CHAPTER 10

Transformations on Words and q-Calculus

10.0. Introduction

When sorting was systematically studied in the sixties and seventies, in particular for comparing the different methods used in practice, it was essential to go back to the classics, to the works by MacMahon and especially to his treatise on Combinatory Analysis. He had made an extensive study of the distributions of several *statistics* on permutations, or more generally, on "permutations" with repeated elements, simply called *words* in the sequel. The most celebrated of those statistics is probably the classical *number of inversions* which stands for a very natural measurement of how far a permutation is from the identity. There are several other statistics relevant to sorting or to statistical theory, such as the *number of descents*, the *number of excedances*, the *major index*, and more recently the *Denert statistic*.

MacMahon had already calculated the distributions of the early statistics and proved that some of them were equally distributed on each class of rearrangements of a given word. Let us state one of his basic results. To this end suppose that X is a finite non-empty set, referred to as an *alphabet*. For convenience, take X to be the subset $\{1, 2, \ldots, r\}$ $(r \geq 1)$ of the positive integers, equipped with its standard ordering. Let $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ be a sequence of r nonnegative integers and v be the nondecreasing word $v = 1^{c_1} 2^{c_2} \ldots r^{c_r}$, i.e., $v = y_1 y_2 \ldots y_m$ with $m = c_1 + c_2 + \cdots + c_r$ and $y_1 = \cdots = y_{c_1} = 1$, $y_{c_1+1} = \cdots = y_{c_1+c_2} = 2, \ldots, y_{c_1+\cdots+c_{r-1}+1} = \cdots = y_m = r$. The class of all rearrangements of v, i.e., the class of all the words w that can be obtained from v by permuting its letters in some order will be denoted by $R(\mathbf{c})$.

If $w = x_1 x_2 \ldots x_m$ is such a word, the *number of excedances*, $\operatorname{exc} w$, and the *number of descents*, $\operatorname{des} w$, and also the *major index* of w are classically defined as

$$\begin{aligned}
\operatorname{exc} w &= \#\{i : 1 \leq i \leq m, x_i > y_i\}, \\
\operatorname{des} w &= \#\{i : 1 \leq i \leq m-1, x_i > x_{i+1}\}, \\
\operatorname{maj} w &= \sum \{i : 1 \leq i \leq m-1, x_i > x_{i+1}\}.
\end{aligned} \tag{10.0.1}$$

Let $A^{\rm exc}_{\bf c}(t)$ (resp. $A^{\rm des}_{\bf c}(t)$) be the generating polynomial for the class $R({\bf c})$ by the statistic "exc" (resp. "des"), i.e.,

$$A^{\rm exc}_{\bf c}(t) = \sum_w t^{\operatorname{exc} w}, \qquad A^{\rm des}_{\bf c}(t) = \sum_w t^{\operatorname{des} w} \quad (w \in R({\bf c})).$$

MacMahon showed that those two polynomials were equal for every $\bf c$. More explicitly he showed that the generating functions for those two families of polynomials had the same analytic expression. This raises the question of providing methods for deriving those analytic expressions. This will be done in the first part of this chapter in the more general set-up of q-calculus, as not only single statistics will be considered, but pairs of statistics.

Now saying that the previous two polynomials are equal for every $\bf c$ implies that the two statistics "exc" and "des" are *equidistributed* on each rearrangement class $R({\bf c})$. Proving this equidistribution property in a *bijective* manner means that a bijection ϕ on each rearrangement class $R({\bf c})$ is to be constructed with the property that

$$\operatorname{exc} w = \operatorname{des} \phi(w) \tag{10.0.2}$$

holds for every w.

This brings up the matter of the second part of this chapter: does there exist a systematic way for constructing those bijections? We shall see that a large class of those bijections can be constructed by means of a *straightening* algorithm on biwords which is based on a *commutation rule* itself defined on the biwords. Although any commutation rule can be integrated in the algorithm, our attention will be focused on the *contextual* commutation that serves to the construction of a bijection Φ mapping a *pair* of statistics onto another pair. Instead of property (10.0.2) we shall have

$$(\operatorname{exc}, \operatorname{den})\, w = (\operatorname{des}, \operatorname{maj})\, \Phi(w), \tag{10.0.3}$$

where "maj" and "den" are the *major index* and the *Denert statistic* (further defined in section 10.11), respectively.

For every class $R({\bf c})$ introduce the two generating polynomials

$$A^{\rm exc,den}_{\bf c}(t,q) = \sum_{w \in R({\bf c})} t^{\operatorname{exc} w} q^{\operatorname{den} w}, \quad A^{\rm des,maj}_{\bf c}(t,q) = \sum_{w \in R({\bf c})} t^{\operatorname{des} w} q^{\operatorname{maj} w}.$$

An analytical expression for $A^{\rm des,maj}_{\bf c}(t,q)$ was already derived by MacMahon (see section 10.2). But there is no direct way for proving that the polynomial $A^{\rm exc,den}_{\bf c}(t,q)$ is equal to that analytical expression. Thus the construction of the bijection Φ is crucial.

After recalling the fundamental material on q-calculus in section 10.1 we present the *MacMahon Verfahren* which is a rearrangement method that has been generalized in various contexts. In section 10.3 we discuss an insertion technique that makes possible the derivation of a recurrence relation for generating polynomials for words and in section 10.4 we show how to go from a recurrence relation to an identity between q-series. All the calculations made in sections 10.2–4 involve the generating polynomials $A_{\mathbf{c}}^{\mathrm{des},\mathrm{maj}}(t,q)$.

The second part of the paper (section 10.5–11) is devoted to the construction of the main algorithm. It involves the introduction of commutation rules on biwords that serve to the constructions of both bijections ϕ and Φ. We conclude with the proofs of the equidistribution properties (10.0.2), (10.0.3).

10.1. The q-binomial coefficients

We use the following notations on q-calculus. First, $(a;q)_n$ denotes the *q-ascending factorial*

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n\ge 1.\end{cases}$$

Here a and q are any symbols, variables, or real or complex numbers. The *q-binomial coefficient* (or the Gaussian polynomial) is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0\le k\le n;\\ 0, & \text{otherwise.}\end{cases} \tag{10.1.1}$$

The following properties of the q-binomial coefficients are straightforward and given without proof:

$$\begin{bmatrix} n\\ 0\end{bmatrix} = \begin{bmatrix} n\\ n\end{bmatrix} = 1; \qquad \begin{bmatrix} n\\ k\end{bmatrix} = \begin{bmatrix} n\\ n-k\end{bmatrix}; \tag{10.1.2}$$

$$\begin{bmatrix} n\\ k\end{bmatrix} = \begin{bmatrix} n-1\\ k\end{bmatrix} + q^{n-k}\begin{bmatrix} n-1\\ k-1\end{bmatrix}; \tag{10.1.3}$$

$$\begin{bmatrix} n\\ k\end{bmatrix} = \begin{bmatrix} n-1\\ k-1\end{bmatrix} + q^{k}\begin{bmatrix} n-1\\ k\end{bmatrix}; \tag{10.1.4}$$

$$\lim_{q\to 1}\begin{bmatrix} n\\ k\end{bmatrix} = \binom{n}{k}. \tag{10.1.5}$$

The q-binomial coefficient has a combinatorial interpretation in terms of non-decreasing sequences of integers, as stated in the next proposition, where $\mathbf{a} = (a_1,\dots,a_n)$ denotes a nonincreasing sequence of nonnegative integers and where $\|\mathbf{a}\| = a_1+\dots+a_n$.

Proposition 10.1.1. *For each pair of nonnegative integers (k,n) we have*

$$\begin{bmatrix} k+n\\ k\end{bmatrix} = \sum_{k\ge a_1\ge\dots\ge a_n\ge 0} q^{\|\mathbf{a}\|} = \sum_{n\ge b_1\ge\dots\ge b_k\ge 0} q^{\|\mathbf{b}\|}. \tag{10.1.6}$$

Proof. The fact that the above two summations are equal follows from the symmetry of the q-binomial coefficient $\left[{k+n \atop k}\right]$ in k and n. Denote the first summation by $D(k,n)$ and let $D(0,0)=1$. Then $D(n,0)=D(0,k)$ for every $n\ge 1$ and $k\ge 1$. Next, for k and $n\ge 1$

$$D(k,n)=\sum_{\mathbf{a},\,a_n=0} q^{\|\mathbf{a}\|}+\sum_{\mathbf{a},\,a_n\ge 1} q^{\|\mathbf{a}\|}.$$

Let $b_i=a_i-1$ $(i=1,\ldots,n)$ in the second summation. Then

$$\begin{aligned} D(k,n)&=\sum_{k\ge a_1\ge\cdots\ge a_{n-1}\ge 0} q^{\|\mathbf{a}\|}+\sum_{k-1\ge b_1\ge\cdots\ge b_n\ge 0} q^{n+\|\mathbf{b}\|}\\ &=D(k,n-1)+q^n\,D(k-1,n). \end{aligned}$$

This shows that $D(k,n)$ satisfies the recurrence relation (10.1.2), (10.1.3) for the q-binomial coefficient $\left[{k+n \atop k}\right]$. ■

Proposition 10.1.1 provides the generating function for the nonincreasing sequences of integers bounded from above. There is also a formula for sequences without upper bound, as explained next. For each integer $n\ge 0$ consider the expansion

$$\frac{1}{(t;q)_{1+n}}=\sum_{s\ge 0}\sum_{m\ge 0} t^s\,q^m\,p(s,m). \tag{10.1.7}$$

The coefficient $p(s,m)$ is equal to the number of sequences of nonnegative integers $(i_0,i_1,\ldots,i_n)$ such that $i_0+i_1+\cdots+i_n=s$ and $1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=m$. Consequently, $p(s,m)$ is equal to the number of nonincreasing sequences $\mathbf{a}=(a_1,a_2,\ldots,a_s)$ such that $n\ge a_1\ge\cdots\ge a_s\ge 0$ and $\|a\|=m$. It follows from Proposition 10.1.1 that for each $s\ge 0$

$$\sum_{m\ge 0} q^m\,p(s,m)=\sum_{n\ge a_1\ge\cdots\ge a_s\ge 0} q^{\|\mathbf{a}\|}=\sum_{s\ge a_1\ge\cdots\ge a_n\ge 0} q^{\|\mathbf{a}\|}. \tag{10.1.8}$$

so that

$$\frac{1}{(t;q)_{1+n}}=\sum_{s\ge 0} t^s\sum_{s\ge a_1\ge\cdots\ge a_n\ge 0} q^{\|\mathbf{a}\|}=\sum_{s\ge 0} t^s\begin{bmatrix} n+s\\ n\end{bmatrix}. \tag{10.1.9}$$

10.2. The MacMahon Verfahren

Let $A_{\mathbf{c}}(t,q)=A_{\mathbf{c}}^{\mathrm{des},\mathrm{maj}}(t,q)$ be the generating polynomial for the class $R(\mathbf{c})$ by the pair $(\mathrm{des},\mathrm{maj})$. Those two statistics have been defined in (10.0.1). By convention, $A_{\mathbf{c}}(t,q)=1$, if $\mathbf{c}$ is the null sequence. In this section we shall derive the identity

$$\frac{1}{(t;q)_{1+\|c\|}}A_{\mathbf{c}}(t,q)=\sum_{s\ge 0} t^s\begin{bmatrix} c_1+s\\ s\end{bmatrix}\cdots\begin{bmatrix} c_r+s\\ s\end{bmatrix}, \tag{10.2.1}$$

by means of the so-called *MacMahon Verfahren.*

First let us derive a symmetry property for the polynomials $A_{\mathbf{c}}(t,q)$. For each permutation σ of the set of letters $\{1,2,\ldots,r\}$ denote by $\sigma\mathbf{c}$ the sequence $(c_{\sigma(1)},c_{\sigma(2)},\ldots,c_{\sigma(r)})$, so that $R(\sigma\mathbf{c})$ is the class of all the rearrangements of the word $1^{c_{\sigma(1)}}2^{c_{\sigma(2)}}\ldots r^{c_{\sigma(r)}}$.

THEOREM 10.2.1. *For each permutation σ of the set $\{1,2,\ldots,r\}$ the distributions of the pair* (des, maj) *over $R(\mathbf{c})$ and over $R(\sigma\mathbf{c})$ are identical. In other words, $A_{\mathbf{c}}(t,q)=A_{\sigma\mathbf{c}}(t,q)$.*

Proof. It suffices to prove the property when σ is a transposition $(i,i+1)$ of two adjacent integers $(1\le i\le r-1)$. Consider a word w in $R(\mathbf{c})$ and write all its factors of the form $(i+1)i$ in bold-face; then replace all the maximal factors of the form $i^a(i+1)^b$, with $a\ge 0$, $b\ge 0$, that do not involve any bold-face letters by $i^b(i+1)^a$. Finally, rewrite all the bold-face letters in roman type. Clearly, the transformation is a bijection that maps each word w in $R(\mathbf{c})$ onto a word w' in $R((i,i+1)\mathbf{c})$ with the property that $(\mathrm{des},\mathrm{maj})\,w=(\mathrm{des},\mathrm{maj})\,w'$. ∎

To derive identity (10.2.1) we proceed as follows. By (10.1.9) the left-hand side of (10.2.1) is equal to the sum of the series

$$\sum t^{s'+\operatorname{des} w}q^{\|\mathbf{a}\|+\operatorname{maj} w},$$

extended over the triples $(s',\mathbf{a},w)$, where s' is a nonnegative integer, where $\mathbf{a}$ is a nonincreasing sequence of length $\|\mathbf{c}\|$ such that $s'\ge a_1\ge\cdots\ge a_{\|\mathbf{c}\|}\ge 0$ and where $w\in R(\mathbf{c})$.

