{2^\ell-1}^{(\ell)}) = 0, \quad \nu_2(v_{2^\ell}^{(\ell)}) = \nu_2(v_{2^{\ell-1}}^{(\ell-1)}) = -1.$$

Then, $\nu_2(v_{2^\ell+1}^{(\ell)}) = -1$. Thus, $v_{2^\ell+1}^{(\ell)}$ is nonzero and $\nu_2(v_{2^\ell+2}^{(\ell)}) = -1 - (-1) = 0$. We derive that $\nu_2(v_{2^\ell+3}^{(\ell)}) = 0$, thus, $v_{2^\ell+3}^{(\ell)}$ is nonzero and $\nu_2(v_{2^\ell+4}^{(\ell)}) = 0$.

Inductively, we get that $v^{(\ell)}_{2^\ell+5}, \dots, v^{(\ell)}_{2^{\ell+1}-1}$ are all nonzero and

$$\nu_2(v^{(\ell)}_{2^\ell+5}) = \dots = \nu_2(v^{(\ell)}_{2^{\ell+1}-1}) = 0.$$

This completes the induction step. □

Set

$$g_\ell(z) = \frac{1}{z} f_\ell(z) = \frac{1}{z} \prod_{h=0}^{\ell} (1 - z^{-2^h}),$$

and

$$g_\ell(z) = [0; a_1(z), a_2(z), \dots, a_m(z)] = \cfrac{1}{a_1(z)} + \cfrac{1}{a_2(z)} + \cdots + \cfrac{1}{a_m(z)},$$

where $a_i(z)$ is a polynomial with rational coefficients for $1 \le i \le m$. The following theorem, which can be seen as a finite version of [3, Prop. 3.3], shows that all the $a_i(z)$ are polynomials of degree one.

Theorem 2.2. *Let $v^{(\ell)}_j$ $(\ell \ge 0, 0 \le j \le 2^{\ell+1} - 1)$ be the family of rational numbers defined in Proposition* 2.1*. Then,*

$$g_\ell(z) = \cfrac{v^{(\ell)}_0}{z+1} + \cfrac{v^{(\ell)}_1}{z-1} + \cfrac{v^{(\ell)}_2}{z+1} + \cdots + \cfrac{v^{(\ell)}_{2^{\ell+1}-1}}{z-1}, \quad \ell \ge 0. \tag{2.1}$$

Moreover, for every $\ell \ge 0$, all the partial quotients in the continued fraction expansion of $g_\ell(z)$ are polynomials of degree one.

Proof. The last assertion of Theorem 2.2 immediately follows from Proposition 2.1. We prove identity (2.1) by induction on ℓ. Since

$$\frac{1}{z}(1 - z^{-1}) = \cfrac{1}{z+1} + \cfrac{1}{z-1}$$

and

$$\frac{1}{z}(1 - z^{-1})(1 - z^{-2}) = \cfrac{1}{z+1} + \cfrac{2}{z-1} - \cfrac{1/2}{z+1} + \cfrac{1/2}{z-1},$$

identity (2.1) is true for $\ell = 0, 1$. Let $k \ge 1$ be an integer and suppose that (2.1) is true for $\ell \le k$. We set

$$h_{k+1}(z) = \cfrac{v^{(k+1)}_0}{z+1} + \cfrac{v^{(k+1)}_1}{z-1} + \cfrac{v^{(k+1)}_2}{z+1} + \cdots + \cfrac{v^{(k+1)}_{2^{k+2}-1}}{z-1}. \tag{2.2}$$

It suffices prove that $g_{k+1}(z) = h_{k+1}(z)$. By applying the even contraction theorem (see, for example, [9, Theorem 2.1(1)]) to $h_{k+1}(z)$, we have

$$h_{k+1}(z) = b_0 + \cfrac{a_1}{b_1} + \cfrac{a_2}{b_2} + \cfrac{a_3}{b_3} + \cdots, \tag{2.3}$$

where

$$a_1 = v^{(k+1)}_0 (z-1),$$

$$
\begin{aligned}
a_2 &= -v_1^{(k+1)} v_2^{(k+1)}(z-1),\\
a_j &= -v_{2j-3}^{(k+1)} v_{2j-2}^{(k+1)}(z-1)^2, \quad 3 \le j \le 2^{k+1},\\
a_j &= 0, \quad j > 2^{k+1},\\
b_0 &= 0,\\
b_1 &= (z+1)(z-1) + v_1^{(k+1)},\\
b_j &= (z-1)\big((z-1)(z+1) + v_{2j-1}^{(k+1)} + v_{2j-2}^{(k+1)}\big), \quad 2 \le j \le 2^{k+1},\\
b_j &= 0, \quad j > 2^{k+1}.
\end{aligned}
$$

