THE q-SERIES IN COMBINATORICS; PERMUTATION STATISTICS

(Preliminary version)

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Dominique Foata and Guo-Niu Han

Dominique Foata
Département de mathématique
Université Louis Pasteur
7, rue René-Descartes
F-67084 Strasbourg, France
foata@math.u-strasbg.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université Louis Pasteur et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu@math.u-strasbg.fr

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Introduction

The inverse Laplace transform maps each formal power series $\sum_{n\geq 0} a(n)u^n$ in one variable u into another series $\sum_{n\geq 0}(a(n)/n!)u^n$, whose coefficient of order n is *normalized* by the factor $n!$ We then obtain the so-called *exponential* generating function for the sequence $(a(n))$ $(n \geq 0)$. The normalization has numerous advantages: the derivative is obtained by a simple shift of the coefficients; the exponential of a series can be explicitly calculated; there are closed formulas for the *exponential* generating functions for several classical orthogonal polynomials, ... However the algebra of exponential generating functions cannot be regarded as the universal panacea. Further kinds of series are needed, for instance to express some generating series for the symmetric groups by certain statistics.

In the middle of the eighteenth century Heine introduced a new class of series in which the normalized factor $n!$ was replaced by a polynomial of degree n in another variable, more precisely, the series where the coefficient of order n is normalized by the polynomial denoted by $(q;q)_n$, in another variable q, defined by

$$(q;q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-q)(1-q^2)\cdots(1-q^n), & \text{if } n \geq 1. \end{cases} \tag{0.1}$$

The algebra of those series has been largely developed by Jackson in the beginning of the twentieth century. It has then fallen into disuse, except perhaps in the field of Partition Theory, but has vigorously come back in several mathematical domains, in particular in the theory of Quantum Groups and naturally in Combinatorics.

Those series have been named *q-series.* They are simply formal power series in two variables, say, u and q, where the latter variable q, used for the normalization, plays a privileged role. Let $\Omega[[u,q]]$ denote the algebra of formal series in two variables u and q, with coefficients in a ring Ω. Each element of that algebra can be expressed as a series

$$a = \sum_{n\geq 0, m\geq 0} a(n,m)\, u^n\, q^m, \tag{0.2}$$

where, for each ordered pair (n,m), the symbol $a(n,m)$ belongs to Ω. Such a series can be seen as a series *in one variable* u, with coefficients in the ring $\Omega[[q]]$ of series in one variable q, *i. e.*,

$$a=\sum_{n\ge 0}u^n\Bigl(\sum_{m\ge 0}a(n,m)\,q^m\Bigr). \tag{0.3}$$

For each integer $n\ge 0$ the expression $(q;q)_n$ is a polynomial in q, which is invertible in $\Omega[[q]]$, since its constant coefficient is 1. Thus the series a can be rewritten as

$$a=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}\,b(n;q), \tag{0.4}$$

where $b(n;q)$ is the formal series in the variable q

$$b(n;q):=(q;q)_n\cdot\Bigl(\sum_{m\ge 0}a(n,m)\,q^m\Bigr). \tag{0.5}$$

Each formal series a written in the form (0.4) is called a q*-series.* The coefficient $u^n/(q;q)_n$ is then a formal series in the unique variable q.

The purpose of this memoir is to give a basic description of the algebra of those series and describe the use that has been made of them in Combinatorics, in particular for expressing the generating functions for certain statistics defined on permutations, words, multipermutations, signed permutations and other finite structures. It has been customary to regroup all the techniques that have been developed under the name of *Permutation Statistic Study*, even though the group of permutations is not the only group structure involved. The statistics themselves can be uni- or several-variable, or even set-valued. As will be seen, the q-series enter into the picture in a very natural way.

The q*-binomial theorem*, which is stated and proved in the first section, is the basic tool in q-Calculus. It opens the door to all the q-series identities and also gives rise to two expansions of the q*-exponential*, as a q-series itself, and also as an infinite product.

The polynomial $(q;q)_n$, defined above, is next studied in a combinatorial context. This leads to a discussion of the so-called *Mahonian* statistics, especially the *Major Index* and the *Inversion Number* that will play an essential role in this memoir. One of our goals, indeed, is to try to understand why the so-called *natural* q*-analogs* of various numbers or polynomials can be derived by means of either one of those statistics.

The Major Index is strongly related to the combinatorial theory of the representation of finite groups, particularly when dealing with various tableaux (standard, semi-standard, …). The inversion number requires other techniques, in particular the so-called q-iteration method, that will be used on several occasions.

The *q-binomial coefficients* or *Gaussian polynomials* that appear in many identities on q-series are studied in several combinatorial environments, as is done in section 4. With the study of the q-multinomial coefficient we are led to introduce the statistic "number of inversions" for classes of permutations with repetitions. We prefer to use the term "rearrangement" (of a given finite sequence) or "word." This is the content of section 5.

The *MacMahon Verfahren*, introduced in section 6, is the first tool that makes possible the transcription of properties of certain statistics defined on the symmetric group or on some classes of rearrangements to the algebra of q-series. As a first application, it is shown that the Major Index has the same distribution as the number of inversions on each class of rearrangements.

A careful study of the MacMahon *Verfahren* serves to find a q-extension of the traditional Eulerian polynomials, namely the *Euler-Mahonian polynomials* $A_{\mathbf{m}}(t,q)$, that appear to be generating polynomials for classes of rearrangements by the bivariable statistic $(\mathrm{des}, \mathrm{maj})$. The statistic "des" is the *number of descents* that has been studied in several combinatorial contexts and "maj" is the Major Index.

As shown in section 8, there are four equivalent definitions of the Euler-Mahonian polynomials. The proofs of those equivalences are based on fundamental techniques in q-Calculus, finite difference and iterative methods. The insertion technique that looks so natural when dealing with univariable statistics on the symmetric group becomes intricate for several-variable statistics. A *marked word* technique is presented in section 9 and appears to be successful for deriving a recurrence relation for the Euler-Mahonian polynomials $A_{\mathbf{m}}(t,q)$.

When the class of rearrangements is reduced to the symmetric group, the Euler-Mahonian polynomials become the so-called *q-maj-Eulerian polynomials* ${}^{\mathrm{maj}}A_n(t,q)$, as they form a first q-analog of the traditional Eulerian polynomials $A_n(t)$ in one variable t. However, when the exponential generating function for the latter polynomials is *q-analogized* in a proper way, another q-analog of those polynomials, namely the q-inv Eulerian polynomial ${}^{\mathrm{inv}}A_n(t,q)$ appears. As shown in the notations, "inv" plays for ${}^{\mathrm{inv}}A_n(t,q)$ the role that "maj" does for ${}^{\mathrm{maj}}A_n(t,q)$.

The *Major Index* and the *Inversion Number*, that can be defined for each rearrangement of a given finite sequence of integers, are equidistributed over each rearrangement class. Section 11 contains the construction of a bijection Φ of the class onto itself such that $\mathrm{inv}\,\Phi(w) = \mathrm{maj}\,w$.

The bijection has several other properties, in particular when the underlying class is the symmetric group.

With section 12 we start the study of permutation statistics that involve both the permutation and its inverse. Besides "maj" we are led to introduce "imaj" that is nothing but the Major Index of the *inverse* permutation. We then see the first occurrence of the classical infinite product $\prod_{i\ge 1, j\ge 1} \frac{1}{1-ux_iy_j}$, that is to be expanded, once the substitutions $x_i \leftarrow q_1^{i-1}$, $y_j \leftarrow q_j^{j-1}$ are made. The resulting series are series in two bases, normalized by denominators of the form $(q_1;q_1)_n\,(q_2;q_2)_n$.

In section 13 a further extension of the MacMahon *Verfahren* leads to the derivation of the distribution of a four-variable statistic on the symmetric group.

The theory of symmetric functions hides too many useful identities and too many combinatorial algorithms not to appear in this memoir. In particular, the infinite product mentioned above has a celebrated expansion in terms of products of *Schur functions.* As those functions have a handy combinatorial interpretation—as shown in section 17—it was essential to give the main properties of those functions. This is the content of sections 15, 16, and 17.

In the expansion of a Schur function we find monomials that are coded by the so-called *semi-standard tableaux.* In their turn, those tableaux can be further coded by *standard tableaux* and sequences of numbers. This coding has several applications. In particular, it serves to express a Schur function, in which variables are replaced by powers of a variable q, as a generating function for standard tableaux by a certain statistic. This is the content of section 18.

In section 19 we find an overview of the *Robinson-Schensted correspondence* that enables the transfer of geometric properties on tableaux to analogous properties on permutations. As an application, a bibasic generating function for polynomials in several variables defined on symmetric groups is derived.

The next four sections 20, 21, 22 and 23 deal with *Eulerian Calculus*, that is, the study of geometric properties of the Eulerian polynomials and its various extensions. By "extensions" we mean three aspects: (i) extension to the group of the *signed permutations*, (ii) q-extension, that is, the introduction of a suitable Mahonian statistic "inv" or "maj," (iii) the study of generating polynomials for *pairs* or finite sequences of permutations or signed permutations. The combination of those three extensions leads to the combinatorial study of some Bessel functions, q-Bessel functions and finite analogs of Bessel functions.

The crucial step in Eulerian Calculus is to find the appropriate q-analog for the generating polynomial for the signed permutations by their number of descents. Our *analytic choice* (see section 21) forces us to find a suitable definition of inversions for signed permutations. This leads to a coherent study of all the extensions described above. Notice that the *length*, as defined in the theory of Coxeter groups, does not conduct to an elegant derivation in the algebra of q-series.

In our last section on Eulerian Calculus (section 23) we introduce the bi-indexed Eulerian polynomials and explain how the Désarménien *Verfahren* makes possible the study of congruences of those polynomials with respect to the cyclotomic polynomials. The section ends with a short study on signed Eulerian Numbers.

We end these Lecture Notes with a combinatorial study of the basic trigonometric functions, especially the tangent and secant functions. The coefficients in their q-expansions are the generating polynomials for the so-called alternating permutations by number of inversions. The same combinatorial set-up is used to interpret the coefficients in the p, q-expansions of the bibasic tangent and secant.

1. The q-Binomial Theorem

Take up again the notations (0.3)–(0.5). When, for each $n \geq 0$, the ratio $b(n+1; q)/b(n; q)$ is a *rational fraction* in q^n, equal to 1 for $q = 0$ and such that $b(0; q) = 1$, we get what is called a *basic hypergeometric series.* In the analytic expression of such a series the following notation is used that extends the notation (0.1): for each element ω in the ring define the *q-ascending factorial in ω* by

$$(1.1) \qquad (\omega; q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-\omega)(1-\omega q)\ \ldots\ (1-\omega q^{n-1}), & \text{if } n \geq 1; \end{cases}$$

in its *finite* version and

$$(1.2) \qquad (\omega; q)_\infty := \lim_n (\omega; q)_n = \prod_{n\geq 0} (1-\omega q^n);$$

in its *infinite* version.

When the underlying ring Ω is the complex field, the rational fraction $b(n+1; q)/b(n; q)$ can be written as

$$\frac{b(n+1;q)}{b(n;q)} = \frac{(1-\alpha_1 q^n)\ldots(1-\alpha_r q^n)}{(1-\beta_1 q^n)\ldots(1-\beta_s q^n)},$$

where $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are complex numbers. By iteration,

$$\begin{aligned} b(n;q) &= \frac{b(n;q)}{b(n-1;q)} \cdots \frac{b(2;q)}{b(1;q)} \frac{b(1;q)}{b(0;q)} \\ &= \frac{(1-\alpha_1 q^{n-1}) \ldots (1-\alpha_r q^{n-1})}{(1-\beta_1 q^{n-1}) \ldots (1-\beta_s q^{n-1})} \\ &\qquad \times \cdots \times \frac{(1-\alpha_1 q) \ldots (1-\alpha_r q)}{(1-\beta_1 q) \ldots (1-\beta_s q)} \frac{(1-\alpha_1) \ldots (1-\alpha_r)}{(1-\beta_1) \ldots (1-\beta_s)}, \end{aligned}$$

that is

$$b(n;q) = \frac{(\alpha_1;q)_n \ldots (\alpha_r;q)_n}{(\beta_1;q)_n \ldots (\beta_s;q)_n}. \tag{1.3}$$

For an arbitrary ring Ω the expression of $b(n;q)$ given in (1.3) is taken as a definition. As each q-ascending factorial $(\beta_i;q)_n$ has a constant coefficient equal to 1, it is invertible in the ring $\Omega[[q]]$, so that $b(n;q)$ as shown in (1.3) is well-defined. Then call *basic hypergeometric series* each q-series of the form

$${}_r\varphi_s\left(\begin{matrix}\alpha_1, \ldots, \alpha_r\\ \beta_1, \ldots, \beta_s\end{matrix}; q, u\right) := \sum_{n\ge 0} \frac{(\alpha_1;q)_n \ldots (\alpha_r;q)_n}{(\beta_1;q)_n \ldots (\beta_s;q)_n} \frac{u^n}{(q;q)_n}. \tag{1.4}$$

Such a series can be defined in each algebra $\Omega[[u,q]]$ of formal series in two variables u and q, whatever the underlying ring Ω is. When $r=0$ (resp. $s=0$, resp. $r=0$ and $s=0$), the following notations are used: ${}_0\varphi_s\left(\begin{matrix}\overline{}\\ \beta_1, \ldots, \beta_s\end{matrix}; q, u\right)$ (resp. ${}_r\varphi_0\left(\begin{matrix}\alpha_1, \ldots, \alpha_r\\ \overline{}\end{matrix}; q, u\right)$, resp. ${}_0\varphi_0\left(\begin{matrix}-\\ -\end{matrix}; q, u\right)$).

In the *q-binomial theorem* which is stated next, the series ${}_1\varphi_0\left(\begin{smallmatrix}\alpha\\ -\end{smallmatrix}; q, u\right)$ has a closed expression in terms of an infinite product.

Theorem 1.1 (q-Binomial Theorem)**.** *The following identity holds:*

$$\sum_{n\ge 0} (\alpha;q)_n \frac{u^n}{(q;q)_n} = \frac{(\alpha\, u;q)_\infty}{(u;q)_\infty} = \prod_{n\ge 0} \frac{1-\alpha\, u\, q^n}{1-uq^n}; \tag{1.5}$$

or, equivalently:

$${}_1\varphi_0\left(\begin{matrix}\alpha\\ -\end{matrix}; q, u\right) = \frac{(\alpha\, u;q)_\infty}{(u;q)_\infty}. \tag{1.6}$$

Before giving the proof of the theorem it matters to make several remarks.

(a) The *order* $o(a)$ of a formal series $a = \sum_{n\ge 0, m\ge 0} a(n,m)\, u^n\, q^m$ is defined to be the smallest integer $k \ge 0$ such that the polynomial

$$\sum_{n+m=k} a(n,m)\, u^n\, q^m,$$

called the *homogeneous polynomial of degree* k *of* a (in u and q), is not zero. Consider a countable family (a_s) $(s \ge 0)$ of formal series in two variables. As for the series in one variable, it is readily seen that if the order of a_s tends to infinity with s, then the infinite product $\prod_{s\ge 0}(1-a_s)$ is a well-defined series. In the infinite product $(\alpha u; q)_\infty$, with $\alpha \ne 0$, the term $\alpha\, u\, q^n$ is a series (reduced to a monomial) of order $(n+1)$. As $o(\alpha\, u\, q^n)$ tends to infinity with n, the infinite product $(\alpha\, u; q)_\infty$ is well-defined. The same property holds for the product $(u;q)_\infty$ occurring in the denominator.

(b) Within the coefficient $(\alpha;q)_n/(q;q)_n$ of u^n in formula (1.5), make the substitution $\alpha \leftarrow q^\alpha$ to obtain $(q^\alpha;q)_n/(q;q)_n$; then let q tend to 1. We get $(\alpha)_n/n!$. The ratio $(q^\alpha;q)_n/(q;q)_n$ is said to be the *q-analog* of the ascending factorial $(\alpha)_n/n!$, where

$$(\alpha)_n = \begin{cases} 1, & \text{if } n=0;\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n\ge 1.\end{cases}$$

But the series $\sum\limits_{n\ge 0} u^n\, (\alpha)_n/n!$ is the *hypergeometric series* ${}_1F_0\bigl({\alpha \atop -};u\bigr)$, that safisfies the identity

$$(1.7)\qquad {}_1F_0\Bigl({\alpha \atop -};u\Bigr) = (1-u)^{-\alpha},$$

which is known to be the *binomial identity.* Notice that (1.7) is used to extend the definition of $(1-u)^{-\alpha}$ when α is not an integer (positive or negative). Identity (1.6) is said to be the *q-analog* of (1.7).

