PERMUTATIONS WITH EXTREMAL NUMBER OF FIXED POINTS

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Abstract. We extend Stanley's work on alternating permutations with extremal number of fixed points in two directions: first, alternating permutations are replaced by permutations with a prescribed descent set; second, instead of simply counting permutations we study their generating polynomials by number of excedances. Several techniques are used: Désarménien's desarrangement combinatorics, Gessel's hook-factorization and the analytical properties of two new permutation statistics "DEZ" and "lec". Explicit formulas for the maximal case are derived by using symmetric function tools.

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1. Introduction

Let $J = \{j_1, j_2, \ldots, j_r\}_<$ be a set of integers arranged increasingly and let $\mathfrak{S}_J$ denote the set of all permutations on J. For each permutation $\sigma = \sigma(j_1)\sigma(j_2)\cdots\sigma(j_r) \in \mathfrak{S}_J$ define the *number of excedances*, the *number of fixed points* and the *descent set* of σ to be

$$\begin{aligned} \operatorname{fix}\sigma &= |\{i : 1 \le i \le r, \sigma(j_i) = j_i\}|, \\ \operatorname{exc}\sigma &= |\{i : 1 \le i \le r, \sigma(j_i) > j_i\}|, \\ \operatorname{DES}\sigma &= \{i : 1 \le i \le r-1, \sigma(j_i) > \sigma(j_{i+1})\}, \end{aligned} \tag{1}$$

respectively. A permutation without fixed points is called a *derangement*. When $J = [n] := \{1, 2, \ldots, n\}$, we recover the classical definitions. The set $\mathfrak{S}_{[n]}$ is abbreviated by $\mathfrak{S}_n$, and for $\sigma \in \mathfrak{S}_n$ we write σ_i for $\sigma(i)$. Our main results are the following Theorems 1 and 2.

Theorem 1. *Let J be a subset of $[n-1]$.*
(i) *If $\sigma \in \mathfrak{S}_n$ and $\operatorname{DES}\sigma = J$, then*

$$\operatorname{fix}\sigma \le n - |J|.$$

(ii) *Let $F_n(J)$ be the set of all permutations σ of $[n]$ such that $\operatorname{DES}\sigma = J$ and $\operatorname{fix}\sigma = n - |J|$. Furthermore, let $G(J)$ be the set of all derangements τ on J such that $\tau(i) > \tau(i+1)$ whenever i and $i+1$ belong to J. Then*

$$\sum_{\sigma \in F_n(J)} s^{\operatorname{exc}\sigma} = \sum_{\tau \in G(J)} s^{\operatorname{exc}\tau}.$$

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Example. For $n = 8$ and $J = \{1, 2, 3, 6\}$, there are two permutations in $F_n(J)$, both having two excedances: 74315628 and 74325618. On the other hand, there are two derangements in $G(J)$, both having two excedances: 6321 and 6312.

Theorem 2. *Let $D_0^J(n)$ be the set of all derangements σ on $[n]$ such that $\operatorname{DES}\sigma = J$, and let $D_1^J(n)$ be the set of all permutations σ on $[n]$ such that $\operatorname{DES}\sigma = J$ with exactly one fixed point. If J is a proper subset of $[n-1]$, then there is a polynomial $Q_n^J(s)$ with positive integral coefficients such that*

$$\sum_{\sigma\in D_0^J(n)} s^{\operatorname{exc}\sigma} - \sum_{\sigma\in D_1^J(n)} s^{\operatorname{exc}\sigma} = (s-1)Q_n^J(s).$$

Example. For $n = 6$ and $J = \{1, 3, 4, 5\}$ there are six derangements in $D_0^J(n)$:

$$216543, 316542, 416532, 436521, 546321, 645321;$$

and there are six permutations in $D_1^J(n)$:

$$326541, 426531, 516432, 536421, 615432, 635421.$$

The numbers of excedances are respectively 3, 3, 3, 4, 3, 3 and 3, 3, 2, 3, 2, 3, so that

$$\sum_{\sigma\in D_0^J(n)} s^{\operatorname{exc}\sigma} - \sum_{\sigma\in D_1^J(n)} s^{\operatorname{exc}\sigma} = (5s^3+s^4)-(4s^3+2s^2) = (s-1)(s^3+2s^2).$$

Theorem 1 extends Stanley's work on alternating permutations (that we explain next) with maximal number of fixed points, and Theorem 2 extends the corresponding minimal case. The extensions are in two directions: first, alternating permutations are replaced by permutations with a prescribed descent set; second, instead of simply counting permutations we study their generating polynomials by number of excedances.

A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is said to be *alternating* (resp. *reverse alternating*) if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$ (resp. if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots$); or equivalently, if $\operatorname{DES}\pi$ is $\{1, 3, 5, \dots\} \cap [n-1]$ (resp. $\{2, 4, 6, \dots\} \cap [n-1]$). Therefore, results on permutations with a prescribed descent set apply to alternating permutations. Let $D_k(n)$ be the set of permutations in $\mathfrak{S}_n$ with exactly k fixed points. Then $D_0(n)$ is the set of derangements of order n. Write $d_k(n)$ (resp. $d_k^*(n)$) for the number of alternating (resp. reverse alternating) permutations in $D_k(n)$. The next two corollaries are immediate consequences of Theorems 1 and 2.