By removing the common factors in numerators and denominators, we obtain

$$
h_{k+1}(z) = b_0 + \frac{a_1'}{b_1'} + \frac{a_2'}{b_2'} + \frac{a_3'}{b_3'} + \cdots \tag{2.4}
$$

where

$$
\begin{aligned}
a_1' &= v_0^{(k+1)}(z-1),\\
a_2' &= -v_1^{(k+1)} v_2^{(k+1)},\\
a_j' &= -v_{2j-3}^{(k+1)} v_{2j-2}^{(k+1)}, \quad 3 \le j \le 2^{k+1},\\
a_j' &= 0, \quad j > 2^{k+1},\\
b_0' &= 0,\\
b_1' &= (z+1)(z-1) + v_1^{(k+1)},\\
b_j' &= (z-1)(z+1) + v_{2j-1}^{(k+1)} + v_{2j-2}^{(k+1)}, \quad 2 \le j \le 2^{k+1},\\
b_j' &= 0, \quad j > 2^{k+1}.
\end{aligned}
$$

Using the recurrence relations defined in the statement of Theorem 2.2, a quick calculation shows that we have

$$
h_{k+1}(z) = \frac{z-1}{z^2+1} + \frac{v_1^{(k)}}{z^2-1} + \frac{v_2^{(k)}}{z^2+1} + \cdots + \frac{v_{2^{k+1}-1}^{(k)}}{z^2-1}.
$$

This implies that $h_{k+1}(z) = (z-1)g_k(z^2) = g_{k+1}(z)$. □

Theorem 2.2 shows that the sequence of partial denominators of the continued fraction (2.1) is given by $z+1, z-1, z+1, z-1, \ldots$, that is, by the alternating sequence over $\{z+1, z-1\}$. For a more general $(z+b, z-b)$ phenomenon, see [8, Lemma 3.1].

3. Proof of Theorem 1.1

The key new ingredient for the proof of Theorem 1.1 is the fact that all the partial quotients of the rational functions $f_\ell(z)$ are linear. This allows

us to follow the argument of [4], with some minor changes. For the sake of readability, we keep most of the notation of [4] and we sketch how to adapt the proof of [4, Th. 11]. Instead of working with the (infinite) power series $g_{\mathbf{u}}(z)$ used in [4], we fix a positive integer ℓ and work with the (finite) power series

$$g_\ell(z) = z^{-1} f_\ell(z) = z^{-1} - z^{-2} - z^{-3} + z^{-4} + \ldots + (-1)^\ell z^{-2^\ell}.$$

Since, by Theorem 2.2, all the partial quotients of $g_\ell(z)$ are linear, the auxiliary results in [4] hold. Furthermore, for $m \ge 1$, we have

$$\begin{aligned} g_{\ell+m}(z) &= z^{-1} \prod_{h=0}^{\ell+m} (1 - z^{-2^h}) \\ &= g_\ell(z^{2^m})\, z^{2^m-1} \prod_{h=0}^{m-1} (1 - z^{-2^h}) = g_\ell(z^{2^m}) \prod_{h=0}^{m-1} (z^{2^h} - 1). \end{aligned}$$

Denoting by $p_{k,\ell}(z)/q_{k,\ell}(z)$ the convergents to $q_\ell(z)$, where $k = 1, \ldots, 2^{\ell+1}$, and defining $p_{k,\ell,m}(z)$ and $q_{k,\ell,m}(z)$ by

$$\begin{aligned} q_{k,\ell,m}(z) &= q_{k,\ell}(z^{2^m}), \\ p_{k,\ell,m}(z) &= \prod_{h=0}^{m-1} (z^{2^h} - 1)\, p_{k,\ell}(z^{2^m}), \end{aligned}$$

the analogue of [4, (2.25)] holds, namely, we have

$$\left| g_{\ell+m}(b) - \frac{p_{k,\ell,m}(b)}{q_{k,\ell,m}(b)} \right| \le \frac{2(k+1)k^{k/2}2^m}{q_{k,\ell,m}(b) \cdot b^{k2^m+1}}, \tag{3.1}$$

for $k = 1, \ldots, 2^{\ell+1}$. By [4, Lemma 9], for m large enough, the integer $q_{k,\ell,m}(b)$ is controlled and is comparable to b^{k2^m}.