The main difference between (1.6) and (1.7) is the following: when dealing with the algebra of formal series, formula (1.7) is a *definition* of $(1-u)^{-\alpha}$ whenever α is not an integer, while (1.6) is an *identity.*

However, when considering the series ${}_1F_0\bigl({\alpha \atop -};u\bigr)$ and ${}_1\varphi_0\bigl({\alpha \atop -};q,u\bigr)$ as power series of the *complex* variables u and q, the two formulas (1.6) and (1.7) are *identities*, if the moduli of u and q are less than 1. The first proof of (1.6) given below is directly inspired from the classical proof of (1.7) in the analytic case. However, the end of the proof uses a topological argument on formal series and not on analytic series.

(c) In view of the proof of Theorem 1.1 let us mention the following identity, easy to derive:

$$(1.8)\qquad (\alpha;q)_n - (\alpha\, q;q)_n = (\alpha\, q;q)_{n-1}\, \alpha(q^n-1) \quad (n\ge 1).$$

There exist numerous relations on the q-ascending factorials. Among the most frequent let us quote the *associativity* property

$$(\alpha;q)_{n+k}=(\alpha;q)_n\,(\alpha q^n;q)_k \tag{1.9}$$

and the *reverse formula*

$$(\alpha^{-1}q^{1-n};q)_n=(\alpha;q)_n\,(-\alpha^{-1})^n\,q^{-n(n-1)/2}. \tag{1.10}$$

No comment for (1.9). The latter can be proved as follows. For $n\ge 1$ we have

$$\begin{aligned}(\alpha;q)_n&=(1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})\\&=(-\alpha)(1-\alpha^{-1})(-\alpha q)(1-\alpha^{-1}q^{-1})\cdots(-\alpha q^{n-1})(1-\alpha^{-1}q^{-(n-1)})\\&=(-\alpha)^nq^{n(n-1)/2}(1-\alpha^{-1})(1-\alpha^{-1}q^{-1})\cdots(1-\alpha^{-1}q^{1-n})\\&=(-\alpha)^nq^{n(n-1)/2}(\alpha^{-1}q^{1-n};q)_n,\end{aligned}$$

a formula that still holds for $n=0$.

Finally, notice that the relation

$$(\alpha;q)_n=\frac{(\alpha;q)_\infty}{(\alpha q^n;q)_\infty} \tag{1.11}$$

can be used to define the q-ascending factorial $(\alpha;q)_n$ for *every real number* n.

(d) Let $\alpha=q^N$ in (1.5). We get

$$\frac{(q^Nu;q)_\infty}{(u;q)_\infty}=(u;q)_N^{-1}=\sum_{n\ge0}(q^N;q)_n\frac{u^n}{(q;q)_n}. \tag{1.12}$$

But, if u and q are regarded as complex variables with modulus less than 1 and if we let q tend to 1, we obtain the identity

$$(1-u)^{-N}=\sum_{n\ge0}(N)_n\,\frac{u^n}{n!},$$

that is, the usual binomial identity.

First proof of Theorem 1.1. Start with the series ${}_1\varphi_0\big({\alpha \atop -};q,u\big)=\sum\limits_{n\ge0}u^n\,(\alpha;q)_n/(q;q)_n$ and evaluate the *q-difference*

$$\begin{aligned}{}_1\varphi_0\Big({\alpha\atop -};q,u\Big)-{}_1\varphi_0\Big({\alpha\atop -};q,qu\Big)&=\sum_{n\ge1}\frac{(\alpha;q)_n}{(q;q)_n}u^n(1-q^n)=\sum_{n\ge1}\frac{(\alpha;q)_n}{(q;q)_{n-1}}u^n\\&=(1-\alpha)u\Big(1+\sum_{n\ge2}\frac{(\alpha\,q;q)_{n-1}}{(q;q)_{n-1}}u^{n-1}\Big)\\&=(1-\alpha)u\,{}_1\varphi_0\Big({\alpha\,q\atop -};q,u\Big).\end{aligned} \tag{1.13}$$

By using (1.8) we get:

$$\begin{aligned}
{}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\Big)-{}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha\,q}{-};q,u\Big)&=\sum_{n\ge1}\frac{(\alpha;q)_n-(\alpha\,q;q)_n}{(q;q)_n}u^n\\
&=-\alpha\sum_{n\ge1}\frac{(\alpha\,q;q)_{n-1}}{(q;q)_{n-1}}u^n\\
&=-\alpha\,u\,{}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha\,q}{-};q,u\Big). \qquad (1.14)
\end{aligned}$$

From (1.13) and (1.14) it follows that

$${}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\Big)=\frac{1-\alpha\,u}{1-u}\,{}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,q\,u\Big),$$

and by iteration

$$(1.15)\qquad {}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\Big)=\frac{(\alpha\,u;q)_m}{(u;q)_m}\,{}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,q^m\,u\Big)\quad(m\ge0).$$

If ${}_1\varphi_0\big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\big)$ is considered as an analytic series of the complex variable u, it suffices to say that ${}_1\varphi_0\big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\big)$ is continuous inside the unit disk. As the series is equal to 1 for $u=0$, identity (1.6) follows from (1.15) by letting m tend to infinity.

With the topology of formal series we may use the following argument: consider a pair (i,j) of nonnegative integers such that $i+j\ge1$. As soon as $m\ge j+1$ the coefficients of $u^i\,q^j$ in

$$\frac{(\alpha\,u;q)_\infty}{(u;q)_\infty}\quad\text{and in}\quad\frac{(\alpha\,u;q)_m}{(u;q)_m}$$

are equal. But ${}_1\varphi_0\big(\genfrac{}{}{0pt}{}{\alpha}{-};q,q^m\,u\big)$ is of the form $1+q^m\,u\,a$, where a is a formal series in the two variables u and q. It follows that for every i the coefficients of $u^i\,q^j$ in

$$\frac{(\alpha\,u;q)_m}{(u;q)_m}\quad\text{and if}\quad\frac{(\alpha\,u;q)_m}{(u;q)_m}(1+q^m\,u\,a)={}_1\varphi_0\Big(\genfrac{}{}{0pt}{}{\alpha}{-};q,u\Big)$$

are the same. Hence, identity (1.6) is proved. ∎

Second proof of Theorem 1.1. The right-hand side of identity (1.6) is a formal series that can be written as

$$b(u,q):=\sum_{n\ge0}c_n(q)\,u^n,$$

where, for every $n \ge 0$, the coefficient $c_n(q)$ is a formal series in the variable q. But

$$b(u,q) = \frac{1-\alpha\, u}{1-u} \prod_{n\ge 0} \frac{(1-\alpha\, u\, q\, q^n)}{(1-u\, q\, q^n)} = \frac{1-\alpha\, u}{1-u} b(uq,q);$$

therefore

$$(1-\alpha\, u) b(uq,q) = (1-u) b(u,q),$$

and then

$$(1-\alpha u) \sum_{n\ge 0} c_n(q)\, q^n u^n = (1-u) \sum_{n\ge 0} c_n(q)\, u^n.$$

Looking for the coefficient of u^{n+1} on each side provides:

$$c_{n+1}(q) q^{n+1} - \alpha\, q^n\, c_n(q) = c_{n+1}(q) - c_n(q)\,;$$

so that

$$c_{n+1}(q) = c_n(q) \frac{1-\alpha\, q^n}{1-q^{n+1}}.$$

As $c_0(q)=1$, the right expression

$$c_n(q) = \frac{(\alpha;q)_n}{(q;q)_n}$$

is found by induction on n. $\square$

Let $\alpha = 0$ in (1.6). We get

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty}. \tag{1.16}$$

Now consider the infinite product $(-u;q)_\infty$ and again take the argument developed in the second proof. If we let $(-u;q)_\infty = \sum_{n\ge 0} c_n(q)\, u^n$, we find $c_{n+1}(q) q^{n+1} + c_n(q) q^n = c_{n+1}(q)$. Hence,

$$c_{n+1}(q) = \frac{q^n c_n(q)}{(1-q^{n+1})}.$$

As $c_0(q)=1$, we get

$$c_{n+1}(q) = \frac{q^{(n+1)n/2}}{(q;q)_{n+1}};$$

and then the identity

$$\sum_{n\ge 0} q^{n(n-1)/2} \frac{u^n}{(q;q)_n} = (-u;q)_\infty. \tag{1.17}$$

The two series appearing in (1.16) and (1.17) are respectively denoted by $e_q(u)$ and $E_q(u)$ and are referred to as the *first* and the *second q-exponential.*

2. Mahonian Statistics

For each integer $n \geq 0$ let S_n be a set of cardinality $n!$ (for example, the permutation group $\mathfrak{S}_n$). By *statistic* on S_n we simply mean a mapping $f : S_n \to \mathbb{N}$ with nonnegative integral values. The polynomial

$$a(n) := \sum_{s \in S_n} q^{f(s)}$$

is called the *generating polynomial for* S_n *by the statistic* f; or, sometimes, the generating polynomial for f. The series

$$a := \sum_{n \geq 0} u^n \frac{a(n)}{(q;q)_n} \tag{2.1}$$

is called the q-*generating function* for the polynomials $(a(n))$ $(n \geq 0)$. If, for each $n \geq 0$, the polynomial $a(n)$ has the form

$$a(n) = \frac{(q;q)_n}{(1-q)^n} = (1 + q + \cdots + q^{n-1}) \cdots (1 + q + q^2)(1 + q), \tag{2.2}$$

we say that f is a *Mahonian* statistic on the family (S_n) $(n \geq 0)$.

With each positive integer n is associated its q-*analog* defined by

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \tag{2.3}$$

and its q-*factorial*

$$\begin{aligned} [n]_q! &:= [n]_q \, [n-1]_q \cdots [2]_q \, [1]_q \\ &= \frac{(1-q^n)}{(1-q)} \frac{(1-q^{n-1})}{(1-q)} \cdots \frac{(1-q^2)}{(1-q)} \frac{(1-q)}{(1-q)} \\ &= \frac{(q;q)_n}{(1-q)^n} \\ &= (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1+q). \end{aligned} \tag{2.4}$$

Consequently, the generating polynomial defined in (2.2) is equal to the q-factorial of n and the q-generating function has the simple form

$$a = \sum_{n \geq 0} u^n \frac{1}{(q;q)_n} \frac{(q;q)_n}{(1-q)^n} = \Big(1 - \frac{u}{1-q}\Big)^{-1}. \tag{2.5}$$

In this section our purpose is to introduce several Mahonian statistics that are of constant use in the study of the q-series. Some of their properties are being derived.

The first one of these statistics is denoted by "tot" ("tot" for "total"). Although its definition is straightforward, it is very useful in many calculations. For each $n \geq 0$ let SE_n denote the set of the *subexcedent* sequences $x = (x_1, x_2, \ldots, x_n)$. By *subexcedent* we mean a sequence of integers x_i, of length n, that satisfy the inequalities $0 \leq x_i \leq i-1$ for all $i = 1, 2, \ldots, n$. The cardinality of SE_n is of course $n!$ For each sequence $x = (x_1, x_2, \ldots, x_n) \in \mathrm{SE}_n$, define

$$\operatorname{tot} x := x_1 + x_2 + \cdots + x_n. \tag{2.6}$$

Proposition 2.1. *The statistic* "tot" *on* SE_n *is Mahonian, that is, for each* $n \geq 1$ *we have:*

$$\sum_{x \in SE_n} q^{\operatorname{tot} x} = \frac{(q;q)_n}{(1-q)^n}. \tag{2.7}$$

Proof. The result is banal for $n = 1$. By induction on n:

$$\begin{aligned}\sum_{x \in SE_n} q^{\operatorname{tot} x} &= \sum_{x' \in SE_{n-1}} q^{\operatorname{tot} x'} \sum_{0 \leq x_n \leq n-1} q^{x_n}\\ &= \frac{(q;q)_{n-1}}{(1-q)^{n-1}}(1 + q + \cdots + q^{n-1})\\ &= \frac{(q;q)_n}{(1-q)^n}. \quad \square\end{aligned}$$

The next three Mahonian statistics that are being introduced are defined on the permutation group $\mathfrak{S}_n$; they are called the *Inversion Number* "inv", the *Major Index* "maj" and the *Denert statistic* "den".

Let $\sigma = \sigma(1) \ldots \sigma(n)$ be a permutation, written as linear word. It is traditional to define the *Inversion Number*, $\operatorname{inv} \sigma$, of the permutation σ as the number of ordered pairs of integers (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.

The *Major Index* $\operatorname{maj} \sigma$ of σ is defined to be the *sum of the positions* i where a descent $\sigma(i) > \sigma(i+1)$ occurs. We can also write

$$\operatorname{maj} \sigma := \sum_{1 \leq i \leq n-1} i\, \chi\{\sigma(i) > \sigma(i+1)\}, \tag{2.8}$$

by making use of the χ-notation. Remember that for each statement A we write $\chi(A) = 1$ or 0 depending on whether A is true or not.

The definition of "den" is based on the notion of *cyclic interval*. Let i, j be two positive integers; the *cyclic interval* $\rrbracket i, j \rrbracket$ is then

$$\rrbracket i, j \rrbracket := \begin{cases}]i, j], & \text{if } i \le j; \\ [1, j] +]i, +\infty[, & \text{if } i > j. \end{cases}$$

The *Denert statistic*, $\operatorname{den} \sigma$, of the permutation σ is defined as the number of ordered pairs (i, j) such that $1 \le i < j \le n$ and $\sigma(i) \in \rrbracket \sigma(j), j \rrbracket$.

To show that those three statistics are Mahonian on $\mathfrak{S}_n$, we construct three *bijections* $\sigma \mapsto x$ of $\mathfrak{S}_n$ onto SE_n having the properties

$$\operatorname{inv} \sigma = \operatorname{tot} x, \quad \operatorname{maj} \sigma = \operatorname{tot} x, \quad \operatorname{den} \sigma = \operatorname{tot} x,$$

respectively. The construction of those three bijections makes use of three different *codings* of the permutations. The image x of σ under each of those bijections is called the inv-*coding*, the maj-*coding* and the den-*coding* of σ, respectively.

2.1. *The* inv-*coding (also called Lehmer coding).* Let $\sigma = \sigma(1) \dots \sigma(n)$ be a permutation. For each $i = 1, \dots, n$ define x_i as being the number of terms $\sigma(j)$ to the left of $\sigma(i)$ which are greater than $\sigma(i)$, that is,

$$x_i := \sum_{j : 1 \le j \le i-1} \chi\big(\sigma(j) > \sigma(i)\big).$$

The sequence $x := (x_1, \dots, x_n)$ just defined is obviously subexcedent. Furthermore, the correspondence $\sigma \mapsto x$ is bijective. Moreover, the sum $\operatorname{tot} x$ of the x_i's is precisely equal to the *Inversion Number* $\operatorname{inv} \sigma$ of the permutation σ.

In the following example, under every element $\sigma(i)$ is written the corresponding x_i of the inv-coding.

$$\begin{aligned} \sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 5 & 4 & 9 & 2 & 6 & 3 & 8 \end{pmatrix} \\ x &= \;\;\; \begin{matrix} 0 & 1 & 1 & 2 & 0 & 4 & 2 & 5 & 1 \end{matrix} \end{aligned}$$

In particular, $\operatorname{inv} \sigma = \operatorname{tot} x = 16$.

To reconstruct σ from its inv-coding x, proceed as follows: first let $\sigma(n) := n - x_n$. Once the elements $\sigma(k+1), \dots, \sigma(n)$ have been obtained, eliminate all the terms of the sequence $(n, n-1, \dots, 2, 1)$ equal to one of the $\sigma(l)$'s for a certain $l \ge k+1$. Then, $\sigma(k)$ is equal to be the $(x_k + 1)$-st term of the sequence $(n, n-1, \dots, 2, 1)$, when reading that sequence from left to right.

For example, start from the subexcedent sequence

$$x = (0, 1, 1, 2, 0, 4, 2, 5, 1)$$

of length $n = 9$; first, get $\sigma(9) := n - x_n = 9 - 1 = 8$. Then, form the sequence $(9, 7, 6, 5, 4, 3, 2, 1)$, where 8 has been deleted. Then, $\sigma(8)$ is equal to the $(x_8 + 1) = (5 + 1) = 6$-th term of the sequence, that is, $\sigma(8) := 3$. The running sequence becomes $(9, 7, 6, 5, 4, 2, 1)$, whose $(x_7 + 1) = (2 + 1) = 3$-rd term is 6; hence, $\sigma(7) := 6$. Next, consider $(9, 7, 5, 4, 2, 1)$ whose $(x_6 + 1) = (4 + 1) = 5$-th term is 2; then $\sigma(6) := 2$, and so on.

