THE THUE–MORSE CONTINUED FRACTIONS IN CHARACTERISTIC 2 ARE ALGEBRAIC

YANN BUGEAUD AND GUO-NIU HAN

Abstract. Let a, b be distinct, non-constant polynomials in $\mathbb{F}_2[z]$. Let $\xi_{a,b}$ be the power series in $\mathbb{F}_2((z^{-1}))$ whose sequence of partial quotients is the Thue–Morse sequence over $\{a, b\}$. We establish that $\xi_{a,b}$ is algebraic of degree 4.

1. Introduction

An infinite sequence $(a_n)_{n\geq 0}$ over a finite alphabet $\mathcal{A}$ is *k-automatic* for an integer $k \geq 2$ if it can be generated by a finite automaton which reads the representation in base k of a non-negative integer n from right to left and outputs an element a_n in $\mathcal{A}$. It is called *automatic* if it is k-automatic for some integer $k \geq 2$. In 1968, Cobham [14] conjectured that, for any integer $b \geq 2$, the base-b expansion of an irrational real number never forms an automatic sequence. This was confirmed in 2007 by Adamczewski and Bugeaud [1] (see [3, 30] for alternative proofs), who also established that if the Hensel expansion of an irrational p-adic number ξ is an automatic sequence, then ξ is transcendental. In a similar spirit, Bugeaud [10] proved that the sequence of partial quotients of an algebraic real number of degree at least 3 never forms an automatic sequence.

Analogous questions can be asked for power series over a finite field, but the answers are different. Throughout the paper, we let q denote a power of a prime number p and $\mathbb{F}_q$ denote the field with q elements. In 1979 Christol [12, 13] established that a power series in $\mathbb{F}_q((z^{-1}))$ is algebraic over $\mathbb{F}_q(z)$ if and only if the sequence of its coefficients is q-automatic (or, equivalently, is p-automatic).

In analogy with the continued fraction algorithm for real numbers, there is a well-studied continued fraction algorithm for power series in $\mathbb{F}_q((z^{-1}))$, the partial quotients being nonconstant polynomials in $\mathbb{F}_q[z]$. In both settings, eventually periodic expansions correspond to quadratic elements, but much more is known on the continued fraction expansion of algebraic power series than on that of algebraic real numbers. In 1949 Mahler [26] established that, for $p \geq 3$, the root

$$z^{-1} + z^{-p} + z^{-p^2} + \dots$$

in $\mathbb{F}_q((z^{-1}))$ of the polynomial $zX^p - zX + 1$ has unbounded partial quotients, in the sense that their degrees are unbounded. In the opposite direction, there exist power series which are algebraic of degree at least 3 and have bounded partial quotients (recall that analogous statements have not yet been proved, nor disproved, for real numbers): in 1976, Baum and Sweet [9] proved that the continued

---

2010 *Mathematics Subject Classification.* 11J70, 11A55, 11J61, 11T55.

Key words and phrases. Power series field, Continued fraction, Transcendence.

fraction expansion of the unique solution ξ_{BS} in $\mathbb{F}_2((z^{-1}))$ of the equation

$$zX^3 + X + z = 0$$

has all its partial quotients of degree at most 2.

At the end of the 80s, Mendès France asked whether the sequence of partial quotients of an algebraic power series in $\mathbb{F}_q((z^{-1}))$ is q-automatic, as soon as it takes only finitely many different values. A positive answer has been given by Allouche [5] and Allouche, Bétréma, and Shallit [6] for the algebraic power series of degree at least 3 in $\mathbb{F}_p((z^{-1}))$ (here, $p \geq 3$) which have been constructed by Mills and Robbins [27] and whose partial quotients are polynomials of degree one; see also Lasjaunias and Yao [23–25]. However, in 1995 Mkaouar [28] (see also Yao [33] for an alternative proof) gave a negative answer to the question of Mendès France by establishing that the sequence of partial quotients of the Baum–Sweet power series ξ_{BS} is morphic, but not automatic. Recall that a sequence is called morphic if it is the image under a coding of a fixed point of a substitution. If the substitution can be chosen of constant length k, then the sequence is k-automatic. The equivalence between the two definitions of automatic sequences given here was established by Cobham [15]; see [7].

All this shows that the sequence of partial quotients of an algebraic power series may or may not be automatic. Conversely, very little is known about the Diophantine nature of a power series whose sequence of partial quotients is automatic, but not ultimately periodic. The only contribution to this question is a recent work of Hu and Han [19]. They proved that, for any distinct non-constant polynomials a and b in $\mathbb{F}_2[z]$ whose sum of degrees is at most 7, the power series whose sequence of partial quotients is the Thue–Morse sequence written over $\{a, b\}$ is algebraic of degree 4. Recall that the Thue–Morse word

$$\mathbf{t} = t_0 t_1 t_2 \ldots = abbabaabbaababbabaababbaabbabaab \ldots$$

over $\{a, b\}$ is defined by $t_0 = a$, $t_{2k} = t_k$ and $t_{2k+1} = a$ (resp., b) if $t_k = b$ (resp., a), for $k \geq 0$. The Thue–Morse sequence is the most famous automatic sequence and is not ultimately periodic.

To establish their result, Hu and Han made use of a Guess 'n' Prove method and implemented a program which takes a pair (a, b) of distinct non-constant polynomials as input and outputs its minimal defining polynomial and a complete proof. The time needed grows with the degrees of a and b. Hu and Han conjectured that their result holds more generally for every pair (a, b) of distinct non-constant polynomials.

In the present paper, built on [19], we confirm this conjecture. Our main results are stated in Section 2 and proved in Sections 4 and 5. We consider higher degree exponents of approximation in Section 3. Additional remarks are gathered in Section 6.

2. Results

Let a and b be distinct non-constant polynomials in $\mathbb{F}_q[z]$ and let $\xi_{q,a,b}$ denote the power series in $\mathbb{F}_q((z^{-1}))$ whose sequence of partial quotients is the Thue–Morse sequence $\mathbf{t}$ over the alphabet $\{a, b\}$. Since $\mathbf{t}$ is not ultimately periodic, $\xi_{q,a,b}$

is transcendental or algebraic of degree at least 3 over $\mathbb{F}_q(z)$. Furthermore, as $\mathbf{t}$ begins with infinitely palindromes, an argument given in the proof of Theorem 3.2 shows that $\xi_{q,a,b}$ and its square are very well simultaneously approximable by rational fractions with the same denominator and that, consequently, $\xi_{q,a,b}$ cannot be algebraic of degree 3. This proves that $\xi_{q,a,b}$ is transcendental or algebraic of degree at least 4 over $\mathbb{F}_q(z)$.

Our main result asserts that, in the case $q = 2$, any Thue–Morse continued fraction $\xi_{2,a,b} = \xi_{a,b}$ is algebraic of degree 4 over $\mathbb{F}_2(z)$.

Theorem 2.1. *Let a, b be non-constant, distinct polynomials in $\mathbb{F}_2[z]$ and let*

$$\xi_{a,b} = [0; a, b, b, a, b, a, a, b, b, a, a, b, a, b, b, a, \ldots]$$

be the power series in $\mathbb{F}_2((z^{-1}))$ whose sequence of partial quotients is the Thue–Morse sequence $\mathbf{t}$ over the alphabet $\{a, b\}$. Then, $\xi_{a,b}$ is algebraic of degree 4 over $\mathbb{F}_2(z)$. More precisely, setting

$$\begin{aligned} A_0 &= b(a+b)(a^2b^2 + a^2 + b^2) + a^2b^4, \\ A_1 &= ab(a+b)(a^2b^2 + a^2 + b^2), \\ A_2 &= a^2b^2(a^2b^2 + a^2 + b^2), \\ A_3 &= A_1, \\ A_4 &= a(a+b)(a^2b^2 + a^2 + b^2) + a^2b^4, \end{aligned}$$

we have

$$A_4\xi_{a,b}^4 + A_3\xi_{a,b}^3 + A_2\xi_{a,b}^2 + A_1\xi_{a,b} + A_0 = 0.$$

Remark. In Theorem 2.1 the ground field $\mathbb{F}_2$ can be replacd by any field K of characteristic 2, and a and b by any non-constant, distinct polynomials in $K[z]$.