We now take a large integer L (which corresponds to the integer ℓ in the statement of the theorem) and study the rate of approximation to the rational number $g_L(b)$ of denominator $b^{2^{L+1}}$ by rational numbers p/q of denominator less than $b^{2^{L+1}}$. We follow the argument of [4] and look for a power of b close to q. We use the fact that every integer n less than 2^{L+1} is rather close to a product $k2^m$, where $L = m + \ell$ and $k \le 2^{\ell+1}$. The latter constraint comes from the construction of our finite sequence of good rational approximations to $g_L(b)$. It is not required to hold in [4].

Let p/q be a rational number with $q < b^{2^{L+1}}$ and q sufficiently large (it is sufficient to assume that q exceeds b^{κ_1}, for some absolute constant κ_1) to guarantee that the real number x defined in [4, (3.2)] satisfies $x > b^2$.

As in [4, (3.8)], we set $t = (\log x)/(\log b)$; note that $t > 2$. Let $\tau \ge 2$ be a real number and denote by $\log_2$ the logarithm in base 2. Assume that

$$t + 2\tau\sqrt{t \log_2 t} < 2^{L+1}. \tag{3.2}$$

This inequality holds if, for a suitable positive κ_2, depending only on τ, we have

$$q < \exp\bigl(-\kappa_2 \log b \sqrt{\log_2 q \log_2 \log_2 q}\,\bigr) b^{2^{L+1}}. \tag{3.3}$$

There exist integers n and m and a real number α such that 2^m divides n and

$$t \le n \le t + 2\tau\sqrt{t \log_2 t}, \quad 2^m = \alpha\tau\sqrt{t \log_2 t}, \quad 1 \le \alpha \le 2.$$

It follows from (3.2) that the integer

$$k = \frac{n}{2^m} \le \frac{t}{\alpha\tau\sqrt{t \log_2 t}} + \frac{2}{\alpha}$$

is less than 2^{L-m+1}, as required.

To make use of [4, Lemmas 8 and 9], we also have to check [4, (2.26)] and [4, (2.34)], that is, that the inequalities

$$2^{2^m} \ge 4(k+1)k^{k/2}, \quad 2^{2^m} > 3k^{k/2}$$

hold. Since we have

$$\alpha\tau\sqrt{t \log_2 t} \ge \Bigl(\frac{\sqrt{t}}{\alpha\tau\sqrt{\log_2 t}} + \frac{2}{\alpha}\Bigr) \log\Bigl(\frac{\sqrt{t}}{\alpha\tau\sqrt{\log_2 t}} + \frac{2}{\alpha}\Bigr)$$

for τ large enough, both inequalities are satisfied if τ is large enough.

We conclude that we have an inequality similar to [4, (3.1)] provided that q exceeds b^{κ_1} and satisfies (3.3). Namely, there exists an absolute positive real number C such that, for every rational number p/q with

$$b^{\kappa_1} < q < b^{2^{L+1} - \kappa_2 \sqrt{\log_2 q \log_2 \log_2 q}},$$

we have

$$\left| g_L(b) - \frac{p}{q} \right| \ge \frac{1}{q^2 \exp(C \log b \sqrt{\log q \, \log\log(3q)})}.$$

Since $g_L(b)$ is a rational number of denominator $b^{2^{L+1}}$, the last inequality, with possibly a different value of C, holds if q satisfies

$$b^{2^{L+1} - \kappa_2 \sqrt{\log_2 q \log_2 \log_2 q}} \le q < b^{2^{L+1}}.$$

It also holds (again with possibly a different value of C) when $q \le b^{\kappa_1}$, by using rational approximations coming from (3.1) combined with triangle inequalites. This proves the theorem.

4. A further example

The method developed in Section 2 is not specific to the Thue–Morse sequence and may be used to derive a similar conclusion for other sequences. In this section, we give a further example, whose corresponding infinite product was considered in [2, 4].