By (10.1.6) the right-hand side of (10.2.1) is the sum of the series

$$\sum t^s q^{\|\mathbf{a}^{(1)}\|+\cdots+\|\mathbf{a}^{(r)}\|}$$

extended over all sequences $(s,\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(r)})$, where s is a nonnegative integer and where $\mathbf{a}^{(1)}=(a_{1,1},\ldots,a_{1,c_1})$, … , $\mathbf{a}^{(r)}=(a_{r,1},\ldots,a_{r,c_r})$ are nonincreasing sequences of integers all comprised between s and 0.

To prove that the sums of those two series are equal it suffices to build a bijection $(s,\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(r)})\mapsto(s',\mathbf{a},w)$ having the properties

$$s=s'+\operatorname{des} w \quad\text{and}\quad \|\mathbf{a}^{(1)}\|+\cdots+\|\mathbf{a}^{(r)}\|=\|\mathbf{a}\|+\operatorname{maj} w. \tag{10.2.2}$$

The construction of the bijection is an updated version of a bijection already derived by MacMahon that has been generalized in several contexts. The rearrangement method described below is usually referred to as the *MacMahon Verfahren.*

Form the two-row matrix

$$\begin{pmatrix} a_{1,1} & \ldots & a_{1,c_1} & a_{2,1} & \ldots & a_{2,c_2} & \ldots & a_{r,1} & \ldots & a_{r,c_r} \\ 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots & r & \ldots & r \end{pmatrix}$$

and rearrange its columns in such a way that the mutual orders of the columns with the same bottom entries are preserved and the entire top row is *nonincreasing.* Let

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} y_1\ y_2\ \cdots\ y_{\|\mathbf{c}\|} \\ x_1\ x_2\ \ldots\ x_{\|\mathbf{c}\|} \end{pmatrix} \tag{10.2.3}$$

be the resulting matrix (remember that $c_1 + \cdots + c_r = \|\mathbf{c}\|$.) From the previous method of rearrangement we have $y_k = y_{k+1} \Rightarrow x_k \leq x_{k+1}$, or equivalently

$$x_k > x_{k+1} \Rightarrow y_k > y_{k+1}. \tag{10.2.4}$$

The top row of the matrix (10.2.3) is a word $v = y_1 y_2 \ldots y_{\|\mathbf{c}\|}$ of length $\|\mathbf{c}\|$ which is the unique nonincreasing rearrangement of the juxtaposition product $\mathbf{a}^{(1)} \ldots \mathbf{a}^{(r)}$. The bottom row of the matrix (10.2.3) is a word $w = x_1 x_2 \ldots x_{\|\mathbf{c}\|}$ that belongs to $R(\mathbf{c})$.

For $i = 1, 2, \ldots, \|\mathbf{c}\|$ let z_i be the number of descents in the right factor $x_i x_{i+1} \ldots x_{\|\mathbf{c}\|}$ of w, that is to say, the number of indices j such that $i \leq j \leq \|\mathbf{c}\| - 1$ and $x_j > x_{j+1}$. In particular,

$$z_1 = \operatorname{des} w. \tag{10.2.5}$$

Also, by the very definition of the major index,

$$\operatorname{maj} w = z_1 + z_2 + \cdots + z_{\|\mathbf{c}\|}. \tag{10.2.6}$$

Now condition (10.2.5) implies that the word $\mathbf{a} = a_1 a_2 \ldots a_{\|\mathbf{c}\|}$ defined by

$$a_i = y_i - z_i \quad (i = 1, 2, \ldots, \|\mathbf{c}\|), \tag{10.2.7}$$

is *nonincreasing*; moreover, its letters are *nonnegative.* Then define

$$z' = s - \operatorname{des} w.$$

As $s \geq y_1 = \max a_{i,j}$ and $z_1 = \operatorname{des} w$, we deduce that:

$$s' = s - \operatorname{des} w \geq y_1 - z_1 \geq 0$$

and also

$$\|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\| = \sum_i y_i = \sum_i a_i + \sum_i z_i = \|\mathbf{a}\| + \operatorname{maj} w.$$

The two conditions (10.2.2) are fulfilled. The bijection

$$(s, \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(r)}) \mapsto (s', \mathbf{a}, w)$$

is fully described and is completely reversible. Identity (10.2.1) is then established. ∎

Example 10.2.2. Illustrate the previous construction with an example. Start with the sequence $(s, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)})$ defined by $r = 3$; $\mathbf{a}^{(1)} = 6,5,1,1,0,0$; $\mathbf{a}^{(2)} = 5,4,1,1$; $\mathbf{a}^{(3)} = 3,1$ and $s = 7$. The rearrangement of the matrix

$$\begin{pmatrix} 6\,5\,1\,1\,0\,0\,5\,4\,1\,1\,3\,1 \\ 1\,1\,1\,1\,1\,1\,2\,2\,2\,2\,3\,3 \end{pmatrix}$$

as in (10.2.3) yields

$$\begin{pmatrix} 6\,5\,5\,4\,3\,1\,1\,1\,1\,1\,0\,0 \\ 1\,1\,2\,2\,3\,1\,1\,2\,2\,3\,1\,1 \end{pmatrix}.$$

Hence

$$\begin{aligned} v &= 6,5,5,4,3,1,1,1,1,1,0,0;\\ w &= 1,1,2,2,3,1,1,2,2,3,1,1;\\ z &= 2,2,2,2,2,1,1,1,1,1,0,0;\\ \mathbf{a} &= 4,3,3,2,1,0,0,0,0,0,0,0;\\ \operatorname{des} w &= 2; \qquad s' = s - \operatorname{des} w = 5. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \mathbf{a}^{(3)} &= 6+5+1+1+5+4+1+1+3+1 = 28\\ &= \|\mathbf{a}\| + \operatorname{maj} w = (4+3+3+2+1) + (5+10) = 28. \end{aligned}$$

If $u_1, u_2, \dots, u_r$ are r commuting variables, it is convenient to use the notations $\mathbf{u}^{\mathbf{c}} = u_1^{c_1} u_2^{c_2} \dots u_r^{c_r}$ and $(\mathbf{u}; q)_{s+1} = (u_1; q)_{s+1} \cdots (u_r; q)_{s+1}$. Below the summations of the form $\sum_{\mathbf{c}}$ are extended to all sequences $\mathbf{c} = (c_1, \dots, c_r)$ of r nonnegative integers, including the null sequence.

Form the following *factorial* generating function

$$A(t,q;\mathbf{u}) = \sum_{\mathbf{c}} A_{\mathbf{c}}(t,q) \frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+|\mathbf{c}\|}} \tag{10.2.8}$$

for the polynomials $A_{\mathbf{c}}(t,q)$. It follows from (10.2.1) that

$$\begin{aligned} A(t,q;\mathbf{u}) &= \sum_{s\ge 0} t^s \sum_{\mathbf{c}} \mathbf{u}^{\mathbf{c}} \begin{bmatrix} c_1+s \\ s \end{bmatrix} \cdots \begin{bmatrix} c_r+s \\ s \end{bmatrix}\\ &= \sum_{s\ge 0} t^s \Big(\sum_{c_1} u_1^{c_1} \begin{bmatrix} c_1+s \\ s \end{bmatrix}\Big) \cdots \Big(\sum_{c_r} u_r^{c_r} \begin{bmatrix} c_r+s \\ s \end{bmatrix}\Big), \end{aligned}$$

so that by (10.1.9)

$$A(t,q;\mathbf{u}) = \sum_{s\ge 0} \frac{t^s}{(\mathbf{u};q)_{s+1}}. \tag{10.2.9}$$

Conversely, it is clear that (10.2.9) implies (10.2.1). We then have two ways for expressing the polynomials $A_{\mathbf{c}}(t,q)$. In the next section we will see another expression for those polynomials by means of a recurrence relation.

10.3. The insertion technique

When deriving a recurrence relation for generating polynomials over permutation groups of order $n = 1, 2, \ldots$, the insertion technique is of frequent use: starting with a permutation of order n we study the modification brought to the underlying statistic when the letter $(n+1)$ is *inserted* into the $(n+1)$ slots of the permutation. With words with repetitions some transformations called *word marking* in the sequel must be made on the initial word.

Write

$$A_{\mathbf{c}}(t,q) = \sum_{s\ge 0} A_{\mathbf{c},s}(q)t^s, \tag{10.3.1}$$

so that $A_{\mathbf{c},s}(q)$ is the generating polynomial for the words $w \in R(\mathbf{c})$ such that $\operatorname{des} w = s$ by the major index. It will be conventient to use the notations $[s]_q = 1 + q + q^2 + \cdots + q^{s-1}$ and $\mathbf{c} + 1_j = (c_1, \ldots, c_j + 1, \ldots, c_r)$ for each $j = 1, 2, \ldots, r$ and each sequence $\mathbf{c} = (c_1, c_2, \ldots, c_r)$.

PROPOSITION 10.3.1. *With $\|\mathbf{c}\| = c_1 + \cdots + c_r$ and $1 \le j \le r$ the following relations hold*

$$\begin{aligned}
&(1 - q^{c_j+1})A_{\mathbf{c}+1_j}(t,q) = \\
&\qquad (1 - tq^{1+\|\mathbf{c}\|})A_{\mathbf{c}}(t,q) - q^{c_j+1}(1-t)A_{\mathbf{c}}(tq,q); &(10.3.2)\\
&[c_j + 1]_q A_{\mathbf{c}+1_j,s}(q) = \\
&\qquad [c_j + 1 + s]_q\, A_{\mathbf{c},s}(q) + q^{s+c_j}[1 + \|\mathbf{c}\| - s - c_j]_q\, A_{\mathbf{c},s-1}(q). &(10.3.3)
\end{aligned}$$

Proof. The latter identity is equivalent to the former one, so that only (10.3.4) is to be proved. By Theorem 10.2.1 this relation is equivalent to the relation formed when j is replaced by any integer in $\{1, \ldots, r\}$. It is convenient to prove the relation for $j = 1$ which reads

$$\begin{aligned}
&(1 + q + \cdots + q^{c_1})A_{\mathbf{c}+1_1,s}(q) = \\
&\quad (1 + q + \cdots + q^{c_1+s})A_{\mathbf{c},s}(q) + (q^{c_1+s} + \cdots + q^{\|\mathbf{c}\|})A_{\mathbf{c},s-1}(q). \quad (10.3.4)
\end{aligned}$$

Consider the set $R^*(\mathbf{c} + 1_1, s)$ of 1-*marked* words, i.e., rearrangements w^* of $1^{c_1+1} \ldots r^{c_r}$ with s descents such that exactly one letter equal to 1 has been marked. Each word $w \in R(\mathbf{c}+1_1)$ that has s descents gives rise to c_1+1 marked words $w^{(0)}, \ldots, w^{(c_1)}$. Define

$$\operatorname{maj}^* w^{(i)} = \operatorname{maj} w + n_1,$$

where n_1 is the number of letters equal to 1 to the *right* of the marked 1. Then clearly

$$\sum_{i=0}^{c_1} \operatorname{maj}^* w^{(i)} = (1 + q + \cdots + q^{c_1}) \operatorname{maj} w.$$

Hence

$$(1+q+\cdots+q^{c_1})A_{\mathbf{c}+1_1,s}(q)=\sum_{w\in R^*(\mathbf{c}+1_1,s)} q^{\operatorname{maj}^* w}.$$

Let $m=\|\mathbf{c}\|$ and let the word $w=x_1x_2\dots x_m\in R(\mathbf{c})$ have s descents. Say that w has $m+1$ *slots* x_ix_{i+1}, $i=0,\dots,m$ (where $x_0=0$ and $x_{m+1}=\infty$ by convention). Call the slot x_ix_{i+1} *green* if either x_ix_{i+1} is a descent, $x_i=1$, or $i=0$. Call the other slots *red*. Then there are $1+s+c_1$ green slots and $m-s-c_1$ red slots. Label the green slots $0,1,\dots,c_1+s$ from right to left, and label the red slots $c_1+s+1,\dots,m$ from left to right.