2.2. *The* maj-*coding.* Starting with a permutation $\sigma' \in \mathfrak{S}_{n-1}$, written as a word $\sigma'(1) \ldots \sigma'(n-1)$, we can generate n permutations $\sigma \in \mathfrak{S}_n$ by inserting the letter n to the left of the word, or between two letters $\sigma'(i)$ and $\sigma'(i+1)$ for $1 \le i \le n-2$, or to the right of the word, say, *in position* $i = 0, 1, 2, \ldots, (n-1)$, respectively. Thus, every permutation $\sigma \in \mathfrak{S}_n$ is characterized by an ordered pair (σ', i), where $\sigma' \in \mathfrak{S}_{n-1}$ and $0 \le i \le n-1$. The surjection $\psi : \sigma \mapsto \sigma'$ of $\mathfrak{S}_n$ onto $\mathfrak{S}_{n-1}$ consists of removing the letter n from the word $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n)$.

For describing the maj-coding we *relabel* the n possible positions where n can be inserted into $\sigma' = \sigma'(1) \ldots \sigma'(n-1)$ in the following manner: label $j = 0$ is given to the insertion of n to the right of the word σ'; suppose that σ' has d *descents*, that is to say, d positions $\sigma'(i)\sigma'(i+1)$ such that $\sigma'(i) > \sigma'(i+1)$. We label those descents $j = 1$, $j = 2$, … , $j = d$, *from right to left*; the insertion of n to the left of σ' is given label $j = d+1$ and the labels $j = d+2, d+3, \ldots, n-1$ are given to the $(n-2-d)$ insertions into the other positions when reading the word σ' *from left to right.*

If the letter n in the original permutation σ is in position j for the *relabelling* just described, we adopt the notations:

$$\begin{gathered} \sigma_{n-1} := \sigma' = \psi(\sigma); \quad x_n := j; \\ \sigma_n := \sigma := [\sigma_{n-1}, x_n]. \end{gathered} \tag{2.9}$$

In the same manner, to σ_{n-1} there corresponds a pair $[\sigma_{n-2}, x_{n-1}]$ and by iteration we obtain a sequence of pairs $[\sigma_{n-3}, x_{n-2}]$, … , $[\sigma_0, x_1]$, where σ_0 is the void permutation and $x_1 = 0$. This yields a sequence, necessarily subexcedent $x = (x_1, x_2, \ldots, x_n)$, that is called the *maj-coding* of σ.

Example. Consider the permutation $\sigma = 7\,1\,5\,4\,9\,2\,6\,3\,8$. The permutations σ_1, σ_2, … , σ_8, σ_9 are simply the *subwords* reduced to the letter 1, to the letters $1, 2$, … , to the letters $1, 2, \ldots, 8$, finally to the letters $1, 2, \ldots, 9$.

At each step of the construction we have to *maj-label* the inserting positions as was indicated before. In Table 2.1 the maj-labellings appear as subscripts of the permutations written in the third column. The maj-coding of σ is then $x = (0,0,0,2,3,5,4,0,2)$. Notice that $\operatorname{maj}\sigma = \operatorname{tot} x = 16$.

Permutation	maj-coding	maj-labelling
$\sigma_1 = 1$	$x_1 = 0$	${}_1 1_0$
$\sigma_2 = 1\,2$	$x_2 = 0$	${}_1 1_2 2_0$
$\sigma_3 = 1\,2\,3$	$x_3 = 0$	${}_1 1_2 2_3 3_0$
$\sigma_4 = 1\,4\,2\,3$	$x_4 = 2$	${}_2 1_3 4_1 2_4 3_0$
$\sigma_5 = 1\,5\,4\,2\,3$	$x_5 = 3$	${}_3 1_4 5_2 4_1 2_5 3_0$
$\sigma_6 = 1\,5\,4\,2\,6\,3$	$x_6 = 5$	${}_4 1_5 5_3 4_2 2_6 6_1 3_0$
$\sigma_7 = 7\,1\,5\,4\,2\,6\,3$	$x_7 = 4$	${}_5 7_4 1_6 5_3 4_2 2_7 6_1 3_0$
$\sigma_8 = 7\,1\,5\,4\,2\,6\,3\,8$	$x_8 = 0$	${}_5 7_4 1_6 5_3 4_2 2_7 6_1 3_8 8_0$
$\sigma_9 = 7\,1\,5\,4\,9\,2\,6\,3\,8$	$x_9 = 2$	

Table 2.1

To reconstruct the permutation σ from a subexcedent sequence $x = (x_1, x_2, \ldots, x_n)$, we put $\sigma_1 := 1$, then obtain $\sigma_2 := [\sigma_1, x_2]$ (with the notations (2.9)), … , until we reach $\sigma := \sigma_n = [\sigma_{n-1}, x_n]$.

Proposition 2.2. *Let $\psi : \sigma \mapsto \sigma'$ be the surjection of $\mathfrak{S}_n$ onto $\mathfrak{S}_{n-1}$ that consists of removing the letter n from the word $\sigma(1)\ldots\sigma(n)$. For $n \geq 2$ and for each permutation $\sigma' \in \mathfrak{S}_{n-1}$ the generating polynomial for the class $\psi^{-1}(\sigma')$ by the Major Index "maj" is given by*

$$\sum_{\sigma\in\psi^{-1}(\sigma')} q^{\operatorname{maj}\sigma} = q^{\operatorname{maj}\sigma'}(1 + q + q^2 + \cdots + q^{n-1}). \tag{2.10}$$

Moreover, with the notation (2.9)

$$\operatorname{maj}[\sigma_{n-1}, x_n] = \operatorname{maj}\sigma_{n-1} + x_n. \tag{2.11}$$

Finally, if $x = (x_1, x_2, \ldots, x_n)$ is the maj-coding of σ, then

$$\operatorname{maj}\sigma = x_1 + x_2 + \cdots + x_n = \operatorname{tot} x. \tag{2.12}$$

Proof. Identities (2.10) and (2.12) follow from (2.11) that is now being proved. When n is inserted to the right of $\sigma' = \sigma_{n-1}$, into the position maj-labelled 0, we have $\operatorname{maj}[\sigma', 0] = \operatorname{maj}\sigma'$. If n is inserted into the x_n-th descent ($1 \leq x_n \leq d = \operatorname{des}\sigma'$) (labelled from right to left), the x_n descents occurring on the right are shifted one position to the right; the other descents remain alike. Hence, (2.11) holds. In the same manner, (2.11) holds for $x_n = d + 1$, since the maj-labelling corresponds to an insertion of n at the beginning of the word. Now if $\sigma'(i) < \sigma'(i+1)$ is the k-th non-descent when σ' is read from left to right $(1 \le k \le n-d-2)$, the left factor $\sigma'(1)\sigma'(2)\ldots\sigma'(i)$ contains $i-k$ descents and the right factor $\sigma'(i+1)\sigma'(i+2)\ldots\sigma'(n-1)$ exactly $d-i+k$ descents. The insertion of n between $\sigma'(i)$ and $\sigma'(i+1)$, into a position maj-labelled $d+k+1$, increases the Major Index by $(i+1)+(d-i+k)=d+k+1$, since a new descent is created between $\sigma'(i)$ and $\sigma'(i+1)$ and the $(d-i+k)$ descents of the right factor $\sigma'(i+1)\sigma'(i+2)\ldots\sigma'(n-1)$ are shifted one position to the right. $\square$

2.3. *The den-coding.* The Denert statistic, $\operatorname{den}\sigma$, of a permutation $\sigma \in \mathfrak{S}_n$ can be calculated by means of its *den-coding*, defined as follows. For each integer j $(1 \le j \le n)$ define x_j as the number of integers i such that

$$1 \le i \le j-1 \quad \text{and} \quad \sigma(i) \in \,]\!]\sigma(j), j]\!]. \tag{2.13}$$

The *den-coding* of σ is defined to be the sequence $x := (x_1, x_2, \ldots, x_n)$, which is obviously subexcedent. Clearly, $\operatorname{den}\sigma = x_1+x_2+\cdots+x_n = \operatorname{tot} x$ and the mapping $\sigma \mapsto x$ is injective and then bijective. Let us illustrate the calculation of the den-coding with an example.

In the example below the first row shows the integers j from 1 to 9, the second row the value of $\sigma(j)$, the third row the value of the cyclic interval $]\!]\sigma(j), j]\!]$, the fourth row the value of x_j (which is the number of integers i such that $1 \le i \le j-1$ and $\sigma(i) \in \,]\!]\sigma(j), j]\!]$).

j	1	2	3	4	5	6	7	8	9
$\sigma(j)$	7	1	5	4	9	2	6	3	8
$]\!]\sigma(j), j]\!]$	$\{1,8,9\}$	$\{2\}$	$\{6,7,8,9,1,2,3\}$	$\emptyset$	$\{1,2,3,4,5\}$	$\{3,4,5,6\}$	$\{7\}$	$\{4,5,6,7,8\}$	$\{9\}$
x_j	0	0	2	0	3	2	1	4	1

The den-coding of σ is then $x = (0,0,2,0,3,2,1,4,1)$. In particular, $\operatorname{den}\sigma = 13$.

To recover σ from x, first define $\sigma(n) := n - x_n$. Suppose that $\sigma(j+1)$, $\ldots$, $\sigma(n)$ have been determined from $x_{j+1}, \ldots, x_n$. Write the sequence

$$j, (j-1), \ldots, 2, 1, n, (n-1), \ldots, (j+1).$$

From that list remove all the elements equal to $\sigma(l)$ for a given $l \ge j+1$. Then $\sigma(j)$ is the (x_j+1)-st letter in the sequence when reading it from left to right.

Three bijections ϕ^{inv}, ϕ^{maj}, ϕ^{den} of $\mathfrak{S}_n$ onto SE_n have so been constructed with the following properties:

$$\operatorname{inv}\sigma = \operatorname{tot}\phi^{\mathrm{inv}}(\sigma), \quad \operatorname{maj}\sigma = \operatorname{tot}\phi^{\mathrm{maj}}(\sigma), \quad \operatorname{den}\sigma = \operatorname{tot}\phi^{\mathrm{den}}(\sigma).$$

By taking the composition products of ϕ^{inv}, ϕ^{maj} and ϕ^{den}, together with their inverse bijections, we can explicitly construct one-to-one correspondences of $\mathfrak{S}_n$ onto itself, that map "inv" onto "maj", "inv" onto "den" and "maj" onto "den."

3. The algebra of the q-binomial coefficients

Consider the product $c = a \cdot b$ of two formal series in the variable u written in the form

$$a = \sum_{i \geq 0} \frac{u^i}{i!}\, a(i) \quad \text{and} \quad b = \sum_{j \geq 0} \frac{u^j}{j!}\, b(j).$$

If we want to express the product c in the form

$$c = \sum_{n \geq 0} \frac{u^n}{n!}\, c(n),$$

we are led to the identity

$$c(n) = \sum_{\substack{i \geq 0,\, j \geq 0 \\ i+j=n}} \binom{n}{i} a(i)\, b(j) \quad (n \geq 0),$$

where $\displaystyle\binom{n}{i}$ is the binomial coefficient $\displaystyle\frac{n!}{i!\,(n-i)!}$.

Now if we replace the factorials $i!$, $j!$, $n!$ occurring in the denominators by their q-analogs $(q;q)_i$, $(q;q)_j$, $(q;q)_n$, as they were defined in (0.1) and if the coefficients $a(i)$, $b(j)$, $c(n)$ are replaced by formal series $a(i,q)$, $b(j,q)$, $c(n,q)$ in the variable q, we obtain the identity

$$c(n,q) = \sum_{\substack{i \geq 0,\, j \geq 0 \\ i+j=n}} \begin{bmatrix} n \\ i \end{bmatrix} a(i,q)\, b(j,q),$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} := \frac{(q;q)_n}{(q;q)_i\,(q;q)_{n-i}} \quad (0 \leq i \leq n). \tag{3.1}$$

We can also write :

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q^{i+1};q)_{n-i}}{(q;q)_{n-i}} = \frac{(q^{n-i+1};q)_i}{(q;q)_i}. \tag{3.2}$$

The expression $\begin{bmatrix} n \\ i \end{bmatrix}$ is called *q-binomial coefficient* or *Gaussian polynomial.* It is a remarkable fact that this coefficient is a polynomial in q, with nonnegative integral coefficient. This can be derived in an algebraic manner.

In definition (3.1) make the convention that the q-binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$ is zero when condition $0 \le i \le n$ does *not* hold. First, we have

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1; \tag{3.3}$$

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n \\ n-i \end{bmatrix}. \tag{3.4}$$

We also have two Pascal Triangle formulas

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}; \tag{3.5}$$

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i \end{bmatrix}; \tag{3.6}$$

that can be derived by mimicking the traditional calculus of the binomial coefficients:

$$\begin{aligned}
\begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n-1 \\ i \end{bmatrix} &= \frac{(q;q)_{n-1}}{(q;q)_i\,(q;q)_{n-i}}((1-q^n)-(1-q^{n-i})) \\
&= \frac{(q;q)_{n-1}\,q^{n-i}(1-q^i)}{(q;q)_i\,(q;q)_{n-i}} \\
&= \frac{q^{n-i}(q;q)_{n-1}}{(q;q)_{i-1}\,(q;q)_{n-i}} = q^{n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}.
\end{aligned}$$

In the same manner,

$$\begin{aligned}
\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n \\ n-i \end{bmatrix} &= \begin{bmatrix} n-1 \\ n-i \end{bmatrix} + q^{n-(n-i)} \begin{bmatrix} n-1 \\ n-i-1 \end{bmatrix} \\
&= \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i \end{bmatrix}.
\end{aligned}$$

Finally, the limit $\lim\limits_{q\to 1} \begin{bmatrix} n \\ i \end{bmatrix} = \begin{pmatrix} n \\ i \end{pmatrix}$ is straightforward.

Relation (3.3) and one of the relations (3.5), (3.6) show that the q-binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$ is a *polynomial* in q, with nonnegative coefficients, of degree $i(n-i)$. The first values of the q-binomial coefficients $\begin{bmatrix} n \\ i \end{bmatrix}$ are shown in Table 3.1.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$; $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1$; $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1+q$; $\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 1$;
$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1+q+q^2$; $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1$; $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1+q+q^2+q^3$;
$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1+q+2q^2+q^3+q^4$; $\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 1$; $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 1+q+q^2+q^3+q^4$;
$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 1+q+2q^2+2q^3+2q^4+q^5+q^6$.

Table 3.1

In (1.16) and (1.17) we have obtained two expressions for each one of the two q-exponentials $e_q(u)$ and $E_q(u)$, first as infinite products, then as q-series :

$$e_q(u) = \frac{1}{(u;q)_\infty} = \sum_{n\ge 0} \frac{u^n}{(q;q)_n}; \tag{3.7}$$

$$E_q(u) = (-u;q)_\infty = \sum_{n\ge 0} q^{n(n-1)/2} \frac{u^n}{(q;q)_n}. \tag{3.8}$$

By means of the q-binomial coefficients we can obtain the expansions of the *finite* products $1/(u;q)_N$ and $(-u;q)_N$, where N is a nonnegative integer. Those two products can be regarded as *finite versions* of the two q-exponentials $e_q(u)$ and $E_q(u)$.

