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# THE $k$-EXTENSION OF A MAHONIAN STATISTIC

BY

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ABSTRACT. — Clarke and Foata have recently studied the $k$-extension of several Mahonian statistics. There is an alternate definition for the $k$-Denert statistic that is derived in the present paper.

RÉSUMÉ. — Récemment, Clarke et Foata ont étudié la $k$-extension de plusieurs statistiques mahoniennes. Dans cet article, on introduit une autre définition pour la $k$-statistique de Denert.

## 1. Introduction

Let

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0\,; \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n \geq 1\,; \end{cases}$$

denote the *q-ascending factorial* and for each sequence $\mathbf{c} = (c_1, c_2, \dots, c_r)$ of non-negative integers, of sum $m$, let

$$\begin{bmatrix} m \\ c_1, c_2, \dots, c_r \end{bmatrix} = \frac{(q;q)_m}{(q;q)_{c_1}(q;q)_{c_2}\dots(q;q)_{c_r}}$$

denote the *q-multinomial coefficient*. Also denote by $R(\mathbf{c})$ the class of all $m!\,/(c_1!\,c_2!\,\dots c_r!)$ rearrangements of the (non-decreasing) word $1^{c_1}2^{c_2}\dots r^{c_r}$. By *Mahonian statistic* it is meant an integer-valued mapping "stat" defined on each class $R(\mathbf{c})$ that satisfies the identity

$$\begin{bmatrix} m \\ c_1, c_2, \dots, c_r \end{bmatrix} = \sum_w q^{\operatorname{stat} w} \qquad (w \in R(\mathbf{c})).$$

The *inversion number* "inv" and the *major index* "maj" are classical examples of Mahonian statistics (see [M]). Another example of such a statistic is provided with the *Denert statistic*, "den", introduced recently (see [Den]) in the study of hereditary orders in central simple algebras. The statistics "maj" and "den" have two definitions, one "natural" or

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more direct, the other one involving the notion of *cyclic interval.* The natural definitions refer to *descents* and *excedances*, two other statistics that have ben extensively studied since MacMahon (see [L, chap. 10]). We first recall the definitions of $k$-descent and of $k$-excedance, as they were introduced in the work by Clarke and Foata [CF1]).

Let $r$, $l$, $k$ be three non-negative integers such that $1 \le r$ and $l + k = r$. Let $X$ denote the *alphabet* $[\,r\,] = \{1, 2, \ldots, r\}$ equipped with the natural order of the integers. Call the letters $1, 2, \ldots, l$ *small* and the letters $l + 1, l + 2, \ldots, l + k = r$ *large.* Also let $S = S_k = \{1, 2, \ldots, l\}$ and $L = L_k = \{l + 1, l + 2, \ldots, r\}$. Let $w = x_1 x_2 \ldots x_m$ be a word in the alphabet $X$ and denote by $\overline{w} = y_1 y_2 \ldots y_m$ its *non-decreasing* rearrangement. Let $1 \le i \le m$; say that $i$ is a *place of $k$-descent* (resp. *place of $k$-excedance*) for $w$, if $i \ne m$ and $x_i > x_{i+1}$ or $x_i = x_{i+1} \ge l + 1$, or if $i = m$ and $x_i \ge l + 1$ (resp. if $x_i > y_i$ or $x_i = y_i \ge l + 1$). The *number* of places of $k$-descent (resp. of places of $k$-excedance) in $w$ is denoted by "$\mathrm{des}_k\, w$" (resp. "$\mathrm{exc}_k\, w$"). Clarke and Foata [CF1] proved that "$\mathrm{des}_k$" and "$\mathrm{exc}_k$" were *equidistributed* on each rearrangement class $R(\mathbf{c})$. In their second paper [CF2] they constructed a bijection $\rho$ of $R(\mathbf{c})$ onto itself satisfying

$$(\mathrm{des}_k, \mathrm{maj}_k)(w) = (\mathrm{exc}_k, \mathrm{den}_k)(\rho(w)),$$