We define an absolute value $|\cdot|$ on the field $\mathbb{F}_q((z^{-1}))$ by setting $|0| = 0$ and, for any non-zero power series $\xi = \xi(z) = \sum_{h=-m}^{+\infty} a_h z^{-h}$ with $a_{-m} \neq 0$, by setting $|\xi| = q^m$. We write $||\xi||$ for the norm of the fractional part of ξ, that is, of the part of the series which comprises only the negative powers of z.

For a power series ξ in $\mathbb{F}_q((z^{-1}))$, let

$$C(\xi) = \liminf_{Q \in \mathbb{F}_q[z] \setminus \{0\}} |Q| \cdot \|Q\xi\|$$

denote its Lagrange constant. Clearly, $C(\xi)$ is an element of

$$\mathcal{L}_q = \{q^{-k} : k \geq 1\} \cup \{0\},$$

and it is positive if and only if the degrees of the partial quotients of ξ are bounded.

If $[a_0; a_1, a_2, \ldots]$ denotes the continued fraction expansion of ξ and $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ for $n \geq 1$, then

$$|q_n\xi - p_n| = |q_{n+1}|^{-1} = |a_{n+1}q_n|^{-1}.$$

Consequently, we have

$$C(\xi) = 2^{-\limsup_{n\to+\infty} \deg a_n}.$$

Exactly as in [26], we can check that the root $z^{-1}+z^{-4}+z^{-4^2}+\dots$ in $\mathbb{F}_2((z^{-1}))$ of zX^4+zX+1 is algebraic of degree 4 and has unbounded partial quotients, thus its Lagrange constant is 0. Since the degrees of a and b can be arbitrarily chosen in Theorem 2.1, we derive at once the following corollary.

Corollary 2.2. *There exist algebraic power series of degree* 4 *in* $\mathbb{F}_2((z^{-1}))$ *with an arbitrarily prescribed Lagrange constant in* $\mathcal{L}_2$.

A power series ξ in $\mathbb{F}_q((z^{-1}))$ is called badly approximable when the degrees of its partial quotients are uniformly bounded. We define the spectrum of a badly approximable power series ξ in $\mathbb{F}_q((z^{-1}))$ as the set of positive integers k such that ξ has infinitely many partial quotients of degree k.

Problem 2.3. *Let* $\mathcal{N}$ *be a finite set of positive integers. Does there exist a badly approximable power series* ξ *in* $\mathbb{F}_q((z^{-1}))$ *which is algebraic of degree at least* 3 *and whose spectrum is equal to* $\mathcal{N}$*?*

Theorem 2.1 answers positively Problem 2.3 for $q=2$ and any set $\mathcal{N}$ of cardinality 2.

For a power q of a prime number p the class $H(q)$ of hyperquadratic power series in $\mathbb{F}_q((z^{-1}))$ has attracted special attention; see e.g. [21, 31]. It is composed of the irrational power series ξ in $\mathbb{F}_q((z^{-1}))$ for which there exist polynomials A, B, C, D in $\mathbb{F}_q[z]$ such that $AD-BC\neq 0$ and an integer s with

$$\xi=\frac{A\xi^{p^s}+B}{C\xi^{p^s}+D}.$$

Once a power series is known to be algebraic, a natural question is to determine whether it belongs to the restricted class of hyperquadratic power series.

In another direction, since the pioneering works of Kolchin [20] and Osgood [29], formal derivation has been used to study rational approximation to power series. By differentiating with respect to z the minimal defining polynomial satisfied by an algebraic power series ξ in $\mathbb{F}_q((z^{-1}))$ of degree d, we see that ξ', where the $'$ indicates the derivation on $\mathbb{F}_q((z^{-1}))$ with respect to z extending the derivation on $\mathbb{F}_q[z]$, can be expressed as a polynomial in ξ of degree at most $d-1$.

We say that a power series ξ in $\mathbb{F}_q((z^{-1}))$ satisfies a Riccati differential equation if there are A, B and C in $\mathbb{F}_q(z)$ such that

$$\xi'=A\xi^2+B\xi+C.$$

Clearly, any cubic power series satisfies a Riccati equation. This is also the case for any hyperquadratic power series, but the converse does not hold; see e.g. [22, Section 4].

Proposition 2.4. *Let* a, b *be non-constant, distinct polynomials in* $\mathbb{F}_2[z]$*. Then, the Thue–Morse continued fraction* $\xi_{a,b}$ *in* $\mathbb{F}_2((z^{-1}))$ *is differential-quadratic: it satisfies the Riccati equation*

$$[(ab(a+b))x]'=(ab)'(1+x^2).$$

Furthermore, $\xi_{a,b}$ *is not hyperquadratic.*

Remark. The Riccati equation in Proposition 2.4 is equivalent to $[(ab(a+b))x]' = [ab(1+x^2)]'$. Its solutions are the power series ξ such that $ab(a+b)\xi + ab(1+\xi^2)$ is a square, that is, such that $1 + (a+b)\xi + \xi^2$ is a linear combinaison of terms $a^i b^j$, with i and j odd.

The strategy of the proof of Theorem 2.1 is the following. After a careful study of the minimal defining polynomials $P_{a,b}(X)$ of $\xi_{a,b}$ found by Hu and Han [19] in the case where the sum of the degrees of a and b is at most 7, we have guessed the coefficients, expressed in terms of a and b, of the quartic polynomial $P_{a,b}(X)$ which vanishes at $\xi_{a,b}$, for *any* distinct, non-constant polynomials a and b. It then only remained for us to check that $P_{a,b}(\xi_{a,b}) = 0$. This step is, however, much more difficult than it may seem to be. Denoting by p_ℓ/q_ℓ (we drop the letters a, b) the ℓ-th convergent of $\xi_{a,b}$ for $\ell \geq 1$, it is sufficient to check that $P_{a,b}(p_\ell/q_\ell)$ tends to 0 as ℓ tends to infinity along a subsequence of the integers. We focus on the indices ℓ which are powers of 4. We heavily use the properties of symmetry of the Thue–Morse sequence and we proceed by induction to show that $|P_{a,b}(p_{4^k}/q_{4^k})|$ is, for $k \geq 1$, less than $|q_{4^k}|^{-2}$ times some constant independent of k.

3. Higher degree exponents of approximation

Beside the rational approximation to a power series ξ in $\mathbb{F}_q((z^{-1}))$, we often consider the simultaneous rational approximation of $\xi, \xi^2, \ldots, \xi^n$ by rational fractions with the same denominator, as well as small values of the linear form $b_0 + b_1\xi + \ldots + b_n\xi^n$ with coefficients in $\mathbb{F}_q[z]$. This leads us to introduce the exponents of approximation w_n and λ_n, defined below. For a survey of recent results on these exponents evaluated at real numbers, the reader is directed to [11].

The height $H(P)$ of a polynomial $P(X) = b_n(z)X^n + \ldots + b_1(z)X + b_0(z)$ over $\mathbb{F}_q[z]$ is the maximum of the absolute values of its coefficients, that is, of $|b_0|, |b_1|, \ldots, |b_n|$. Recall that the 'fractional part' $\|\cdot\|$ is defined by

$$\Big\| \sum_{h=-m}^{+\infty} a_h z^{-h} \Big\| = \Big| \sum_{h=1}^{+\infty} a_h z^{-h} \Big|,$$

for every power series $\xi = \sum_{h=-m}^{+\infty} a_h z^{-h}$ in $\mathbb{F}_q((z^{-1}))$.