For example, with $r=3$, the word $w=2,2,1,3,2,1,2,3,3$ has three descents and ten slots. As $c_1=2$, there are eight green slots and two red slots, labelled as follows

slot	0 |		2 |		2 |		1 |		3 |		2 |		1 |		2 |		3 |		3 | ∞	
label		5		6		4		3		2		1		0		7		8		9

Denote by $w^{(i)}$ the word obtained from w by inserting a marked 1 into the i-th slot. Then it may be verified that

$$\operatorname{des} w^{(i)}=\begin{cases}\operatorname{des} w, & \text{if } i\le c_1+s;\\ \operatorname{des} w+1, & \text{otherwise.}\end{cases} \tag{10.3.5}$$

$$\operatorname{maj}^* w^{(i)}=\operatorname{maj} w+i. \tag{10.3.6}$$

Example 10.3.2. Consider the above word w. The following table shows the values of "des" and "maj*" on $w^{(i)}$. Descents are indicated by $\frown$ and the marked **1** is written in boldface.

i	$w^{(i)}$	$\operatorname{des} w^{(i)}$	$\operatorname{maj}^* w^{(i)}$
0	$2\ \ 2\frown1\ \ 3\frown2\frown1\ \ \mathbf{1}\ \ 2\ \ 3\ \ 3$	3	11
1	$2\ \ 2\frown1\ \ 3\frown2\frown\mathbf{1}\ \ 1\ \ 2\ \ 3\ \ 3$	3	12
2	$2\ \ 2\frown1\ \ 3\frown\mathbf{1}\ \ 2\frown1\ \ 2\ \ 3\ \ 3$	3	13
3	$2\ \ 2\frown1\ \ \mathbf{1}\ \ 3\frown2\frown1\ \ 2\ \ 3\ \ 3$	3	14
4	$2\ \ 2\frown\mathbf{1}\ \ 1\ \ 3\frown2\frown1\ \ 2\ \ 3\ \ 3$	3	15
5	$\mathbf{1}\ \ 2\ \ 2\frown1\ \ 3\frown2\frown1\ \ 2\ \ 3\ \ 3$	3	16
6	$2\frown\mathbf{1}\ \ 2\frown1\ \ 3\frown2\frown1\ \ 2\ \ 3\ \ 3$	4	17
7	$2\ \ 2\frown1\ \ 3\frown2\frown1\ \ 2\frown\mathbf{1}\ \ 3\ \ 3$	4	18
8	$2\ \ 2\frown1\ \ 3\frown2\frown1\ \ 2\ \ 3\frown\mathbf{1}\ \ 3$	4	19
9	$2\ \ 2\frown1\ \ 3\frown2\frown1\ \ 2\ \ 3\ \ 3\frown\mathbf{1}$	4	20

So each word $w\in R(\mathbf{c})$ with s descents and $\operatorname{maj} w=n$ gives rise to c_1+s+1 marked words in $R^*(\mathbf{c}+1_1,s)$ with maj^* equal to $n,n+1,\dots,n+c_1+s$; and to $m-s-c_1$ marked words in $R^*(\mathbf{c}+1_1,s+1)$ with maj^* equal to $n+c_1+s+1,\dots,n+m$. Hence a word w in $R(\mathbf{c})$ with $s-1$ descents gives rise to $m-s+1-c_1$ marked words in $R^*(\mathbf{c}+1_1,s)$ with maj^* equal to $\operatorname{maj} w+c_1+s,\dots,\operatorname{maj} w+m$. This now proves relation (3.4). ∎

10.4. The (t,q)-factorial generating functions

In the previous section we have seen that formulas (10.2.1) and (10.2.9) implied each other. The purpose of this section is to show that the recurrence formula (10.3.2) is also equivalent to (10.2.1) and (10.2.9). This is achieved by a manipulation of q-series we shall describe in full details.

As defined in (10.2.8) consider the factorial generating function

$$A(t,q;\mathbf{u}) = \sum_{\mathbf{c}} \frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}} A_{\mathbf{c}}(t,q) \tag{10.4.1}$$

and consider the partial q-difference

$$D_{u_r} = A(t,q;u_1,\ldots,u_r) - A(t,q;u_1,\ldots,u_{r-1},u_r q).$$

Directly from (10.4.1) we obtain

$$\begin{aligned} D_{u_r} &= \sum_{\mathbf{c}} (1-q^{c_r+1}) \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{2+\|\mathbf{c}\|}} A_{\mathbf{c}+1_r}(t,q) \\ &= \sum_{\mathbf{c}} (1-tq^{\|\mathbf{c}\|+1}) \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{2+\|\mathbf{c}\|}} A_{\mathbf{c}}(t,q) - \sum_{\mathbf{c}} q^{c_r+1}(1-t) \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{2+\|\mathbf{c}\|}} A_{\mathbf{c}}(tq,q). \end{aligned}$$

Now use the recurrence relation (10.3.2). We get

$$\begin{aligned} \sum_{\mathbf{c}} (1-tq^{\|\mathbf{c}\|+1}) \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{2+\|\mathbf{c}\|}} A_{\mathbf{c}}(t,q) &= \sum_{\mathbf{c}} \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{1+\|\mathbf{c}\|}} A_{\mathbf{c}}(t,q) \\ &= u_r A(t,q;\mathbf{u},) \end{aligned}$$

and

$$\begin{aligned} \sum_{\mathbf{c}} q^{c_r+1}(1-t) \frac{\mathbf{u}^{\mathbf{c}+1_r}}{(t;q)_{2+\|\mathbf{c}\|}} A_{\mathbf{c}}(tq,q) &= \sum_{\mathbf{c}} \frac{\mathbf{u}^{\mathbf{c}+1_r} q^{c_r+1}}{(tq;q)_{1+\|\mathbf{c}\|}} A_{\mathbf{c}}(tq,q) \\ &= u_r q A(tq,q;u_1,\ldots,u_{r-1},u_r q,). \end{aligned}$$

Hence

$$\begin{aligned} A(t,q;\mathbf{u}) &- A(t,q;u_1,\ldots,u_{r-1},u_r q,) \\ &= u_r A(t,q;\mathbf{u}) - u_r q A(tq,q;u_1,\ldots,u_{r-1},u_r q,). \end{aligned} \tag{10.4.2}$$

The (partial) q-difference equation with respect to each u_i $(i=1,\ldots,r)$ has the form

$$\begin{aligned} A(t,q;\mathbf{u}) &- A(t,q;u_1,\ldots,u_i q,\ldots,u_r) \\ &= u_i A(t,q;\mathbf{u}) - u_i q A(tq,q;u_1,\ldots,u_i q,\ldots,u_r). \end{aligned} \tag{10.4.3}$$

Now let

$$A(t,q;\mathbf{u}) = \sum_{s\ge 0} t^s G_s(\mathbf{u},q).$$

From (10.4.3) we get

$$\sum_{s\ge 0} t^s(1-u_i)G_s(\mathbf{u},q) = \sum_{s\ge 0} t^s(1-u_iq^{s+1})G_s(u_1,\dots,u_iq,\dots,u_r,q).$$

Taking the coefficient of t^s in both members yields the relation

$$G_s(\mathbf{u},q) = \frac{1-u_iq^{s+1}}{1-u_i} G_s(u_1,\dots,u_iq,\dots,u_r,,q), \tag{10.4.4}$$

for $i=1,\dots,r$. Now put

$$F_s(\mathbf{u},q) = G_s(\mathbf{u},q)(\mathbf{u};q)_{s+1}. \tag{10.4.5}$$

From equation (10.4.4) we deduce that for $i=1,\dots,r$

$$F_s(\mathbf{u},q) = F_s(u_1,\dots,u_iq,\dots,u_r,q). \tag{10.4.6}$$

But $F_s(\mathbf{u},q)$ can be expressed as $F_s(\mathbf{u},q) = \sum_{\mathbf{c}} \mathbf{u}^{\mathbf{c}} F_{s,\mathbf{c}}(q)$, where $F_{s,\mathbf{c}}(q)$ is a power series in non-negative powers of q. Fix $\mathbf{c}$ and let a be a non-zero component of $\mathbf{c}$. Then relation (10.4.6) implies that $F_{s,\mathbf{c}}(q) = q^a F_{s,\mathbf{c}}(q)$. Therefore, $F_{s,\mathbf{c}}(q)=0$. Hence $F_s(\mathbf{u},q) = F_{s,0}(q)$. It remains to evaluate $F_{s,0}(q)$. But from (10.4.5)

$$F_{s,0}(q) = F_s(\mathbf{u},q)\Big|_{\mathbf{u}=0} = G_s(\mathbf{u},q)(\mathbf{u};q)_{s+1}\Big|_{\mathbf{u}=0} = G_s(0,q) = 1,$$

as $\sum_{s\ge 0} t^s G_s(0,q) = A(t,q;0) = \dfrac{1}{(t;q)_1} = \sum_{s\ge 0} t^s$. Thus $G_s(\mathbf{u},q) = \dfrac{1}{(\mathbf{u};q)_{s+1}}$ by (10.4.5). This proves identity (10.2.9). Conversely showing that (10.2.9) $\Rightarrow$ (10.3.2) is much simpler, for (10.2.9) implies (10.4.3) in an easy manner and from (10.4.3) the recurrence relation (10.3.2) can be reached without any difficulty.

10.5. Words and biwords

The rest of this chapter is devoted to the construction of a class of bijections on each class $R(\mathbf{c})$ based on specific commutation rules. We will see that by means of the so-called Cartier-Foata rule and the contextual rule two bijections ϕ and Φ can be constructed having properties (10.0.2) and (10.0.3), respectively.

Keep the same alphabet $X=\{1,2,\dots,r\}$. A *biword* is an ordered pair of words of the same length, written as $\alpha=(h,b)$ ("h" stands for "high" and "b" for "bottom") or as

$$\alpha = \begin{pmatrix} h \\ b \end{pmatrix} = \begin{pmatrix} h_1h_2\dots h_m \\ b_1b_2\dots b_m \end{pmatrix}.$$

For easy reference we shall sometimes indicate the *places* 1, 2, … , m of the letters on the top of the biword:

$$\begin{bmatrix} \mathrm{id} \\ h \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & \ldots & m \\ h_1 & h_2 & \ldots & h_m \\ b_1 & b_2 & \ldots & b_m \end{bmatrix}.$$

The word h (resp. b) is the *top* (resp. *bottom*) word of the biword (h, b). Each biword $\binom{h}{b}$ can also be seen as a word whose letters are the *biletters* $\binom{h_1}{b_1}, \ldots,$ $\binom{h_m}{b_m}$. The integer m is the *length* of the biword w. A triple $(h, b; i)$ where i is an integer satisfying $1 \le i \le m-1$ is called a *pointed biword.* When h and b are rearrangements of each other, the biword (h, b) is said to be a *circuit.*

Two classes of *circuits* will play a special role. First, we introduce the *standard circuits* $\Gamma(b)$ which are circuits of the form $\binom{\bar b}{b}$, where $\bar b$ is the nondecreasing rearrangement of the word b with respect to the standard ordering. Clearly Γ maps each word onto a standard circuit in a bijective manner.

The second class of circuits is defined as follows. A nonempty word $b = b_m b_1 \ldots b_{m-2} b_{m-1}$ is said to be *dominated,* if $b_m > b_1$, $b_m > b_2, \ldots, b_m > b_{m-1}$. The *right to left cyclic shift* of b is defined to be the word $\delta b = b_1 b_2 \ldots b_{m-1} b_m$. A biword of the form $\binom{\delta b}{b}$ with b dominated is called a *dominated cycle.*

As it is known or easily verified, each word b is the juxtaposition product $u^1 u^2 \ldots$ of dominated words whose first letters $\mathrm{pre}(u^1)$, $\mathrm{pre}(u^2), \ldots$ are in nondecreasing order:

$$\mathrm{pre}(u^1) \le \mathrm{pre}(u^2) \le \cdots \tag{10.5.1}$$

That factorization, called the *increasing factorization* of b, is unique.

Given the increasing factorization $u^1 u^2 \ldots$ of a word b, we can form the juxtaposition product

$$\Delta(b) = \begin{pmatrix} \delta u^1 & \delta u^2 & \ldots \\ u^1 & u^2 & \ldots \end{pmatrix} \tag{10.5.2}$$

of the dominated cycles. Clearly Δ maps each word onto a product of dominated cycles satisfying inequalities (10.5.1), in a bijective manner. Such a product, written as a biword (10.5.2), will be called a *well-factorized circuit.*

Example 10.5.1. Consider the word $b = 2, 2, 1, 3, 5, 3, 4, 5, 1$. The standard circuit associated with b reads

$$\Gamma(b) = \begin{pmatrix} 1\,1\,2\,2\,3\,3\,4\,5\,5 \\ 2\,2\,1\,3\,5\,3\,4\,5\,1 \end{pmatrix}.$$

It has an increasing factorization given by: $2\,|\,2\,1\,|\,3\,|\,5\,3\,4\,|\,5\,1$, so that the corresponding well-factorized circuit reads:

$$\Delta(b) = \begin{pmatrix} 2 & 1\,2 & 3 & 3\,4\,5 & 1\,5 \\ 2 & 2\,1 & 3 & 5\,3\,4 & 5\,1 \end{pmatrix}.$$

As will be seen the next bijections on words can be viewed as composition products

$$b \mapsto \Gamma(b) \mapsto \Delta(c) \mapsto c,$$

where the mapping $\Gamma(b) \mapsto \Delta(c)$ will be described as a sequence of *commutations* on circuits.