Corollary 3 ([14], Conjecture 6.3)**.** *Let D_n denote the number of derangements. Then, for $n \ge 2$ we have*

$$d_n(2n) = d_{n+1}(2n+1) = d^*_{n+1}(2n+1) = d^*_{n+2}(2n+2) = D_n.$$

Corollary 4 ([14], Corollary 6.2)**.** *For $n \ge 2$ we have $d_0(n) = d_1(n)$ and $d_0^*(n) = d_1^*(n)$.*

Stanley enumerated $D_k(n)$ and came up with Corollaries 3 and 4 on alternating permutations with extremal number of fixed points. He then asked for combinatorial proofs of them. This is the motivation of the paper. The results in Corollary 3, conjectured by Stanley, was recently proved by Chapman and Williams [16] in two ways, one directly and the other using the newly developed concept of permutation tableaux [15]. In Section 3 we give a direct proof of a generalized form of Corollary 3. Corollary 4 is actually a special case of a more general result due to Gessel and Reutenauer, which itself can be derived from Theorem 2 by setting $s = 1$, as stated in the next corollary.

Corollary 5 ([10], Theorem 8.3)**.** *Let J be a proper subset of $[n-1]$. Then, the number of derangements in $\mathfrak{S}_n$ with descent set J is equal to the number of permutations in $\mathfrak{S}_n$ with exactly one fixed point and descent set J.*

The paper is organized as follows. In Section 2 we give the proof of Theorem 1 that contains the results for the maximal case. Section 3 includes a direct proof of an extension of Corollary 3. Section 4 introduces the necessary part of Gessel and Reutenauer's work for enumerating the maximal case. Section 5 is devoted to the proof of Theorem 2 dealing with the minimal case. We conclude the paper by making several remarks of analytic nature (see Section 6). In particular, Corollary 18, proved combinatorially, should deserve an analytic proof. Several techniques are used: Désarménien's desarrangement combinatorics [1], Gessel's hook-factorization [9] and the analytical properties of two new permutation statistics "DEZ" and "lec" [5, 6].

2. Permutations with maximal number of fixed points

Our task in this section is to prove Theorem 1. The proof relies on the properties of the new statistic "DEZ" introduced by Foata and Han [5]. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ let $\sigma^0 = \sigma_1^0\sigma_2^0\cdots\sigma_n^0$ be the word derived from σ by replacing each fixed point $\sigma_i = i$ by 0. The set-valued statistic "DEZ" is defined by

$$\operatorname{DEZ}\sigma = \operatorname{DES}\sigma^0 := \{i : 1 \le i \le n-1, \sigma_i^0 > \sigma_{i+1}^0\}.$$

For example, if $\sigma = 8\,2\,1\,3\,5\,6\,4\,9\,7$, then $\operatorname{DES}\sigma = \{1, 2, 6, 8\}$, $\sigma^0 = 8\,0\,1\,3\,0\,0\,4\,9\,7$ and $\operatorname{DEZ}\sigma = \operatorname{DES}\sigma^0 = \{1, 4, 8\}$. The basic property of the statistic "DEZ" is given in the following proposition.

Proposition 6 ([5], Theorem 1.4)**.** *The two three-variable statistics* $(\operatorname{fix}, \operatorname{exc}, \operatorname{DEZ})$ *and* $(\operatorname{fix}, \operatorname{exc}, \operatorname{DES})$ *are equi-distributed on the symmetric group $\mathfrak{S}_n$.*

More precisely, Proposition 6 asserts that there is a bijection $\Phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ such that

$$\operatorname{fix}\pi = \operatorname{fix}\Phi(\pi), \quad \operatorname{exc}\pi = \operatorname{exc}\Phi(\pi), \quad \operatorname{DES}\pi = \operatorname{DEZ}\Phi(\pi), \text{ for all } \pi \in \mathfrak{S}_n.$$

By Proposition 6 Theorem 1 is equivalent to the following Theorem **??**$'$, where the statistic "DES" has been replaced by "DEZ".

Theorem 1$'$. *Let J be a subset of $[n-1]$.*

(i) *If $\sigma \in \mathfrak{S}_n$ and $\operatorname{DEZ}\sigma = J$, then*

$$\operatorname{fix}\sigma \le n - |J|.$$

(ii) *Let $F_n'(J)$ be the set of all permutations σ of order n such that $\operatorname{DEZ}\sigma = J$ and $\operatorname{fix}\sigma = n - |J|$. Furthermore, let $G(J)$ be the set of all derangements τ on J such that*

$\tau(i) > \tau(i+1)$ *whenever* i *and* $i+1$ *belong to* J. *Then*

$$\sum_{\sigma \in F'_n(J)} s^{\operatorname{exc}\sigma} = \sum_{\tau \in G(J)} s^{\operatorname{exc}\tau}.$$

Proof of Theorem **??**$'$. Let σ be a permutation such that $\operatorname{DEZ}\sigma = J$ and let $i \in J$. Then $\sigma_i^0 > \sigma_{i+1}^0 \geq 0$, so that i is not a fixed point of σ. It follows that σ has at least $|J|$ non-fixed points. This proves (i).