where "$\mathrm{maj}_k$" and "$\mathrm{den}_k$" are two $k$-extensions of Mahonian statistics in the following sense. Let $\mathrm{SPD}_k\, w$ (resp. $\mathrm{SPE}_k\, w$) be the *sum* of the places of $k$-descents (resp. of $k$-excedances) in $w$. Clarke and Foata [CF2] defined the *$k$-major index*, "$\mathrm{maj}_k\, w$," of $w$ as

$$\mathrm{maj}_k\, w = \mathrm{SPD}_k\, w.$$

It would have seemed natural to define "$\mathrm{den}_k\, w$" as $\mathrm{den}_k\, w = \mathrm{SPE}_k\, w$, but $\mathrm{SPD}_k$ is *not* equidistributed with $\mathrm{maj}_k$ on each class $R(\mathbf{c})$. However, the following inequality

$$\mathrm{den}_k\, w \ge \mathrm{SPE}_k\, w$$

holds. Therefore, the natural question arises : can well-defined predicates be found to characterize the (non-negative) difference "$\mathrm{den}_k - \mathrm{SPE}_k$" and then provide the "natural" definition of "$\mathrm{den}_k$" ? For $k = 0$ the question had been answered successfully by Han [H] (see the solutions by [FZ] and [C] in the case of words without repetitions of letters). It is the purpose of this paper to provide the predicates for an arbitrary $k$.

## 2. The $k$-extensions

Recall the definition of a *$k$-cyclic interval*, a notion introduced in [CF2]. First, for $a, b \in X$ the *cyclic interval* $\rrbracket a, b\rrbracket$ is defined by (see [H]) :

$$\rrbracket a, b\rrbracket = \begin{cases} \,]a, b], & \text{if } a \le b\,; \\ X \setminus\, ]b, a], & \text{otherwise.} \end{cases}$$

Notice that $\rrbracket a,a \rrbracket = \emptyset$. Secondly, the *k-cyclic interval* $\rrbracket a,b \rrbracket_k$ is defined as

$$\rrbracket a,b \rrbracket_k = \begin{cases} \rrbracket a,b \rrbracket, & \text{if } a,b \in S\,; \\ \rrbracket a,b \rrbracket \cup \{a\}, & \text{if } a \in L,\ b \in S\,; \\ \rrbracket a,b \rrbracket \setminus \{b\}, & \text{if } a \in S,\ b \in L\,; \\ \rrbracket a,b \rrbracket \cup \{a\} \setminus \{b\}, & \text{if } a,b \in L,\ a \neq b\,; \\ X, & \text{if } a = b \in L. \end{cases}$$

Let $w = x_1 x_2 \ldots x_m$ be a word in the alphabet $X$. For $i = 1, \ldots, m$ denote by $\operatorname{Fact}_i w = x_1 \ldots x_{i-1}$ its *left factor* of length $(i-1)$. For each subset $B$ of $X$ let $\operatorname{Fact}_i w \cap B$ be the *subword* of $\operatorname{Fact}_i w$ consisting of all the letters belonging to $B$ and let $|\operatorname{Fact}_i w \cap B|$ be the length of that subword.

We still denote by $\overline{w} = y_1 y_2 \ldots y_m$ the non-decreasing rearrangement of $w$. The $\operatorname{den}_k$*-coding* of $w$ is defined to be the sequence $(s_i)_{1 \leq i \leq m+1}$ (see the example at the end of this section), where

$$s_i = \begin{cases} \left|\operatorname{Fact}_i w \cap \rrbracket x_i, y_i \rrbracket_k\right|, & \text{if } 1 \leq i \leq m\,; \\ |w \cap L|, & \text{if } i = m+1\,; \end{cases}$$

and the statistic "$\operatorname{den}_k w$" to be

$$\operatorname{den}_k w = \sum_{i=1}^{m+1} s_i.$$

Furthermore, the *inversion number* and the *weak inversion number* are $k$-updated as follows :

$$\begin{aligned} \operatorname{inv}_k w =& \#\{1 \leq i < j \leq m \mid x_i > x_j \ \text{ ou } \ x_i = x_j \geq l+1\} \\ &+ \#\{1 \leq i \leq m \mid x_i \geq l+1\}\ ; \\ \operatorname{imv}_k w =& \#\{1 \leq i < j \leq m \mid x_i > x_j \ \text{ ou } \ x_i = x_j \leq l\}\ . \end{aligned}$$

The definition "$\operatorname{inv}_k$" was introduced in an unpublished note by Foata [F]. Clearly, $\operatorname{inv}_0 = \operatorname{inv}$ (the usual number of inversions) and $\operatorname{imv}_0 = \operatorname{imv}$ (the number of (weak) inversions, as already used in [H]).