Proposition 3.1. *We have the identities:*

$$\frac{1}{(u;q)_N} = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix} u^n; \tag{3.9}$$

$$(-u;q)_N = \sum_{0\le n\le N} \begin{bmatrix} N \\ n \end{bmatrix} q^{n(n-1)/2}\, u^n. \tag{3.10}$$

Proof. To derive (3.9) go back to (1.12). We have:

$$\begin{aligned} \frac{1}{(u;q)_N} &= \frac{(q^N u;q)_\infty}{(u;q)_\infty} = \sum_{n\ge 0} (q^N;q)_n \frac{u^n}{(q;q)_n} \\ &= 1 + \sum_{n\ge 1} \frac{(q;q)_{N+n-1}}{(q;q)_{N-1}\,(q;q)_n} u^n = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix} u^n. \end{aligned}$$

To derive (3.10) make use of the q-binomial theorem:

$$\begin{aligned} (-u;q)_N &= \frac{(-u;q)_\infty}{(-uq^N;q)_\infty} = \frac{(q^{-N}\,(-uq^N);q)_\infty}{(-uq^N;q)_\infty} \\ &= \sum_{n\ge 0} (q^{-N};q)_n \frac{(-uq^N)^n}{(q;q)_n} = \sum_{0\le n\le N} (q^{-N};q)_n \frac{(-uq^N)^n}{(q;q)_n}. \end{aligned}$$

The summation is finite, since $(q^{-N};q)_n$ is zero for every $n \ge N+1$. Now use (1.10) for $\alpha = q^{-N}$, that is,

$$(q^{N+1-n};q)_n = (q^{-N};q)_n\,(-q^N)^n\, q^{-n(n-1)/2}.$$

We get

$$(-u;q)_N = \sum_{0\le n\le N} (q^{N+1-n};q)_n\, q^{n(n-1)/2} \frac{u^n}{(q;q)_n}$$
$$= \sum_{0\le n\le N} \begin{bmatrix} N \\ n \end{bmatrix} q^{n(n-1)/2}\, u^n,$$

because of (3.2) with the substitutions $n \leftarrow N$ and $i \leftarrow n$. ∎

We can also prove (3.9) and (3.10) by induction on N, using the Pascal Triangle formulas (3.5) and (3.6).

4. q-Binomial Combinatorial Structures

For an easy handling of the q-binomial coefficients in Combinatorics it is essential to be familiar with the basic combinatorial structures that admit those coefficients as generating polynomials. For each pair of integers (N,n) we give the description of several pairs (A,f), where A is a finite set of cardinality $\binom{N}{n}$ and f is a statistic defined on A having the property that

$$\sum_{a\in A} q^{f(a)} = \begin{bmatrix} N+n \\ n \end{bmatrix}. \tag{4.1}$$

Four of those structures are introduced below: the *partitions of integers*, the *nondecreasing sequences of integers*, the *binary words*, the *ordered partitions in two blocks*. Each of these structures has its own geometry and its specific underlying statistic.

4.1. *Partitions of integers.* Formula (3.9) reads

$$\frac{1}{(1-u)(1-uq)\cdots(1-uq^N)} = \sum_{n\ge 0} \begin{bmatrix} N+n \\ n \end{bmatrix} u^n. \tag{4.2}$$

The left-hand side of (4.2) can be expressed as a formal series in the two variables q and u

$$\sum_{n\ge 0} u^n \sum_{m\ge 0} p(m,n,N)\, q^m,$$

where $p(m,n,N)$ is equal to the number of sequences $(m_0, m_1, m_2, \ldots, m_N)$ of nonnegative integers such that

$$m_0 + m_1 + \cdots + m_N = n \quad \text{and} \quad 1.m_1 + 2.m_2 + \cdots + N.m_N = m, \tag{4.3}$$

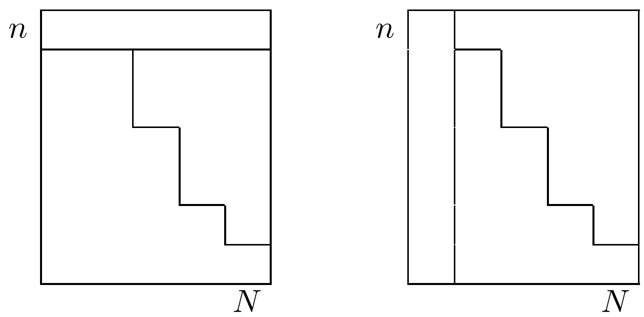
or, in an equivalent manner, to the number of partitions $1^{m_1}2^{m_2}\dots N^{m_N}$ of the integer m whose number of parts is *at most equal* to n (because of the occurrence of the coefficient m_0). Hence,

$p(m,n,N)$ is equal to the number of partitions of m in at most n parts, all the parts being at most equal to N.

Notice that $p(m,n,N)=0$ for $m\ge nN+1$. Let $\mathcal{P}(n,N)$ be the set of partitions in at most n parts, all of them being at most equal to N (their Ferrers diagrams are then contained in a rectangle of basis N and height n). Let $\|\pi\|$ denote the *weight* of a partition $\pi\in\mathcal{P}(n,N)$, that is, $\|\pi\|=m$ if π is a partition of m. Accordingly,

$$\begin{bmatrix}N+n\\ n\end{bmatrix}=\sum_{0\le m\le nN}p(m,n,N)\,q^m=\sum_{\pi\in\mathcal{P}(n,N)}q^{\|\pi\|}. \tag{4.4}$$

The first q-binomial model is then the pair $(\mathcal{P}(n,N),\|\cdot\|)$.



First kind Second kind

Fig. 4.1

There is another way to derive identity (4.4), by using induction on $N+n$, the formula being trivial for $N+n=1$. We use (3.6), that can be rewritten as

$$\begin{bmatrix}N+n\\ n\end{bmatrix}=\begin{bmatrix}N+(n-1)\\ n-1\end{bmatrix}+q^n\begin{bmatrix}(N-1)+n\\ n\end{bmatrix}.$$

But the factor $\left[{N+(n-1)\atop n-1}\right]$ is the generating polynomial for the partitions in at most $(n-1)$ parts, all of them being at most equal to N. Call them *of the first kind*. The factor $\left[{(N-1)+n\atop n}\right]$ is the generating polynomial for the partitions π whose Ferrers diagram is contained in the rectangle $(N-1)\times n$. Add a column of height n to the left of the Ferrers diagram of each π. We obtain the Ferrers diagram of a partition π' having n parts exactly, all of them being at most equal to N. Say that those partitions π' are *of* *the second kind.* Their generating polynomial is equal to $q^n \left[{(N-1)+n \atop n}\right]$ by induction. But, every partition in at most n parts, all at most equal to N, is either of the first kind, or of the second kind (see Fig. 4.1). $\square$

4.2. *Nondecreasing sequences of integers.* This model will appear to be extremely convenient, for a great many of combinatorial objects can be easily coded by sequences of integers. For each pair (N, n) of integers let $\mathrm{NDS}(N, n)$ (resp. $\mathrm{IS}(N, n)$) be the set of the nondecreasing (resp. increasing) sequences of nonnegative integers $b = (b_1, b_2, \ldots, b_N)$ such that $0 \le b_1 \le b_2 \le \cdots \le b_N \le n$ (resp. $0 \le b_1 < b_2 < \cdots < b_N \le n$). As above, let $\operatorname{tot} b := b_1 + b_2 + \cdots + b_N$.

Proposition 4.1. *For each pair of integers (N, n) we have:*

$$\begin{bmatrix} N+n \\ n \end{bmatrix} = \sum_{b \in \mathrm{NDS}(N,n)} q^{\operatorname{tot} b} = \sum_{b \in \mathrm{NDS}(n,N)} q^{\operatorname{tot} b}; \tag{4.5}$$

$$q^{N(N-1)/2} \begin{bmatrix} n+1 \\ N \end{bmatrix} = \sum_{B \in \mathrm{IS}(N,n)} q^{\operatorname{tot} B}. \tag{4.6}$$

Proof. Notice the symmetry of formula (4.5) in N and n. To derive (4.5) we construct a bijection $\pi \mapsto b$ of $\mathcal{P}(n, N)$ onto $\mathrm{NDS}(n, N)$ that has the property that $\|\pi\| = \operatorname{tot} b$. Let $\pi = (\pi_1 \ge \pi_2 \ge \cdots \ge \pi_n \ge 0)$ be a partition in at most n parts, all of them at most equal to N. The bijection is simply given by

$$\pi \mapsto (\pi_n, \ldots, \pi_2, \pi_1) = b.$$

Suppose $n \ge N - 1$. To prove (4.6) we use the traditional bijection $B \mapsto b$ that maps each increasing sequence $B \in \mathrm{IS}(N, n)$ onto a nondecreasing sequence $b \in \mathrm{NDS}(N, n - N + 1)$, defined by

$$(0 \le B_1 < B_2 < \cdots < B_N \le n) \mapsto (0 \le b_1 \le b_2 \le \cdots \le b_N \le n - N + 1),$$

where $b_1 := B_1$, $b_2 := B_2 - 1$, $b_3 := B_3 - 2$, $\ldots$, $b_N := B_N - N + 1$. It follows that

$$\operatorname{tot} B = \frac{N(N-1)}{2} + \operatorname{tot} b$$

and

$$\begin{aligned} \sum_{B \in \mathrm{IS}(N,n)} q^{\operatorname{tot} B} &= q^{N(N-1)/2} \sum_{b \in \mathrm{NDS}(N, n-N+1)} q^{\operatorname{tot} b} \\ &= q^{N(N-1)/2} \begin{bmatrix} N + (n - N + 1) \\ N \end{bmatrix} = q^{N(N-1)/2} \begin{bmatrix} n+1 \\ N \end{bmatrix}. \quad \square \end{aligned}$$

The second q-binomial model is $(\mathrm{NDS}(N, n), \operatorname{tot})$.

4.3. *Binary words.* Let $\mathrm{BW}(N,n)$ denote the set of all words of length $(N+n)$ having exactly N letters equal to 1 and n letters equal to 0. If $x = x_1x_2\dots x_{N+n}$ is such a word, the *inversion number*, $\operatorname{inv} x$, of the word x is defined as the number of *subwords* (not simply factors) 10 of the word x.

Example. We can also write the number of 1's that appear to the left of each letter equal to 0, as shown below for the word x.

$$\begin{array}{rl} x = & 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1 \\ & \cdot\ 1\ 1\ \cdot\ 2\ \cdot\ 3\ 3\ \cdot \end{array}$$

Hence $\operatorname{inv} x = 1+1+2+3+3 = 10$.

Proposition 4.2. *For each pair of integers (N,n) we have:*

$$\begin{bmatrix} N+n \\ n \end{bmatrix} = \sum_{x\in \mathrm{BW}(N,n)} q^{\operatorname{inv} x}. \tag{4.17}$$

Proof. Again, we construct a bijection $\pi \mapsto x$ of $\mathcal{P}(N,n)$ onto $\mathrm{BW}(N,n)$, such that $\|\pi\| = \operatorname{inv} x$. Every partition $\pi \in \mathcal{P}(N,n)$ can be, in its multiplicative version, described as a monomial $i_1^{n_1} i_2^{n_2} \dots i_r^{n_r}$, where $0 \le i_1 < i_2 < \cdots < i_r \le N$, $n_1 \ge 1$, $n_2 \ge 1$, … , $n_r \ge 1$ and $n_1+n_2+\cdots+n_r = n$. If the number of parts $l(\pi)$ of π is strictly less than n, let $n_1 := n - l(\pi)$ and $i_1 := 0$. Then the partition π has its parts only equal to i_2, … , i_r, repeated n_2, … , n_r times, respectively. If $l(\pi) = n$, then $1 \le i_1$ and π has its parts equal to i_1, … , i_r, repeated n_1, … , n_r times, respectively.

With the partition π associate the word x

$$x := 1^{i_1}0^{n_1}1^{i_2-i_1}0^{n_2}1^{i_3-i_2}0^{n_3}\dots 0^{n_{r-1}}1^{i_r-i_{r-1}}0^{n_r}1^{N-i_r}.$$

The word x has $i_1 + (i_2-i_1) + (i_3-i_2) + \cdots + (i_r - i_{r-1}) + (N - i_r) = N$ letters equal to 1 and $n_1+n_2+n_3+\cdots+n_{r-1}+n_r = n$ letters equal to 0. Moreover, $\|\pi\| = i_1.n_1 + i_2.n_2 + \cdots + i_r.n_r = i_1.n_1 + (i_1 + (i_2-i_1)).n_2 + \cdots + (i_1 + (i_2 - i_1) + \cdots + (i_r - i_{r-1})).n_r = \operatorname{inv} x$. Finally, the mapping $\pi \mapsto x$ is obviously injective, and then bijective. $\square$

Remark. There is a geometric manner to see the bijection $\pi \mapsto x$ described in the previous proof. Put the Ferrers diagram of the partition π inside a triangle of basis N and height n. The rim of the Ferrers diagram is a polygonal line, made of vertical and horizontal steps of length 1, starting from the point whose coordinates are $(0,n)$ down to the point with coordinates $(N,0)$. The rim has exactly n vertical steps and N horizontal

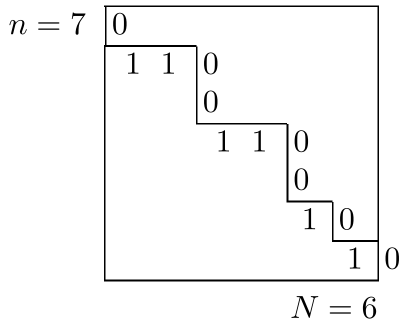


Fig. 4.2

steps. Now read the rim of π from top to bottom and from left to right and give label 0 (resp. label 1) to each vertical (resp. horizontal) step. The word x thereby obtained is the binary word described in the previous bijection (see Fig. 4.2).

Example. Consider the partition $\pi = (6,5,4,4,2,2)$ belonging to $\mathcal{P}(6,7)$. In its multiplicative version it can be expressed as the monomial $0^1 2^2 4^2 5^1 6^1$. The word x corresponding to that monomial is the word $x = 1^0 0^1 1^{2-0} 0^2 1^{4-2} 0^2 1^{5-4} 0^1 1^{6-5} 0^1 1^{6-6} = 0,1,1,0,0,1,1,0,0,1,0,1,0$, which is the word we can read on the rim of its Ferrers diagram using the previous labelling.

The third q-binomial model is then $(\mathrm{BW}(N,n), \mathrm{inv})$.

4.4. *Ordered Partitions into two blocks.* The word "partition" used in this subsection will refer to (set) partitions. The (set) partition of a finite set S is a collection of subsets (called *blocks*) of S, two by two disjoint, whose union is S. By *ordered partition* of a set it is meant a (set) partition into blocks, together with a linear ordering of those blocks. For convenience, we may assume that some of those blocks are empty.

Now consider the set $[N+n]$ of the $(N+n)$ integers $1,2,\dots,N+n$. If (A,B) is an ordered partition of $[N+n]$ into two blocks, $\gamma(A)$ (resp. $\gamma(B)$) will designate the *increasing* word whose letters are the elements of A (resp. of B) written in increasing order. There is no inversion of letters in each of the words $\gamma(A)$, $\gamma(B)$, so that the number of inversions $\mathrm{inv}(\gamma(A)\gamma(B))$ in the juxtaposition product $\gamma(A)\gamma(B)$ is equal to the number of pairs (a,b) such that $a \in A$, $b \in B$ and $a > b$.

Proposition 4.3. *For each pair of integers (N,n) we have*

$$\begin{bmatrix} N+n \\ n \end{bmatrix} = \sum_{(A,B)} q^{\mathrm{inv}(\gamma(A)\gamma(B))},$$

where the sum is over the set $\mathrm{OP}(N,n)$ of all the ordered partitions (A,B) of the set $[N+n]$ into two blocks such that $|A| = N$ and $|B| = n$.

Proof. It follows from Proposition 4.2 that the q-binomial coefficient $\begin{bmatrix}N+n\\ n\end{bmatrix}$ is the generating function for the binary words $x = x_1x_2\dots x_{N+n}$ having N letters equal to 1 and n letters equal to 0, by the inversion number. With such a binary word x associate the ordered partition (A, B) of $[N + n]$ defined by: $i \in A$ or $i \in B$ depending on whether $x_i = 0$ or $x_i = 1$. The inversion $x_j = 1$, $x_{j'} = 0$, $j < j'$ in the word x will then correspond to the inversion $j' > j$ between the element $j' \in A$ and the element $j \in B$. □

The fourth q-binomial model is then $(\mathrm{OP}(N, n), \mathrm{inv})$.

5. The q-multinomial coefficients

They form a natural q-extension of the multinomial coefficients and can be introduced as follows. For each integer $r \geq 1$ and each sequence of nonnegative integers $(m_1, m_2, \dots, m_r)$ let

$$\begin{bmatrix}m_1 + m_2 + \cdots + m_r\\ m_1, m_2, \dots, m_r\end{bmatrix}_q := \frac{(q;q)_{m_1+m_2+\cdots+m_r}}{(q;q)_{m_1}(q;q)_{m_2}\dots(q;q)_{m_r}}. \tag{5.1}$$

If there is no ambiguity the subscript q is suppressed. In case $r = 2$ we recover the expression of the Gaussian polynomial studied in the previous section. The fact that the q-multinomial coefficient is a polynomial in q with positive integral coefficients follows from the combinatorial interpretations given in the sequel

When q tends to 1, the q-multinomial coefficient tends to the multinomial coefficient $\binom{m_1+m_2+\cdots+m_r}{m_1,m_2,\dots,m_r}$, as is readily verified. We can then expect that the q-multinomial coefficient is the generating polynomial for a set of cardinality $\binom{m_1+m_2+\cdots+m_r}{m_1,m_2,\dots,m_r}$ by a certain statistic.