For nonzero integers u, v with $u^2 \neq v$, let us consider, for $\ell \geq 0$,

$$\widetilde{g}_{u,v,\ell}(z) = \frac{1}{z} \prod_{h=0}^{\ell} (1 + uz^{-3^h} + vz^{-2\cdot 3^h}).$$

Inspired by [2, Theorem 1.2] where similar infinite products are studied, we define

$$\begin{aligned}
&\alpha_1^{(0)} = -u, \quad \alpha_2^{(0)} = \frac{u^3 - 2uv}{u^2 - v}, \quad \alpha_3^{(0)} = \frac{uv}{u^2 - v},\\
&\beta_1^{(0)} = 1, \quad \beta_2^{(0)} = u^2 - v, \quad \beta_3^{(0)} = \frac{v^3}{u^4 - 2u^2v + v^2},\\
&\alpha_1^{(\ell)} = -u, \quad \alpha_2^{(\ell)} = \frac{u^3 - 2uv + u}{u^2 - v}, \quad \alpha_3^{(\ell)} = \frac{uv - u}{u^2 - v},\\
&\beta_1^{(\ell)} = 1, \quad \beta_2^{(\ell)} = u^2 - v, \quad \beta_3^{(\ell)} = \frac{u^4 - 3u^2v + v^3 + u^2}{u^4 - 2u^2v + v^2},\\
&\alpha_{3k+4}^{(\ell)} = -u, \quad \beta_{3k+4}^{(\ell)} = \frac{\beta_{k+2}^{(\ell-1)}}{\beta_{3k+3}^{(\ell)}\beta_{3k+2}^{(\ell)}}, \quad \beta_{3k+5}^{(\ell)} = u^2 - v - \beta_{3k+4}^{(\ell)},\\
&\alpha_{3k+5}^{(\ell)} = u - \frac{\alpha_{k+2}^{(\ell-1)} + uv - \alpha_{3k+2}^{(\ell)}\beta_{3k+4}^{(\ell)}}{\beta_{3k+5}^{(\ell)}}, \quad \alpha_{3k+6}^{(\ell)} = u - \alpha_{3k+5}^{(\ell)},\\
&\beta_{3k+6}^{(\ell)} = v - \alpha_{3k+5}^{(\ell)}\alpha_{3k+6}^{(\ell)}.
\end{aligned}$$

We claim that

$$\widetilde{g}_{u,v,\ell}(z) = \cfrac{\beta_1^{(\ell)}}{z + \alpha_1^{(\ell)}} + \cfrac{\beta_2^{(\ell)}}{z + \alpha_2^{(\ell)}} + \cfrac{\beta_3^{(\ell)}}{z + \alpha_3^{(\ell)}} + \cdots + \cfrac{\beta_{3^{\ell+1}}^{(\ell)}}{z + \alpha_{3^{\ell+1}}^{(\ell)}}, \quad \ell \geq 0,$$

where some of the rational numbers $\beta_j^{(\ell)}$ may vanish.

In the sequel, we consider only the case $u = v = -1$, that is,

$$\widetilde{g}_{\ell}(z) = \frac{1}{z} \prod_{h=0}^{\ell} (1 - z^{-3^h} - z^{-2\cdot 3^h}),$$

and establish the non-vanishing result, also by using the 2-adic valuation. Set

$$\alpha_1^{(0)} = 1, \quad \alpha_2^{(0)} = -\frac{3}{2}, \quad \alpha_3^{(0)} = \frac{1}{2},$$

$$\begin{aligned}
&\beta_1^{(0)} = 1, \quad \beta_2^{(0)} = 2, \quad \beta_3^{(0)} = -\frac{1}{4},\\
&\alpha_1^{(\ell)} = 1, \quad \alpha_2^{(\ell)} = -2, \quad \alpha_3^{(\ell)} = 1,\\
&\beta_1^{(\ell)} = 1, \quad \beta_2^{(\ell)} = 2, \quad \beta_3^{(\ell)} = 1,\\
&\alpha_{3k+4}^{(\ell)} = 1, \quad \beta_{3k+4}^{(\ell)} = \frac{\beta_{k+2}^{(\ell-1)}}{\beta_{3k+3}^{(\ell)}\beta_{3k+2}^{(\ell)}}, \quad \beta_{3k+5}^{(\ell)} = 2 - \beta_{3k+4}^{(\ell)},\\
&\alpha_{3k+5}^{(\ell)} = -1 - \frac{\alpha_{k+2}^{(\ell-1)} + 1 - \alpha_{3k+2}^{(\ell)}\beta_{3k+4}^{(\ell)}}{\beta_{3k+5}^{(\ell)}}, \quad \alpha_{3k+6}^{(\ell)} = -1 - \alpha_{3k+5}^{(\ell)},\\
&\beta_{3k+6}^{(\ell)} = -1 - \alpha_{3k+5}^{(\ell)}\alpha_{3k+6}^{(\ell)}.
\end{aligned}$$