Now, consider the case where σ has exactly $n - |J|$ fixed points. Then J is the set of all the non-fixed points of σ. By removing the fixed points from σ we obtain a derangement τ on J. If $i, i+1 \in J$, then $\tau(i) = \sigma(i) > \sigma(i+1) = \tau(i+1)$. It follows that $\tau \in G(J)$. On the other hand, take any derangement $\tau \in G(J)$ and let σ be the permutation defined by

$$\sigma(i) = \begin{cases} \tau(i), & \text{if } i \in J, \\ i, & \text{if } i \notin J. \end{cases}$$

Then $\operatorname{DEZ}\sigma = J$. It is easy to see that $\sigma \in F'_n(J)$ and $\operatorname{exc}\sigma = \operatorname{exc}\tau$. This proves the second part of Theorem **??**$'$. □

Example. Suppose $n = 8$ and $J = \{1, 2, 3, 6\}$. Let us search for the permutations $\sigma \in \mathfrak{S}_8$ such that $\operatorname{fix}\sigma = 8 - |J| = 4$ and $\operatorname{DEZ}\sigma = J$. There are two derangements τ in $G(J)$, namely, 6321 and 6312, both having two excedances, so that the two corresponding elements σ in $F'_n(J)$ are 63245178 and 63145278, both having two excedances.

Remarks. (i) For permutations with descent set J it is easy to show that the maximum number of fixed points is $n-|J|$, except when J consists of an odd number of consecutive integers. In the latter exceptional case the only decreasing permutation has exactly one fixed point and therefore is not a derangement.

(ii) The first part of Theorem 1 can also be proved directly by using the fact that in any consecutive decreasing subsequence of π, say $\pi_i > \pi_{i+1} > \cdots > \pi_{i+k}$, there is at most one fixed point in $\{i, i+1, \ldots, i+k\}$. However the "DEZ" statistic is an essential tool in the proof of the second part.

3. An extension of Corollary 3

Stanley's conjectured result in Corollary 3 was first proved by Williams [16] using the newly developed concept of permutation tableaux. A direct proof without using permutation tableaux was later included in her updated version with Chapman. Our direct proof was independently derived just after Williams' first proof. It has the advantage of automatically showing the following extension (Proposition 7). We only give the generalized form for $d_n(2n) = D_n$, since the other cases are similar. All of the three proofs are bijective, and the bijections are all equivalent. Note that Proposition 7 is still a corollary of Theorem 1.

Proposition 7. *The number of alternating permutations in* $\mathfrak{S}_{2n}$ *with* n *fixed points and* k *excedances is equal to the number of derangements in* $\mathfrak{S}_n$ *with* k *excedances.*

Let π be an alternating permutation. Then, each doubleton $\{\pi_{2i-1}, \pi_{2i}\}$ contains at most one fixed point. This proves the following lemma.

Lemma 8. *Each alternating permutation $\pi \in \mathfrak{S}_n$ has at most $\lceil n/2 \rceil$ fixed points. When this maximum is reached, either $2i-1$, or $2i$ is a fixed point of π $(2 \le 2i \le n+1)$.*

When the underlying set of the permutation π is not necessarily $[n]$, we use $\pi(i)$ instead of π_i for convenience. An integer i is called an *excedance*, a *fixed point*, or a *subcedance* of π if $\pi(i) > i$, $\pi(i) = i$, or $\pi(i) < i$, respectively.

Proof of Proposition 7. Let $\pi \in \mathfrak{S}_{2n}$ be alternating and have exactly n fixed points. It follows from Lemma 8 that for each i we have the following property: either $2i-1$ is a fixed point and $2i$ a subcedance, or $2i-1$ is an excedance and $2i$ a fixed point. Conversely, if the property holds, the permutation π is necessarily alternating, because $\pi(2i) \le 2i < 2i+1 \le \pi(2i+1)$; $\pi(2i-1) \ge 2i-1 \ge \pi(2i)-1$. Those inequalities imply that $\pi(2i-1) > \pi(2i)$, since $2i-1$ and $2i$ cannot be both fixed points.

By removing all fixed points of π we obtain a derangement σ on an n-subset of $[2n]$. The standardization of σ, which consists of replacing the i-th smallest element of σ by i, yields a derangement τ on $[n]$. We claim that the map $\varphi : \pi \mapsto \tau$ is the desired bijection. Since the standardization preserves excedances, subcedances and fixed points, it maps one element of $\{\pi(2i-1), \pi(2i)\}$ to $\tau(i)$. It follows that $\tau(i) > i$ if and only if $2i-1$ is an excedance and $2i$ is a fixed point of π, and that $\tau(i) < i$ if and only if $2i-1$ is a fixed point and $2i$ is a subcedance of π. Thus, the set of all fixed points of π can be constructed from τ. The map φ is then reversible.

The proposition then follows since the bijection preserves the number of excedances. $\square$

Example. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & \bar{2} & 6 & \bar{4} & \bar{5} & 1 & 10 & \bar{8} & \bar{9} & 7 \end{pmatrix}$. Removing all the fixed points gives $\sigma = \begin{pmatrix} 1 & 3 & 6 & 7 & 10 \\ 3 & 6 & 1 & 10 & 7 \end{pmatrix}$, standardized to $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$. Conversely, τ has excedances at positions $1, 2, 4$ and subcedances at positions $3, 5$. This implies that $2, 4, 8$ and $5, 9$ are fixed points of π and hence we can construct π. Furthermore, we have $\operatorname{exc} \pi = \operatorname{exc} \sigma = \operatorname{exc} \tau = 3$.