The two combinatorial interpretations of the q-binomial coefficient in terms of classes of partitions and nondecreasing sequences are difficult to be extended to the multinomial case. However, when going from the *binary* words, studied in the previous subsection, to the words whose letters belong to an alphabet of cardinality r $(r \geq 2)$ and when the statistic "inv" is extended to those words, the q-multinomial coefficient can easily be interpreted in a combinatorial way. In the sequel the word "rearrangement" of a word w, with or without repeated letters, means any word derived from w by permuting its letters in some order.

For $r \geq 1$ and for each sequence $\mathbf{m} = (m_1, m_2, \dots, m_r)$ of nonnegative integers let $R(\mathbf{m})$ denote the class of all the words of length $m = m_1 + m_2 + \cdots + m_r$ which are *rearrangements* of the nondecreasing word $1^{m_1}2^{m_2}\dots r^{m_r}$. The number of such rearrangements is equal to the multinomial coefficient $\binom{m_1+m_2+\cdots+m_r}{m_1,m_2,\dots,m_r}$.

Let $w = x_1x_2\dots x_m$ be a word belonging to the class $R(\mathbf{m})$. The *number of inversions*, $\operatorname{inv} w$, of w is defined to be the number of pairs (i, j) such that $1 \le i < j \le m$ and $x_i > x_j$. For each word $w = x_1x_2\dots x_m$ it is convenient, for each $j = 1, \dots, m$, to determine the number z_j of letters x_i lying *to the left of* x_j such that $x_i > x_j$. Then $\operatorname{inv} w = z_1 + \cdots + z_m$.

In the following example the number of inversions of w is determined from the sequence of the z_j's:

$$\begin{aligned} w &= \ 3\ 1\ 3\ 4\ 1\ 2\ 5\ 4\ 3 \\ z &= \ 0\ 1\ 0\ 0\ 3\ 3\ 0\ 1\ 3 \end{aligned}$$

so that $\operatorname{inv} w = \operatorname{tot} z = 1 + 3 + 3 + 1 + 3 = 11$.

Theorem 5.1. *The q-multinomial coefficient* $\begin{bmatrix} m_1+m_2+\cdots+m_r \\ m_1,m_2,\dots,m_r \end{bmatrix}$ *is the generating polynomial for the set* $R(\mathbf{m})$ *by the number of inversions. In other words,*

$$\begin{bmatrix} m_1 + m_2 + \cdots + m_r \\ m_1, m_2, \dots, m_r \end{bmatrix} = \sum_{w \in R(\mathbf{m})} q^{\operatorname{inv} w}. \tag{5.2}$$

Proof. Relation (5.2) is banal for $r = 1$ and holds for $r = 2$ by Proposition 4.2. Consider the factorization

$$\begin{aligned} \begin{bmatrix} m_1 + m_2 + \cdots + m_{r+1} \\ m_1, m_2, \dots, m_{r+1} \end{bmatrix} &= \frac{(q;q)_{m_1+m_2+\cdots+m_{r+1}}}{(q;q)_{m_1}(q;q)_{m_2}\dots(q;q)_{m_{r+1}}} \\ &= \frac{(q;q)_{m_1+m_2+\cdots+m_{r+1}}}{(q;q)_{m_1+m_2+\cdots+m_r}(q;q)_{m_{r+1}}} \frac{(q;q)_{m_1+m_2+\cdots+m_r}}{(q;q)_{m_1}(q;q)_{m_2}\dots(q;q)_{m_r}} \\ &= \begin{bmatrix} m_1 + m_2 + \cdots + m_{r+1} \\ m_1 + m_2 + \cdots + m_r, m_{r+1} \end{bmatrix} \begin{bmatrix} m_1 + m_2 + \cdots + m_r \\ m_1, m_2, \dots, m_r \end{bmatrix} \end{aligned}$$

and take a word $w = x_1x_2\dots x_{m'}$ from the set $R(m_1, m_2, \dots, m_{r+1})$, so that its length is $m' = m_1+m_2+\cdots+m_{r+1}$. The inversions $x_i > x_j$ $(i < j)$ of w fall into two classes: (i) the inversions of the form $x_i = r+1 > s = x_j$, where s is one of the integers 1, 2, ... , r; (ii) the inversions of the form $x_i = s > t = x_j$, where $r \ge s > t \ge 1$.

Let w_1 denote the word of length m' derived from w by replacing all the letters less than or equal to r by 1 and all the letters equal to $(r+1)$ by 2. Likewise, let w_2 denote the *subword* of length $m = m_1 + m_2 + \cdots + m_r$ obtained from w by deleting *all* the letters equal to $(r + 1)$.

The map $w \mapsto (w_1, w_2)$ is obviously a bijection of $R(m_1, m_2, \dots, m_{r+1})$ onto $R(m_1 + m_2 + \cdots + m_r, m_{r+1}) \times R(m_1, m_2, \dots, m_r)$. Moreover,

$$\operatorname{inv} w = \operatorname{inv} w_1 + \operatorname{inv} w_2. \tag{5.3}$$

But by induction on r we have:

$$\sum_{w_1} q^{\operatorname{inv} w_1} = \begin{bmatrix} m_1+m_2+\cdots+m_{r+1} \\ m_1+m_2+\cdots+m_r, m_{r+1} \end{bmatrix};$$
$$\sum_{w_2} q^{\operatorname{inv} w_2} = \begin{bmatrix} m_1+m_2+\cdots+m_r \\ m_1, m_2, \ldots, m_r \end{bmatrix}.$$

The identity (5.2) is then a consequence of those identities and of (5.3). ∎

6. The MacMahon Verfahren

The German word *Verfahren* means "procedure," "way of doing," ... This term applies to the rearrangement method of sequences of numbers, imagined by MacMahon when he was dealing with q-series in a combinatorial context.

Obviously, the fraction $1/(q;q)_m$ is the generating function for the partitions of integers in at most m parts. If we write such a partition in its classical form $\pi = (\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m \geq 0)$, then the sequence $b = (b_1, \ldots, b_{m-1}, b_m) := (\pi_m, \ldots, \pi_2, \pi_1)$ is a nondecreasing sequence of m nonnegative integers. Extending our previous notation we write $b \in \mathrm{NDS}(m)$, so that

$$\frac{1}{(q;q)_m} = \sum_{b \in \mathrm{NDS}(m)} q^{\operatorname{tot} b}, \tag{6.1}$$

where $\operatorname{tot} b = b_1 + \cdots + b_m$.

In section 2 we have introduced the Mahonian statistic on a set of cardinality $n!$ and more essentially on the symmetric group $\mathfrak{S}_n$. We now extend the definition of that statistic to arbitrary rearrangement classes. A statistic "stat" is said to be *Mahonian*, if for every class $R(\mathbf{m})$ the following identity holds:

$$\frac{1}{(q;q)_{m_1+\cdots+m_r}} \sum_{w \in R(\mathbf{m})} q^{\operatorname{stat} w} = \frac{1}{(q;q)_{m_1} \cdots (q;q)_{m_r}}. \tag{6.2}$$

Theorem 5.1 says nothing but that the inversion number "inv" is a Mahonian statistic. By using (6.1), identity (6.2) can be rewritten in the form

$$\sum_{b \in \mathrm{NDS}(m),\, w \in R(\mathbf{m})} q^{\operatorname{tot} b + \operatorname{stat} w} = \sum_{b^{(1)}, \ldots, b^{(r)}} q^{\operatorname{tot} b^{(1)} + \cdots + \operatorname{tot} b^{(r)}}, \tag{6.3}$$

where $b^{(1)}\in \mathrm{NDS}(m_1),\ldots ,b^{(r)}\in \mathrm{NDS}(m_r)$. Hence, we can also say that a statistic "stat" is *Mahonian*, if to every pair $(b,w)\in \mathrm{NDS}(m)\times R(\mathbf{m})$ there corresponds a unique sequence $(b^{(1)},\ldots,b^{(r)})\in \mathrm{NDS}(m_1)\times\cdots\times\mathrm{NDS}(m_r)$ such that

$$\operatorname{tot} b+\operatorname{stat} w=\operatorname{tot} b^{(1)}+\cdots+\operatorname{tot} b^{(r)}. \tag{6.4}$$

The purpose of this section is to use that definition for making the Major Index appear as a Mahonian statistic, not only for permutations, but for rearrangements of arbitrary words.

Each sequence $(b^{(1)},\ldots,b^{(r)})$ can be mapped, in a bijective way, onto a two-row matrix

$$\begin{pmatrix} b^{(1)}_{m_1} & \ldots & b^{(1)}_1 & b^{(2)}_{m_2} & \ldots & b^{(2)}_1 & \ldots & b^{(r)}_{m_r} & \ldots & b^{(r)}_1\\ 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots & r & \ldots & r\end{pmatrix}, \tag{6.5}$$

where, on the first row, the *nonincreasing rearrangements* $b^{(1)}_{m_1}\ldots b^{(1)}_1$, $b^{(2)}_{m_2}\ldots b^{(2)}_1$, ... , $b^{(r)}_{m_r}\ldots b^{(r)}_1$ of the sequences $b^{(1)}$, $b^{(2)}$, ... , $b^{(r)}$ have been in that order.

The idea of the MacMahon *Verfahren* is to rearrange the *columns* of the latter matrix in such a way that the elements on the top row will be in *nonincreasing* order (when read from left to right), this being made in a one-to-one manner. The bottom row will then go from $1^{m_1}2^{m_2}\ldots r^{m_r}$ to a rearrangement of that word. To realize the rearrangement of the columns we make use of the following *commutation rule*:

(6.6) *two columns* $\begin{pmatrix} c\\ d\end{pmatrix}$ *and* $\begin{pmatrix} c'\\ d'\end{pmatrix}$ *commute if and only if* $c\neq c'$.

The commutation rule being given, to each matrix of type (6.5) there corresponds, *in a bijective manner*, a matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_m\\ x_1 & x_2 & \ldots & x_m\end{pmatrix}, \tag{6.7}$$

whose top row is *nonincreasing* and if $y_k=y_{k+1}$, then $x_k\le x_{k+1}$, or, in an equivalent way,

$$x_k>x_{k+1}\Longrightarrow y_k>y_{k+1}. \tag{6.8}$$

In other words, if there is a *descent* on the bottom row, there is also a descent on the top row, the converse being not necessarily true.

For example, let $r = 3$, $m_1 = 6$, $m_2 = 2$, $m_3 = 4$, $b^{(1)} = (0,0,1,1,5,6)$, $b^{(2)} = (1,3)$, $b^{(3)} = (1,1,4,5)$, so that $m = m_1 + m_2 + m_3 = 12$. The matrix of type (6.5) reads:

$$\begin{pmatrix} 6 & 5 & 1 & 1 & 0 & 0 & 3 & 1 & 5 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

Using the commutation rule (6.6) the matrix is transformed into a matrix of type (6.8):

$$\begin{pmatrix} 6 & 5 & 5 & 4 & 3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & \mathbf{3} & \mathbf{2} & 1 & 1 & 2 & 3 & \mathbf{3} & 1 & 1 \end{pmatrix}.$$

The coefficients on the bottom row that are greater than their successors are written in bold-face. We see that the corresponding coefficients on the top row are greater than their successors (property (6.8)).

Go back to the general case and let $v := y_1 y_2 \ldots y_m$ denote the word appearing on the top row of the matrix (6.7). It is the *unique nonincreasing* rearrangement of the juxtaposition product $b^{(1)} \ldots b^{(r)}$. Next, let $w := x_1 x_2 \ldots x_m$ be the word appearing on the bottom row of (6.7). It is a well-defined word belonging to $R(\mathbf{m})$.

For $k = 1, 2, \ldots, m$ let z_k be the number of descents in the right factor $x_k x_{k+1} \ldots x_m$ of w, that is, the number of subscripts j such that $k \le j \le m-1$ and $x_j > x_{j+1}$; next let $b_k := y_k - z_k$ $(1 \le k \le m)$. If $x_k > x_{k+1}$, then $z_k = z_{k+1} + 1$ by definition of z_k and also $y_k \ge y_{k+1} + 1$ by (6.8). It follows that $b_k = y_k - z_k \ge y_{k+1} + 1 - (z_{k+1} + 1) = b_{k+1}$. However, if $x_k \le x_{k+1}$, we always have $y_k \ge y_{k+1}$, since v is nonincreasing and also $z_k = z_{k+1}$. Hence $b_k = y_k - z_k \ge y_{k+1} - z_{k+1} = b_{k+1}$.

We conclude that the sequence b defined by $b := (b_m, \ldots, b_2, b_1)$ satisfies the relations $0 \le b_m \le \cdots \le b_2 \le b_1$, so that $b \in \mathrm{NDS}(m)$. Finally, if $z(w)$ designates the sequence $(z_1, z_2, \ldots, z_m)$, we have:

$$\begin{aligned} \operatorname{tot} b^{(1)} + \cdots + \operatorname{tot} b^{(r)} &= y_1 + y_2 + \cdots + y_m \\ &= (b_1 + z_1) + (b_2 + z_2) + \cdots + (b_m + z_m) \\ &= \operatorname{tot} b + \operatorname{tot} z(w). \end{aligned} \tag{6.9}$$

Comparing the last identity with (6.4) we see that $\operatorname{tot} z(w)$ is a new *Mahonian statistic*, if it can be verified that the mapping $(b^{(1)}, \ldots, b^{(r)}) \mapsto (b, w)$ is bijective. But the construction that has just been made is perfectly reversible: starting with a word $w \in R(\mathbf{m})$ and a sequence $b = (b_m, \ldots, b_2, b_1) \in \mathrm{NDS}(m)$, we first determine the sequence $z(w) = (z_1, \ldots, z_m)$. We know that the word $v = y_1 \ldots y_{m-1} y_m$ defined by $y_i := b_i + z_i$ $(1 \le i \le m)$ is nonincreasing. We next form the two-row matrix $\binom{v}{w}$ and by applying the commutation rule (6.6) we define the sequences $b^{(1)}, \ldots , b^{(r)}$ by (6.5). Relation (6.9) obviously holds.

Let us take again the previous example. We had obtained:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 6\ 5\ 5\ 4\ 3\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\ 1\ 1\ 3\ \mathbf{3}\ \mathbf{2}\ 1\ 1\ 2\ 3\ \mathbf{3}\ 1\ 1 \end{pmatrix}.$$

We now get

$$\begin{aligned} v &= 6\ 5\ 5\ 4\ 3\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\ w &= 1\ 1\ 3\ 3\ 2\ 1\ 1\ 2\ 3\ 3\ 1\ 1 \\ z(w) &= 3\ 3\ 3\ 3\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\ &\ 3\ 2\ 2\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 = b \text{ (read from right to left)} \end{aligned}$$

and can verify that

$$\begin{aligned} \operatorname{tot} b^{(1)} + \cdots + \operatorname{tot} b^{(r)} &= (6+5+1+1) + (3+1) + (5+4+1+1) = 28 \\ &= \operatorname{tot} b + \operatorname{tot} z(w) = (3+2+2+1+1) + (12+2+5) = 28. \end{aligned}$$

Going back to the general case the problem is to characterize the new Mahonian statistic "$\operatorname{tot} z(w)$" in a more direct way. It is, indeed, the *Major Index*, "maj", already introduced in section 2 in the case of permutations. Its definition can be extended to the case of arbitrary words.

Definition. Let $w = x_1x_2\ldots x_m$ be a word whose letter are taken from the alphabet $\{1, 2, \ldots , r\}$. The *Major Index*, "$\operatorname{maj} w$", of the word w is defined by

$$\operatorname{maj} w := \sum_{1\le i\le m-1} i\,\chi(x_i > x_{i+1}). \tag{6.10}$$

Thus, for calculating the Major Index of a word, we determine its descents and their positions. The Major Index is the sum of the positions of its descents.

Proposition 6.2. *For every word w we have:* $\operatorname{maj} w = \operatorname{tot} z(w)$.