Let $i_1 < i_2 < \cdots < i_e$ be the increasing sequence of the places of $k$-excedance in $w$ and $j_1 < j_2 < \cdots < j_{m-e}$ the complementary sequence, i.e., the increasing sequence of the places of *non-k*-excedance. Form the two subwords : $\operatorname{Exc}_k w = x_{i_1} x_{i_2} \cdots x_{i_e}$ and $\operatorname{Nexc}_k w = x_{j_1} x_{j_2} \cdots x_{j_{m-e}}$. The main result of this paper is the following result, a true $k$-extension of Theorem 2.1 (ii) in [H].

THEOREM 1. — *For each word $w$ in the alphabet $X$ we have :*

$$\mathrm{den}_k\, w = \mathrm{SPE}_k\, w + \mathrm{imv}_k(\mathrm{Exc}_k\, w) + \mathrm{inv}_k(\mathrm{Nexc}_k\, w).$$

For example, let $r = 5$, $k = 2$, $l = r - k = 3$, so that $X = \{1, 2, 3, 4, 5\}$ with the small letters 1, 2, 3 and the large letters 4 and 5. First, $\mathrm{inv}_k(21321144) = 3+0+3+2+0+0+1+0+2 = 11$ and $\mathrm{imv}_k(3253345445) = 3+0+5+1+0+0+2+0+0+0 = 11$. With the word $w = 3253231432514414 54$ and $\overline{w} = 111222333344444555$ we obtain the following table, where the values of $k$-excedance are printed in bold face on the third row.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| id = | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $\overline{w}$ = | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | |
| $w$ = | **3** | **2** | **5** | **3** | 2 | **3** | 1 | **4** | 3 | 2 | **5** | 1 | **4** | **4** | 1 | 4 | **5** | 4 | |
| **s** = | 0 | 1 | 0 | 2 | 0 | 3 | 5 | 7 | 0 | 4 | 9 | 7 | 12 | 13 | 7 | 3 | 16 | 4 | 8 |

Hence $\mathrm{den}_k = \sum_i s_i = 101$. On the other hand, $\mathrm{SPE}_k\, w = 79$, $\mathrm{imv}_k(\mathrm{Exc}_k\, w) = 11$ and $\mathrm{inv}_k(\mathrm{Nexc}_k\, w) = 11$.

## 3. Cardinality properties

It will be convenient to make use of abridged notations for cardinalities of sets that are associated with biwords. Those notations speak for themselves. They all apply to biwords $\binom{\overline{w}}{w} = \binom{y_1 y_2 \cdots y_m}{x_1 x_2 \cdots x_m}$, where $w = x_1 x_2 \cdots x_m$ is a word in the alphabet $X$ and where $\overline{w} = y_1 y_2 \cdots y_m$ is its non-decreasing rearrangement. If $z$ is a letter in $X$, define :

$$\begin{aligned}
\begin{bmatrix} \cdot \\ z \end{bmatrix} &:= \#\{1 \le i \le m \mid x_i = z\};\\
\begin{bmatrix} z & \ne \\ z & z \end{bmatrix} &:= \#\{1 \le i < j \le m \mid x_i = x_j = y_i = z, y_j \ne z\};\\
\begin{bmatrix} \cdot & z \\ > & z \end{bmatrix} &:= \#\{1 \le i < j \le m \mid x_i > z, x_j = y_j = z\}.
\end{aligned}$$

The following lemma already appears in Dumont [D].