Definition 3.1. *Let ξ be in $\mathbb{F}_q((z^{-1}))$. Let $n \geq 1$ be an integer. We let $w_n(\xi)$ denote the supremum of the real numbers w for which*

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in polynomials $P(X)$ over $\mathbb{F}_q[z]$ of degree at most n. We let $\lambda_n(\xi)$ denote the supremum of the real numbers λ for which

$$0 < \max\{\|Q(z)\xi\|, \ldots, \|Q(z)\xi^n\|\} < q^{-\lambda \deg(Q)}$$

has infinitely many solutions in polynomials $Q(z)$ in $\mathbb{F}_q[z]$. For positive real numbers w, λ, set

$$B_n(\xi, w) = \liminf_{H(P)\to+\infty} H(P)^w \cdot |P(\xi)|$$

and

$$B'_n(\xi,\lambda) = \liminf_{|Q|\to+\infty} |Q|^\lambda \cdot \max\{\|Q(z)\xi\|, \dots, \|Q(z)\xi^n\|\}.$$

Let ξ be an algebraic power series in $\mathbb{F}_q((z^{-1}))$ of degree $d \geq 2$. Let $n \geq 1$ be an integer. We briefly show how a Liouville-type argument allows us to bound $w_n(\xi)$ from above. Denote by $\xi_1 = \xi, \xi_2, \dots, \xi_d$ the Galois conjugates of ξ. Let $P(X)$ be a non-zero polynomial in $\mathbb{F}_q[z](X)$. Then, the product $P(\xi_1)\dots P(\xi_d)$ is a nonzero element of $\mathbb{F}_q(z)$, whose absolute value is bounded from below by $c_1(\xi)$, which (as $c_2(\xi)$ and $c_3(\xi)$ below) is positive and depends only on ξ. Since $|P(\xi_j)|$, $j = 2, \dots, d$, are bounded from above by $c_2(\xi)$ times $H(P)$, we get that

$$|P(\xi)| > c_3(\xi) H(P)^{-d+1}$$

and we derive that

$$w_n(\xi) \leq d - 1, \quad n \geq 1. \tag{3.1}$$

In the particular case of $\xi_{a,b}$, this gives $w_3(\xi_{a,b}) \leq 3$, by Theorem 2.1. Actually, this inequality is an equality.

Theorem 3.2. *The Thue–Morse power series $\xi_{a,b}$ in $\mathbb{F}_2((z^{-1}))$ satisfies*

$$\lambda_2(\xi_{a,b}) = 1, \quad w_2(\xi_{a,b}) = 3, \quad w_n(\xi_{a,b}) = 3, \quad \lambda_n(\xi_{a,b}) = \frac{1}{3}, \quad n \geq 3.$$

Moreover, there are positive constants c_4, c_5, depending only on a and b, such that

$$|P(\xi_{a,b})| > c_4 H(P)^{-3}, \quad \textit{for every nonzero } P(X) \textit{ in } \mathbb{F}_2[z](X) \textit{ of degree} \leq 3,$$

and there are polynomials $P(X)$ in $\mathbb{F}_2[z](X)$ of degree 2 of arbitrarily large height such that

$$|P(\xi_{a,b})| < c_5 H(P)^{-3}.$$

Proof. Since the continued fraction expansion of $\xi_{a,b}$ begins with arbitrarily large palindromes, $\xi_{a,b}$ and its square are simultaneously well approximable by rational fractions with the same denominator. This argument appeared in [2] and we recall it below for the sake of completeness. For $k \geq 0$, let p_k/q_k denote the k-th convergent to $\xi_{a,b}$. Recall that the Thue–Morse word $\mathbf{t} = t_0t_1t_2\dots$ over $\{a,b\}$ is the fixed point starting with a of the uniform morphism τ defined by $\tau(a) = ab$ and $\tau(b) = ba$. Since the words $\tau^2(a) = abba$ and $\tau^2(b) = baab$, and the prefix of length 4 of $\mathbf{t} = abba\dots$ are palindromes, every prefix of $\mathbf{t}$ of length a power of 4 is a palindrome. Put

$$\mathcal{M}_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{M}_b = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, for $k \geq 1$, we have

$$\mathcal{M}_{t_0}\mathcal{M}_{t_1}\dots\mathcal{M}_{t_{k-1}} = \begin{pmatrix} q_k & q_{k-1} \\ p_k & p_{k-1} \end{pmatrix}.$$

Take $k = 4^\ell$ for some positive integer ℓ. The matrix $\mathcal{M}_{t_0}\mathcal{M}_{t_1}\dots\mathcal{M}_{t_{k-1}}$ is symmetric since $\mathcal{M}_a$ and $\mathcal{M}_b$ are symmetric and $t_0t_1\dots t_{k-1}$ is a palindrome. Consequently, $p_k = q_{k-1}$. It then follows from $t_{k-1} = a$, $t_k = b$ and the theory of continued fractions in $\mathbb{F}_2((z^{-1}))$ that

$$\Big|\xi_{a,b} - \frac{p_k}{q_k}\Big| = \frac{1}{|b||q_k|^2}, \quad \Big|\xi_{a,b} - \frac{p_{k-1}}{q_{k-1}}\Big| = \frac{1}{|a||q_{k-1}|^2} = \frac{1}{|q_{k-1}|\cdot|q_k|},$$

thus

$$\Big|\xi_{a,b}^2 - \frac{p_{k-1}}{q_k}\Big| = \Big|\Big(\xi_{a,b} - \frac{p_k}{q_k}\Big)\Big(\xi_{a,b} + \frac{p_{k-1}}{q_{k-1}}\Big) + \frac{\xi_{a,b}}{q_{k-1}q_k}\Big| = \frac{1}{|q_k|^2},$$

since $|\xi_{a,b}| = 1/|a|$. We conclude that

$$\max\{\|q_k\xi_{a,b}\|, \|q_k\xi_{a,b}^2\|\} = \frac{1}{|q_k|}.$$

This gives $\lambda_2(\xi_{a,b}) \geq 1$ and there is equality since $\lambda_1(\xi_{a,b}) = 1$. Furthermore, the quantity $B_2'(\xi_{a,b}, 1)$ is finite, since $\xi_{a,b}$ has bounded partial quotients.

By the power series field analogue of a classical transference inequality established by Aggarwal [4], we immediately obtain that $B_2(\xi_{a,b}, 3)$ is finite, thus $w_2(\xi_{a,b}) \geq 3$.

This lower bound holds as well for $w_2(\xi_{q,a,b})$. Combined with (3.1), this shows that $\xi_{q,a,b}$ is transcendental or algebraic of degree at least 4, thereby establishing the assertion stated at the end of the first paragraph of Section 2.

Since $\xi_{a,b}$ is algebraic of degree 4, by Theorem 2.1, the Liouville-type result obtained below Definition 3.1 combined with the lower bound $w_2(\xi_{a,b}) \geq 3$ implies that $w_2(\xi_{a,b}) = w_3(\xi_{a,b}) = 3$ and

$$w_n(\xi_{a,b}) = 3, \quad \lambda_n(\xi_{a,b}) = \frac{1}{3}, \quad n \geq 3,$$

the value of $\lambda_3(\xi_{a,b})$ being a consequence of a transference inequality established in [4]. □

4. Proof of Theorem 2.1

We keep the notation of Theorem 2.1 and check that $\xi_{a,b}$ is a root of the equation

$$A_4X^4 + \ldots + A_1X + A_0 = 0.$$

In view of the discussion at the beginning of Section 2, this implies that $\xi_{a,b}$ is algebraic of degree 4.

The computation is easier if we replace a and b by their inverses, that is, if we consider the continued fraction

$$\zeta = [0; a^{-1}, b^{-1}, b^{-1}, a^{-1}, b^{-1}, a^{-1}, a^{-1}, b^{-1}, \ldots]. \tag{4.1}$$

Definition 4.1. *Set*

$$M_0(a,b) = a\begin{pmatrix} a^{-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix}$$

and

$$M_k(a,b) = M_{k-1}(a,b)\cdot M_{k-1}(b,a)^2 \cdot M_{k-1}(a,b), \quad k \geq 1,$$

$$N_k(a,b) = M_{k-1}(a,b)\cdot M_{k-1}(b,a), \quad k \geq 1.$$

Observe that

$$M_1(a,b) = \begin{pmatrix} a^2b^2+b^2+1 & a^2b+ab^2+a \\ a^2b+ab^2+a & a^2b^2+a^2 \end{pmatrix}$$

and that, for $k \ge 1$, the matrix $M_k(a,b)$ is symmetric and of the form

$$\begin{pmatrix} 1+\cdots+(ab)^{2^{2k-1}} & a+\cdots+(a+b)^{(2^{2k-1}+1)/3}(ab)^{2^{2k-2}} \\ a+\cdots+(a+b)^{(2^{2k-1}+1)/3}(ab)^{2^{2k-2}} & a^2+\cdots+(ab)^{2^{2k-1}} \end{pmatrix}.$$

An immediate induction shows that its entries are polynomials in a and b, whose degrees in a (resp., in b) are at most equal to 2^{2k-1}.