Proof. If $w = x_1x_2\ldots x_m$ and $1 \le i \le m$, we have defined z_i as being the number of descents in the right factor $x_ix_{i+1}\ldots x_m$, and $z(w)$ as being the sequence of the z_i's. If w is of length 1, obviously $\operatorname{maj} w = \operatorname{tot} z(w) = 0$. If w is of length greater than 1, define $w' := x_1x_2\ldots x_{m-1}$ and let $z(w') = (z'_1, \ldots , z'_{m-1})$. If $x_{m-1} \le x_m$, then $\operatorname{maj} w = \operatorname{maj} w'x_m = \operatorname{maj} w' = \operatorname{tot} z(w') = \operatorname{tot} z(w'x_m) = \operatorname{tot} z(w)$. If $x_{m-1} > x_m$, then $\operatorname{maj} w = \operatorname{maj} w' + (m-1)$, since there is a descent in position $(m-1)$. On the other hand, $z(w) = ((z'_1 + 1), \ldots , (z'_{m-1} + 1))$; hence $\operatorname{tot} z(w) = \operatorname{tot} z(w') + (m-1)$. ∎

In the next theorem we state the results derived in this section about the Major Index.

Theorem 6.3. *The Major Index is a q-multinomial statistic; that is, for each rearrangement class $R(\mathbf{m})$ the identity*

$$\begin{bmatrix} m_1 + m_2 + \cdots + m_r \\ m_1, m_2, \ldots, m_r \end{bmatrix} = \sum_{w \in R(\mathbf{m})} q^{\operatorname{maj} w} \tag{6.11}$$

holds.

The right-hand side of (6.11) is the generating polynomial for the class $R(\mathbf{m})$ by the Major Index, a polynomial that will be denoted by $A_{\mathbf{m}}(q)$. Relation (6.11) can also be rewritten as

$$\frac{1}{(q;q)_m} A_{\mathbf{m}}(q) = \frac{1}{(q;q)_{m_1} \cdots (q;q)_{m_r}}, \tag{6.12}$$

which is (6.2) with "maj" replacing "stat."

Now introduce the algebra of power formal series in the variables u_1, u_2, ... , u_r. Let $\mathbf{u}^{\mathbf{m}} := u_1^{m_1} u_2^{m_2} \ldots u_r^{m_r}$, also $(\mathbf{u};q)_\infty := (u_1;q)_\infty \cdots (u_r;q)_\infty$ and remember that $m = m_1 + \cdots + m_r$. Next, multiply (6.12) by $\mathbf{u}^{\mathbf{m}}$ and sum the two sides of the equation with respect to all the sequences $\mathbf{m} = (m_1, \ldots, m_r)$ of r nonnegative integers. By using the q-Binomial Theorem (Theorem 1.1) we get the equivalent identity

$$\sum_{\mathbf{m}} A_{\mathbf{m}}(q) \frac{\mathbf{u}^{\mathbf{m}}}{(q;q)_m} = \frac{1}{(\mathbf{u};q)_\infty}. \tag{6.13}$$

7. A refinement of the MacMahon Verfahren

Re-examine the *inverse* mapping of the bijection

$$(b^{(1)}, \ldots, b^{(r)}) \mapsto (b, w) \tag{7.1}$$

of $\mathrm{NDS}(m_1) \times \cdots \times \mathrm{NDS}(m_r)$ onto $\mathrm{NDS}(m) \times R(\mathbf{m})$, described in the previous section. Each term z_i in the sequence $z(w) = (z_1, z_2, \ldots, z_m)$ counts the number of descents in the right factor $x_i x_{i+1} \ldots x_m$ of the word w. Let $\operatorname{des} w := \sum_{1 \le i \le m-1} \chi(x_i > x_{i+1})$ denote the *number of descents* of the word w, so that

$$z_1 = \operatorname{des} w. \tag{7.2}$$

As $y_1 = b_m + z_1$ and since y_1 is the maximum term in the sequence $b^{(1)}$, ... , $b^{(r)}$, we also have:

$$b^{(1)}_{m_1} \le b_1 + \operatorname{des} w, \ \ldots\ , b^{(r)}_{m_r} \le b_1 + \operatorname{des} w. \tag{7.3}$$

Take a nonnegative integer s' and a nondecreasing sequence $b = b_m \dots b_2 b_1$ such that $0 \le b_m \le \dots \le b_1 \le s'$, i.e., $b \in \mathrm{NDS}(m, s')$; further take a word $w \in R(\mathbf{m})$ and let

$$s := s' + \operatorname{des} w. \tag{7.4}$$

It follows from the inequalities (7.3) that

$$b^{(1)} \in \mathrm{NDS}(m_1, s), \dots, b^{(r)} \in \mathrm{NDS}(m_r, s). \tag{7.5}$$

The bijection constructed in the previous section also has the property stated in the next proposition.

Proposition 7.1. *To each triple (s', b, w) such that $s' \ge 0$, $b \in \mathrm{NDS}(m, s')$ and $w \in R(\mathbf{m})$ there corresponds, in a bijective manner, a sequence $(s, b^{(1)}, \dots, b^{(r)})$, where $s = s' + \operatorname{des} w$ and where $b^{(1)} \in \mathrm{NDS}(m_1, s)$, ... , $b^{(r)} \in \mathrm{NDS}(m_r, s)$, having the property:*

$$\operatorname{tot} b^{(1)} + \dots + \operatorname{tot} b^{(r)} = \operatorname{tot} b + \operatorname{maj} w.$$

Let $A_{\mathbf{m}}(t, q)$ denote the generating polynomial for $R(\mathbf{m})$ by the bistatistic (des, maj):

$$A_{\mathbf{m}}(t, q) = \sum_{w \in R(\mathbf{m})} t^{\operatorname{des} w} \, q^{\operatorname{maj} w}. \tag{7.6}$$

Then

$$\begin{aligned}
\frac{1}{(t;q)_{m+1}} A_{\mathbf{m}}(t,q) &= \sum_{s' \ge 0} t^{s'} \begin{bmatrix} m+s' \\ s' \end{bmatrix} A_{\mathbf{m}}(t,q) && \text{[by (3.9)]}\\
&= \sum_{s' \ge 0} t^{s'} \sum_{b \in \mathrm{NDS}(m,s')} q^{\operatorname{tot} b} A_{\mathbf{m}}(t,q) && \text{[by (4.5)]}\\
&= \sum_{\substack{s' \ge 0,\, b \in \mathrm{NDS}(m,s'),\\ w \in R(\mathbf{m})}} t^{s' + \operatorname{des} w} q^{\operatorname{tot} b + \operatorname{maj} w}\\
&= \sum_{s \ge 0} t^s \sum_{\substack{s' \ge 0,\, b \in \mathrm{NDS}(m,s'),\\ w \in R(\mathbf{m}),\, s' + \operatorname{des} w = s}} q^{\operatorname{tot} b + \operatorname{maj} w}\\
&= \sum_{s \ge 0} t^s \sum_{\substack{b^{(1)} \in \mathrm{NDS}(m_1,s), \dots,\\ b^{(r)} \in \mathrm{NDS}(m_r,s)}} q^{\operatorname{tot} b^{(1)} + \dots + \operatorname{tot} b^{(r)}},
\end{aligned}$$

[in view of Proposition 7.1]

so that

$$\frac{1}{(t;q)_{m+1}} A_{\mathbf{m}}(t,q) = \sum_{s \ge 0} t^s \begin{bmatrix} m_1+s \\ s \end{bmatrix} \dots \begin{bmatrix} m_r+s \\ s \end{bmatrix} \quad \text{[by (4.5)]}. \tag{7.7}$$

As at the end of the previous section, we can express (7.7) as an identity between formal power series in r variables $u_1, u_2, \ldots, u_r$. However, those series will be normalized by denominators of the form $(t;q)_{m+1}$. Again, let $\mathbf{u}^{\mathbf{m}} := u_1^{m_1} u_2^{m_2} \ldots u_r^{m_r}$ and let $(\mathbf{u};q)_{s+1} := (u_1;q)_{s+1} \cdots (u_r;q)_{s+1}$. Next, multiply (7.7) by $\mathbf{u}^{\mathbf{m}}$ and sum over all sequences $\mathbf{m} = (m_1, \ldots, m_r)$ of r nonnegative integers. Then,

$$\begin{aligned}\sum_{\mathbf{m}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}}}{(t;q)_{1+m}} &= \sum_{s\geq 0} t^s \sum_{\mathbf{m}} \mathbf{u}^{\mathbf{m}} \begin{bmatrix} m_1+s \\ s \end{bmatrix} \cdots \begin{bmatrix} m_r+s \\ s \end{bmatrix} \\ &= \sum_{s\geq 0} t^s \Bigl(\sum_{m_1} u_1^{m_1} \begin{bmatrix} m_1+s \\ s \end{bmatrix}\Bigr) \cdots \Bigl(\sum_{m_r} u_r^{m_r} \begin{bmatrix} m_r+s \\ s \end{bmatrix}\Bigr),\end{aligned}$$

so that by using (3.9),

$$\sum_{\mathbf{m}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}}}{(t;q)_{1+m}} = \sum_{s\geq 0} \frac{t^s}{(\mathbf{u};q)_{s+1}}. \tag{7.8}$$

Remark. Identities (7.7) and (7.8) appear to be "t-extensions" of identities (6.12) and (6.13) derived in the previous section.

8. The Euler-Mahonian polynomials

In the previous section the polynomials $A_{\mathbf{m}}(t,q)$ have been introduced as generating polynomials for the class $R(\mathbf{m})$ by the bi-statistic $(\mathrm{des}, \mathrm{maj})$. In formula (7.7) they appear as *numerators* of rational fractions whose series expansion in t has an explicit form. In fact, formula (7.7) is only another way of looking at their (t,q)-generating function, as obtained in (7.8). However there are other ways of expressing the polynomials *without any reference to any combinatorial interpretation*, as shown in this section. First, we state a definition that will be made valid, once we prove that the four items (1)–(4) are equivalent. For proving the equivalence of those statements we use two methods: a *finite difference q-calculus* and a *q-iteration* that are developed afterwards.

For each multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_{r-1}, m_r)$ let $|\mathbf{m}| := m_1 + m_2 + \cdots + m_{r-1} + m_r$ (a quantity that has been denoted by m in the previous section) and $\mathbf{m}+1_r := (m_1, m_2, \ldots, m_{r-1}, m_r+1)$. Also keep the notations $\mathbf{u}$, $\mathbf{u}^{\mathbf{m}}$, $(\mathbf{u};q)_{s+1}$; also $[0]_q := 0$ and $[m]_q := 1 + q + \cdots + q^{m-1}$ $(m \geq 1)$.

Definition 8.1. Let r be a fixed positive integer. A sequence

$$\Bigl(A_{\mathbf{m}}(t,q) = \sum_{s\geq 0} t^s A_{\mathbf{m},s}(q)\Bigr)$$

of polynomials in two variables t and q, indexed by a multi-index $\mathbf{m} = (m_1, \ldots, m_r)$ of r nonnegative integers, is said to be *Euler-Mahonian*, if one of the following *equivalent* four conditions holds:

(1) For all $\mathbf{m}$ we have:

$$\frac{1}{(t;q)_{m+1}} A_{\mathbf{m}}(t,q) = \sum_{s\ge 0} t^s \begin{bmatrix} m_1+s \\ s \end{bmatrix} \ldots \begin{bmatrix} m_r+s \\ s \end{bmatrix}. \tag{8.1}$$

(2) The (t,q)-generating function for those polynomials $A_{\mathbf{m}}(t,q)$ is given by:

$$\sum_{\mathbf{m}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}}}{(t;q)_{1+|\mathbf{m}|}} = \sum_{s\ge 0} \frac{t^s}{(\mathbf{u};q)_{s+1}}. \tag{8.2}$$

(3) The recurrence relation holds:

$$\begin{aligned} &(1-q^{m_r+1})A_{\mathbf{m}+1_r}(t,q) \\ &\qquad = (1-tq^{1+|\mathbf{m}|})A_{\mathbf{m}}(t,q) - q^{m_r+1}(1-t)A_{\mathbf{m}}(tq,q). \end{aligned} \tag{8.3}$$

(4) The recurrence relation holds for the coefficients $A_{\mathbf{m},s}(q)$:

$$\begin{aligned} &[m_r+1]_q\, A_{\mathbf{m}+1_r,s}(q) \\ &\quad = [m_r+1+s]_q\, A_{\mathbf{m},s}(q) + q^{s+m_r}[1+|\mathbf{m}|-s-m_r]_q\, A_{\mathbf{m},s-1}(q). \end{aligned} \tag{8.4}$$

For the equivalence (1) $\Leftrightarrow$ (2) see the previous section. To verify (3) $\Leftrightarrow$ (4) notice that (8.3) is equivalent, for each $s \ge 0$, to

$$\begin{aligned} &(1-q^{m_r+1})A_{\mathbf{m}+1_r,s}(q) \\ &= A_{\mathbf{m},s}(q) - q^{1+|\mathbf{m}|}A_{\mathbf{m},s-1}(q) - q^{m_r+1+s}A_{\mathbf{m},s}(q) + q^{m_r+1+(s-1)}A_{\mathbf{m},s-1}(q), \end{aligned}$$

which, in turn, is equivalent to (8.4) by dividing by $(1-q)$. We next prove (2) $\Rightarrow$ (3) (resp. (3) $\Rightarrow$ (2)) by means of the *finite difference q-calculus* (resp. the *q-iteration*) given next.

8.1. *A finite difference q-calculus.* Let

$$A(t,q;\mathbf{u}) = A(t,q;u_1,\ldots,u_r) := \sum_{\mathbf{m}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}}}{(t;q)_{1+|\mathbf{m}|}} \tag{8.5}$$

denote the left-hand side of (8.2) and form the *q-finite difference* applied to the sole variable u_r:

$$D_{u_r} := A(t,q;u_1,\ldots,u_r) - A(t,q;u_1,\ldots,u_{r-1},u_rq).$$

We get:

$$D_{u_r} = \sum_{\substack{\mathbf{m} \\ m_r \geq 1}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}}}{(t;q)_{1+|\mathbf{m}|}} - \sum_{\substack{\mathbf{m} \\ m_r \geq 1}} A_{\mathbf{m}}(t,q) \frac{u_1^{m_1} \dots u_{r-1}^{m_{r-1}} (u_r q)^{m_r}}{(t;q)_{1+|\mathbf{m}|}}$$

$$= \sum_{\mathbf{m}} A_{\mathbf{m}+1_r}(t,q) \frac{\mathbf{u}^{\mathbf{m}+1_r}}{(t;q)_{2+|\mathbf{m}|}}$$

$$- \sum_{\mathbf{m}} A_{\mathbf{m}+1_r}(t,q) \frac{u_1^{m_1} \dots u_{r-1}^{m_{r-1}} (u_r q)^{m_r+1}}{(t;q)_{2+|\mathbf{m}|}},$$

so that

$$D_{u_r} = \sum_{\mathbf{m}} (1 - q^{m_r+1}) A_{\mathbf{m}+1_r}(t,q) \frac{\mathbf{u}^{\mathbf{m}+1_r}}{(t;q)_{2+|\mathbf{m}|}}. \tag{8.2}$$

Now use the *right-hand side* of (8.2):

$$D_{u_r} = \sum_{s \geq 0} \frac{t^s}{(u_1;q)_{s+1} \dots (u_r;q)_{s+1}} - \sum_{s \geq 0} \frac{t^s}{(u_1;q)_{s+1} \dots (u_r q;q)_{s+1}}$$

$$= \sum_{s \geq 0} \frac{t^s}{(\mathbf{u};q)_{s+1}} \Big[1 - \frac{1-u_r}{1-u_r q^{s+1}} \Big]$$

$$= u_r \sum_{s \geq 0} \frac{t^s}{(\mathbf{u};q)_{s+1}} \Big[1 - q^{s+1} \frac{1-u_r}{1-u_r q^{s+1}} \Big]$$

$$= u_r \big(A(t,q;u_1,\dots,u_r) - qA(tq,q;u_1,\dots,u_{r-1},u_r q) \big).$$

This yields:

$$A(t,q;u_1,\dots,u_r) - A(t,q;u_1,\dots,u_{r-1},u_r q) = u_r \big(A(t,q;u_1,\dots,u_r) - qA(tq,q;u_1,\dots,u_{r-1},u_r q) \big). \tag{8.6}$$

Re-write every term on the right-hand side by means of the polynomials $A_{\mathbf{m}}(t,q)$. We get:

$$u_r\, A(t,q;\mathbf{u}) = \sum_{\mathbf{m}} A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}+1_r}}{(t;q)_{1+|\mathbf{m}|}}$$

$$= \sum_{\mathbf{m}} (1 - tq^{1+|\mathbf{m}|}) A_{\mathbf{m}}(t,q) \frac{\mathbf{u}^{\mathbf{m}+1_r}}{(t;q)_{2+|\mathbf{m}|}}. \tag{8.7}$$

In the same manner,

$$u_r q\, A(tq,q;u_1,\dots,u_r q) = \sum_{\mathbf{m}} A_{\mathbf{m}}(tq,q) \frac{u_1^{m_1} \dots u_{r-1}^{m_{r-1}} (u_r q)^{m_r+1}}{(tq;q)_{1+|\mathbf{m}|}}$$

$$= \sum_{\mathbf{m}} q^{m_r+1} (1-t) A_{\mathbf{m}}(tq,q) \frac{\mathbf{u}^{\mathbf{m}+1_r}}{(t;q)_{2+|\mathbf{m}|}}. \tag{8.5}$$

Taking (8.5)—(8.7) into account we deduce the recurrence relation (8.4).