10.6. Commutations

Suppose given a four-variable Boolean function $Q(x, y; z, t)$ (also written as $Q\left(\begin{matrix} x, y \\ z, t \end{matrix}\right)$) defined on quadruples of letters in X. The *commutation* "Com" induced by the Boolean function $Q(x, y; z, t)$ is defined to be a mapping that maps each pointed biword $(h, b; i)$ onto a biword $(h', b') = \mathrm{Com}(h, b; i)$ with the following properties: if

$$\begin{pmatrix} h \\ b \end{pmatrix} = \begin{pmatrix} h_1 h_2 \dots h_m \\ b_1 b_2 \dots b_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h' \\ b' \end{pmatrix} = \begin{pmatrix} h'_1 h'_2 \dots h'_{m'} \\ b'_1 b'_2 \dots b'_{m'} \end{pmatrix},$$

then

(C0) $m' = m$;

(C1) $h'_j = h_j$, $b'_j = b_j$ for every $j \neq i, i+1$;

(C2) $h'_{i+1} = h_i$, $h'_i = h_{i+1}$ (the i-th and $(i+1)$-st letters of the top word are transposed);

(C3) $b'_i = b_i$ and $b'_{i+1} = b_{i+1}$ if $Q(h_i, h_{i+1}; b_i, b_{i+1})$ true; $b'_i = b_{i+1}$ and $b'_{i+1} = b_i$ if $Q(h_i, h_{i+1}; b_i, b_{i+1})$ false.

We can also describe the commutation by the following pair of mappings

$$\begin{aligned} h = h_1 \dots h_{i-1} h_i h_{i+1} h_{i+2} \dots h_m &\mapsto h' = h_1 \dots h_{i-1} h_{i+1} h_i h_{i+2} \dots h_m; \\ b = b_1 \dots b_{i-1} b_i b_{i+1} b_{i+2} \dots b_m &\mapsto b' = b_1 \dots b_{i-1}\, z\, t\, b_{i+2} \dots b_m; \end{aligned}$$

where either $z = b_i$, $t = b_{i+1}$ if $Q(h_i, h_{i+1}; b_i, b_{i+1})$ true, or $z = b_{i+1}$, $t = b_i$ if $Q(h_i, h_{i+1}; b_i, b_{i+1})$ false.

Definition 10.6.1. A Boolean function $Q(x, y; z, t)$ is said to be *bi-symmetric* if it is symmetric in the two sets of parameters $\{x, y\}$, $\{z, t\}$.

Lemma 10.6.2. *The commutation "Com" induced by a bi-symmetric Boolean function $Q(x, y; z, t)$ is involutive, i.e., if $(h', b') = \mathrm{Com}(h, b; i)$, then $(h, b) = \mathrm{Com}(h', b'; i)$.*

The proof of the lemma is a simple verification and will be omitted. In the rest of the chapter we will assume that all the four-variable Boolean functions $Q(x, y; z, t)$ are bi-symmetric.

Two extreme cases are worth being mentioned, when Q is the Boolean function Q_{true} "always true" (resp. Q_{false} "always false"). The commutation $\mathrm{Com}_{\text{true}}$, associated with Q_{true}, permutes only the i-th and $(i+1)$-st *letters* of the *top word* b, while $\mathrm{Com}_{\text{false}}$, associated with Q_{false}, permutes the i-th and $(i+1)$-st *biletters* of the *biword* (h, b).

Sorting a biword is defined as follows. Again consider a biword (h, b) of length m and let $(h', b') = \mathrm{Com}(h, b; i)$ with $1 \le i \le m-1$. If $1 \le j \le m-1$, we can form the pointed biword $(h', b'; j)$ and further apply the commutation "Com" to $(h', b'; j)$. We obtain the biword $\mathrm{Com}(h', b'; j) = \mathrm{Com}(\mathrm{Com}(h, b; i); j)$, we shall denote by $\mathrm{Com}(h, b; i, j)$. By induction $\mathrm{Com}(h, b; i_1, \ldots, i_n)$ can be defined, where $(i_1, \ldots, i_n)$ is a given sequence of integers less than m.

As each commutation always permutes two adjacent letters within the *top* word (condition (C2)), we can transform each biword (h, b) into a biword (h', b') whose top word h' is *non-decreasing* by applying a sequence of commutations. We can also say that for each biword (h, b) there exists a sequence $(i_1, \ldots, i_n)$ of integers such that the top word in the resulting biword $\mathrm{Com}(h, b; i_1, \ldots, i_n)$ is non-decreasing. Such a biword is called a *minimal* biword and the sequence $(i_1, \ldots, i_n)$ a *commutation sequence.*

When using the commutations $\mathrm{Com}_{\text{true}}$ or $\mathrm{Com}_{\text{false}}$ we always reach the same minimal biword, but the commutation sequence is not unique. With an arbitrary commutation "Com" neither the minimal biword, nor the commutation sequence are necessarily unique. We then define a particular commutation sequence $(i_1, \ldots, i_n)$ called the *minimal sequence* by the following two conditions:

(i) it is of minimum length;

(ii) it is minimal with respect to the lexicographic order.

Clearly the minimal sequence is uniquely defined by those two conditions and depends only on the top word h in (h, b). The minimal biword derived from (h, b) by using the minimal sequence is called the *straightening* of the biword (h, b). The derivation is described in the following algorithm `SORTB`.

Algorithm `SORTB`: sorting a biword: Given a biword (h, b) and a commutation "Com" the following algorithm transforms (h, b) into its straightening (h', b').

Prototype $(h', b') :=$ `SORTB`(h, b, Com).

(1) Let $(h', b') := (h, b)$.

(2) If h' is non-decreasing, `RETURN` (h', b').

(3) Else, let j be the smallest integer such that $h'(j) > h'(j+1)$. Then let $(h', b') := \mathrm{Com}(h', b'; j)$. Go to (2).

Example 10.6.3. Consider the biword

$$\begin{bmatrix} \mathrm{id} \\ h \\ b \end{bmatrix} = \begin{bmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9 \\ 2\ 1\ 2\ 3\ 3\ 4\ 5\ 1\ 5 \\ 2\ 2\ 1\ 3\ 5\ 3\ 4\ 5\ 1 \end{bmatrix}.$$

The sequence of the indices j that occur in Algorithm `SORTB` applied to the biword is:

1, that transforms h' into		1,2,2,3,3,4,5,1,5
7,	*ibid.*	1,2,2,3,3,4,1,5,5
6,	*ibid.*	1,2,2,3,3,1,4,5,5
5,	*ibid.*	1,2,2,3,1,3,4,5,5
4,	*ibid.*	1,2,2,1,3,3,4,5,5
3,	*ibid.*	1,2,1,2,3,3,4,5,5
2,	*ibid.*	1,1,2,2,3,3,4,5,5

so that the minimal sequence is: 1,7,6,5,4,3,2, and accordingly the final word h' is 1,1,2,2,3,3,4,5,5. Notice that the final word b' depends on the commutation rule Com.

10.7. The two commutations

We shall introduce two commutations associated with two specific Boolean functions Q.

7.1. *The Cartier-Foata commutation.* We denote by Com_{CF} the commutation induced by the following Boolean function Q_{CF}:

$$Q_{CF}\binom{x,y}{z,t} \text{ true} \quad \text{if and only if} \quad x=y. \tag{10.7.1}$$

7.2. *The contextual commutation.* For each letter x let $x^+ = x + \frac{1}{2}$ and denote by Com_H the commutation induced by the following Boolean function Q_H:

$$Q_H\binom{x,y}{z,t} \text{ true iff } (z-x^+)(z-y^+)(t-x^+)(t-y^+) > 0. \tag{10.7.2}$$

Notice that both Q_{CF} and Q_H are bi-symmetric, so that $Q_{CF}^2 = Q_H^2 =$ the identity map.

The second commutation can also be defined by means of the following "cyclic intervals." Place the r elements $1, 2, \ldots, x,\ (x+1), \ldots, (r-1), r$ on a circle or on a square (!) counterclockwise and place a bracket on each of those elements as shown in Fig. 10.1. For $x, y \in X$ $(x \neq y)$ the cyclic interval $]\!]x,y]\!]$ is the subset of all the elements that lie between x and y when the circle is read counterclockwise. The brackets (in the French notation) indicate if the extremities of the interval are to be included or not.

For instance, suppose $1 < x < r$. Then $]\!]1,x]\!] = \{2,\ldots,x\}$ (the origin 1 excluded, but the end x included), while $]\!]x,1]\!] = \{x+1,\ldots,r,1\}$ (x excluded but 1 included); finally, let $]\!]x,x]\!] = \emptyset$.

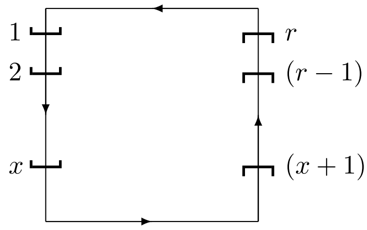

Figure 10.1. Cyclic Interval

PROPOSITION 10.7.1. *The Boolean function* $Q_H\binom{x,y}{z,t}$ *is true if and only if both* z,t *are in* $\rrbracket x,y \rrbracket$ *or neither in* $\rrbracket x,y \rrbracket$.

The proof is a lengthy but easy verification and is omitted. Notice that condition (10.7.2) is efficient in programming while the other condition involving cyclic intervals is more adapted for human beings!

10.8. The main algorithm

It is denoted by $\mathbf{T}$ and is defined for any Boolean function Q. Let Com be the commutation induced by Q. Then $\mathbf{T}$ transforms each word b into a rearrangement c of b.

Prototype $c := \mathbf{T}(b, \mathrm{Com})$.

(1) Let h be the nondecreasing rearrangement of b. Form the standard circuit $\Gamma(b) = (h, b)$:

$$\begin{bmatrix} \mathrm{id} \\ h \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & m \\ h_1 & h_2 & \cdots & h_m \\ b_1 & b_2 & \cdots & b_m \end{bmatrix},$$

let $c := b$ and α be the empty cycle $\alpha = \begin{bmatrix} . \\ . \\ . \end{bmatrix}$.

(2a) If all the places $1, 2, \cdots, m$ occur in α, `RETURN` c (the juxtaposition product of the bottom words in α.)

(2b) Else, let D be the greatest *place* not occurring in α.

(2c) Let M be the greatest *letter* in h not in α, so that $h_D = M$ and the initial biword has been changed into:

$$\begin{bmatrix} \mathrm{id} \\ h \\ c \end{bmatrix} = \begin{bmatrix} * \cdots * & D \\ * \cdots * & M \\ * \cdots * & * \end{bmatrix} \overbrace{\begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix} \cdots \begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix}}^{\alpha}.$$

(3a) Let $B := c_D$.

(3b) If $B = M$, terminate the dominated cycle $\begin{bmatrix} * & \cdots & * \\ * & \cdots & M \\ M & \cdots & * \end{bmatrix}$ and add it to the left of α, so that the new α reads $\begin{bmatrix} * & \cdots & * \\ * & \cdots & M \\ M & \cdots & * \end{bmatrix} \alpha$. Go to (2a).

(3c) Else, look for the greatest *place* $j \leq D-1$ such that $B = h_j$; in short

$$\begin{bmatrix} \mathrm{id} \\ h \\ c \end{bmatrix} = \begin{bmatrix} \cdots & j & \cdots & D & \cdots & * \\ \cdots & B & \cdots & * & \cdots & M \\ \cdots & * & \cdots & B & \cdots & * \end{bmatrix} \begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix} \cdots \begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix}.$$

If $j \leq D-2$, apply the commutation:

$$(h, c) = \mathrm{Com}(h, c; j, j+1, \cdots, D-2),$$

so that $h_{D-1} = B$ after running the commutation; in short:

$$\begin{bmatrix} \mathrm{id} \\ h \\ c \end{bmatrix} = \begin{bmatrix} \cdots & j & \cdots & D-1 & D & \cdots & * \\ \cdots & * & \cdots & B & * & \cdots & M \\ \cdots & * & \cdots & * & B & \cdots & * \end{bmatrix} \begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix} \cdots \begin{bmatrix} * \cdots * \\ * \cdots * \\ * \cdots * \end{bmatrix}.$$

(3d) Let $D := D - 1$ and go to (3a).

We can verify that each step in the previous algorithm is feasible. For example, the place j in step (3c) is well-defined: at this stage $\binom{h}{c}$ is the product of the left factor (in square brackets) $\left[\begin{smallmatrix} h' \\ c' \end{smallmatrix}\right]$ by α and h' is necessarily a rearrangement of c'.

Define the two transformations:

$$\phi(b) = \mathbf{T}(b, \mathrm{Com}_{CF}) \quad \text{and} \quad \Phi(b) = \mathbf{T}(b, \mathrm{Com}_H). \tag{10.8.1}$$

Example 10.8.1. Consider the word $b = 2, 1, 2, 3, 3, 5, 4, 5, 1$ and the circuit

$$\Gamma(b) = \begin{bmatrix} \mathrm{id} \\ h \\ b \end{bmatrix} = \begin{bmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,1\,2\,2\,3\,3\,4\,5\,5 \\ 2\,1\,2\,3\,3\,5\,4\,5\,1 \end{bmatrix}$$

and calculate the image of b under ϕ and Φ.

For the first transformation we easily obtain

$$\begin{bmatrix} \mathrm{id} \\ h \\ c \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 3 & 4 & 5\,6\,7\,8\,9 \\ 2 & 3 & 4 & 5 & 3\,2\,1\,1\,5 \\ 2 & 3 & 4 & 5 & 5\,3\,2\,1\,1 \end{bmatrix},$$

so that $c = \phi(b) = 2\,3\,4\,5\,5\,3\,2\,1\,1$.