LEMMA 2. — *Let $w$ be a permutation of $X$ having the fixed point $z$. Then, the following identity holds :*

$$\begin{bmatrix} \cdot & z \\ > & z \end{bmatrix} = \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix}.$$

*Proof.* — It suffices to consider the graph of the permutation $w$ represented in a square $r \times r$ (as shown in Fig. 1). The above identity simply says that rectangles $B$ and $C$ contain the same numbers of points of the graph. $\square$

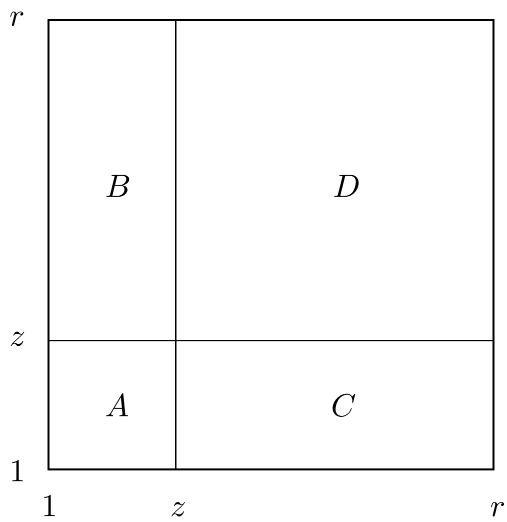


Fig. 1

This result can be generalized to arbitrary words as follows.

LEMMA 3. — *Let $w$ be an arbitrary word in the alphabet $X$ and $z$ a letter of that alphabet. Then*

$$\begin{bmatrix} \cdot & z \\ > & z \end{bmatrix} = \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix} + \begin{bmatrix} z & \neq \\ z & z \end{bmatrix} - \begin{bmatrix} z & z \\ z & \neq \end{bmatrix}.$$

*Proof.* — For each integer $j$ such that $1 \le j \le m$ let

$$\begin{aligned} \begin{bmatrix} \cdot & z \\ > & z \end{bmatrix}_j &:= \#\{i < j \mid x_i > z, x_j = y_j = z\}; \\ \begin{bmatrix} z & \neq \\ z & z \end{bmatrix}_j &:= \#\{j < i \mid x_i = z, y_i \neq z, x_j = y_j = z\}; \\ \begin{bmatrix} z & z \\ z & \cdot \end{bmatrix}_j &:= \#\{j < i \mid y_i = z, x_j = y_j = z\}; \\ &\dots \end{aligned}$$

so that, for instance, $\begin{bmatrix} \cdot & z \\ > & z \end{bmatrix}_j$ is zero if $(y_j, x_j) \neq (z, z)$. The biword $\left[\begin{smallmatrix} \overline{w} \\ w \end{smallmatrix}\right]$ can be represented as the graph of the mapping $i \mapsto x_i (1 \le i \le m)$. Such a graph is contained in an rectangle $m \times r$ as shown in Fig. 2.

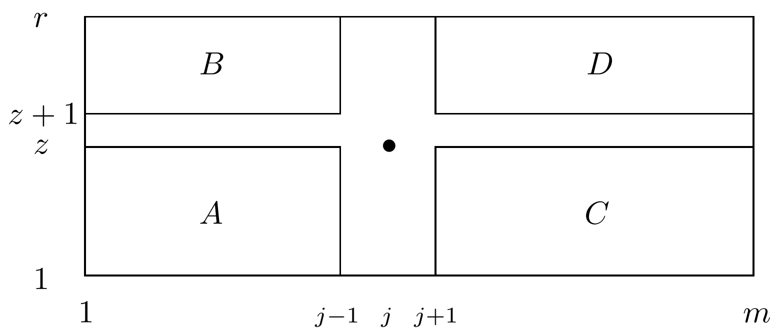


Fig. 2

Keep the same notations for the word $w \in R(\mathbf{c})$ and its non-decreasing rearrangement $\overline{w} = 1^{c_1}2^{c_2}\cdots r^{c_r}$ and let $(y_j, x_j) = (z, z)$. We then have $j \le c_1 + c_2 + \cdots + c_z$. Let $d = c_1 + c_2 + \cdots + c_z - j = \left[\begin{smallmatrix} z & z \\ z & \cdot \end{smallmatrix}\right]_j$. Now as the number of points of the graph contained in rectangle $B$ is equal to the number of points of the same graph contained in $C$, minus $d$, we get :