8.2. *A q-iteration method.* By using the notation (8.5) we see that the recurrence relation (8.3) can be rewritten:

$$
\begin{aligned}
(8.8)\quad A(t,q;\mathbf{u}) - A(t,q;u_1,\ldots,u_iq\ldots,u_r)&\\
= u_i\,A(t,q;\mathbf{u}) - u_iq\,A(tq,q;u_1,\ldots,u_iq,\ldots,u_r).&
\end{aligned}
$$

Let $A(t,q;\mathbf{u}) := \sum\limits_{s\ge 0} t^s G_s(\mathbf{u},q)$. We deduce:

$$\sum_{s\ge 0} t^s(1-u_i)G_s(\mathbf{u},q) = \sum_{s\ge 0} t^s(1-u_iq^{s+1})G_s(u_1,\ldots,u_iq,\ldots,u_r,q).$$

Now take the coefficient of t^s on each side. We obtain:

$$(8.9)\qquad G_s(\mathbf{u},q) = \frac{1-u_iq^{s+1}}{1-u_i}G_s(u_1,\ldots,u_iq,\ldots,u_r,,q),$$

for $i=1,\ldots,r$. Then, let $F_s(\mathbf{u},q) := G_s(\mathbf{u},q)(\mathbf{u};q)_{s+1}$ and use (8.9). For $i=1,\ldots,r$ we obtain the equation

$$(8.10)\qquad F_s(\mathbf{u},q) = F_s(u_1,\ldots,u_iq,\ldots,u_r,q).$$

But we can write $F_s(\mathbf{u},q) = \sum_{\mathbf{m}} \mathbf{u}^{\mathbf{m}} F_{s,\mathbf{m}}(q)$, where $F_{s,\mathbf{m}}(q)$ is a (positive) power series in q. Fix the multi-index $\mathbf{m}$ and let m_i be a nonzero component of $\mathbf{m}$. Relation (8.10) implies: $F_{s,\mathbf{m}}(q) = q^{m_i}F_{s,\mathbf{m}}(q)$. Hence, $F_{s,\mathbf{m}}(q) = 0$ and $F_s(\mathbf{u},q) = F_{s,0}(q)$, a quantity that remains to be evaluated. But, by definition of $F_s(u,q)$, we have:

$$F_{s,0}(q) = F_s(\mathbf{u},q)\,\Big|_{\mathbf{u}\,=\,0} = G_s(\mathbf{u},q)(\mathbf{u};q)_{s+1}\,\Big|_{\mathbf{u}\,=\,0} = G_s(0,q) = 1,$$

since $\displaystyle\sum_{s\ge 0} t^s G_s(0,q) = A(t,q;0) = \frac{1}{(t;q)_1} = \sum_{s\ge 0} t^s$. Thus $G_s(\mathbf{u},q) = \dfrac{1}{(\mathbf{u};q)_{s+1}}$. □

Remark 8.2. If σ is a permutation of the set $\{1,\ldots,r\}$, denote by $\sigma\mathbf{m}$ the sequence $(m_{\sigma(1)},\ldots,m_{\sigma(r)})$. In particular, $R(\sigma\mathbf{m})$ is the class of all the rearrangements of the word $1^{m_{\sigma(1)}}\ldots r^{m_{\sigma(r)}}$. As the product of the binomial coefficient on the right-hand side of (8.1) is *symmetric* in m_1, … , m_r, we conclude that for every permutation σ we have:

$$(8.11)\qquad A_{\sigma\mathbf{m}}(t,q) = A_{\mathbf{m}}(t,q).$$

Remark 8.3. The relations (8.3) and (8.4) provide the evaluations of the first values of the polynomials $A_{\mathbf{m}}(t,q)$. Because of the previous remark it suffices to make the calculations for the nonincreasing sequences $\mathbf{m}$.

Table of the polynomials $A_{\mathbf{m}}(t,q)$:
$A_{(1)} = 1$; $A_{(1,1)} = 1 + tq$; $A_{(2)} = 1$; $A_{(1,1,1)} = 1 + t(2q + 2q^2) + t^2q^3)$;
$A_{(2,1)} = 1 + t(q + q^2)$; $A_{(3)} = 1$;
$A_{(1,1,1,1)} = 1 + t(3q + 5q^2 + 3q^3) + t^2(3q^3 + 5q^4 + 3q^3) + t^3q^6$;
$A_{(2,1,1)} = 1 + t(2q + 3q^2 + 2q^3) + t^2(q^2 + 2q^4 + q^5)$;
$A_{(2,2)} = 1 + t(q + 2q^2 + q^3) + t^2q^4$; $A_{(3,1)} = 1 + t(q + q^2 + q^3)$; $A_{(4)} = 1$.

9. The insertion technique

In section 7 we have shown that the Euler-Mahonian polynomial $A_{\mathbf{m}}(t,q)$ was the generating polynomial for the class $R(\mathbf{m})$ by the bi-statistic $(\mathrm{des}, \mathrm{maj})$. To derive the result we made use of the MacMahon *Verfahren* and obtained identity (8.1). The natural question is whether we can prove the same result by using one of the two *recurrence relations* (8.3), (8.4). If we dealt with the symmetric group (i.e., with all the m_i's equal to 1), we would try the traditional *insertion technique*: start with a permutation of order r and study the modification brought to the underlying statistic when the letter $(r+1)$ is *inserted* into the $(r+1)$ slots of the permutation. The technique can be applied without any difficulty. Identity (8.4)—with all the m_i's equal to 1— is then easily derived.

With words with repeated letters the derivation is not straightforward. A transformation called *word marking* must be made on the initial word. The word marking goes as follows. This time we consider the polynomial

$$A_{\mathbf{m}}(t,q) = \sum_{s\ge 0} A_{\mathbf{m},s}(q)t^s \tag{9.1}$$

as being the generating polynomial for $R(\mathbf{m})$ by the bi-statistic $(\mathrm{des}, \mathrm{maj})$, so that $A_{\mathbf{m},s}(q)$ is the generating polynomial for the words $w \in R(\mathbf{m})$ such that $\mathrm{des}\, w = s$ by the Major Index. Again use the notations $[s]_q := 1 + q + q^2 + \cdots + q^{s-1}$, $|\mathbf{m}| := m_1 + \cdots + m_r$ and $\mathbf{m} + 1_j := (m_1, \ldots, m_j + 1, \ldots, m_r)$ for each $j = 1, 2, \ldots, r$ and each sequence $\mathbf{m} = (m_1, m_2, \ldots, m_r)$.

Proposition 9.1. *Let* $1 \le j \le r$ *and let* $A_{\mathbf{m}}(t,q)$ *be the generating polynomial for* $R(\mathbf{m})$ *by the bi-statistic* $(\mathrm{des}, \mathrm{maj})$. *Then the following* *relations hold:*

$$(9.2)\quad (1-q^{m_j+1})A_{\mathbf{m}+1_j}(t,q) = (1-tq^{1+|\mathbf{m}|})A_{\mathbf{m}}(t,q) - q^{m_j+1}(1-t)A_{\mathbf{m}}(tq,q);$$

$$(9.3)\quad [m_j+1]_q A_{\mathbf{m}+1_j,s}(q) = [m_j+1+s]_q\, A_{\mathbf{m},s}(q) + q^{s+m_j}[1+|\mathbf{m}|-s-m_j]_q\, A_{\mathbf{m},s-1}(q).$$

Proof. The latter identity is equivalent to the former one, so that only (9.3) is to be proved. From Remark 8.3 this relation is equivalent to the relation formed when j is replaced by any integer in $\{1,\ldots,r\}$. It is convenient to prove the relation for $j=1$ which reads

$$(9.4)\quad (1+q+\cdots+q^{m_1})A_{\mathbf{c}+1_1,s}(q) = (1+q+\cdots+q^{m_1+s})A_{\mathbf{m},s}(q) + (q^{m_1+s}+\cdots+q^{|\mathbf{m}|})A_{\mathbf{m},s-1}(q).$$

Consider the set $R^*(\mathbf{m}+1_1,s)$ of 1-*marked* words, i.e., rearrangements w^* of $1^{m_1+1}\ldots r^{m_r}$ with s descents such that exactly one letter equal to 1 has been marked. Each word $w\in R(\mathbf{m}+1_1)$ that has s descents gives rise to m_1+1 marked words $w^{(0)},\ldots,w^{(m_1)}$. Define

$$\operatorname{maj}^* w^{(i)} := \operatorname{maj} w + n_1,$$

where n_1 is the number of letters equal to 1 to the *right* of the marked 1. Then clearly

$$\sum_{i=0}^{m_1} \operatorname{maj}^* w^{(i)} = (1+q+\cdots+q^{m_1})\operatorname{maj} w.$$

Hence

$$(1+q+\cdots+q^{m_1})A_{\mathbf{m}+1_1,s}(q) = \sum_{w\in R^*(\mathbf{m}+1_1,s)} q^{\operatorname{maj}^* w}.$$

Let $m=|\mathbf{m}|$ and let the word $w=x_1x_2\ldots x_m\in R(\mathbf{m})$ have s descents. Say that w has $m+1$ *slots* x_ix_{i+1}, $i=0,\ldots,m$ (where $x_0=0$ and $x_{m+1}=\infty$ by convention). Call the slot x_ix_{i+1} *green* if either x_ix_{i+1} is a descent, $x_i=1$, or $i=0$. Call the other slots *red.* Then there are $1+s+m_1$ green slots and $m-s-m_1$ red slots. Label the green slots $0,1,\ldots,m_1+s$ from right to left, and label the red slots $m_1+s+1,\ldots,m$ from left to right.

For example, with $r=3$, the word $w=2,2,1,3,2,1,2,3,3$ has three descents and ten slots. As $m_1=2$, there are eight green slots and two red slots, labelled as follows

$$\begin{array}{lccccccccccccccccccc}
\text{slot} & 0 & | & 2 & | & 2 & | & 1 & | & 3 & | & 2 & | & 1 & | & 2 & | & 3 & | & 3 & | & \infty\\
\text{label} & & 5 & & 6 & & 4 & & 3 & & 2 & & 1 & & 0 & & 7 & & 8 & & 9 &
\end{array}$$

i	$w^{(i)}$	$\operatorname{des} w^{(i)}$	$\operatorname{maj}^* w^{(i)}$
0	$2\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ \mathbf{1}\ \ 2\ \ 3\ \ 3$	3	11
1	$2\ \ 2\frown 1\ \ 3\frown 2\frown \mathbf{1}\ \ 1\ \ 2\ \ 3\ \ 3$	3	12
2	$2\ \ 2\frown 1\ \ 3\frown \mathbf{1}\ \ 2\frown 1\ \ 2\ \ 3\ \ 3$	3	13
3	$2\ \ 2\frown 1\ \ \mathbf{1}\ \ 3\frown 2\frown 1\ \ 2\ \ 3\ \ 3$	3	14
4	$2\ \ 2\frown \mathbf{1}\ \ 1\ \ 3\frown 2\frown 1\ \ 2\ \ 3\ \ 3$	3	15
5	$\mathbf{1}\ \ 2\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ 2\ \ 3\ \ 3$	3	16
6	$2\frown \mathbf{1}\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ 2\ \ 3\ \ 3$	4	17
7	$2\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ 2\frown \mathbf{1}\ \ 3\ \ 3$	4	18
8	$2\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ 2\ \ 3\frown \mathbf{1}\ \ 3$	4	19
9	$2\ \ 2\frown 1\ \ 3\frown 2\frown 1\ \ 2\ \ 3\ \ 3\frown \mathbf{1}$	4	20

Table 9.1

Denote by $w^{(i)}$ the word obtained from w by inserting a marked 1 into the i-th slot. Then it may be verified that

$$\operatorname{des} w^{(i)} = \begin{cases} \operatorname{des} w, & \text{if } i \le m_1 + s; \\ \operatorname{des} w + 1, & \text{otherwise.} \end{cases} \tag{9.5}$$

$$\operatorname{maj}^* w^{(i)} = \operatorname{maj} w + i. \tag{9.6}$$

Example. Consider the above word w. In Table 9.1 the values of "des" and "maj*" on $w^{(i)}$. Descents are indicated by $\frown$ and the marked **1** is written in boldface.

So each word $w \in R(\mathbf{m})$ with s descents and $\operatorname{maj} w = n$ gives rise to $m_1 + s + 1$ marked words in $R^*(\mathbf{m} + 1_1, s)$ with maj^* equal to $n, n+1, \ldots, n+m_1+s$; and to $m-s-m_1$ marked words in $R^*(\mathbf{m}+1_1, s+1)$ with maj^* equal to $n+m_1+s+1, \ldots, n+m$. Hence a word w in $R(\mathbf{m})$ with $s-1$ descents gives rise to $m-s+1-m_1$ marked words in $R^*(\mathbf{m}+1_1, s)$ with maj^* equal to $\operatorname{maj} w+m_1+s, \ldots, \operatorname{maj} w+m$. This now proves relation (9.4). $\square$

10. The two forms of the q-Eulerian polynomials

When the multi-index $\mathbf{m}$ is of the form $(1^r) = (1, 1, \ldots, 1)$, the Euler-Mahonian polynomial $A_{\mathbf{m}}(t, q)$ will be denoted by ${}^{\mathrm{maj}}A_r(t, q)$ and referred to as the *q-maj-Eulerian polynomial.* It also follows from the previous two sections that ${}^{\mathrm{maj}}A_r(t, q)$ is the generating polynomial for the symmetric group $\mathfrak{S}_r$ by the bi-statistic $(\operatorname{des}, \operatorname{maj})$. As for the polynomial $A_{\mathbf{m}}(t, q)$ for an arbitrary $\mathbf{m}$, the new polynomial ${}^{\mathrm{maj}}A_r(t, q)$ can be characterized in four different ways, as shown in the next definition.

Definition 10.1. A sequence $({}^{\mathrm{maj}}A_r(t,q))$ of polynomials in two variables t and q, indexed by the integers $r\ge 0$, is said to be *q-maj-Eulerian*, if one of the following *equivalent* four conditions holds:

(1) For every integer $r\ge 0$ we have:

$$\frac{1}{(t;q)_{r+1}}\,{}^{\mathrm{maj}}A_r(t,q)=\sum_{s\ge 0} t^s\,\bigl([s+1]_q\bigr)^r. \tag{10.1}$$

(2) The ordinary (resp. exponential) generating function for the ratios $\dfrac{{}^{\mathrm{maj}}A_r(t,q)}{(t;q)_{r+1}}$ is given by:

$$\sum_{r\ge 0} u^r\,\frac{{}^{\mathrm{maj}}A_r(t,q)}{(t;q)_{r+1}}=\sum_{s\ge 0} t^s\,\frac{1}{1-u\,[s+1]_q}; \tag{10.2a}$$

$$\sum_{r\ge 0} \frac{u^r}{r!}\,\frac{{}^{\mathrm{maj}}A_r(t,q)}{(t;q)_{r+1}}=\sum_{s\ge 0} t^s\,\exp(u\,[s+1]_q). \tag{10.2b}$$

(3) The recurrence relation holds:

$$(1-q)\,{}^{\mathrm{maj}}A_r(t,q)=(1-tq^r)\,{}^{\mathrm{maj}}A_{r-1}(t,q)-q(1-t)\,{}^{\mathrm{maj}}A_{r-1}(tq,q). \tag{10.3}$$

(4) With ${}^{\mathrm{maj}}A_r(t,q):=\sum\limits_{s\ge 0} t^s\,{}^{\mathrm{maj}}A_{r,s}(q)$ the coefficients ${}^{\mathrm{maj}}A_{r,s}(q)$ satisfy the recurrence:

$${}^{\mathrm{maj}}A_{r,s}(q)=[s+1]_q\,{}^{\mathrm{maj}}A_{r-1,s}(q)+q^s[r-s]_q\,{}^{\mathrm{maj}}A_{r-1,s-1}(q). \tag{10.4}$$

In the above definition (10.1), is the specialization of (8.1). However (8.2a) and (8.2b) have no immediate counterparts, but the exponential generating function for the ratios $\dfrac{1}{(t;q)_{r+1}}\,{}^{\mathrm{maj}}A_r(t,q)$ has an interesting closed form, as written in (10.2). Finally, (10.3) and (10.4) are straightforward specializations of (8.3) and (8.4) when $\mathbf{m}=1^{r-1}$.