For a polynomial $M(a,b)$ (or a matrix $M(a,b)$ whose coefficients are polynomials) in the variables a and b, write $\widetilde{M(a,b)}$ the image of $M(a,b)$ under the involution which exchanges a and b, that is, $\widetilde{M(a,b)} = M(b,a)$.

It follows from the definition of $N_k(a,b)$ that its upper left and lower right entries are symmetrical in a and b, while its upper right entry is obtained from its lower left one by exchanging a and b. We introduce some further notation.

Definition 4.2. *Write*

$$M_k(a,b) = \begin{pmatrix} Q_k(a,b) & P_k(a,b) \\ P_k(a,b) & R_k(a,b) \end{pmatrix}, \quad k \ge 0,$$

and

$$N_k(a,b) = \begin{pmatrix} U_k(a,b) & V_k(a,b) \\ V_k(b,a) & W_k(a,b) \end{pmatrix}, \quad k \ge 1,$$

To shorten the notation, we simply write

$$M_k = \begin{pmatrix} Q_k & P_k \\ P_k & R_k \end{pmatrix}, \quad N_k = \begin{pmatrix} U_k & V_k \\ \widetilde{V}_k & W_k \end{pmatrix}, \quad k \ge 1,$$

and observe that

$$U_k = \widetilde{U}_k, \quad W_k = \widetilde{W}_k, \quad k \ge 1.$$

Set

$$\tau = 1 + a + b$$

and define

$$n_k = 2^{2k-1}, \quad k \ge 1.$$

First, we establish several relations between the entries of M_k and $\widetilde{M}_k$.

Proposition 4.3. *For $k \ge 1$, we have*

$$Q_k + R_k = \tau^{(2n_k+2)/3},$$
$$P_k + \widetilde{P}_k = (1+\tau)\tau^{(2n_k-4)/3},$$
$$Q_k + \widetilde{R}_k = \tau^{(2n_k-4)/3}.$$

Proof. We proceed by induction. Check that

$$Q_1 + R_1 = a^2 + b^2 + 1 = \tau^2,$$
$$P_1 + \widetilde{P}_1 = a + b = 1 + \tau = (1+\tau)\tau^0,$$

and

$$Q_1 + \widetilde{R}_1 = 2a^2b^2 + 2b^2 + 1 = 1 = \tau^0.$$

Set

$$m_k = (2n_k + 2)/3 = (2^{2k} + 2)/3, \quad k \geq 0,$$

and observe that

$$m_k = 4m_{k-1} - 2, \quad k \geq 1.$$

Since $M_k = N_k \widetilde{N}_k$, we have

$$Q_k = U_k^2 + V_k^2, \quad R_k = \widetilde{V}_k^2 + W_k^2, \quad P_k = U_k \widetilde{V}_k + V_k W_k, \quad k \geq 1.$$

Let $k \geq 2$ be an integer and assume that the proposition holds for the index $k-1$. Since τ is invariant by exchanging a and b, it follows from the induction hypothesis that

$$R_{k-1} + Q_{k-1} = \widetilde{R}_{k-1} + \widetilde{Q}_{k-1} = \tau^{m_{k-1}}.$$

Consequently, we get

$$\begin{aligned} V_k + \widetilde{V}_k &= Q_{k-1}\widetilde{P}_{k-1} + P_{k-1}\widetilde{R}_{k-1} + \widetilde{Q}_{k-1}P_{k-1} + \widetilde{P}_{k-1}R_{k-1} \\ &= \widetilde{P}_{k-1}(Q_{k-1} + R_{k-1}) + P_{k-1}(\widetilde{Q}_{k-1} + \widetilde{R}_{k-1}) \\ &= (P_{k-1} + \widetilde{P}_{k-1})(Q_{k-1} + R_{k-1}) \\ &= (1+\tau)\tau^{m_{k-1}-2}\tau^{m_{k-1}}. \end{aligned}$$

Likewise, we have

$$\begin{aligned} U_k + W_k &= Q_{k-1}\widetilde{Q}_{k-1} + R_{k-1}\widetilde{R}_{k-1} \\ &= Q_{k-1}(\tau^{m_{k-1}} + \widetilde{R}_{k-1}) + \widetilde{R}_{k-1}(Q_{k-1} + \tau^{m_{k-1}}) \\ &= \tau^{m_{k-1}}(Q_{k-1} + \widetilde{R}_{k-1}) = \tau^{2m_{k-1}-2}. \end{aligned}$$

This gives

$$P_k + \widetilde{P}_k = (V_k + \widetilde{V}_k)(U_k + W_k) = (1+\tau)\tau^{4m_{k-1}-4} = (1+\tau)\tau^{m_k-2},$$

as expected.

We get

$$\begin{aligned} Q_k + R_k &= U_k^2 + V_k^2 + \widetilde{V}_k^2 + W_k^2 \\ &= (U_k + W_k)^2 + (V_k + \widetilde{V}_k)^2 \\ &= \tau^{4m_{k-1}-4} + (1+\tau^2)\tau^{4m_{k-1}-4} = \tau^{4m_{k-1}-2} = \tau^{m_k}. \end{aligned}$$

Finally, we check that

$$\begin{aligned} Q_k + \widetilde{R}_k &= U_k^2 + W_k^2 \\ &= (U_k + W_k)^2 = \tau^{4m_{k-1}-4} = \tau^{m_k-2}, \end{aligned}$$

and the proof is complete. □

In the next lemma, we express P_k, Q_k, R_k in terms of a new auxiliary quantity denoted by Z_k.

Lemma 4.4. *Let k be a positive integer and define*

$$Z_k = U_k + V_k.$$

Then, we have

$$Q_k = Z_k^2, \quad R_k = Z_k^2 + \tau^{(2n_k+2)/3}, \quad P_k = Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k},$$

and

$$\widetilde{Z}_k = Z_k + (1+\tau)\tau^{(n_k-2)/3}.$$

Proof. Since $Q_k = U_k^2 + V_k^2$, we get immediately that $Q_k = Z_k^2$. The expression of R_k in terms of Z_k follows then from Proposition 4.3.

The matrix M_k is the product, in a suitable order, of n_k copies of $M_0(a,b)$ and n_k copies of $M_0(b,a)$, whose determinants are respectively equal to a^2 and b^2. The computation of the determinant of M_k then gives the relation

$$Q_k R_k + P_k^2 = (ab)^{2n_k}.$$

Consequently,

$$P_k^2 = Q_k(Q_k + \tau^{(2n_k+2)/3}) + (ab)^{2n_k},$$

thus

$$P_k = Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}.$$

It follows from Proposition 4.3 that

$$\widetilde{Q}_k + Q_k = \tau^{(2n_k+2)/3} + \tau^{(2n_k-4)/3} = (1+\tau^2)\tau^{(2n_k-4)/3},$$

which implies the last equality of the lemma. □

We are now in position to express Z_{k+1} in terms of n_k and Z_k.