For the second we indicate all the commutations needed in bold-face:

$$\begin{aligned}
\begin{bmatrix} h \\ b \end{bmatrix} &= \begin{bmatrix} 1\,\mathbf{1}\,\mathbf{2}\,2\,3\,3\,4\,5\,5 \\ 2\,1\,2\,3\,3\,5\,4\,5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 1\,2\,\mathbf{1}\,\mathbf{2}\,3\,3\,4\,5\,5 \\ 2\,2\,1\,3\,3\,5\,4\,5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 1\,2\,2\,\mathbf{1}\,\mathbf{3}\,3\,4\,5\,5 \\ 2\,2\,1\,3\,3\,5\,4\,5\,1 \end{bmatrix} \\
&\mapsto \begin{bmatrix} 1\,2\,2\,3\,\mathbf{1}\,\mathbf{3}\,4\,5\,5 \\ 2\,2\,1\,3\,3\,5\,4\,5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 1\,2\,2\,3\,3\,\mathbf{1}\,\mathbf{4}\,5\,5 \\ 2\,2\,1\,3\,5\,3\,4\,5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 1\,2\,2\,3\,3\,4\,\mathbf{1}\,\mathbf{5}\,5 \\ 2\,2\,1\,3\,5\,3\,4\,5\,1 \end{bmatrix} \\
&\mapsto \begin{bmatrix} 1\,2\,2\,3\,3\,4\,5\ \ 1\,5 \\ 2\,2\,1\,3\,5\,3\,4\ \ 5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 1\,2\,2\,3\ \ 3\,4\,5\ \ 1\,5 \\ 2\,2\,1\,3\ \ 5\,3\,4\ \ 5\,1 \end{bmatrix} \\
&\mapsto \begin{bmatrix} \mathbf{1}\,\mathbf{2}\,2\ \ 3\ \ 3\,4\,5\ \ 1\,5 \\ 2\,2\,1\ \ 3\ \ 5\,3\,4\ \ 5\,1 \end{bmatrix} \mapsto \begin{bmatrix} 2\ \ 1\,2\ \ 3\ \ 3\,4\,5\ \ 1\,5 \\ 2\ \ 2\,1\ \ 3\ \ 5\,3\,4\ \ 5\,1 \end{bmatrix}
\end{aligned}$$

so that $c = \Phi(b) = 2\,2\,1\,3\,5\,3\,4\,5\,1$.

10.9. The inverse of the algorithm

Given a commutation Com, the following algorithm denoted by $\mathbf{T}^{-1}$ transforms a word c into a word b such that $b = \mathbf{T}^{-1}(c, \mathrm{Com})$.

Prototype $b := \mathbf{T}^{-1}(c, \mathrm{Com})$.

(1) Let $i := 1$, $S := c_1$;

(2a) If $i = \mathrm{length}(c)$, let $h_i := S$, $(h, b) :=$ `SORTB`(h, c, Com). `RETURN` b.

(2b) Else, let $B := c_{i+1}$.

(2c) If $B \geq S$, let $h_i := S$, $S := B$. Else, let $h_i := B$.

(3) Let $i := i + 1$. Go to (2a).

Now examine algorithm $\mathbf{T}$. Before returning c in step (2a) the algorithm provides the juxtaposition product $\alpha = \gamma^1\gamma^2\ldots$ of cycles. Let u^1, u^2, … be the bottom words of those cycles and let $\mathrm{pre}(u^1)$, $\mathrm{pre}(u^2)$, … be the first letters of those bottom words. Steps (2c) and (3b) say that each cycle γ^i was terminated as soon as $\mathrm{pre}(u^i)$ was greater than all the other letters in the cycle. Accordingly, all the cycles γ^i are *dominated*. Furthermore, $\mathrm{pre}(u^1) \leq \mathrm{pre}(u^2) \leq \cdots$

Thus $u^1u^2\ldots$ is the increasing factorization of c (in the terminology of section 6), while $\alpha = \gamma^1\gamma^2\ldots = \begin{pmatrix} \delta u^1 & \delta u^2 & \ldots \\ u^1 & u^2 & \ldots \end{pmatrix}$ is the *increasing* product of dominated cycles, i.e., α is equal to the well-factorized circuit $\Delta(c)$. We can say that the algorithm $\mathbf{T}$ maps b onto $\Gamma(b)$, then transforms each *standard circuit* $\Gamma(b)$ into a *well-factorized circuit* $\Delta(c)$, the word c being a rearrangement of b. Let U be the mapping $U : \Gamma(b) \mapsto \Delta(c)$, so that $\mathbf{T}$ is the composition product

$$b \mapsto \Gamma(b) \overset{U}{\mapsto} \Delta(c) \mapsto c \tag{10.9.1}$$

As each commutation applied to a pointed biword is involutive, U and therefore $\mathbf{T}$ are bijective.

Further examine Algorithm $\mathbf{T}^{-1}$ and let $u^1 u^2 \cdots$ be the increasing factorization of c as a product of dominated words. Once we have reached step (2a), verified that the test $i = \text{length}(c)$ was positive and executed $h_i := S$, the biword (h, c) is exactly the well-factorized circuit

$$\Delta(c) = \begin{pmatrix} h \\ c \end{pmatrix} = \begin{pmatrix} \delta u^1 & \delta u^2 & \ldots \\ u^1 & u^2 & \ldots \end{pmatrix}.$$

Thus Algorithm $\mathbf{T}^{-1}$ first builds up the well-factorized circuit $\Delta(c)$ and applies algorithm `SORTB` to $\Delta(c)$ to produce a standard circuit $\Gamma(b)$, so that $\mathbf{T}^{-1}$ may be represented as the sequence

$$c \mapsto \Delta(c) \xmapsto{\texttt{SORTB}} \Gamma(b) \mapsto b. \tag{10.9.2}$$

Again as each local commutation applied to a pointed biword is involutive, $\mathbf{T}^{-1}$ is a bijection.

Finally, to prove that $\mathbf{T}$ and $\mathbf{T}^{-1}$ are inverse of each other, we simply examine Algorithm $\mathbf{T}$. The commutations are made only in steps (2c) and (3c). In both steps the reverse operation can be written as

$$(h, c) := \texttt{SORTB}(h, c, \text{Com}).$$

We have then proved the following property

PROPERTY 10.9.1. *Algorithms $\mathbf{T}$ and $\mathbf{T}^{-1}$ are inverse of each other, i.e., for each word b we have*

$$\mathbf{T}^{-1}(\mathbf{T}(b, \text{Com}), \text{Com}) = \mathbf{T}(\mathbf{T}^{-1}(b, \text{Com}), \text{Com}) = b.$$

REMARK 10.9.2. Algorithms $\mathbf{T}$ and $\mathbf{T}^{-1}$ are valid for each bi-symmetric Boolean function Q. However only the Cartier-Foata and the contextual commutations will be used to derive the next results on statistics on words.

10.10. Statistics on circuits

Let $C(X)$ denote the set of all circuits. Remember that a circuit is a pair of words $\alpha = \binom{h}{b}$, where $h = y_1 y_2 \ldots y_m$ and $b = x_1 x_2 \ldots x_m$ are rearrangements of each other and h is *not* necessarily non-decreasing. Two circuits α and β are said to be *H-equivalent*, written $\alpha \sim \beta$, if one can be obtained from the other by a sequence of commutations Com_H (see paragraphe 7.2).

The two statistics "des" and "maj" for each circuit $\alpha = \binom{h}{b}$ are defined as follows. They depend only on the bottom word b. First let $\operatorname{des}\alpha = \operatorname{des} b$. Then the statistic "maj" is based on the notion of *cyclic interval*, as introduced in section 10.7. Put $x_{m+1} = \infty$ (an auxiliary letter greater than every letter of X). Then for each $i = 1, 2, \ldots, m$ define q_i to be the number of j such that $1 \le j \le i-1$ and $x_j \in \rrbracket x_i, x_{i+1} \rrbracket$. The sequence $(q_1, q_2, \ldots, q_m)$ is said to be the *maj-coding* of α. Define

$$\operatorname{maj}\alpha = q_1 + q_2 + \cdots + q_m. \tag{10.10.1}$$

Now given the commutation Com_H we can apply Algorithm `SORTB` of section 10.6 to each circuit α. It produces a standard circuit β to which the rearrangement U defined in (10.9.1) can be further applied to derive a well-factorised circuit γ:

$$\alpha \xmapsto{\texttt{SORTB}} \beta \overset{U}{\mapsto} \gamma \tag{10.10.2}$$

Let Ψ denote the mapping $\alpha \mapsto \gamma$. Because of (10.9.1) and (10.9.2) we have $\Psi(\alpha) = \alpha$ if α is well-factorized. In particular, Ψ is surjective.

Theorem 10.10.1. *There exists at most one bivariate statistic (f, g) defined on $C(X)$ having the following two properties:*

(1) $\alpha \sim \alpha' \Rightarrow (f, g)\,\alpha = (f, g)\,\alpha'$;

(2) if α is well-factorized, then

$$(f, g)\,\alpha = (\operatorname{des}, \operatorname{maj})\,\alpha. \tag{10.10.3}$$

Proof. Both algorithms `SORTB` and U involve sequences of commutations Com_H, so that if $\gamma = \Psi(\alpha)$, we have $(f, g)\,\alpha = (f, g)\,\gamma = (\operatorname{des}, \operatorname{maj})\,\gamma$. ∎

Our next task is to give an explicit definition of the pair (f, g). For each circuit $\alpha = \binom{h}{b}$ with $h = y_1 y_2 \ldots y_m$ and $b = x_1 x_2 \ldots x_m$ define $\operatorname{exc}\alpha$ to the the number of integers i such that $1 \le i \le m$ and $x_i > y_i$. For each place i ($1 \le i \le m$) define p_i to be the number of j such that $1 \le j \le i-1$ and $x_j \in \rrbracket x_i, y_i \rrbracket$. The sequence $(p_1, p_2, \ldots, p_m)$ is said to be the *den-coding* of α. Furthermore, define

$$\operatorname{den}\alpha = p_1 + p_2 + \cdots + p_m. \tag{10.10.4}$$

Example 10.10.2. The following circuit

$$\alpha = \begin{bmatrix} \mathrm{id} \\ h' \\ b' \end{bmatrix} = \begin{bmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 2\,1\,2\,3\,3\,4\,5\,1\,5 \\ 2\,2\,1\,3\,5\,3\,4\,5\,1 \end{bmatrix}.$$

was already considered in Example 10.6.3. It has an excedance at places 2, 5, 8, so that $\operatorname{exc}\alpha = 4$. For its den-coding we first have $p_1 = p_2 = 0$. As $2 \in \rrbracket 1, 2 \rrbracket$, we have $p_3 = 2$. Then $p_4 = 0$. As $]\!]5,3]\!] = \{1,2,3\}$, $p_5 = 4$. Next $p_6 = 0$. Also $]\!]4,5]\!] = \{5\}$, so that $p_7 = 1$ and $]\!]5,1]\!] = \{1\}$, so that $p_8 = 1$. As $]\!]1,5]\!] = \{2,3,4,5\}$, we get $p_9 = 7$. Thus $\operatorname{den}\alpha = 0+0+2+0+4+0+1+1+7 = 15$.

THEOREM 10.10.3. *The pair* $(\operatorname{exc}, \operatorname{den})$ *is the unique bivariate statistic defined on* $C(X)$ *having properties* (1) *and* (2) *of Theorem* 10.10.1.

Proof. Proving that $\alpha \sim \alpha' \Rightarrow (\operatorname{exc}, \operatorname{den})\,\alpha = (\operatorname{exc}, \operatorname{den})\,\alpha'$ is lengthy but easy, as the property is to be proved only when α and α' differ by a commutation Com_H. The proof is omitted.

To show that $(\operatorname{exc}, \operatorname{den})\,\alpha = (\operatorname{des}, \operatorname{maj})\,\alpha$ when α is well-factorized proceed as follows.

Let $\begin{pmatrix} a_2 & \dots & a_{i+1} & a_{i+2} & \dots & a_k & a_1 \\ a_1 & \dots & a_i & a_{i+1} & \dots & a_{k-1} & a_k \end{pmatrix}$ and $\begin{pmatrix} b_2 & \dots \\ b_1 & \dots \end{pmatrix}$ be two successive dominated cycles in the increasing factorization of α, so that

$$\alpha = \begin{pmatrix} \dots & a_2 & \dots & a_{i+1} & a_{i+2} & \dots & a_k & a_1 & b_2 & \dots \\ \dots & a_1 & \dots & a_i & a_{i+1} & \dots & a_{k-1} & a_k & b_1 & \dots \end{pmatrix}.$$

Inside each dominated cycle a pair like (a_i, a_{i+1}) occurs horizontally and vertically, so that there is a descent $a_i a_{i+1}$ if and only if there is an excedance $\binom{a_{i+1}}{a_i}$. Furthermore, the letters in w to the left of a_i that fall into the cyclic interval $]\!]a_i, a_{i+1}]\!]$ bring the same contribution to both $\operatorname{maj}\alpha$ and $\operatorname{den}\alpha$. If $\binom{a_{i+1}}{a_i}$ is the j-th biletter of α (when read from left to right), we have $p_j = q_j$ in the notations used in (10.10.4)) and (10.10.1).