4. Enumeration for the maximal case

In this section we will use Theorem 1 to enumerate the number of permutations with a prescribed descent set and having the maximal number of fixed points. Every descent set $J \subseteq [n-1]$ can be partitioned into blocks of consecutive integers, such that numbers from different blocks differ by at least 2. Let $J^b = (a_1, a_2, \ldots, a_k)$ denote the sequence of the size of the blocks. For instance, if $J = \{1, 2, 3, 6\}$, then $1, 2, 3$ form a block and 6 itself forms another block. Hence $J^b = (3, 1)$. Let M_J denote the number of permutations in $\mathfrak{S}_n$ with descent set J having $n - |J|$ fixed points. By Theorem 1 the number M_J depends only on J^b. Thus, we can denote M_J by $M(a_1, a_2, \ldots, a_k)$.

Theorem 9. *The number $M(a_1,\ldots,a_k)$ is the coefficient of $x_1^{a_1}\cdots x_k^{a_k}$ in the expansion of*

$$\frac{1}{(1+x_1)(1+x_2)\cdots(1+x_k)(1-x_1-x_2-\cdots-x_k)}.$$

An immediate consequence of Theorem 9 is the following Corollary 10, which says that $M(a_1,a_2,\ldots,a_k)$ is symmetric in the a_i's.

Corollary 10. *For each permutation $\tau\in\mathfrak{S}_k$ we have*

$$M(a_1,a_2,\ldots,a_k)=M(a_{\tau_1},a_{\tau_2},\ldots,a_{\tau_k}).$$

For example, $M(3,1)$ counts two derangements 4312 and 4321; $M(1,3)$ counts two derangements 3421 and 4321. This symmetry seems not easy to prove directly. Using Theorem 9 an explicit formula for $M(a_1,a_2,\ldots,a_k)$ can be obtained when $k=1,2$. We have $M(a)=1$ if a is even, and $M(a)=0$ if a is odd; also

$$M(a,b)=\sum_{j=2}^{a+b}\sum_{i=0}^{j}(-1)^j\binom{a+b-j}{a-i}.$$

To prove Theorem 9 we need some notions from [10], where Gessel and Reutenauer represented the number of permutations with given cycle structure and descent set by the scalar product of two special characters of the symmetric group introduced by Foulkes [7, 8]. Their results were also key ingredients in [14] for the enumeration of alternating permutations by number of fixed points. In what follows, we assume the basic knowledge of symmetric functions (see, e.g., [11, 12, 13]). The scalar product $\langle\ ,\ \rangle$ of two symmetric functions is a bilinear form defined for all partitions λ and μ by

$$\langle m_\lambda,h_\mu\rangle=\langle h_\mu,m_\lambda\rangle=\delta_{\lambda\mu},\tag{2}$$

where m_λ is the monomial symmetric function, h_μ is the complete symmetric function, and δ is the usual Kronecker symbol. Moreover, if ω is the homomorphism defined by $\omega e_i=h_i$ and $\omega h_i=e_i$, where e_i is the elementary symmetric function, then for any symmetric functions f and g we have

$$\langle f,g\rangle=\langle\omega f,\omega g\rangle.\tag{3}$$

Associate the function

$$S_J=\sum_{\mathrm{DES}\,w=J}x_{w_1}x_{w_2}\cdots x_{w_n}\tag{4}$$

with each subset $J\subseteq[n-1]$, where the sum ranges over all words on positive integers with descent set J. We claim that S_J is a symmetric function whose shape is a border strip (see [13, p. 345]). In particular, $S_{[n-1]}$ is equal to e_n, the elementary symmetric function of order n. On the other hand, every partition λ of n has an associate symmetric function L_λ related to a Lie representation. The definition of L_λ is omitted here (see

[10]); just remember that the symmetric function corresponding to derangements of order n is given by

$$\mathcal{D}_n = \sum_{\lambda} L_\lambda = \sum_{j=0}^{n} (-1)^j e_j h_1^{n-j}, \tag{5}$$

where the sum ranges over all partitions λ having no part equal to 1 [10, Theorem 8.1]. We need the following result from [10] for our enumeration.

Proposition 11 (Gessel-Reutenauer)**.** *The number of permutations having descent set J and cycle structure λ is equal to the scalar product of the symmetric functions S_J and L_λ.*

Proof of Theorem 9. For each fixed integer sequence $(a_1, a_2, \ldots, a_k)$ let $s_i = a_1 + a_2 + \cdots + a_i$ for $i = 1, \ldots, k$ and $\ell = s_k$. Then $M(a_1, a_2, \ldots, a_k)$ is the number of *derangements* $\pi \in \mathfrak{S}_\ell$ such that s_i with $i = 1, 2, \ldots, k-1$ may or may not be a descent of π, and such that all the other numbers in $[\ell - 1]$ are descents of π. There is then a set T of 2^{k-1} descent sets J to consider, depending on whether each s_i is a descent or not (for $i = 1, \ldots, k-1$). By Proposition 11 and linearity we have

$$M(a_1, a_2, \ldots, a_k) = \langle \sum_{J \in T} S_J, \mathcal{D}_\ell \rangle. \tag{6}$$