Proposition 4.5. *For $k \geq 1$, we have*

$$Z_{k+1} = \tau^{n_k} Z_k + \tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k}.$$

Proof. We proceed by induction. Observe that

$$Z_1 = U_1 + V_1 = ab + b + 1$$

and

$$Z_2 = a^4b^4 + a^3b + a^2b^2 + a^2 + ab + ab^3 + b^2 + 1 + a^3b^2 + a^2b + a^2b^3 + b + b^3.$$

Since

$$\tau^2 Z_1 + \tau(ab)^2 + (ab)^4 = (1 + a^2 + b^2)(ab + b + 1) + (1 + a + b)a^2b^2 + a^4b^4,$$

we see that the proposition holds for $k = 1$. Assuming that it holds for a positive integer k, we get

$$\begin{aligned}
Z_{k+1} &= P_k\widetilde{P}_k + Q_k\widetilde{Q}_k + Q_k\widetilde{P}_k + P_k\widetilde{R}_k \\
&= P_k(\widetilde{Q}_k + \widetilde{R}_k) + (P_k + Q_k)(\widetilde{Q}_k + \widetilde{P}_k) \\
&= \Big(Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}\Big)\tau^{(2n_k+2)/3} \\
&\quad + \Big(\tau^{(n_k+1)/3} Z_k + (ab)^{n_k}\Big)\Big(\tau^{(n_k+1)/3}\widetilde{Z}_k + (ab)^{n_k}\Big) \\
&= \tau^{(2n_k+2)/3} Z_k^2 + \tau^{n_k+1} Z_k + (ab)^{n_k}\tau^{(2n_k+2)/3}
\end{aligned}$$

$$
\begin{aligned}
&+ \tau^{(n_k+1)/3} Z_k \Big(\tau^{(n_k+1)/3} Z_k + (1+\tau)\tau^{(2n_k-1)/3} + (ab)^{n_k} \Big) \\
&+ (ab)^{n_k} \Big(\tau^{(n_k+1)/3} Z_k + (1+\tau)\tau^{(2n_k-1)/3} + (ab)^{n_k} \Big) \\
= {} & \tau^{n_k+1} Z_k + (ab)^{n_k} \tau^{(2n_k+2)/3} + \tau^{n_k} Z_k (1+\tau) \\
&+ (ab)^{n_k} \Big((1+\tau)\tau^{(2n_k-1)/3} + (ab)^{n_k} \Big) \\
= {} & \tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}.
\end{aligned}
$$

This completes the proof. □

We deduce from Proposition 4.5 an equation of degree 4 satisfied by Z_k.

Proposition 4.6. *For $k \geq 1$, we have*

$$
\begin{aligned}
Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3} \\
+ (a+b)(ab)^{n_k} \tau^{(2n_k-4)/3} + (ab)^{2n_k} = 0.
\end{aligned}
$$

Proof. For $k \geq 1$, set

$$
\Delta_k = Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3}.
$$

Note that

$$
\begin{aligned}
\Delta_1 &= (1 + b^4 + a^4 b^4) + (1+a+b)(1 + b^2 + a^2 b^2 + (a+b)(1+b+ab)) \\
&\quad + b(a+b)(1 + a^2 + b^2) + a^2 \\
&= a^4 b^4 + a^2 b^3 + a^3 b^2 = (a+b)(ab)^{n_1} \tau^{(2n_1-4)/3} + (ab)^{2n_1}.
\end{aligned}
$$

Thus, Proposition 4.6 holds for $k = 1$. Let k be a positive integer. Since $n_{k+1} = 4n_k$, it follows from Proposition 4.5 that

$$
\begin{aligned}
\Delta_{k+1} = {} & Z_{k+1}^4 + \tau^{(8n_k-1)/3} Z_{k+1}^2 + (a+b)\tau^{4n_k-1} Z_{k+1} \\
&+ (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\
= {} & \Big(\tau^{n_k} Z_k + \tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k} \Big)^4 \\
&+ \tau^{(8n_k-1)/3} \Big(\tau^{n_k} Z_k + \tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k} \Big)^2 \\
&+ (a+b)\tau^{4n_k-1} \Big(\tau^{n_k} Z_k + \tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k} \Big) \\
&+ (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\
= {} & \tau^{4n_k} Z_k^4 + \tau^{(8n_k-4)/3}(ab)^{4n_k} + (ab)^{8n_k} \\
&+ \tau^{(8n_k-1)/3} \Big(\tau^{2n_k} Z_k^2 + \tau^{(4n_k-2)/3}(ab)^{2n_k} + (ab)^{4n_k} \Big) \\
&+ (a+b)\tau^{4n_k-1} \Big(\tau^{n_k} Z_k + \tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k} \Big) \\
&+ (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\
= {} & \tau^{4n_k} Z_k^4 + \tau^{(8n_k-1)/3}(\tau^{2n_k} Z_k^2) + (a+b)\tau^{4n_k-1}(\tau^{n_k} Z_k) \\
&+ (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} + \tau^{(8n_k-4)/3}(ab)^{4n_k}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tau^{(8n_k-1)/3}(\tau^{(4n_k-2)/3}(ab)^{2n_k} + (ab)^{4n_k}) \\
&+ (a+b)\tau^{4n_k-1}(\tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k}) + (ab)^{8n_k} \\
= {}& \tau^{4n_k}\Big(Z_k^4 + \tau^{(2n_k-1)/3}Z_k^2 + (a+b)\tau^{n_k-1}Z_k \\
&+ (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3}\Big) + \tau^{(8n_k-4)/3}(1+\tau)(ab)^{4n_k} \\
&+ \tau^{(8n_k-1)/3}\tau^{(4n_k-2)/3}(ab)^{2n_k} \\
&+ (a+b)\tau^{4n_k-1}(\tau^{(2n_k-1)/3}(ab)^{n_k} + (ab)^{2n_k}) + (ab)^{8n_k} \\
= {}& \tau^{4n_k}\Delta_k + \tau^{(8n_k-4)/3}(a+b)(ab)^{4n_k} + \tau^{4n_k}(ab)^{2n_k} \\
&+ (a+b)\tau^{4n_k-1}(\tau^{(2n_k-1)/3}(ab)^{n_k}) + (ab)^{8n_k}.
\end{aligned}
$$

Assuming that

$$\Delta_k = (a+b)(ab)^{n_k}\tau^{(2n_k-4)/3} + (ab)^{2n_k},$$

we deduce that

$$
\begin{aligned}
\Delta_{k+1} &= \tau^{4n_k}(a+b)(ab)^{n_k}\tau^{(2n_k-4)/3} + \tau^{4n_k}(ab)^{2n_k} \\
&\quad + \tau^{(8n_k-4)/3}(a+b)(ab)^{4n_k} + \tau^{4n_k}(ab)^{2n_k} \\
&\quad + (a+b)\tau^{4n_k-1}(\tau^{(2n_k-1)/3}(ab)^{n_k}) + (ab)^{8n_k} \\
&= \tau^{(8n_k-4)/3}(a+b)(ab)^{4n_k} + (ab)^{8n_k}.
\end{aligned}
$$

Since $n_{k+1} = 4n_k$, this shows that Proposition 4.6 holds for every positive integer k. □

Set

$$
\begin{aligned}
a_2 &= (1+a+b)^2, \\
a_0 &= a(a+b)a_2 + a^2, \\
a_1 = a_3 &= (a+b)a_2, \\
a_4 &= b(a+b)a_2 + a^2.
\end{aligned}
$$

To prove that

$$a_4\zeta^4 + a_3\zeta^3 + a_2\zeta^2 + a_1\zeta + a_0 = 0,$$

with ζ as in (4.1), it is sufficient to show that the absolute value of

$$\delta_k := a_4P_k^4 + a_3P_k^3Q_k + a_2P_k^2Q_k^2 + a_1P_kQ_k^3 + a_0Q_k^4$$

tends to 0 as k tends to infinity. The next proposition is more precise.