$$\begin{aligned}
\begin{bmatrix} \cdot & z \\ > & z \end{bmatrix}_j &= \begin{bmatrix} z & \cdot \\ z & \le \end{bmatrix}_j - \begin{bmatrix} z & z \\ z & \cdot \end{bmatrix}_j \\
&= \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix}_j + \begin{bmatrix} z & \cdot \\ z & z \end{bmatrix}_j - \begin{bmatrix} z & z \\ z & \cdot \end{bmatrix}_j \\
&= \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix}_j + \begin{bmatrix} z & \ne \\ z & z \end{bmatrix}_j - \begin{bmatrix} z & z \\ z & \ne \end{bmatrix}_j .
\end{aligned}$$

Summing over all $j$ yields the identity of the lemma. ∎

Lemma 4. — *Let $w$ be a word and $z$ a letter. Then*

$$\begin{bmatrix} < \\ z \end{bmatrix} + \begin{bmatrix} \ne & \ne \\ z & z \end{bmatrix} - \begin{bmatrix} \ne & z \\ z & \ne \end{bmatrix} - \begin{bmatrix} > & > \\ z & z \end{bmatrix} + \begin{bmatrix} < & < \\ z & z \end{bmatrix} = 0.$$

*Proof.* — As there is no biletter $\binom{z}{z}$ on the left-hand side of the previous relation, it suffices to prove the identity when the biword $\binom{\overline{w}}{w}$ contains no biletter $\binom{z}{z}$. In other words, we add the condition :

$$\begin{bmatrix} z \\ z \end{bmatrix} = 0.$$

Under this condition the biword $\binom{\overline{w}}{w}$ has a unique factorization

$$\binom{\overline{w}}{w} = \binom{\alpha}{\alpha'} \cdot \binom{\beta}{\beta'} \cdot \binom{\gamma}{\gamma'},$$

where $\alpha$, $\alpha'$, $\beta$, $\beta'$, $\gamma$, $\gamma'$ are words having the following properties :

(i) all the letters in $\beta$ are equal to $z$;

(ii) there is no occurrence of the letter $z$ in the words $\alpha$, $\gamma$ and $\beta'$.

Let $a$ (resp. $b$) be the number of occurrences of $z$ in $\beta$ (resp. in $\alpha'$). The number of occurrences of $z$ in $\gamma'$ is then $a-b$. Therefore, the left-hand side of the relation in the statement of the Lemma is equal to :

$$b + a(a-1)/2 - ab - (a-b)(a-b-1)/2 + b(b-1)/2 = 0. \quad \square$$

## 4. Proof of Theorem 1

By induction on $k$. For $k=0$ it is true, as shown in [H]. Let $k \geq 1$ and $k' = k-1$. It then suffices to prove

$$\Delta = \Delta\,\mathrm{den} - \Delta\,\mathrm{SPE} - \Delta\,\mathrm{imv} - \Delta\,\mathrm{inv} = 0\,;$$

where

$$\begin{aligned}
\Delta\,\mathrm{den} &= \mathrm{den}_k\, w - \mathrm{den}_{k'}\, w\,;\\
\Delta\,\mathrm{SPE} &= \mathrm{SPE}_k\, w - \mathrm{SPE}_{k'}\, w\,;\\
\Delta\,\mathrm{imv} &= \mathrm{imv}_k(\mathrm{Exc}_k\, w) - \mathrm{imv}_{k'}(\mathrm{Exc}_{k'}\, w)\,;\\
\Delta\,\mathrm{inv} &= \mathrm{inv}_k(\mathrm{Nexc}_k\, w) - \mathrm{inv}_{k'}(\mathrm{Nexc}_{k'}\, w).
\end{aligned}$$

Again let $w = x_1 x_2 \cdots x_m$ and $\overline{w} = y_1 y_2 \cdots y_m$. Also let $z = l+1$ and denote by $L_k = \{z, z+1, \cdots, z+k-1 = r\}$ and $L_{k'} = \{z+1, z+2, \cdots, z+k' = r\}$ the sets of the large letters associated with $k$ and $k'$, respectively. The basic fact is that the letter $z$ is *large* in the $k$-extension, but *small* in the $k'$-extension.