From (4) it follows that

$$\sum_{J \in T} S_J = \sum_{\mathrm{DES}\, w \in T} x_{w_1} x_{w_2} \cdots x_{w_n} = \sum_{[\ell-1] \setminus \{s_1, s_2, \ldots, s_{k-1}\} \subseteq \mathrm{DES}\, w} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

Each word w occurring in the latter sum is the juxtaposition product $w = u^{(1)} u^{(2)} \cdots u^{(k)}$, where each $u^{(i)}$ is a decreasing word of length a_i $(i = 1, 2, \ldots, k)$. Hence $\sum_{J \in T} S_J = e_{a_1} e_{a_2} \cdots e_{a_k}$. In (6) replace $\sum_{J \in T} S_J$ by $e_{a_1} e_{a_2} \cdots e_{a_k}$ and $\mathcal{D}_\ell$ by the second expression in (5). We obtain

$$M(a_1, a_2, \ldots, a_k) = \langle e_{a_1} e_{a_2} \cdots e_{a_k}, \sum_{j=0}^{n} (-1)^j e_j h_1^{n-j} \rangle.$$

The image under ω yields

$$\begin{aligned} M(a_1, a_2, \ldots, a_k) &= \langle \omega e_{a_1} \cdots e_{a_k}, \omega \sum_{j=0}^{\ell} (-1)^j e_j h_1^{\ell-j} \rangle \\ &= \langle h_{a_1} \cdots h_{a_k}, \sum_{j=0}^{\ell} (-1)^j h_j e_1^{\ell-j} \rangle. \end{aligned}$$

Notice that $\sum_{j=0}^{\ell} (-1)^j h_j e_1^{\ell-j}$ is the coefficient of u^ℓ in

$$\Big(\sum_j h_j (-u)^j \Big) \Big(\sum_i e_1^i u^i \Big) = \frac{1}{(1 + x_1 u)(1 + x_2 u) \cdots (1 + x_k u)(1 - (x_1 + x_2 + \cdots + x_k) u)}.$$

It follows from (2) that $M(a_1, \ldots, a_k)$ is the coefficient of $x_1^{a_1} \cdots x_k^{a_k} u^\ell$ in the expansion of the above fraction. $\square$

5. Permutations with 0 or 1 fixed points

Our objective in this section is to prove Theorem 2. We will establish a chain of equivalent or stronger statements, leading to the final easy one. Further notations are needed. Let $w = w_1 w_2 \cdots w_n$ be a word on the letters $1, 2, \ldots, m$, each letter appearing at least once. The set-statistic $\mathrm{IDES}\, w$ is defined to be the set of all i such that the rightmost i appears to the right of the rightmost $i+1$ in w. Note that if π is a permutation on $[n]$, then $\mathrm{IDES}\, \pi = \mathrm{DES}\, \pi^{-1}$. For every proper subset J of $[n-1]$ let $\mathfrak{S}_n^J$ be the set of permutations $\sigma \in \mathfrak{S}_n$ with $\mathrm{IDES}\, \sigma = J$. Note the difference with the notation of $D_k^J(n)$ for $k = 0, 1$. We will see that it is easier to deal with IDES than with DES directly.

A word $w = w_1 w_2 \cdots w_n$ is said to be a *desarrangement* if $w_1 > w_2 > \cdots > w_{2k}$ and $w_{2k} \le w_{2k+1}$ for some $k \ge 1$. By convention, $w_{n+1} = \infty$. We may also say that the *leftmost trough* of w occurs at an *even* position [6]. This notion was introduced, for permutations, by Désarménien [1] and elegantly used in a subsequent paper [2]. A further refinement is due to Gessel [9]. A desarrangement $w = w_1 w_2 \cdots w_n$ is called a *hook*, if $n \ge 2$ and $w_1 > w_2 \le w_3 \le \cdots \le w_n$. Every nonempty word w on the letters $1, 2, 3, \ldots$ can be written uniquely as a product $u h_1 h_2 \cdots h_k$, where u is a *weakly increasing* word (possibly empty) and each h_i is a hook. This factorization is called the *hook-factorization* of w [6]. For permutations it was already introduced by Gessel [9]. For instance, the hook-factorization of the following word is indicated by vertical bars:

$$w = \mid 1\,2\,4\,5 \mid 6\,4\,5\,6 \mid 4\,1\,3 \mid 6\,5 \mid 5\,4 \mid 6\,1\,1\,4 \mid 5\,1\,1 \mid .$$

Let $u h_1 h_2 \cdots h_k$ be the hook factorization of the word w. The statistic $\mathrm{pix}\, w$ is defined to be the length of u, and the statistic $\mathrm{lec}\, w$ is defined, in terms of inversion statistics "inv", by the sum [6]

$$\mathrm{lec}\, w := \sum_{i=1}^{k} \mathrm{inv}(h_i).$$

In the previous example, $\mathrm{pix}\, w = |1245| = 4$ and $\mathrm{lec}\, w = \mathrm{inv}(6456) + \mathrm{inv}(413) + \mathrm{inv}(65) + \mathrm{inv}(54) + \mathrm{inv}(6114) + \mathrm{inv}(511) = 2 + 2 + 1 + 1 + 3 + 2 = 11$. We say that the word w has k *pixed points* where $k = \mathrm{pix}\, w$. In particular, a desarrangement is a word without any pixed point.