At the end of a dominated cycle we have to compare the contributions of the horizontal pair (a_k, b_1) with the contribution of the vertical pair $\binom{a_1}{a_k}$. But $a_k < a_1 \le b_1$ by definition of the increasing factorization, so that (a_k, a_1) is never an excedance and (a_k, b_1) never a descent.

Now if $a_1 = b_1$, the two cyclic intervals $]\!]a_k, b_1]\!]$ and $]\!]a_k, a_1]\!]$ that serve in the calculation of $\operatorname{maj}\alpha$ and $\operatorname{den}\alpha$ are identical. If $a_1 < b_1$, there is no letter x in w to the left of b_1 such that $a_1 < x \le b_1$. For any two sets A, B let $A + B$ denote the union of A and B when the intersection $A \cap B$ is empty. As

$$]\!]a_k, b_1]\!] =]\!]a_k, a_1]\!] + \{x : a_1 < x \le b_1\},$$

there are as many letters to the left of a_k falling into the interval $]\!]a_k, b_1]\!]$ as letters falling into $]\!]a_k, a_1]\!]$.

Suppose that α is of length m and take up again the notations of (10.1) and (10.2). It remains to compare q_m and p_m. Let $\binom{y_m}{x_m}$ be the rightmost biletter of α. The letter y_m is necessarily equal to the greatest letter occurring in w. Hence the cyclic intervals $]\!]x_m, \infty]\!]$ used for evaluating q_m and $]\!]x_m, y_m]\!]$ for evaluating p_m are equal. ∎

As the transformation U is a sequence of commutations Com_H we have

$$(\operatorname{exc}, \operatorname{den})\,\alpha = (\operatorname{exc}, \operatorname{den})\,U(\alpha) = (\operatorname{des}, \operatorname{maj})\,U(\alpha). \tag{10.10.5}$$

The above development can be reproduced for the commutation Com_{CF}. However the proofs are far simpler. In the same manner, we can prove that the statistic "exc" is the unique statistic having the following properties:

(1) $\alpha \sim \alpha'$ (for Com_{CF}) $\Rightarrow \operatorname{exc}\alpha = \operatorname{exc}\alpha'$;

(2) if α is well-factorized, then $\operatorname{exc}\alpha = \operatorname{des}\alpha$.

Hence, if Com_{CF} is used, we have

$$\operatorname{exc}\alpha = \operatorname{exc}U(\alpha) = \operatorname{des}U(\alpha). \tag{10.10.6}$$

10.11. Statistics on words and equidistribution properties

To get the definitions of $\operatorname{des} w$, $\operatorname{maj} w$, $\operatorname{exc} w$ and $\operatorname{den} w$ for a word w we simply form the *standard circuit* $\Gamma(w)$ and put

$$\begin{aligned} \operatorname{des} w &= \operatorname{des}\Gamma(w), \qquad & \operatorname{maj} w &= \operatorname{maj}\Gamma(w),\\ \operatorname{exc} w &= \operatorname{exc}\Gamma(w), \qquad & \operatorname{den} w &= \operatorname{den}\Gamma(w). \end{aligned} \tag{10.11.1}$$

The definitions given for $\operatorname{des} w$ and $\operatorname{exc} w$ are identical with the definitions given in the introduction. The definition of $\operatorname{den} w$ is new, while that of $\operatorname{maj} w$ differs from the definition given in the introduction. However we have the following result.

THEOREM 10.11.1. *The statistic* $\operatorname{maj} w$ *given in* (10.11.1) *and the statistic* $\operatorname{maj} w$ *given in the introduction are identical.*

This theorem is easy to prove by induction on the length of the word.

The *excedance index* of w is defined as the sum, $\operatorname{excindex} w$, of all i such that i is an excedance in w. When a certain correcting term is added to $\operatorname{excindex} w$, we get the second definition of $\operatorname{den} w$. To fully describe that correcting term we need the further definitions. For each word $w = x_1x_2\ldots x_m$ let

$$\begin{aligned} \operatorname{inv} w &= \#\{1 \le i < j \le m : x_i > x_j\},\\ \operatorname{imv} w &= \#\{1 \le i < j \le m : x_i \ge x_j\}. \end{aligned} \tag{10.11.2}$$

Now if $\operatorname{exc} w = e$, let $i_1 < i_2 < \cdots < i_e$ be the increasing sequence of the excedances of w and let $j_1 < j_2 < \cdots < j_{m-e}$ be the complementary sequence. Form the two subwords

$$\operatorname{Exc} w = x_{i_1}x_{i_2}\ldots x_{i_e}\,; \qquad \operatorname{Nexc} w = x_{j_1}x_{j_2}\ldots x_{j_{m-e}}.$$

Then the *Denert statistic* of w is also defined to be

$$\operatorname{den} w = \operatorname{excindex} w + \operatorname{imv}\operatorname{Exc} w + \operatorname{inv}\operatorname{Nexc} w. \tag{10.11.3}$$

THEOREM 10.11.2. *For every word w the two definitions of* $\operatorname{den} w$ *occurring in* (10.11.1) *and* (10.11.3) *are identical.*

Surprisingly this theorem is not easy to prove, see the Notes below.

THEOREM 10.11.3. *The transformations ϕ and Φ defined in* (10.8.1) *have the equidistribution properties*

$$\operatorname{exc}(w) = \operatorname{des}\,\phi(w) \quad \textit{and} \quad (\operatorname{exc}, \operatorname{den})\, w = (\operatorname{des}, \operatorname{maj})\, \Phi(w).$$

Proof. As shown in (10.8.1) both transformations ϕ and Φ are defined by means of the main algorithm $\mathbf{T}$ which itself is defined by the chain: $b \mapsto \Gamma(b) \overset{U}{\mapsto} \Delta(c) \mapsto c$ (see (10.9.1).)

If Com_{CF} is used, then $\operatorname{exc}\Gamma(b) = \operatorname{des}\Delta(c)$ by (10.10.6). On the other hand, $\operatorname{exc} b = \operatorname{exc}\Gamma(b)$ by (10.11.1) and $\operatorname{des}\Delta(c) = \operatorname{des} c$, as the definition of "des" depends only on the bottom word c of the circuit. Thus $\operatorname{exc} b = \operatorname{des} c$.

If Com_H is used, then $(\operatorname{exc}, \operatorname{den})\,\Gamma(b) = (\operatorname{des}, \operatorname{maj})\,\Delta(c)$ by (10.10.5). Also $(\operatorname{exc}, \operatorname{den})\, b = (\operatorname{exc}, \operatorname{den})\,\Gamma(b)$ by (10.11.1) and $(\operatorname{des}, \operatorname{maj})\,\Delta(c) = (\operatorname{des}, \operatorname{maj})(c)$, as the definition of $(\operatorname{des}, \operatorname{maj})$ depends only on the bottom word c. Hence $(\operatorname{exc}, \operatorname{den})\, b = (\operatorname{des}, \operatorname{maj})(c)$. ■

EXAMPLE 10.11.4. Again consider the word $b = 2, 1, 2, 3, 3, 5, 4, 5, 1$ and its standard circuit $\Gamma(b) = \begin{bmatrix} \mathrm{id} \\ h \\ b \end{bmatrix} = \begin{bmatrix} \mathbf{1}\,2\,3\,\mathbf{4}\,5\,\mathbf{6}\,7\,8\,9 \\ \mathbf{1}\,1\,2\,\mathbf{2}\,3\,\mathbf{3}\,4\,5\,5 \\ \mathbf{2}\,1\,2\,\mathbf{3}\,3\,\mathbf{5}\,4\,5\,1 \end{bmatrix}$. Then $\operatorname{exc} b = 3$. Using the definition (10.11.3) for the Denert statistic we find: $\operatorname{den} b = (1 + 4 + 6) + \operatorname{imv}(2, 3, 5) + \operatorname{inv}(1, 2, 3, 4, 5, 1) = 11 + 0 + 4 = 15$.

The images $\phi(b) = 2, 3, 4, 5, 5, 3, 2, 1, 1$ and $\Phi(b) = 2, 2, 1, 3, 5, 3, 4, 5, 1$ have been determined in Example 10.8.1. Observe that $\operatorname{des}\phi(b) = 3 = \operatorname{exc} b$. The word $\Phi(b)$ has also three descents. Furthermore, its major index is equal to $2 + 5 + 8 = 15$, so that $(\operatorname{exc}, \operatorname{den})\, b = (\operatorname{des}, \operatorname{maj})\,\Phi(b) = (3, 15)$.

Notes

With his treatise on Combinatory Analysis and his numerous papers MacMahon (1915, 1978) may be regarded as the initiator of the study of permutation statistics that includes methods for deriving analytical expressions for generating functions. In particular, he made a clever use of his Master Theorem (see MacMahon 1915, p. 97) that allowed him to show that the generating polynomials for each rearrangement class by the number of descents "des" and by the number of excedances "exc" were equal, so that "des" and "exc" are equidistributed on every rearrangement class. Back in the sixties, as initiated by the late Schützenberger, it was natural to prove such equidistribution properties in a *bijective* manner. The transformation ϕ that satisfies (10.0.2) was constructed in Foata 1965. A further presentation was made in Knuth 1973, p. 24–29, a more algebraic version appeared in Cartier and Foata 1969 and also in Lothaire 1983, chap. 10.

In studying the genus zeta function of local minimal hereditary orders Denert (1990) introduced a new permutation statistic, that was later christened "den". She observed and conjectured that the generating polynomials for each rearrangement class by the pairs $(\mathrm{des}, \mathrm{maj})$ and $(\mathrm{exc}, \mathrm{den})$ were equal. Foata and Zeilberger (1990) proved the conjecture for permutations by making use of the linear partial recurrence operator algebra. Then Han (1990a, 1990b) proved the result combinatorially.

The definition of "den" for arbitrary words (with repetitions) is due to Han (1994) who also constructed a bijection Φ having property (10.0.3) for an arbitrary rearrangement class. In the case of permutations the equivalence between the two definitions (10.11.1) and (10.11.3) of the Denert statistic was given in Foata and Zeilberger 1990. Another proof appeared in Clarke 1995. The general case for arbitrary words was derived by Han (1994), who also introduced the defintion (10.10.1) of "maj", which was basic for constructing the bijection Φ.

When the underlying alphabet X is partitioned into two subalphabets S of *small* letters and L of *large* letters, the classical permutation statistics can be further refined to take large inequalities into account (see Clarke and Foata 1994, 1995a, 1995b). Those statistics are denoted by des_k, $\mathrm{exc}_k, \ldots$ (or by des_U, $\mathrm{exc}_U, \ldots$ in Problem 10.5.2). It is also possible to derive explicit formulas for the generating polynomials by using the techniques developed in sections 10.2–4. Furthermore, a bijection Φ_k of each rearrangement class can be constructed (see Clarke and Foata 1995a) such that $(\mathrm{exc}_k, \mathrm{den}_k)\, w = (\mathrm{des}_k, \mathrm{maj}_k)\, \Phi_k(w)$ holds identically. As shown in Foata and Han 1998 there is a common frame for constructing all the bijections ϕ, Φ, Φ_k based on the concept of biword commutation as presented in sections 10.5–8.

When $\mathbf{c} = 1^r$ the generating polynomial for $(\mathrm{des}, \mathrm{maj})$ is the celebrated q-Eulerian polynomial $A_r(t, q)$ whose first study goes back to Carlitz (1954, 1959, 1975). Also see Problem 10.5.1. There is another class of q-Eulerian polynomials that can be introduced as generating functions for the permutation group by the pair $(\mathrm{des}, \mathrm{inv})$, where "inv" is the number of inversions Stanley 1976.

The basic material on q-calculus can be found in Andrews 1976, Gasper and Rahman 1990. The MacMahon Verfahren takes its rise in MacMahon 1913. Formula (10.2.1) already appeared in MacMahon 1915, vol. 2, p. 211. Stanley (1972) and his disciple Reiner (1993) have extended the MacMahon Verfahren from the linear model used in this chapter and in Problem 10.2.2 to the poset environment and developed an adequate P-partition theory. There have been several papers that propose various techniques to derive analytical expressions for the permutation or word distributions, for example, Gessel 1977, Garsia 1979, Garsia and Gessel 1979, Gessel 1982, Zeilberger 1980 for the "commaed" permutation technique; Fedou and Rawlings 1995 for an adjacency study; Foata and Han 1997 for an iterative method. A systematic permutation statistic study has been undertaken by Clarke, Steingrimsson, and Zeng (1997a, 1997b).