Proposition 4.1. *For $\ell \geq 0$ and $j = 1, \dots, 3^{\ell+1}$, we have*

- *$\nu_2(\alpha_j^{(\ell)}) = -1$ if and only if $j = (3^{\ell+1}+1)/2$ or $j = (3^{\ell+1}+3)/2$; $\alpha_j^{(\ell)} \neq 0$ and $\nu_2(\alpha_j^{(\ell)}) \geq 0$ otherwise;*
- *$\nu_2(\beta_2^{(\ell)}) = 1$; $\nu_2(\beta_j^{(\ell)}) = -2$ if and only if $j = (3^{\ell+1}+3)/2$; $\nu_2(\beta_2^{(\ell)}) = 0$ otherwise.*

As a conseqence, $\beta_j^{(\ell)}$ is nonzero for all $\ell \geq 0$ and $j = 1, 2, \dots, 3^{\ell+1}$.

It follows from Proposition 4.1 that the analogue of Theorem 1.1 holds for the products $z\widetilde{g}_\ell(z)$, namely, there exists a positive real number K such that, for every integer $b \geq 2$ and every integer $\ell \geq 2$, the inequality

$$\Big|\prod_{h=0}^{\ell}(1 - b^{-3^h} - b^{-2\cdot 3^h}) - \frac{p}{q}\Big| > \frac{1}{q^2 \exp(K \log b \sqrt{\log q \, \log\log(3q)})},$$

holds for every rational number p/q different from $\widetilde{g}_\ell(b)$. We omit the details.

Proof. We proceed by induction on ℓ, j. The basis of the induction is clear from the definition of the $\alpha_j^{(\ell)}$. We only display the more complicated steps. We start with the $\alpha_j^{(\ell)}$'s. Since $(3^{\ell+1}+1)/2$ is equal to $3k+5$ with $k = (3^\ell - 3)/2$,

$$\nu_2(\alpha_{(3^{\ell+1}+1)/2}^{(\ell)}) = \nu_2\left(-1 - \frac{\alpha_{k+2}^{(\ell-1)} + 1 - \alpha_{3k+2}^{(\ell)}\beta_{3k+4}^{(\ell)}}{\beta_{3k+5}^{(\ell)}}\right) = -1,$$

because

$$\nu_2(\alpha_{k+2}^{(\ell-1)}) = -1, \quad \nu_2(\alpha_{3k+2}^{(\ell)}) \geq 0, \quad \nu_2(\beta_{3k+4}^{(\ell)}) = 0, \quad \nu_2(\beta_{3k+5}^{(\ell)}) = 0.$$

Then,

$$\nu_2(\alpha_{(3^{\ell+1}+3)/2}^{(\ell)}) = \nu_2\left(-1 - \alpha_{(3^{\ell+1}+1)/2}^{(\ell)}\right) = -1.$$

Since $(3^{\ell+1}+7)/2$ is equal to $3k+5$ with $k=(3^\ell-1)/2$,

$$\nu_2(\alpha^{(\ell)}_{(3^{\ell+1}+7)/2}) = \nu_2\left(-1-\frac{\alpha^{(\ell-1)}_{k+2}+1-\alpha^{(\ell)}_{3k+2}\beta^{(\ell)}_{3k+4}}{\beta^{(\ell)}_{3k+5}}\right) \ge 0,$$

because

$$\nu_2(\alpha^{(\ell-1)}_{k+2}) = -1, \quad \nu_2(\alpha^{(\ell)}_{3k+2}) = -1, \quad \nu_2(\beta^{(\ell)}_{3k+4}) = 0, \quad \nu_2(\beta^{(\ell)}_{3k+5}) = 0.$$