For each permutation σ let $\mathrm{iexc}\, \sigma = \mathrm{exc}\, \sigma^{-1}$. The next proposition was proved in Foata and Han [6].

Proposition 12. *The two three-variable statistics* $(\mathrm{iexc}, \mathrm{fix}, \mathrm{IDES})$ *and* $(\mathrm{lec}, \mathrm{pix}, \mathrm{IDES})$ *are equi-distributed on the symmetric group* $\mathfrak{S}_n$.

Let $K_0^J(n)$ denote the set of all desarrangements in $\mathfrak{S}_n^J$, and $K_1^J(n)$ the set of all permutations in $\mathfrak{S}_n^J$ with exactly one pixed point. Since the map $\sigma \to \sigma^{-1}$ preserves the number of fixed points, Theorem 2 is equivalent to asserting that

$$\sum_{\substack{\sigma \in D_0(n) \\ \mathrm{IDES}(\sigma) = J}} s^{\mathrm{iexc}\, \sigma} - \sum_{\substack{\sigma \in D_1(n) \\ \mathrm{IDES}(\sigma) = J}} s^{\mathrm{iexc}\, \sigma} = (s-1) Q_n^J(s).$$

Then by Proposition 12 this is equivalent to the following Theorem **??**[a].

Theorem 2^a. *We have*
$$\sum_{\sigma\in K_0^J(n)} s^{\operatorname{lec}\sigma} - \sum_{\sigma\in K_1^J(n)} s^{\operatorname{lec}\sigma} = (s-1)Q_n^J(s),$$
where $Q_n^J(s)$ is a polynomial with positive integral coefficients.

The following lemma enables us to prove a stronger result.

Lemma 13. *Let $w = w_1w_2\cdots w_n$ be a desarrangement such that* $\operatorname{IDES} w \neq \{1,2,\ldots,n-1\}$ *and let $w' = w_nw_1w_2\cdots w_{n-1}$. Then, either* $\operatorname{lec} w' = \operatorname{lec} w$, *or* $\operatorname{lec} w' = \operatorname{lec} w - 1$.

Proof. Several cases are to be considered. Say that w belongs to type A if $\operatorname{lec}(w') = \operatorname{lec}(w)$, and say that w belongs to type B if $\operatorname{lec}(w') = \operatorname{lec}(w) - 1$.

Since w is a desarrangement, we may assume $w_1 > w_2 > \cdots > w_{2k} \le w_{2k+1}$ for some k. It follows that w' has one pixed point. Let $h_1\cdots h_k$ be the hook-factorization of w. Then the hook-factorization of w' must have the form $w_n|h_1'\cdots h_\ell'$. Thus, when computing $\operatorname{lec}(w')$, we can simply omit w_n. This fact will be used when checking the various cases. The reader is invited to look at Figures 1–3, where the letters b, c, x, y, z play a critical role.

(1) If the rightmost hook h_k has at least three elements, as shown in Figure 1, then $b \le c$ belongs to type A and $b > c$ belongs to type B. This is because the only possible change for "lec" must come from an inversion containing c. Furthermore, (b,c) forms an inversion for type B and does not form an inversion for type A.

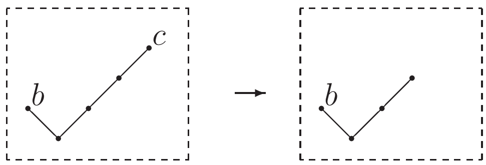

Figure 1. Transformation for case 1.

(2) Suppose the rightmost hook h_k has two elements $b > c$.
   (a) If there is a hook xy followed by several decreasing hooks of length 2 with $y \le z$, as shown in Figure 2, then $x \le z$ belongs to type B and $x > z$ belongs to type A.

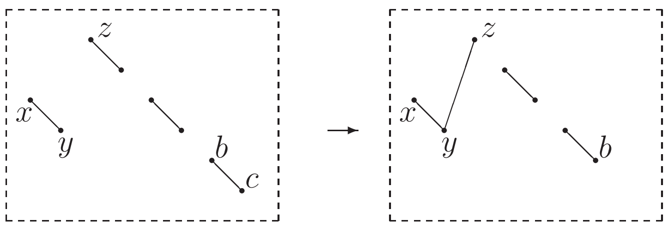

Figure 2. Transformation for case 2a.

   (b) If there is a hook of length at least 3, followed by several decreasing hooks of length 2, then (see Figure 3)

(i) $x > y$ belongs to type B and $x \le y$ belongs to type A in case $y > z$;
(ii) $x \le z$ belongs to type B and $x > z$ belongs to type A in case $y \le z$.