Proposition 4.7. *For $k \geq 1$, there exists a polynomial $T_k(X,Y)$ with coefficients in $\mathbb{F}_2$ such that $\delta_k = (ab)^{2n_k}T_k(a,b)$.*

Proof. We express δ_k in terms of Z_k. A lengthy computation based on Lemma 4.4 gives:

$$
\begin{aligned}
\delta_k &= a_4P_k^4 + a_3P_k^3Q_k + a_2P_k^2Q_k^2 + a_1P_kQ_k^3 + a_0Q_k^4 \\
&= a_4(P_k^4 + Q_k^4) + (a+b)^2\tau^2Q_k^4 + (a+b)\tau^2P_kQ_k(P_k^2 + Q_k^2) + \tau^2P_k^2Q_k^2 \\
&= a_4(\tau^{(n_k+1)/3}Z_k + (ab)^{n_k})^4 + (a+b)^2\tau^2Q_k^4
\end{aligned}
$$

$$
\begin{aligned}
&+ (a+b)\tau^2 P_k Q_k (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 + \tau^2 P_k^2 Q_k^2 \\
&= a_4(\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^4 + (a+b)^2\tau^2 Z_k^8 \\
&+ (a+b)\tau^2 P_k Z_k^2 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 + \tau^2 P_k^2 Z_k^4 \\
&= a_4(\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^4 + (a+b)^2\tau^2 Z_k^8 \\
&+ (a+b)\tau^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}) Z_k^2 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 \\
&+ \tau^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 Z_k^4 \\
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 \\
&+ (a+b)\tau^2 Z_k^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k})(\tau^{(2n_k+2)/3} Z_k^2 + (ab)^{2n_k}) \\
&+ \tau^2 (Z_k^4 + \tau^{(2n_k+2)/3} Z_k^2 + (ab)^{2n_k}) Z_k^4 \\
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 \\
&+ (a+b)\tau^2 Z_k^2 \big(\tau^{(2n_k+2)/3} Z_k^4 + \tau^{3n_k+1} Z_k^3 + \tau^{(2n_k+2)/3} Z_k^2 (ab)^{n_k}\big) \\
&+ (a+b)\tau^2 Z_k^2 \big((ab)^{2n_k} Z_k^2 + (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k + (ab)^{3n_k}\big) \\
&+ \tau^2 (Z_k^8 + \tau^{(2n_k+2)/3} Z_k^6 + (ab)^{2n_k} Z_k^4) \\
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 \\
&+ (a+b)\tau^2 \tau^{(2n_k+2)/3} Z_k^6 + (a+b)\tau^2 \tau^{3n_k+1} Z_k^5 + (a+b)\tau^2 \tau^{(2n_k+2)/3} Z_k^4 (ab)^{n_k} \\
&+ (a+b)\tau^2 (ab)^{2n_k} Z_k^4 + (a+b)\tau^2 (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k^3 + (a+b)\tau^2 Z_k^2 (ab)^{3n_k} \\
&+ \tau^2 Z_k^8 + \tau^2 \tau^{(2n_k+2)/3} Z_k^6 + \tau^2 (ab)^{2n_k} Z_k^4 \\
&= \big(a_4 \tau^{(4n_k+4)/3} + \tau^2 (ab)^{2n_k} + (a+b)\tau^2 \tau^{(2n_k+2)/3} (ab)^{n_k} + (a+b)\tau^2 (ab)^{2n_k}\big) Z_k^4 \\
&+ a_4 (ab)^{4n_k} + [(a+b)^2 \tau^2 + \tau^2] Z_k^8 \\
&+ [(a+b)\tau^2 \tau^{(2n_k+2)/3} + \tau^2 \tau^{(2n_k+2)/3}] Z_k^6 + (a+b)\tau^2 \tau^{3n_k+1} Z_k^5 \\
&+ (a+b)\tau^2 (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k^3 + (a+b)\tau^2 Z_k^2 (ab)^{3n_k} \\
&= \big((b(a+b)\tau^2 + a^2)\tau^{(4n_k+4)/3} + (a+b)\tau^{(2n_k+8)/3} (ab)^{n_k} + \tau^3 (ab)^{2n_k}\big) Z_k^4 \\
&+ (b(a+b)\tau^2 + a^2)(ab)^{4n_k} + \tau^4 Z_k^8 + \tau^{(2n_k+8)/3} Z_k^6 + (a+b)\tau^{3n_k+3} Z_k^5 \\
&+ (a+b)(ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b)\tau^2 Z_k^2 (ab)^{3n_k} \\
&= \tau^4 Z_k^8 + \tau^{(2n_k+11)/3} Z_k^6 + (a+b)\tau^{n_k+3} Z_k^5 \\
&+ [(b(a+b)\tau^2 + a^2)\tau^{(4n_k-5)/3} + (a+b)\tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}] \tau^3 Z_k^4 \\
&+ (a+b)(ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b)\tau^2 (ab)^{3n_k} Z_k^2 \\
&+ (b(a+b)\tau^2 + a^2)(ab)^{4n_k}.
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
\delta_k = \tau^4 \Delta_k Z_k^4 &+ [(a+b)\tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}] \tau^3 Z_k^4 \\
&+ (a+b)(ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b)\tau^2 (ab)^{3n_k} Z_k^2
\end{aligned}
$$

$$+ (b(a+b)\tau^2 + a^2)(ab)^{4n_k},$$

where

$$\Delta_k = Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3}$$

has been already introduced in the proof of Proposition 4.6.

It follows from Proposition 4.6 that

$$\tau^4 \Delta_k + (a+b)\tau^{(2n_k-1)/3}(ab)^{n_k}\tau^3 = \tau^4 (ab)^{2n_k}.$$

Consequently, we get

$$\begin{aligned}\delta_k &= \tau^4 (ab)^{2n_k} Z_k^4 + (ab)^{2n_k}\tau^3 Z_k^4 \\ &\quad + (a+b)(ab)^{2n_k}\tau^{(n_k+1)/3+2} Z_k^3 + (a+b)\tau^2 (ab)^{3n_k} Z_k^2 \\ &\quad + (b(a+b)\tau^2 + a^2)(ab)^{4n_k}.\end{aligned}$$

This shows that δ_k can be written as $(ab)^{2n_k}$ times a polynomial in a and b, which depends on k. □

To conclude the proof of Theorem 2.1, we explain the relationship between P_k, Q_k and the convergents $p_\ell/q_\ell = p_\ell(a,b)/q_\ell(a,b)$ of the Thue–Morse continued fraction

$$\xi_{a,b} = [0; a, b, b, a, b, a, a, b, \ldots].$$

We have

$$q_{4^k}(a,b) = (ab)^{n_k} Q_k(a^{-1}, b^{-1}), \quad p_{4^k}(a,b) = (ab)^{n_k} P_k(a^{-1}, b^{-1}), \quad k \geq 1,$$

and we check that

$$A_j = (ab)^4 a_j(a^{-1}, b^{-1}), \quad 0 \leq j \leq 4.$$

For $k \geq 1$, put

$$\begin{aligned}\varepsilon_{4^k} =& A_4 \Big(\frac{p_{4^k}}{q_{4^k}}\Big)^4 + A_3 \Big(\frac{p_{4^k}}{q_{4^k}}\Big)^3 + A_2 \Big(\frac{p_{4^k}}{q_{4^k}}\Big)^2 + A_1 \frac{p_{4^k}}{q_{4^k}} + A_0 \\ =& \frac{(ab)^{-2n_k+4} T_k(a^{-1}, b^{-1})}{(ab)^{-4n_k} q_{4^k}^4} = \frac{(ab)^{2n_k+4} T_k(a^{-1}, b^{-1})}{q_{4^k}^4}.\end{aligned}$$

Set $d = \deg a + \deg b$. Since

$$|q_{4^k}| = 2^{dn_k}, \quad |T_k(a^{-1}, b^{-1})| \leq 1, \quad |ab|^{2n_k} = 2^{2dn_k},$$

we get

$$|\varepsilon_{4^k}| \leq 2^{-2dn_k} = |q_{4^k}|^{-2}.$$

Recalling that $|\xi_{a,b} - p_{4^k}/q_{4^k}| < |q_{4^k}|^{-2}$, we derive that

$$|A_4 \xi_{a,b}^4 + A_3 \xi_{a,b}^3 + A_2 \xi_{a,b}^2 + A_1 \xi_{a,b} + A_0| \leq \max\{|A_1|, \ldots, |A_4|\} \cdot 2^{-2dn_k}.$$

Since n_k is arbitrarily large, this gives that

$$A_4 \xi_{a,b}^4 + A_3 \xi_{a,b}^3 + A_2 \xi_{a,b}^2 + A_1 \xi_{a,b} + A_0 = 0,$$

and concludes the proof of the theorem.