Then,

$$\nu_2(\alpha^{(\ell)}_{(3^{\ell+1}+9)/2}) = \nu_2\left(-1-\alpha^{(\ell)}_{(3^{\ell+1}+7)/2}\right) \ge 0.$$

Suppose that $j \ne (3^{\ell+1}+1)/2, (3^{\ell+1}+3)/2, (3^{\ell+1}+7)/2, (3^{\ell+1}+9)/2$. If $j=3k+4$, then $\nu_2(\alpha^{(\ell)}_j)=0$. If $j=3k+5$ with $k \ne (3^\ell-3)/2, (3^\ell-1)/2$, then

$$\nu_2(\alpha^{(\ell)}_{3k+5}) = \nu_2\left(-1-\frac{\alpha^{(\ell-1)}_{k+2}+1-\alpha^{(\ell)}_{3k+2}\beta^{(\ell)}_{3k+4}}{\beta^{(\ell)}_{3k+5}}\right) \ge 0,$$

because

$$\nu_2(\alpha^{(\ell-1)}_{k+2}) \ge 0, \quad \nu_2(\alpha^{(\ell)}_{3k+2}) \ge 0, \quad \nu_2(\beta^{(\ell)}_{3k+4}) = 0, \quad \nu_2(\beta^{(\ell)}_{3k+5}) = 0.$$

Then,

$$\nu_2(\alpha^{(\ell)}_{3k+6}) = \nu_2(-1-\alpha^{(\ell)}_{3k+5}) \ge 0.$$

We now deal with the $\beta^{(\ell)}_j$'s. Since $(3^{\ell+1}+3)/2 = 3k+6$ with $k=(3^\ell-3)/2$,

$$\beta^{(\ell)}_{(3^{\ell+1}+3)/2} = \nu_2(-1-\alpha^{(\ell)}_{3k+5}\alpha^{(\ell)}_{3k+6}) = -2,$$

because

$$\nu_2(\alpha^{(\ell)}_{3k+5}) = \nu_2(\alpha^{(\ell)}_{3k+6}) = -1.$$

Suppose that $j \ne (3^{\ell+1}+3)/2$. If $j=3k+4$ with $k \ne (3^{\ell+1}-1)/2$, then

$$\nu_2(\beta^{(\ell)}_{3k+4}) = \nu_2\left(\frac{\beta^{(\ell-1)}_{k+2}}{\beta^{(\ell)}_{3k+3}\beta^{(\ell)}_{3k+2}}\right) = 0,$$

because

$$\nu_2(\beta^{(\ell-1)}_{k+2}) = 0, \nu_2(\beta^{(\ell)}_{3k+3}) = 0, \nu_2(\beta^{(\ell)}_{3k+2}) = 0.$$

If $j=3k+4$ with $k=(3^{\ell+1}-1)/2$, then

$$\nu_2(\beta^{(\ell)}_{3k+4}) = \nu_2\left(\frac{\beta^{(\ell-1)}_{k+2}}{\beta^{(\ell)}_{3k+3}\beta^{(\ell)}_{3k+2}}\right) = 0,$$

because

$$\nu_2(\beta^{(\ell-1)}_{k+2}) = -2, \nu_2(\beta^{(\ell)}_{3k+3}) = -2, \nu_2(\beta^{(\ell)}_{3k+2}) = 0.$$

In both case,

$$\nu_2(\beta^{(\ell)}_{3k+5}) = \nu_2(2 - \beta^{(\ell)}_{3k+4}) = 0.$$

If $j = 3k + 6$ with $k \neq (3^\ell - 3)/2$,

$$\beta^{(\ell)}_{3k+6} = \nu_2(-1 - \alpha^{(\ell)}_{3k+5}\alpha^{(\ell)}_{3k+6}) = 0,$$

because

$$\nu_2(\alpha^{(\ell)}_{3k+5}) \geq 0, \nu_2(\alpha^{(\ell)}_{3k+6}) \geq 0,$$

and if $\nu_2(\alpha^{(\ell)}_{3k+5}) = 0$, then $\nu(\alpha^{(\ell)}_{3k+6}) \geq 1$.

□

Acknowledgements. The authors are very grateful to the referees for their numerous remarks.

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