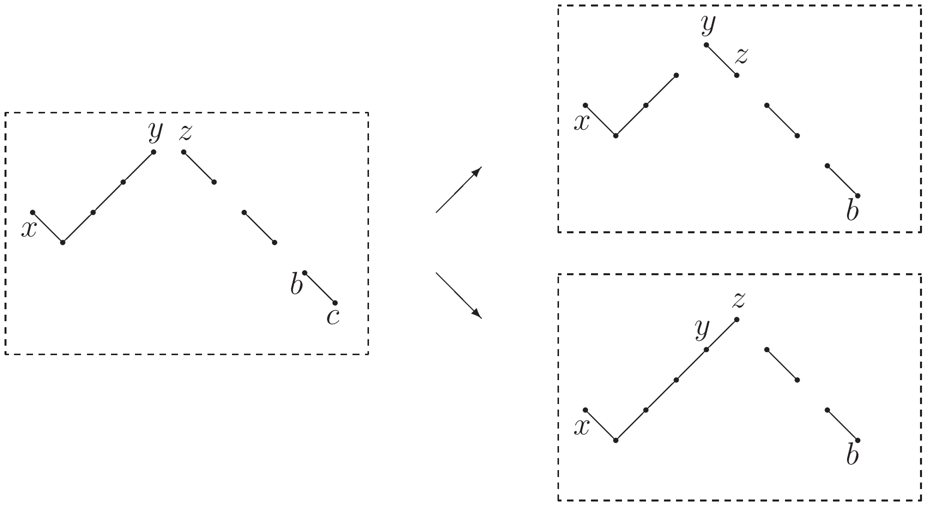

FIGURE 3. Transformations for case 2b.

This achieves the proof of the lemma. □

With the notations of Lemma 13 we say that a desarrangement w is in class A_0 if $\operatorname{lec} w' = \operatorname{lec} w$ and in class B_0 if $\operatorname{lec} w' = \operatorname{lec} w - 1$. A word $w = w_1w_2w_3\cdots w_n$ is said to be in class A_1 (resp. in class B_1) if the word $w_2w_3\cdots w_nw_1$ is in class A_0 (resp. in class B_0). Notice that a word in class A_1 or B_1 has exactly one pixed point. Then, Theorem **??**^{a} is a consequence of the following theorem.

Theorem 2^b. *We have*

$$\sum_{\sigma\in\mathfrak{S}_n^J\cap A_0} s^{\operatorname{lec}\sigma} = \sum_{\sigma\in\mathfrak{S}_n^J\cap A_1} s^{\operatorname{lec}\sigma} \quad \textit{and} \quad \sum_{\sigma\in\mathfrak{S}_n^J\cap B_0} s^{\operatorname{lec}\sigma} = s\sum_{\sigma\in\mathfrak{S}_n^J\cap B_1} s^{\operatorname{lec}\sigma}. \tag{7}$$

Proof. Let $\mathfrak{S}_n^{\subseteq J}$ be the set of all permutations σ of order n such that $\operatorname{IDES}\sigma \subseteq J$. By the inclusion-exclusion principle, equalities (7) are equivalent to the following ones:

$$\sum_{\sigma\in\mathfrak{S}_n^{\subseteq J}\cap A_0} s^{\operatorname{lec}\sigma} = \sum_{\sigma\in\mathfrak{S}_n^{\subseteq J}\cap A_1} s^{\operatorname{lec}\sigma} \quad \text{and} \quad \sum_{\sigma\in\mathfrak{S}_n^{\subseteq J}\cap B_0} s^{\operatorname{lec}\sigma} = s\sum_{\sigma\in\mathfrak{S}_n^{\subseteq J}\cap B_1} s^{\operatorname{lec}\sigma}. \tag{8}$$

If $J = \{j_1, j_2, \dots, j_{r-1}\} \subseteq [n-1]$, define a composition $\mathbf{m} = (m_1, m_2, \dots, m_r)$ by $m_1 = j_1, m_2 = j_2 - j_1, \dots, m_{r-1} = j_{r-1} - j_{r-2}, m_r = n - j_{r-1}$. Let $R(\mathbf{m})$ be the set of all rearrangements of $1^{m_1}2^{m_2}\cdots r^{m_r}$. We construct a bijection ϕ from $R(\mathbf{m})$ to $\mathfrak{S}_n^{\subseteq J}$ by means of the classical *standardization* of words. Let $w \in R(\mathbf{m})$ be a word. From left to right label the letters 1 in w by $1, 2, \dots, m_1$, then label the letters 2 in w by $m_1 + 1, m_1 + 2, \dots, m_1 + m_2$, and so on. Then the standardization of w, denoted by

$\sigma = \phi(w)$, is the permutation obtained by reading those labels from left to right. It is easy to see that ϕ is reversible and $\mathrm{IDES}\,\sigma \subseteq J$ if and only if $w \in R(\mathbf{m})$ (see [3, 6]). Moreover, the permutation σ and the word w have the same hook-factorization *type*. This means that if $ah_1h_2\ldots h_s$ (resp. $bp_1p_2\ldots p_k$) is the hook-factorization of σ (resp. hook-factorization of w), then $k = s$ and $|a| = |b|$. For each $1 \le i \le k$ we have $|h_i| = |p_i|$ and $\mathrm{inv}(h_i) = \mathrm{inv}(p_i)$. Hence $\mathrm{lec}\,w = \mathrm{lec}\,\sigma$ and $\mathrm{pix}\,w = \mathrm{pix}\,\sigma$. Furthermore, σ is in class A_0, A_1, B_0 or B_1 if and only if w is in the same class. Equalities (8) are equivalent to the next ones, whose proofs follow from the definitions of the classes A_0, A_1, B_0, B_1 and Lemma 13.

$$\sum_{\sigma \in R(\mathbf{m})\cap A_0} s^{\mathrm{lec}\,\sigma} = \sum_{\sigma \in R(\mathbf{m})\cap A_1} s^{\mathrm{lec}\,\sigma} \quad \text{and} \quad \sum_{\sigma \in R(\mathbf{m})\cap B_0} s^{\mathrm{lec}\,\sigma} = s \sum_{\sigma \in R(\mathbf{m})\cap B_1} s^{\mathrm{lec}\,\sigma}. \tag{9}$$

□

The following variation of Theorem 2 follows from Theorem **??**^{b}, but cannot be derived from Theorem 2 directly.