4.1. *Calculation of* $\Delta\,$den. — First, reproduce the expressions of "$\mathrm{den}_k$" and "$\mathrm{den}_{k'}$" :

$$\begin{aligned}
s_i &= \begin{cases} |\,\mathrm{Fact}_i\, w \cap \rrbracket x_i, y_i \rrbracket_k|, & \text{if } 1 \leq i \leq m\,;\\ |w \cap L_k|, & \text{if } i = m+1\,;\end{cases}\\
\mathrm{den}_k\, w &= s_1 + s_2 + \cdots + s_{m+1};\\
s'_i &= \begin{cases} |\,\mathrm{Fact}_i\, w \cap \rrbracket x_i, y_i \rrbracket_{k'}|, & \text{if } 1 \leq i \leq m\,;\\ |w \cap L_{k'}|, & \text{if } i = m+1\,;\end{cases}\\
\mathrm{den}_{k'}\, w &= s'_1 + s'_2 + \cdots + s'_{m+1}.
\end{aligned}$$

As $\Delta\,\mathrm{den} = \mathrm{den}_k\, w - \mathrm{den}_{k'}\, w = \sum_i (s_i - s'_i)$, we have to calculate the difference $s_i - s'_i$ for each $i$, and therefore the difference of the two cyclic intervals $I = \rrbracket x_i, y_i \rrbracket_k$ and $I' = \rrbracket x_i, y_i \rrbracket_{k'}$. Using the notation $C = A + B$ to mean that $A \cap B = \emptyset$ and $C = A \cup B$ we see that there are five cases to consider :

(1) —if $x_i=y_i=z$, then $I=X$, $I'=\emptyset$;

(2) —if $x_i=z, y_i\neq z$, then $I=I'+\{z\}$;

(3) —if $x_i\neq z, y_i=z$, then $I'=I+\{z\}$;

(4) —if $x_i\neq z, y_i\neq z$, then $I'=I$;

(5) —for $i=m+1$ we have $L_k=L_{k'}+\{z\}$.

We then get :

$$\Delta\,\mathrm{den}=\sum_i(s_i-s'_i)=\begin{bmatrix}\cdot & z\\ \cdot & z\end{bmatrix}+\begin{bmatrix}\cdot & \neq\\ z & z\end{bmatrix}-\begin{bmatrix}\cdot & z\\ z & \neq\end{bmatrix}+\begin{bmatrix}\cdot\\ z\end{bmatrix}.$$

4.2. *Calculation of* $\Delta\,\mathrm{SPE}$. — We have

$$\Delta\,\mathrm{SPE}=\mathrm{SPE}_k\,w-\mathrm{SPE}_{k'}\,w=\sum_{x_i=y_i=z} i=\begin{bmatrix}\cdot & z\\ \cdot & z\end{bmatrix}+\begin{bmatrix}z\\ z\end{bmatrix}.$$

4.3. *Calculation of* $\Delta\,\mathrm{inv}$ *and* $\Delta\,\mathrm{imv}$. — Call *biword of $k$-excedance* the subword of $\binom{\overline{w}}{w}$ formed with all the biletters whose bottom letters are $k$-excedances. Also denote by $\mathrm{Exc}_k\,w$ that subword. Define the *biword of non-$k$-excedance* in an equivalent manner and denote it by $\mathrm{Nexc}_k\,w$. Those biwords as well as the corresponding biwords defined for $k'$ can be represented as

$$\begin{aligned}\mathrm{Exc}_k\,w&=\binom{w_1zz\cdots zw_2}{w_3zz\cdots zw_4}; &\qquad \mathrm{Exc}_{k'}\,w&=\binom{w_1w_2}{w_3w_4};\\ \mathrm{Nexc}_k\,w&=\binom{w_5w_6}{w_7w_8}; &\qquad \mathrm{Nexc}_{k'}\,w&=\binom{w_5zz\cdots zw_6}{w_7zz\cdots zw_8};\end{aligned}$$