5. Proof of Proposition 2.4

Since
$$A_0 + A_4 = (a+b)^2(ab+a+b)^2$$
is a square, its derivative is 0 and we get that $(A_0)' = (A_4)'$. By differentiating the minimal defining polynomial $A_4X^4 + \ldots + A_1X + A_0$ of $\xi_{a,b}$, we obtain
$$\sum_{j=0}^{4} (A_j)'X^j + A_1X'(1+X^2) = 0,$$
hence
$$(A_0)'(1+X^4) + (A_1)'(X+X^3) = A_1X'(1+X^2)$$
and
$$(A_0)'(1+X^2) + (A_1)'X = A_1X',$$
that is
$$(A_0)'(1+X^2) = (A_1X)'.$$
Since
$$(A_0)' = (ab)'(a+b+ab)^2, \quad A_1 = ab(a+b)(a+b+ab)^2,$$
the last equation becomes
$$[(ab(a+b))X]' = (ab)'(1+X^2).$$
This establishes that $\xi_{a,b}$ is differential-quadratic and satisfies a simple Riccati equation.

Assume now that there are an integer $s \geq 2$ and polynomials A, B, C and D in $\mathbb{F}_2[z]$ such that $\xi_{a,b}$ satisfies
$$A\xi_{a,b}^{2^s+1} + B\xi_{a,b}^{2^s} + C\xi_{a,b} + D = 0.$$
Then, the polynomial $AX^{2^s+1} + BX^{2^s} + CX + D$ must be a multiple of the minimal defining polynomial $A_4X^4 + \ldots + A_1X + A_0$ of $\xi_{a,b}$ and there exist $c_0, c_1, \ldots, c_{2^s-3}$ in $\mathbb{F}_2[z]$ such that
$$(A_4X^4 + \ldots + A_1X + A_0)\,(c_{2^s-3}X^{2^s-3} + \ldots + c_1X + c_0) = AX^{2^s+1} + BX^{2^s} + CX + D.$$
We thus get $2^s - 2$ linear forms in $c_0, \ldots, c_{2^s-3}$ which vanish. The associated matrix is a pentadiagonal (if $s \geq 3$, the case $s = 2$ being immediate) Toeplitz matrix of the form
$$M_s = \begin{pmatrix} A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_3 & A_2 & A_1 & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_4 & A_3 & A_2 & A_1 & A_0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & A_4 & A_3 & A_2 \end{pmatrix}.$$
To compute its determinant, we simply expand it and observe that A_2 is the only coefficient of the minimal defining polynomial of $\xi_{a,b}$ which involves the term a^4b^4. All the other terms involved are of the form a^ib^j, with $0 \leq i, j \leq 4$ and $i + j \leq 7$. Consequently, the determinant of M_s is equal to $(a^4b^4)^{2^s-2}$

plus a linear combination of terms $a^i b^j$, with $0 \le i, j \le 4(2^s - 2)$ and $(i, j) \ne (4(2^s-2), 4(2^s-2))$. In particular, it does not vanish and the system of equations has only the trivial solution $c_0 = \ldots = c_{2^s-3} = 0$. This shows that $\xi_{a,b}$ cannot be hyperquadratic.

6. Concluding remarks

We gather in this concluding section several additional results, without their proofs.

6.1. **Explicit expressions.** By Proposition 4.5, we can derive an explicit formula for Z_k, namely

$$Z_k = \tau^{(2^{2k-1}-2)/3}\Big(1 + b + \sum_{j=0}^{2k-2} \tau^{(2-2^{j+1}-\chi(j))/3}(ab)^{2^j}\Big), \quad k \ge 1,$$

where $\chi(j) := j \pmod 2$. Then, by Lemma 4.4, we obtain the following explicit formula for P_k/Q_k.

Proposition 6.1. *For $k \ge 2$, we have*

$$\frac{P_k}{Q_k} = \frac{a + a^2 b + ab^2 + (a+b)\alpha_k}{1 + b^2 + a^2 b^2 + \alpha_k + \alpha_k^2},$$

where

$$\alpha_k = \sum_{j=1}^{k-1} \tau^{(2-2^{2j+1})/3}(ab)^{2^{2j}}.$$

Notice that α_k satisfies the relation

$$\alpha_k^4 + \tau^2 \alpha_k + \tau^{(8-2^{2k+1})/3}(ab)^{2^{2k}} + a^4 b^4 = 0.$$

Proposition 6.1 implies the following theorem by taking the limit as k goes to infinite.

Theorem 6.2. *Set*

$$\alpha = \sum_{j=1}^{\infty} \tau^{(2-2^{2j+1})/3}(ab)^{2^{2j}}.$$

Then, α is algebraic and it satisfies

$$\alpha^4 + \tau^2 \alpha + a^4 b^4 = 0.$$

Furthermore, the Thue–Morse continued fraction $\xi_{a,b}$ can be expressed as

$$\xi_{a,b} = \frac{a + a^2 b + ab^2 + (a+b)\alpha}{1 + b^2 + a^2 b^2 + \alpha + \alpha^2}.$$

We stress that Theorem 6.2 implies that $\xi_{a,b}$ is algebraic.

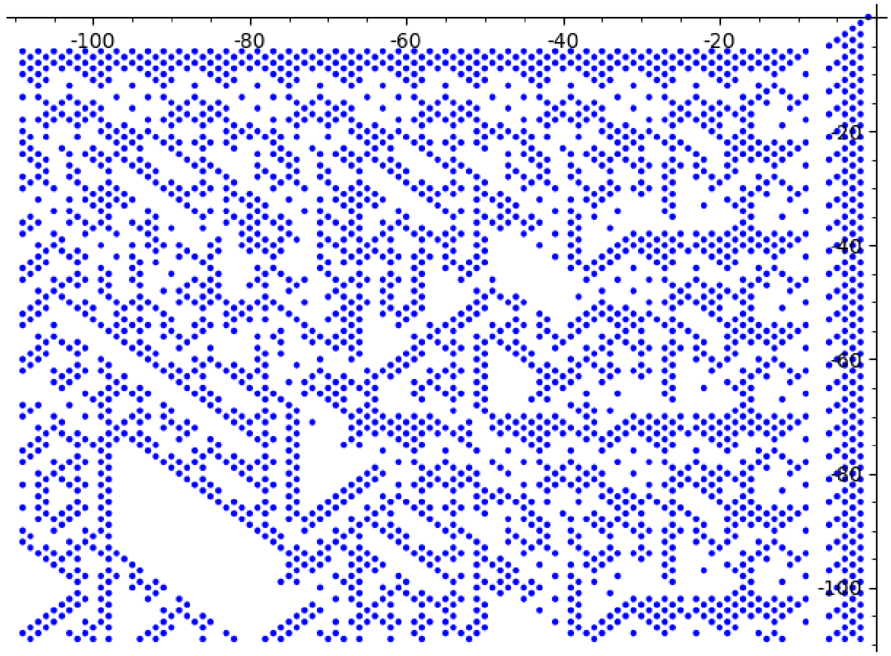

Figure 1. Coefficients of $\xi_{a,b}$

6.2. **Even and odd sections.** To understand the structure of $\xi_{a,b}$, we let

$$\xi_{a,b} = \xi^{ee} + \xi^{eo} + \xi^{oe} + \xi^{oo},$$

where

$$\xi^{ee} = \sum a^{2i} b^{2j}, \quad \xi^{eo} = \sum a^{2i} b^{2j+1}, \quad \xi^{oe} = \sum a^{2i+1} b^{2j}, \quad \xi^{oo} = \sum a^{2i+1} b^{2j+1}.$$

We can derive, by Theorem 2.1, that

$$\xi^{ee} = \xi^{oo} = 0,$$

and ξ^{eo}, ξ^{oe} are algebraic. The coefficients of $\xi_{a,b}$ are reproduced in Figure 1 in the following manner. Let $S = \{(-1,0), (-2,-1), (-2,-3), \ldots\}$ be the set of the blue dots in Figure 1. Then

$$\xi_{a,b} = \sum_{(i,j)\in S} a^i b^j = a^{-1} + a^{-2}b^{-1} + a^{-2}b^{-3} + \cdots$$

6.3. **Jacobi continued fraction for formal power series.** In the field of formal power series $\mathbb{F}((x))$ over a field $\mathbb{F}$, the Jacobi continued fraction $J(\mathbf{u}, \mathbf{v})$ defined by two sequences $\mathbf{u} = (u_n)_{n\geq 1}$ and $\mathbf{v} = (v_n)_{n\geq 0}$ with $v_n \neq 0$ for all $n \geq 0$

is the infinite continued fraction

$$J(\mathbf{u},\mathbf{v}) = \cfrac{v_0}{1+u_1x-\cfrac{v_1x^2}{1+u_2x-\cfrac{v_2x^2}{1+u_3x-\cfrac{v_3x^2}{\ddots}}}}.$$