Theorem 14. *We have*

$$\sum_{\sigma \in D_0^J(n)} s^{\mathrm{iexc}\,\sigma} - \sum_{\sigma \in D_1^J(n)} s^{\mathrm{iexc}\,\sigma} = (s-1)Q_n^J(s)$$

for some polynomial $Q_n^J(s)$ with positive integral coefficients.

6. A power series with positive integral coefficients

The statistics "des" and "maj" are determined by "DES": $\mathrm{des}\,\pi = \#\,\mathrm{DES}\,\pi$ and $\mathrm{maj}\,\pi = \sum_{i\in \mathrm{DES}\,\pi} i$ for $\pi \in \mathfrak{S}_n$. By using Theorem 2 for each proper subset J of $[n-1]$ and by checking the case $J = [n-1]$ directly, we have the following result.

Theorem 15. *There is a polynomial $Q_n(s,t,q)$ with positive integral coefficients such that*

$$\sum_{\sigma \in D_0(n)} s^{\mathrm{exc}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} - \sum_{\sigma \in D_1(n)} s^{\mathrm{exc}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} = (s-1)Q_n(s,t,q) + r_n(s,t,q)$$

where $r_{2k}(s,t,q) = s^k t^{2k-1} q^{k(2k-1)}$ for $k \ge 1$ and $r_{2k+1}(s,t,q) = -s^k t^{2k} q^{k(2k+1)}$ for $k \ge 0$.

A related result is the following, where we use the standard notation for q-series:

$$(z;q)_m = (1-z)(1-zq)\cdots(1-zq^{m-1}).$$

Proposition 16 ([6], Theorem 1.1)**.** *Let $(A_n(s,t,q,Y))_{n\ge 0}$ be the sequence of polynomials in four variables, whose factorial generating function is given by*

$$\sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}.$$

Then $A_n(s,t,q,Y)$ is the generating polynomial for $\mathfrak{S}_n$ according to the four-variable statistic (exc, des, maj, fix). *In other words,*

$$A_n(s,t,q,Y) = \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma}.$$

Since $\sum_{\sigma\in D_0(n)} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}$ is simply $A_n(s,t,q,0)$ and $\sum_{\sigma\in D_1(n)} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}$ is equal to the coefficient of Y in $A_n(s,t,q,Y)$, Theorem 15 and Proposition 16 imply the following theorem.

Theorem 17. *There is a sequence of polynomials $(Q_n(s,t,q))_{n\geq 0}$ with positive integral coefficients such that*

$$\sum_{r\geq 0} t^r \left(1 - u\frac{1-q^{r+1}}{1-q}\right) \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)} - \frac{1}{1-t}$$
$$= (s-1)\sum_{n\geq 1} Q_n(s,t,q)\frac{u^n}{(t;q)_{n+1}} + r(s,t,q),$$

where

$$r(s,t,q) = \sum_{k\geq 1} s^k t^{2k-1} q^{k(2k-1)} \frac{u^{2k}}{(t;q)_{2k+1}} - \sum_{k\geq 0} s^k t^{2k} q^{k(2k+1)} \frac{u^{2k+1}}{(t;q)_{2k+2}}.$$

In the case of $t=1$ and $q=1$ the above theorem yields the following corollary.

Corollary 18. *For each $n\geq 0$ let $Q_n(s)$ be the coefficient of $u^n/n!$ in the Taylor expansion of*

$$H(s) = \frac{u-1}{se^{us} - s^2e^u} - \frac{1}{2s\sqrt{s}}\Big(\frac{e^{u\sqrt{s}}}{\sqrt{s}+1} + \frac{e^{-u\sqrt{s}}}{\sqrt{s}-1}\Big),$$

that is

$$H(s) = \frac{u^3}{3!} + (s+3)\frac{u^4}{4!} + (s^2+17s+4)\frac{u^5}{5!} + (s^3+46s^2+80s+5)\frac{u^6}{6!} + \cdots + Q_n(s)\frac{u^n}{n!} + \cdots$$

Then, the coefficients $Q_n(s)$ are polynomials in s with positive integral coefficients.

It is easy to show that $Q_{2n-1}(1) = D_{2n-1}/2$ and $Q_{2n}(1) = (D_{2n}-1)/2$ for $n\geq 2$. By Formula (6.19) in [4] we have

$$Q_n(1) = \sum_{2\leq 2k\leq n-1} k \times n(n-1)(n-2)\cdots(2k+2).$$

Since $Q_n(1)$ counts the number of desarrangements of type B, Corollary 18 implies that the number of desarrangements of type A equals the number of desarrangements of type B, when excluding the decreasing desarrangement of even length. It would be interesting to have a direct (analytic) proof of Corollary 18 which would not use the combinatorial set-up of this paper.

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