where $|w_1|=|w_3|$, $|w_2|=|w_4|$, $|w_5|=|w_7|$ and $|w_6|=|w_8|$. As $\left[\begin{smallmatrix}\cdot & z\\ > & z\end{smallmatrix}\right]=0$ for the biword $\mathrm{Nexc}_{k'}\,w$ and $\left[\begin{smallmatrix}z & \cdot\\ z & <\end{smallmatrix}\right]=0$ for the biword $\mathrm{Exc}_k\,w$, we derive the following relations :

$$\begin{aligned}\Delta\,\mathrm{inv}&=\mathrm{inv}_k\,\mathrm{Nexc}_k\,w-\mathrm{inv}_{k'}\,\mathrm{Nexc}_{k'}\,w=-\begin{bmatrix}z & \cdot\\ z & <\end{bmatrix}+\begin{bmatrix}> & >\\ z & z\end{bmatrix}+\begin{bmatrix}>\\ z\end{bmatrix};\\ \Delta\,\mathrm{imv}&=\mathrm{imv}_k\,\mathrm{Exc}_k\,w-\mathrm{imv}_{k'}\,\mathrm{Exc}_{k'}\,w=\begin{bmatrix}\cdot & z\\ > & z\end{bmatrix}-\begin{bmatrix}< & <\\ z & z\end{bmatrix}.\end{aligned}$$

4.4. *Calculation of* $\Delta$. — Putting all the above relations together and noting that the term $\left[\begin{smallmatrix}\cdot & z\\ \cdot & z\end{smallmatrix}\right]$ occurs both in $\Delta\,\mathrm{den}$ and in $\Delta\,\mathrm{SPE}$, we obtain :

$$\Delta=\Delta\,\mathrm{den}-\Delta\,\mathrm{SPE}-\Delta\,\mathrm{imv}-\Delta\,\mathrm{inv}$$

$$
\begin{aligned}
&= \begin{bmatrix} \cdot & \neq \\ z & z \end{bmatrix} - \begin{bmatrix} \cdot & z \\ z & \neq \end{bmatrix} + \begin{bmatrix} \cdot \\ z \end{bmatrix} - \begin{bmatrix} z \\ z \end{bmatrix} \\
&\quad + \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix} - \begin{bmatrix} > & > \\ z & z \end{bmatrix} - \begin{bmatrix} > \\ z \end{bmatrix} - \begin{bmatrix} \cdot & z \\ > & z \end{bmatrix} + \begin{bmatrix} < & < \\ z & z \end{bmatrix} \\
&= \begin{bmatrix} < \\ z \end{bmatrix} - \begin{bmatrix} > & > \\ z & z \end{bmatrix} + \begin{bmatrix} < & < \\ z & z \end{bmatrix} \\
&\quad + \begin{bmatrix} \cdot & \neq \\ z & z \end{bmatrix} - \begin{bmatrix} \cdot & z \\ z & \neq \end{bmatrix} + \begin{bmatrix} z & \cdot \\ z & < \end{bmatrix} - \begin{bmatrix} \cdot & z \\ > & z \end{bmatrix} \\
&= \begin{bmatrix} < \\ z \end{bmatrix} - \begin{bmatrix} > & > \\ z & z \end{bmatrix} + \begin{bmatrix} < & < \\ z & z \end{bmatrix} \\
&\quad + \begin{bmatrix} \cdot & \neq \\ z & z \end{bmatrix} - \begin{bmatrix} \cdot & z \\ z & \neq \end{bmatrix} + \begin{bmatrix} z & z \\ z & \neq \end{bmatrix} - \begin{bmatrix} z & \neq \\ z & z \end{bmatrix} && \text{[Lemma 3]} \\
&= \begin{bmatrix} < \\ z \end{bmatrix} - \begin{bmatrix} > & > \\ z & z \end{bmatrix} + \begin{bmatrix} < & < \\ z & z \end{bmatrix} \\
&\quad + \begin{bmatrix} \neq & \neq \\ z & z \end{bmatrix} - \begin{bmatrix} \neq & z \\ z & \neq \end{bmatrix} \\
&= 0 \ . && \text{[Lemma 4]}
\end{aligned}
$$

This completes the proof of Theorem 1.

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