Problems

Section 10.1

10.1.1 (The q-binomial theorem). Using the notation $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$ the q-binomial theorem reads:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} u^n = \frac{(au;q)_\infty}{(u;q)_\infty}.$$

The symbols q and u can be taken as complex numbers such that $|q| < 1$, $|u| < 1$ or as variables. In the latter case the previous identity holds in the algebra of formal power series in two variables with coefficients in a given ring. See Andrews 1976, Theorem 2.1 or Gasper and Rahman 1990, paragraphe 1.3. Consider the following special cases. For $a = 0$,

$$\sum_{n=0}^{\infty} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty} = e_q(u) \text{ (the first } q\text{-exponential.)}$$

For $u \to -u/a$, $a \to \infty$,

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} u^n}{(q;q)_n} = (-u;q)_\infty = E_q(u) \text{ (the second } q\text{-exponential.)}$$

With $a = q^{k+1}$,

$$\sum_{n=0}^{\infty} \begin{bmatrix} k+n \\ n \end{bmatrix} u^n = \frac{1}{(u;q)_{k+1}},$$

and with $a = q^{-k}$, $u \to -uq^k$,

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} \begin{bmatrix} k \\ n \end{bmatrix} u^n = (-u;q)_k.$$

Extraction of coefficients of u^n in the next to the last identity gives

$$\begin{bmatrix} k+n \\ k \end{bmatrix} = \sum_{k \geq a_1 \geq \cdots \geq a_n \geq 0} q^{a_1 + \cdots + a_n},$$

and in the last one

$$q^{\binom{n}{2}} \begin{bmatrix} k \\ n \end{bmatrix} = \sum_{k-1 \geq a_1 > \cdots > a_n \geq 0} q^{a_1 + \cdots + a_n}.$$

This provides another proof of Proposition 10.1.1.

Section 10.2

10.2.1 For each Ferrers diagram λ with m boxes (see section 6.1) and each vector $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ of positive integers such that $c_1+c_2+\cdots+c_r = m$ let $\mathcal{K}(\lambda, \mathbf{c})$ denote the set of Young tableaux containing c_1 1's, c_2 2's, … , c_r r's. Let σ be a permutation of the set $\{1, 2, \ldots, r\}$. The symmetry argument that is carried over the proof of Theorem 10.2.1 can be used to construct a one-to-one correspondence between $\mathcal{K}(\lambda, \mathbf{c})$ and $\mathcal{K}(\lambda, \sigma\,\mathbf{c})$. Proceed as follows. Let $\mathbf{c} = (c_1, \ldots, c_i, c_{i+1}, \ldots, c_r)$ and $\mathbf{c}' = (c_1, \ldots, c_{i+1}, c_i, \ldots, c_r)$ differ only by a transposition of two adjacent terms and consider a tableau T in $\mathcal{K}(\lambda, \mathbf{c})$ in its planar representation (as in section 6.1). Write all the pairs $i, i+1$ in bold-face whenever those two integers occur in the same column with $(i+1)$ just above i. The remaining i's and $(i+1)$'s in T occur as horizontal blocks $i^a\, j^b$ $(a \ge 0,\, b \ge 0)$. We define a bijection $T \mapsto T'$ of $\mathcal{K}(\lambda, \mathbf{c})$ onto $\mathcal{K}(\lambda, \mathbf{c}')$ by replacing each block $i^a\, j^b$ in T by $i^b\, j^a$ and rewriting the vertical pairs $i, i+1$ in roman type. This argument provides another proof of the symmetry of the Schur function (see section 6.4).

10.2.2 (The MacMahon Verfahren revisited). Let $U = (S_<, S_\le, L_<, L_\le)$ be a partition of the alphabet $X = \{1, \ldots, r\}$ such that $S_< \cup S_\le = \{1, \ldots, h\}$ (the *small* letters) and $L_< \cup L_\le = \{h+1, \ldots, r\}$ (the *large* letters) for a certain h $(0 \le h \le r)$. Let $w = x_1 x_2 \ldots x_m$ be a word in the alphabet and let $x_{m+1} = h + \frac{1}{2}$. An integer i such that $1 \le i \le m$ is said to be an U-*descent* in w, if either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and $x_i \in S_\le \cup L_\le$. Let $\operatorname{des}_U w$ (resp. $\operatorname{maj}_U w$) denote the *number* (resp. the *sum*) of the U-descents in w. For each sequence $\mathbf{c} = (c_1, \ldots, c_r)$ consider the generating polynomial for the class $R(\mathbf{c})$ by the pair $(\operatorname{des}_U, \operatorname{maj}_U)$, i.e., $A^U_{\mathbf{c}}(t, q) = \sum_w t^{\operatorname{des}_U w}\, q^{\operatorname{maj}_U w}$ $(w \in R(\mathbf{c}))$. The identity to be proved reads

$$\frac{1}{(t;q)_{1+\|\mathbf{c}\|}} A^U_{\mathbf{c}}(t,q) = \sum_{s\ge 0} t^s \prod_{i\in S_<} \begin{bmatrix} c_i + s \\ c_i \end{bmatrix} \prod_{i\in S_\le} q^{\binom{c_i}{2}} \begin{bmatrix} s+1 \\ c_i \end{bmatrix}$$
$$\times \prod_{i\in L_<} q^{c_i} \begin{bmatrix} c_i + s - 1 \\ c_i \end{bmatrix} \prod_{i\in L_\le} q^{\binom{c_i+1}{2}} \begin{bmatrix} s \\ c_i \end{bmatrix},$$

and can be derived as follows. As in Section 10.2 the left-hand side is equal to the sum of the series $\sum t^{s'+\operatorname{des}_U w} q^{\|\mathbf{a}\|+\operatorname{maj}_U w}$ over all triples $(s', \mathbf{a}, w)$. By using the last two identities of 10.1.1 the right-side is equal to the sum of the series $\sum t^s\, q^{\|\mathbf{a}^{(1)}\|+\cdots+\|\mathbf{a}^{(r)}\|}$, where each $\mathbf{a}^{(i)} = (a_{i,1}, \ldots, a_{i,c_i})$ is a sequence of integers satisfying $s \ge a_{i,1} \ge \cdots \ge a_{i,c_i} \ge 0$, if $i \in S_<$; $s \ge a_{i,1} > \cdots > a_{i,c_i} \ge 0$, if $i \in S_\le$; $s \ge a_{i,1} \ge \cdots \ge a_{i,c_i} \ge 1$, if $i \in L_<$; $s \ge a_{i,1} > \cdots > a_{i,c_i} \ge 1$, if $i \in L_\le$. The bijection $(s', \mathbf{a}, w) \mapsto (s, \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(r)})$ such that $s = s' + \operatorname{des}_U w$ and $\|\mathbf{a}\| + \operatorname{maj}_U w = \|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\|$ can be constructed by rewriting the MacMahon Verfahren developed in Section 10.2 almost verbatim.

(See Foata and Krattenthaler 1995.)

10.2.3 The identity derived in 10.2.2 is equivalent to the following identity between q-series

$$\sum_{\mathbf{c}} A^U_{\mathbf{c}}(t,q)\frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}} = \sum_{s\ge 0} t^s \frac{\prod\limits_{i\in S_{\le}} \big(-u_i;q\big)_{s+1} \prod\limits_{i\in L_{\le}} \big(-q\,u_i;q\big)_s}{\prod\limits_{i\in S_{<}} \big(u_i;q\big)_{s+1} \prod\limits_{i\in L_{<}} \big(qu_i;q\big)_s}.$$

(See Foata and Krattenthaler 1995.)

10.2.4 Write the previous identity as

$$\sum_{\mathbf{c}} A^U_{\mathbf{c}}(t,q)\frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}} = \sum_{s\ge 0} t^s\, a_s(\mathbf{u};q)$$

and let

$$a_\infty(\mathbf{u};q) = \frac{\prod\limits_{i\in S_{\le}} \big(-u_i;q\big)_{\infty} \prod\limits_{i\in L_{\le}} \big(-q\,u_i;q\big)_\infty}{\prod\limits_{i\in S_{<}} \big(u_i;q\big)_{\infty} \prod\limits_{i\in L_{<}} \big(qu_i;q\big)_\infty}.$$

The sequence $(a_s(\mathbf{u};q))$ $(s\ge 0)$ converges to $a_\infty(\mathbf{u};q)$ in the topology of the formal power series in the variables $u_1, \ldots, u_r$. Let $a_{-1}(\mathbf{u};q) = 0$; then the sequence $(a_s(\mathbf{u};q) - a_{s-1}(\mathbf{u};q))$ $(s\ge 0)$ is summable of sum $a_\infty(\mathbf{u};q)$. As we have $(1-t)\sum_s t^s\, a_s(\mathbf{u};q) = \sum_s t^s\,(a_s(\mathbf{u};q) - a_{s-1}(\mathbf{u};q))$, it makes sense to multiply the identity in 10.2.3 by $(1-t)$ and make $t=1$. This yields

$$\sum_{\mathbf{c}} A^U_{\mathbf{c}}(1,q)\frac{\mathbf{u}^{\mathbf{c}}}{(q;q)_{\|\mathbf{c}\|}} = a_\infty(\mathbf{u};q).$$

(See Foata and Krattenthaler 1995.)

Section 10.3

10.3.1 Take up again the notations of 10.2.2 with the further assumption that the subalphabets $S_{\le}$ and $L_{<}$ are empty, so that $\{1,\ldots,h\}$ (resp. $\{(h+1),\ldots,r\}$) is the set of small (resp. large) letters. With $1\le h<r$ the following recurrence relations for the polynomials $A^U_{\mathbf{c}}(t,q)$ can be derived by using the insertion technique:

$$(1-q^{c_h+1})\,A^U_{\mathbf{c}+1_h}(t,q) = (1-tq^{1+\|\mathbf{c}\|})\,A^U_{\mathbf{c}}(t,q) - q^{c_h+1}(1-t)\,A^U_{\mathbf{c}}(tq,q);$$
$$(1-q^{-c_r-1})\,A^U_{\mathbf{c}+1_r}(t,q) = -(1-tq^{1+\|\mathbf{c}\|})\,A^U_{\mathbf{c}}(t,q) - q^{-c_r}(1-t)\,A^U_{\mathbf{c}}(tq,q).$$

(See Clarke and Foata 1995a.)

Section 10.4

10.4.1 With the specializations of 10.3.1 for U the identity written in 10.2.3 becomes

$$\sum_{\mathbf{c}} A^U_{\mathbf{c}}(t,q)\frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}} = \sum_{s\geq 0} t^s \, \frac{\prod\limits_{h+1\leq i\leq r}\big(-q\,u_i;q\big)_s}{\prod\limits_{1\leq i\leq h}\big(u_i;q\big)_{s+1}}.$$

The latter identity can be derived directly from the recurrence relations in 10.3.1. (See Clarke and Foata 1995a.)

10.4.2 (q-Eulerian polynomials). With the specializations of 10.3.1 for U and for $\mathbf{c}=1^r$ the identity in 10.2.2 becomes

$$\frac{1}{(t;q)_{1+r}}\,A^U_{1^r}(t,q) = \sum_{s\geq 0} t^s\,(\,[s+1]_q\,)^h\,q^{r-h}\,(\,[\,s\,]_q\,)^{r-h}.$$

Let $A^h_r(t,q) = A^U_{1^r}(t,q)$ $(0\leq h\leq r)$ and form

$$A_r(x,y;t,q) = \sum_{h=0}^{r}\binom{r}{h}x^{r-h}\,y^h\,A^h_r(t,q).$$

Then

$$\sum_{r\geq 0}\frac{u^r}{r!}\,\frac{A_r(x,y;t,q)}{(t;q)_{r+1}} = \sum_{s\geq 0} t^s\,\exp\big(u(xq\,[\,s\,]_q + y[s+1]_q)\big).$$

For $h=r$ the polynomial $A^r_r(t,q)$ is the traditional q-Eulerian polynomial for the symmetric group $\mathcal{S}_r$ by the pair $(\mathrm{des},\mathrm{maj})$. (See Carlitz 1975.)

10.4.3 (The t-extension of the q-evaluation of a tableau). With the notations of Problem 6.5.1 let $\mathrm{des}\,T$ be the number of the recoils of a tableau T of shape λ with m boxes. The following identity is the t-extension of the identity in 6.5.1, question 4):

$$\sum_{T\in\mathrm{STab}(\lambda)} t^{\mathrm{des}\,T}\,q^{\mathrm{maj}\,T} = (t;q)_{m+1}\sum_{k\geq 0} t^k\,s_\lambda(1,q,q^2,\ldots,q^k).$$

(See Désarménien and Foata 1985, theorem 4.1.)

10.4.4 (The Schur function method). Again let $A_{\mathbf{c}}(t,q) = A^{\mathrm{des},\mathrm{maj}}_{\mathbf{c}}(t,q)$. To each word $w\in R(\mathbf{c})$ there corresponds a unique pair of tableaux (P,Q) such that $\mathrm{ev}(P)=\mathbf{c}$ and Q is a standard tableau such that $\mathrm{des}\,w=\mathrm{des}\,Q$ and $\mathrm{maj}\,w=\mathrm{maj}\,Q$ (see Problem 6.5.1). Hence

$$\sum_{\mathbf{c}} A_{\mathbf{c}}(t,q)\frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}} = \sum_{\mathbf{c}}\frac{\mathbf{u}^{\mathbf{c}}}{(t;q)_{1+\|\mathbf{c}\|}}\sum_{|\lambda|=\|\mathbf{c}\|}\sum_{(P,Q)} t^{\mathrm{des}\,Q}\,q^{\mathrm{maj}\,Q}$$

$$= \sum_{\mathbf{c}} \sum_{|\lambda|=\|\mathbf{c}\|} \sum_{P} \mathbf{u}^{\mathrm{ev}(P)} \times \frac{1}{(t;q)_{1+\|\mathbf{c}\|}} \sum_{Q} t^{\operatorname{des} Q} q^{\operatorname{maj} Q}$$
$$= \sum_{\lambda} s_\lambda(u_1,\dots,u_r) \times \sum_{k\ge 0} t^k\, s_\lambda(1,q,q^2,\dots,q^k),$$

by the definition of a Schur function (see Definition 6.4.1) and Problem 10.4.3. The last product is equal to $\sum_{s\ge 0} t^s/(\mathbf{u};q)_{s+1}$ by using the Cauchy identity (see Theorem 6.4.2) with the alphabets $\xi \leftarrow \{u_1,\dots,u_r\}$ and $\eta \leftarrow \{1,q,\dots,q^k\}$. (See Foata 1995.)