The basic properties of Jacobi continued fractions can be found in [16, 32]. The Hankel determinants $H_n(J(\mathbf{u},\mathbf{v}))$ of $J(\mathbf{u},\mathbf{v})$ can be calculated by means of the following fundamental relation, first stated by Heilermann in 1846 [18]:

$$H_n(J(\mathbf{u},\mathbf{v})) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}.$$

Assume that $\mathbb{F}$ is the field $\mathbb{F}_2$. If the Jacobi continued fraction exists, then $v_j = 1$ for $j \geq 0$. In this case, the Hankel determinants H_n are all equal to 1. By [8, 17], the sequence $(c_n)_{n\geq 0}$ of the coefficients of $J(\mathbf{u},\mathbf{v}) = \sum_{n\geq 0} c_n x^n$ is *apwenian*. This means that the following relations hold:

$$c_0 = 1 \text{ and } \quad c_n \equiv c_{2n+1} + c_{2n+2} \mod 2, \quad n \geq 0.$$

Consider the Jacobi continued fraction $\omega(x)$ defined by the Thue–Morse sequence

$$\omega(x) = \cfrac{1}{1+u_1x+\cfrac{x^2}{1+u_2x+\cfrac{x^2}{1+u_3x+\cfrac{x^2}{\ddots}}}},$$

where $(u_1, u_2, \ldots) = (1, 0, 0, 1, 0, 1, 1, 0, \ldots)$ is the Thue–Morse sequence over $\{0, 1\}$. Since

$$\frac{\omega(z^{-1})}{z} = \xi_{z+1,z},$$

it follows from Theorem 2.1 that

$$g_4\,\omega(x)^4 + g_3\,\omega(x)^3 + g_2\,\omega(x)^2 + g_1\,\omega(x) + g_0 = 0,$$

where

$$\begin{aligned}
g_0 &= x^5 + x^3 + x^2 + x + 1,\\
g_1 &= x^6 + x^5 + x^4 + x^3 + x^2 + x,\\
g_2 &= x^6 + 1,\\
g_3 &= x^8 + x^7 + x^6 + x^5 + x^4 + x^3,\\
g_4 &= x^{10} + x^9 + x^8 + x^7 + x^5 + x^4.
\end{aligned}$$

Consequently, $\omega(x)$ is algebraic of degree 4.

Acknowledgement

The authors thank Alain Lasjaunias for useful discussion.

References

[1] B. Adamczewski and Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, Ann. of Math. 165 (2007), 547–565. 1

[2] B. Adamczewski and Y. Bugeaud, *A short proof of the transcendence of the Thue-Morse continued fractions*, Amer. Math. Monthly 114 (2007), 536–540. 6

[3] B. Adamczewski et C. Faverjon, *Méthode de Mahler : relations linéaires, transcendance et applications aux nombres automatiques*, Proc. London Math. Soc. 115 (2017), 55–90. 1

[4] S. K. Aggarwal, *Transference theorems in the field of formal power series*, Monatsh. Math. 72 (1968), 97–106. 7

[5] J.-P. Allouche, *Sur le développement en fraction continue de certaines séries formelles*, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), 631–633. 2

[6] J.-P. Allouche, J. Bétréma et J.O. Shallit, *Sur des points fixes de morphismes d'un monoïde libre*, RAIRO, Inf. Théor. Appl. 23 (1989), 235–249. 2

[7] J.-P. Allouche and J. Shallit. Automatic sequences. Theory, Applications, Generalizations. Cambridge University Press, Cambridge, 2003. 2

[8] J.-P. Allouche, G.-N. Han, and H. Niederreiter, *Perfect linear complexity profile and apwenian sequences*, Finite Fields Appl. 68 (2020), Article 101761, 13 pages. 18

[9] L. E. Baum and M. M. Sweet, *Continued fractions of algebraic power series in characteristic* 2, Ann. of Math. 103 (1976), 593–610. 1

[10] Y. Bugeaud, *Automatic continued fractions are transcendental or quadratic*, Ann. Sci. École Norm. Sup. 46 (2013), 1005–1022. 1

[11] Y. Bugeaud, *Exponents of Diophantine approximation*. In: Dynamics and analytic number theory, 96–135, London Math. Soc. Lecture Note Ser., 437, Cambridge Univ. Press, Cambridge, 2016. 5

[12] G. Christol, *Ensembles presque périodiques k-reconnaissables*, Theoret. Comput. Sci. 9 (1979), 141–145. 1

[13] G. Christol, T. Kamae, M. Mendès France et G. Rauzy. *Suites algébriques, automates et substitutions*, Bull. Soc. Math. France 108 (1980), 401–419. 1

[14] A. Cobham, *On the Hartmanis-Stearns problem for a class of tag machines*. In: Conference Record of 1968 Ninth Annual Symposium on Switching and Automata Theory, Schenectady, New York (1968), 51–60. 1

[15] A. Cobham, *Uniform tag sequences*, Math. Systems Theory 6 (1972), 164–192. 2

[16] Ph. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. 32(2) (1980), 125–161. 18

[17] Y.-J. Guo, G.-N. Han, and W. Wu, *Criteria for apwenian sequences*, Adv. in Math. 389 (2021), Paper No. 107899, 37 pages. 18

[18] J. B. H. Heilermann, *Über die Verwandlung der Reihen in Kettenbrüche*, J. Reine Angew. Math. 33 (1846), 174–188. 18

[19] Y. Hu and G.-N. Han, *On the algebraicity of Thue-Morse and period-doubling continued fractions*, Acta Arith. 203 (2022), 353–381. 2, 5

[20] E. Kolchin, *Rational approximation to solutions of algebraic differential equations*, Proc. Amer. Math. Soc. 10 (1959), 238–244. 4
[21] A. Lasjaunias, *A survey of diophantine approximation in fields of power series*, Monatsh. Math. 130 (2000), 211–229. 4
[22] A. Lasjaunias and B. de Mathan, *Thue's Theorem in positive characteristic*, J. Reine Angew. Math. 473 (1996), 195–206. 4
[23] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one*, J. Number Theory 149 (2015), 259–284. 2
[24] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions and automatic sequences*, Finite Fields Appl. 40 (2016), 46–60.
[25] A. Lasjaunias and J.-Y. Yao, *On certain recurrent and automatic sequences in finite fields*, J. Algebra 478 (2017), 133–152. 2
[26] K. Mahler, *On a theorem of Liouville in fields of positive characteristic*, Canad. J. Math. 1 (1949), 397–400. 1, 4
[27] W. H. Mills and D. P. Robbins, *Continued fractions for certain algebraic power series*, J. Number Theory 23 (1986), 388–404. 2
[28] M. Mkaouar, *Sur le développement en fraction continue de la série de Baum et Sweet*, Bull. Soc. Math. France 123 (1995), 361–374. 2
[29] C. Osgood, *Effective bounds on the "Diophantine approximation" of algebraic functions over fields of arbitrary characteristic and applications to differential equations*, Nederl. Akad. Wetensch. Proc. Ser. A 78 (1975), 105–119. 4
[30] P. Philippon, *Groupes de Galois et nombres automatiques*, J. London Math. Soc. 92 (2015), 596–614. 1
[31] W. M. Schmidt, *On continued fractions and Diophantine approximation in power series fields*, Acta Arith. 95 (2000), 139–166. 4
[32] Hubert S. Wall. *Analytic Theory of Continued Fractions*. D. Van Nostrand Company, Inc., New York, N. Y., 1948. 18
[33] J.-Y. Yao, *Critères de non-automaticité et leurs applications*, Acta Arith. 80 (1997), 237–248. 2

Université de Strasbourg, Mathématiques, 7, rue René Descartes, 67084 Strasbourg (France)

Institut universitaire de France
Email address: bugeaud@math.unistra.fr

I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France
Email address: guoniu.han@unistra.fr