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Section 10.5+

The remaining problems refer to the last seven sections of this chapter and will be numbered 10.5.n.

10.5.1 (Euler-Mahonian statistics). As seen in Problem 10.4.2, the polynomial $A_r^r(t,q) = A_{1^r}(t,q)$ is the q-Eulerian polynomial that can be interpreted as the generating function for the symmetric group $\mathcal{S}_r$ by the pair (des, maj). Let $A_r^r(t,q) = \sum_{s\ge 0} A_{r,s}^r(q)\, t^s$. With $\mathbf{c} = 1^{r-1}$, $j = r$, $c_r = 0$ the recurrence relation (10.3.3) specializes into

$$A_{r,s}^r(q) = [1+s]_q\, A_{r-1,s}^{r-1}(q) + q^s\, [r-s]_q\, A_{r-1,s-1}^{r-1}(q), \qquad (*)$$

for $1 \le s \le r-1$ with the initial conditions $A_{r,0}^r(q) = 1$ for all $r \ge 0$ and $A_{r,s}^r(q) = 0$ for $s \ge r$. The first values of the polynomials $A_{r,s}^r(q)$ read:

$s =$	0	1	2	3
$r = 0$	1			
1	1			
2	1	q		
3	1	$2q+2q^2$	q^3	
4	1	$3q+5q^2+3q^3$	$3q^3+5q^4+3q^5$	q^6

Let $E = (E_r)$ $(r \ge 0)$ be a family of finite sets such that $\operatorname{Card} E_r = r!$ for every $r \ge 0$. A family $(f,g) = (f_r, g_r)$ $(r \ge 0)$ is said to be *Euler-Mahonian* on E, if $f_0 = g_0 = 0$, $f_1 = g_1 = 1$ and if for every $r \ge 2$ both f_r and g_r are integral-valued functions defined on E_r and there exists a bijection $\psi_r : (w', j) \mapsto w$ of $E_{r-1} \times [0, r-1]$ onto E_r having the properties:

$$g_r(w) = g_{r-1}(w') + j;$$
$$f_r(w) = \begin{cases} f_{r-1}(w'), & \text{if } 0 \le j \le f_{r-1}(w'); \\ f_{r-1}(w') + 1, & \text{if } f_{r-1}(w) + 1 \le j \le r-1. \end{cases} \qquad (**)$$

Each pair (f_r, g_r) is called a *Euler-Mahonian statistic* on E_r.
1) Let (f, g) be Euler-Mahonian on E and for every triple (r, s, l) let $A^r_{r,s,l}$ be the number of elements $w \in E_r$ such that $f_r(w) = l$ and $g_r(w) = s$ and form $A^r_{r,s}(q) = \sum_l A^r_{r,s,l}\, q^l$. Then the family $(A^r_{r,s}(q))$ satisfies the above recurrence relation $(*)$.

2) A word $w = x_1 \dots x_r$ of length r is said to be *subexcedent* if its letters are integral numbers and satisfy $0 \le x_i \le i - 1$ for all $i = 1, \dots, r$. Denote by SE_r the set of those words. Let the *sum* of w be defined by $\operatorname{sum} w = x_1 + \dots + x_r$ and its *Eulerian value*, $\operatorname{eul} w$, by: $\operatorname{eul} = 0$ if w is of length 1, and for $r \ge 2$

$$\operatorname{eul} x_1 \dots x_r = \begin{cases} \operatorname{eul} x_1 \dots x_{r-1}, & \text{if } x_r \le \operatorname{eul} x_1 \dots x_{r-1}; \\ \operatorname{eul} x_1 \dots x_{r-1} + 1, & \text{if } x_r \ge \operatorname{eul} x_1 \dots x_{r-1} + 1. \end{cases}$$

Then the pair $(\operatorname{sum}, \operatorname{eul})$ is a Euler-Mahonian statistic on SE_r for every $r \ge 0$. The bijection ψ_r is obvious to imagine.

3) Let $r \ge 2$ and let $w' = x_1 x_2 \dots x_{r-1}$ be a permutation of $12 \dots (r-1)$ having s descents. Let $x_0 = 0$, $x_r = \infty$ and for each $i = 0, 1, \dots, (r-1)$ label the r slots $x_i x_{i+1}$ as follows: $x_{r-1} x_r$ gets label 0, then reading the permutation *from right to left* label $1, 2, \dots, s$ the s descents $x_i > x_{i+1}$; then reading *from left to right* label $(s+1), \dots, r-1$ the $(r-1-s)$ non-descents $x_i < x_{i+1}$ $(0 \le i \le r-2)$. If the slot $x_i x_{i+1}$ gets label j define $\psi_r(w', j) = x_1 \dots x_i\, r\, x_{i+1} \dots x_{r-1}$. Then with $(f, g) = (\operatorname{des}, \operatorname{maj})$ the mapping ψ_r has the properties $(**)$, so that $(\operatorname{des}, \operatorname{maj})$ is a Euler-Mahonian statistic on each permutation group $\mathcal{S}_r$. (see Carlitz 1954, Rawlings 1981.)

4) Let $r \ge 2$ and let $w' = x_1 x_2 \dots x_{r-1}$ be a permutation of $12 \dots (r-1)$ having s excedances. Let $(x_{i_1} > \dots > x_{i_s})$ be the *decreasing* sequence of the excedance values $x_k > k$ and let $(x_{i_{s+1}} < \dots < x_{i_{r-1}})$ be the *increasing* sequence of the non-excedance values $x_k \le k$. By convention, let $x_{i_0} = r$.
Define $\psi_r(w, 0) = x_1 x_2 \dots x_{r-1} r$. If $1 \le j \le s$ (resp. $s + 1 \le j \le r - 1$) replace each letter x_{i_m} in w' such that $1 \le m \le j$ (resp. such that $1 \le m \le s$) by $x_{i_{m-1}}$, leave the other letters invariant and insert x_{i_j} (resp. x_{i_s}) to the i_j-th place in w'. Let $w = \psi_r(w', j)$ be the permutation thereby obtained.
For example, $w' = 3\,2\,5\,4\,1$ has the $s = 2$ excedances $x_3 = 5 > 3$, $x_1 = 3 > 1$ (in decreasing order) and three non-excedances $x_5 = 1 \le 5$, $x_2 = 2 \le 2$, $x_4 = 4 \le 4$ (in increasing order), so that $(i_1, i_2, i_3, i_4, i_5) = (3, 1, 5, 2, 4)$. With $j = 1$ we have $i_j = 3$ and $x_3 = 5$. To obtain $\psi_6(w', 1)$ replace $x_{i_1} = 5$ by $x_{i_0} = 6$, leave the other letters invariant and insert $x_{i_1} = 5$ to the i_1-th=3rd place. Thus $\psi_6(w', 1) = 3\,2\,5\,6\,4\,1$. For $j = 3$ we have $i_j = 5$ and $x_5 = 1$. As $j = 3 > s = 2$, replace $x_{i_1} = x_3$ by $x_{i_?} = 6$, then $x_{i_2} = x_1 = 3$ by $x_{i_1} = 5$, leave the other letters

invariant and insert $x_{i_s} = x_{i_2} = x_1 = 3$ to the i_3-th=5-th place to yield $\psi_6(w', 3) = 5\,2\,6\,4\,3\,1$.
With $(f, g) = (\mathrm{exc}, \mathrm{den})$ the mapping ψ_r has the properties $(**)$, so that $(\mathrm{exc}, \mathrm{den})$ is a Euler-Mahonian statistic on each permutation group $\mathcal{S}_r$. (see Han 1990b.)

5) Let (f, g) be a Eulerian-Mahonian family on $E = (E_r)$. For each $w \in E_r$ $(r \geq 2)$ let $\psi_r^{-1}(w) = (w', j_r)$, $\psi_{r-1}^{-1}(w') = (w'', j_{r-1})$, $\ldots$, $\psi_2^{-1}(w^{(r-2)}) = (w^{(r-1)}, j_2)$ and $j_1 = 0$; the word $\Psi(w) = j_1 j_2 \ldots j_{r-1} j_r$ is a subexcedant word and Ψ is a bijection of E_r onto SE_r such that $f(w) = \mathrm{sum}\,\Psi(w)$ and $g(w) = \mathrm{eul}\,\Psi(w)$. The bijection Ψ is called the *(f, g)-coding of E_r*.
Let $\Psi_{(\mathrm{des},\mathrm{maj})}$ (resp. $\Psi_{(\mathrm{exc},\mathrm{den})}$) be the $(\mathrm{des}, \mathrm{maj})$-coding (see question 3)) (resp. the $(\mathrm{exc}, \mathrm{den})$-coding (see question 4) of $\mathcal{S}_r$. Then $\Theta = \Psi^{-1}_{(\mathrm{des},\mathrm{maj})} \circ \Psi_{(\mathrm{exc},\mathrm{den})}$ is a bijection of $\mathcal{S}_r$ onto itself that satifies $(\mathrm{exc}, \mathrm{den})\, w = (\mathrm{des}, \mathrm{maj})\, \Theta(w)$. (see Han 1990b.)

10.5.2 With the assumptions of Problem 10.3.1 the alphabet $X = \{1, \ldots, r\}$ is split into two disjoint parts, the set $S = \{1, \ldots, h\}$ of the small letters and $L = \{h + 1, \ldots, r\}$, the set of the large letters. An U-descent of the word $w = x_1 \ldots x_m$ is an integer i such that $1 \leq i \leq m$ and either $x_i > x_{i+1}$, or $x_i = x_{i+1} \in L$ (by convention, $x_{m+1} = h + \frac{1}{2}$.) Denote by $\mathrm{des}_U\, w$ (resp. $\mathrm{maj}_U\, w$) the number (resp. the sum) of the U-descents in w.
Now if $y_1 y_2 \ldots y_m$ is the nondecreasing rearrangement of the word w let $\mathrm{exc}_U\, w$ be the number of i such that $x_i > y_i$, or $x_i = y_i \in L$. The definition of den_U requires the introduction of three further statistics. The *U-excedance index* of w is defined as the sum, $\mathrm{excindex}_U\, w$, of all i such that i is an U-excedance in w. Also let

$$\begin{aligned}\mathrm{inv}_U\, w &= \#\{1 \leq i < j \leq m : x_i > x_j \text{ or } x_i = x_j \geq h + 1\}\\ &\qquad\qquad + \#\{1 \leq i \leq m : x_i \geq h + 1\},\\ \mathrm{imv}_U\, w &= \#\{1 \leq i < j \leq m : x_i > x_j \text{ or } x_i = x_j \leq h\}.\end{aligned}$$

If $\mathrm{exc}_U\, w = e$, let $i_1 < i_2 < \cdots < i_e$ be the increasing sequence of the U-excedances of w and let $j_1 < j_2 < \cdots < j_{m-e}$ be the complementary sequence. Form the two subwords $\mathrm{Exc}_U\, w = x_{i_1} x_{i_2} \ldots x_{i_e}$, $\mathrm{Nexc}_U\, w = x_{j_1} x_{j_2} \ldots x_{j_{m-e}}$. Then the *$U$-Denert statistic*, $\mathrm{den}_U\, w$, of w is defined to be

$$\mathrm{den}_U\, w = \mathrm{excindex}_U\, w + \mathrm{imv}_U\, \mathrm{Exc}_U\, w + \mathrm{inv}_U\, \mathrm{Nexc}_U\, w.$$

When the set L of large letters is empty, all the statistics without any subscript U that were defined in the chapter are recovered.
The algorithm **T** described in section 10.8 can be adequately modified to make up a bijection Φ of each rearrangement class $R(\mathbf{c})$ onto itself having the property

$$(\mathrm{exc}_U, \mathrm{den}_U)\, w = (\mathrm{des}_U, \mathrm{maj}_U)\, \Phi(w)$$

for every word w in $R(\mathbf{c})$. Thus the generating polynomial for $R(\mathbf{c})$ by the pair $(\mathrm{exc}_U, \mathrm{den}_U)$ is the polynomial $A_{\mathbf{c}}^U(t,q)$ whose factorial generating polynomial is shown in Problem 10.4.1. (See Foata and Han 1998 for the details of the construction of Φ, see Clarke and Foata 1994 for an earlier construction and Han 1995 for another equivalent definition for den_U.)

10.5.3 In section 10.10 it is proved that if $\alpha = (h, b)$ and $\alpha' = (h', b')$ are H-equivalent, then $(\mathrm{exc}, \mathrm{den})\,\alpha = (\mathrm{exc}, \mathrm{den})\,\alpha'$. The converse is true whenever the words h, h', b, b' are words without repetitions. (See Clarke 1997.)

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