When $q=1$ in (10.1), (10.2) and (10.4), we recognize some familiar definitions for the so-called *Eulerian polynomials*. Let us introduce them following the same pattern as above.

Definition 10.2. A sequence $(A_r(t))$ of polynomials in one variable t, indexed by the integers $r\ge 0$, is said to be *Eulerian*, if one of the following *equivalent* five conditions holds:

(1) For every integer $r\ge 0$ we have:

$$\frac{1}{(1-t)^{r+1}}\,A_r(t)=\sum_{s\ge 0} t^s\,(s+1)^r. \tag{10.5}$$

(2) The exponential generating function for the ratios $\dfrac{A_r(t)}{(1-t)^{r+1}}$ is given by:

$$\sum_{r\geq 0} \frac{u^r}{r!}\,\frac{A_r(t)}{(1-t)^{r+1}} = \sum_{s\geq 0} t^s \, \exp(u\,(s+1)) = \frac{e^u}{1-te^u}. \tag{10.6}$$

(3) The following recurrence relation holds:

$$A_r(t) = (1+(r-1)t)\,A_{r-1}(t) + t(1-t)\,A'_{r-1}(t), \tag{10.7}$$

where $A'_{r-1}(t)$ denotes the derivative of the polynomial $A_{r-1}(t)$.

(4) With $A_r(t) := \sum\limits_{s\geq 0} t^s\,A_{r,s}$ the coefficients $A_{r,s}$ satisfy the recurrence:

$$A_{r,s} = (s+1)\,A_{r-1,s} + (r-s)\,A_{r-1,s-1}. \tag{10.8}$$

(5) The exponential generating function for the polynomials reads:

$$\sum_{r\geq 0} \frac{u^r}{r!}\,A_r(t) = \frac{1-t}{-t+\exp(u(t-1))}. \tag{10.9}$$

Notice that (10.1), (10.2) and (10.4) that *define* the q-maj Eulerian polynomials are reduced to their counterparts (10.5), (10.6) and (10.8) that *define* the Eulerian polynomials, when q is given the value 1. On the other hand, we go from recurrence (10.8) to the q-recurrence (10.4) by replacing the integers $(s+1)$ and $(r-s)$ occurring in the relation by their q-counterparts $[s+1]_q = 1+q+\cdots+q^s$ and $q^s[r-s]_q = q^{r-s}+q^{r-s+1}+\cdots+q^{r-1}$, respectively. We then say that the q-maj Eulerian polynomial $A_r(t,q)$ is a *q-analog* of the Eulerian polynomial $A_r(t)$.

Also notice that there is no specialization of (10.3) for $q=1$. To obtain a recurrence for the polynomials $A_r(t)$ themselves, as shown in (10.7), we start from (10.8) and make the appropriate identifications. Finally, observe that (10.9) is simply derived from (10.6) with the substitution $u \leftarrow u/(1-t)$. The Eulerian polynomial $A_r(t)$ is the generating polynomial for $\mathfrak{S}_r$ by the number of descents "des."

Another q-extension of the Eulerian polynomials can be achieved by using the defining relation (10.9) or the exponential generating function for the polynomials $t\,A_n(t)$, that can be directly derived from (10.9) and reads

$$1+\sum_{n\geq 1} \frac{u^n}{n!} t\,A_n(t) = \frac{1-t}{1-t\,\exp((1-t)u)}.$$

In the above fraction make the substitution $\exp(u) \leftarrow e_q(u)$ and express the fraction so derived as a q-series

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n} {}^{\mathrm{inv}}A_n(t,q) = \frac{1-t}{1-t\,e_q((1-t)u)}, \tag{10.10}$$

where the ${}^{\mathrm{inv}}A_n(t,q)$'s are coefficients to be determined. Identity (10.10) can be rewritten as

$$\sum_{n\ge 0} {}^{\mathrm{inv}}A_n(t,q)\frac{u^n}{(q;q)_n} = \Bigl(1 - t\sum_{n\ge 1}(1-t)^{n-1}\frac{u^n}{(q;q)_n}\Bigr)^{-1},$$

so that the identity

$$\sum_{n\ge 0} {}^{\mathrm{inv}}A_n(t,q)\frac{u^n}{(q;q)_n} \cdot \Bigl(1 - t\sum_{n\ge 1}(1-t)^{n-1}\frac{u^n}{(q;q)_n}\Bigr) = 1$$

provides the recurrence: ${}^{\mathrm{inv}}A_0(t,q) = 1$ and

$$ {}^{\mathrm{inv}}A_n(t,q) = \sum_{0\le k\le n-1} \begin{bmatrix} n\\ k\end{bmatrix} {}^{\mathrm{inv}}A_k(t,q)\,t\,(1-t)^{n-1-k} \quad (n\ge 1), \tag{10.11}$$

so that the coefficients ${}^{\mathrm{inv}}A_n(t,q)$ are *polynomials* in the two variables t and q with *integral coefficients*. Let q tend to 1 in (10.11) and let $t\,B_n(t) := {}^{\mathrm{inv}}A_n(t,q)\bigm|_{q\,=\,1}$ $(n\ge 1)$. This yields

$$t\,B_n(t) = \sum_{0\le k\le n-1} \binom{n}{k} t\,B_k(t)\,t\,(1-t)^{n-1-k} \quad (n\ge 1),$$

which, in turn, is equivalent to

$$1 + \sum_{n\ge 1} t\,B_n(t)\frac{u^n}{n!} = \frac{1-t}{1-t\,\exp(u(1-t))}.$$

Hence

$$1 + \sum_{n\ge 1} B_n(t)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t-1))},$$

which is the right-hand side of (10.9). Hence $B_n(t) = A_n(t)$ and $t\,A_n(t)$ is the generating polynomial for $\mathfrak{S}_n$ by the statistic $1+\mathrm{des}$. Now what can be said about the polynomial ${}^{\mathrm{inv}}A_n(t,q)$?

Theorem 10.1. *For $n \geq 1$ the polynomial ${}^{\mathrm{inv}}A_n(t,q)$ in the expansion*

$$\frac{1-t}{1-t\,e_q((1-t)u)} = \sum_{n\geq 0} {}^{\mathrm{inv}}A_n(t,q)\frac{u^n}{(q;q)_n} \tag{10.12}$$

is the generating polynomial for $\mathfrak{S}_n$ by the bi-statistic $(1+\mathrm{des},\mathrm{inv})$, *i.e.,*

$${}^{\mathrm{inv}}A_n(t,q) = \sum_{\sigma\in\mathfrak{S}_n} t^{1+\mathrm{des}\,\sigma} q^{\mathrm{inv}\,\sigma}. \tag{10.13}$$

To prove Theorem 10.1 we will show that the induction formula (10.11) holds for the polynomial defined by (10.13). For $k=0,1,\ldots,(n-1)$ let $u_k := \begin{bmatrix} n\\ k\end{bmatrix} {}^{\mathrm{inv}}A_k(t,q)\,t$; then, by iteration on $k=0,1,2,\ldots,(n-1)$, define

$$G_{-1} := 0; \qquad G_k := u_k + (1-t)G_{k-1}. \tag{10.14}$$

We see that for proving the theorem it suffices to show that:

$${}^{\mathrm{inv}}A_n(t,q) = G_{n-1}. \tag{10.15}$$

But (10.15) follows immediately from the following lemma.

Lemma 10.2. *For each $k=0,1,\ldots,(n-1)$ the polynomial G_k defined by the recurrence* (10.14) *is the generating polynomial, by the bi-statistic* $(1+\mathrm{des},\mathrm{inv})$, *for the set of the permutations of order n, whose longest increasing right factor is of length at least equal to $(n-k)$, i.e., permutations $\sigma=\sigma(1)\ldots\sigma(n)$ such that $\sigma(k+1)<\sigma(k+2)<\cdots<\sigma(n)$.*

Proof. By Proposition 4.3 we have

$$\begin{bmatrix} n\\ k\end{bmatrix} = \sum_{(A,B)} q^{\mathrm{inv}(\gamma(A)\gamma(B))},$$

where the sum is over all ordered partitions (A,B) of $[\,n\,]$ into two blocks such that $|A|=k$ and $|B|=n-k$. Recall that $\gamma(A)$ and $\gamma(B)$ designate the *increasing* words whose letters are the elements of A and of B, respectively. With $\mathfrak{S}_A$ denoting the group of the permutations of A we have

$$\begin{aligned}\begin{bmatrix} n\\ k\end{bmatrix} {}^{\mathrm{inv}}A_k(t,q) &= \sum_{(A,B)}\sum_{\tau\in\mathfrak{S}_k} q^{\mathrm{inv}(\gamma(A)\gamma(B))+\mathrm{inv}\,\tau} t^{1+\mathrm{des}\,\tau}\\ &= \sum_{(A,B)}\sum_{\tau\in\mathfrak{S}_A} q^{\mathrm{inv}(\gamma(A)\gamma(B))+\mathrm{inv}\,\tau} t^{1+\mathrm{des}\,\tau},\end{aligned}$$

since A is of cardinality k. As the $(n-k)$ terms of $\gamma(B)$ are in increasing order, the mapping $(\gamma(A)\gamma(B),\tau)\mapsto\sigma$ defined by $\sigma:=\tau\gamma(B)$ is a bijection onto the set of the permutations σ, whose longest increasing right factor is of length at least equal to $(n-k)$. Moreover $\operatorname{inv}\sigma = \operatorname{inv}\gamma(A)\gamma(B)+\operatorname{inv}\tau$ and $\operatorname{des}\sigma=\operatorname{des}\tau+\chi(\sigma(k)>\sigma(k+1))$. Let $\mathfrak{S}_{n,k}$ be the set of the permutations whose longest increasing right factor is *exactly* of length $(n-k)$ and F_k be the generating polynomial for $\mathfrak{S}_{n,k}$ by the bi-statistic $(1+\operatorname{des},\operatorname{inv})$. Then

$$\begin{aligned}\begin{bmatrix}n\\k\end{bmatrix}{}^{\operatorname{inv}}A_k(t,q)&=\sum_{\sigma\in\mathfrak{S}_{n,0}\cup\cdots\cup\mathfrak{S}_{n,k}} q^{\operatorname{inv}\sigma}t^{1+\operatorname{des}\sigma-\chi(\sigma(k)>\sigma(k+1))}\\&=\sum_{\sigma\in\mathfrak{S}_{n,0}\cup\cdots\cup\mathfrak{S}_{n,k-1}} q^{\operatorname{inv}\sigma}t^{1+\operatorname{des}\sigma}+t^{-1}\sum_{\sigma\in\mathfrak{S}_{n,k}} q^{\operatorname{inv}\sigma}t^{1+\operatorname{des}\sigma}\\&=F_0+\cdots+F_{k-1}+t^{-1}F_k.\end{aligned}$$

Hence, by letting $G_k:=F_0+\cdots+F_k$ and by multiplying the identity by t,

$$u_k=tG_{k-1}+(G_k-G_{k-1})=G_k+(t-1)G_{k-1}$$

and then

$$G_k=u_k+(1-t)G_{k-1},$$

which is precisely the induction relation (10.14). $\square$

The polynomials ${}^{\operatorname{maj}}A_n(t,q)$ and ${}^{\operatorname{inv}}A_n(t,q)$ form two q-analogs of the Eulerian polynomial $A_n(t)$. The polynomials $t\,{}^{\operatorname{maj}}A_n(t,q)$ and ${}^{\operatorname{inv}}A_n(t,q)$ already differ for $n=4$. Notice that Theorem 10.1 implies that ${}^{\operatorname{inv}}A_n(t,q)$ is a polynomial with *nonnegative* integral coefficients.

Table of the polynomials ${}^{\operatorname{maj}}A_n(t,q)$ and ${}^{\operatorname{inv}}A_n(t,q)$.

${}^{\operatorname{maj}}A_1(t,q)=1$; ${}^{\operatorname{maj}}A_2(t,q)=1+tq$; ${}^{\operatorname{maj}}A_3(t,q)=1+2tq(q+1)+t^2q^3$;
${}^{\operatorname{maj}}A_4(t,q)=1+tq(3q^2+5q+3)+t^2q^3(3q^2+5q+3)+t^3q^6$.
${}^{\operatorname{inv}}A_1(t,q)=t$; ${}^{\operatorname{inv}}A_2(t,q)=t+t^2q$; ${}^{\operatorname{inv}}A_3(t,q)=t+2t^2q(q+1)+t^3q^3$;
${}^{\operatorname{inv}}A_4(t,q)=t+t^2(q^4+3q^3+4q^2+3q)+t^3(3q^5+4q^4+3q^3+q^2)+t^3q^6$.

11. Major Index and Inversion Number

Again, let $R(\mathbf{m})$ be the set of all rearrangements of $1^{m_1}2^{m_2}\dots r^{m_r}$. In theorems 6.1 and 6.3 it was proved that the generating polynomial for $R(\mathbf{m})$ by the Inversion Number "inv", on the one hand, and by the Major Index "maj", on the other hand, was equal to the q-multinomial coefficient $\begin{bmatrix} m \\ m_1, m_2, \ldots, m_r \end{bmatrix}$, where $m = m_1 + m_2 + \cdots + m_r$. If we write those two generating polynomials under the form

$$\sum_{k \geq 0} q^k \, |\{w \in R(\mathbf{m}) : \operatorname{inv} w = k\}| \quad \text{and} \quad \sum_{k \geq 0} q^k \, |\{w \in R(\mathbf{m}) : \operatorname{maj} w = k\}|,$$

we see that for every $k \geq 0$ we have:

$$|\{w \in R(\mathbf{m}) : \operatorname{inv} w = k\}| = |\{w \in R(\mathbf{m}) : \operatorname{maj} w = k\}|. \tag{11.1}$$

Hence, for every $k \geq 0$ there *exists* a bijection of the set $\{w \in R(\mathbf{m}) : \operatorname{inv} w = k\}$ onto the set $\{w \in R(\mathbf{m}) : \operatorname{maj} w = k\}$; this is equivalent to saying that there *exists* a bijection Φ de $R(\mathbf{m})$ onto $R(\mathbf{m})$ with the property that

$$\operatorname{maj} w = \operatorname{inv} \Phi(w), \tag{11.2}$$

for every $w \in R(\mathbf{m})$. This brings up the problem of *constructing* such a bijection, that is to say, of inventing an explicit algorithm that transforms a word $w \in R(\mathbf{m})$ into a word $w' \in R(\mathbf{m})$ such that $\operatorname{maj} w = \operatorname{inv} w'$ in a one-to-one manner.

Of course, when all the m_i's are equal to 1 and the rearrangement class $R(\mathbf{m})$ is simply the symmetric group $\mathfrak{S}_r$, the construction of such a bijection can be made by means of the maj- and inv-codings introduced in section 2. For arbitrary rearrangement classes we have to follow another route, but the route will be richer, as further properties will be given for free, in particular when we restrict ourselves to the symmetric group.

11.1. *How to construct a bijection.* The proofs of Theorems 6.1 and 6.3 were so different in nature that there was no hint for imagining any immediate construction. However, if we make up a table of the first values of the Major Index and Inversion Number for small classes $R(\mathbf{m})$, we *observe* a further property. Let $L(w)$ ("L" for "last") be the *last* letter of the word w. Then, for every $k \geq 0$ and $x \in X = \{1, 2, \ldots, r\}$ we have:

$$\begin{aligned} |\{w \in R(\mathbf{m}) : \operatorname{inv} w = k,\ L(w) = x\}| \\ = |\{w \in R(\mathbf{m}) : \operatorname{maj} w = k,\ L(w) = x\}|. \end{aligned} \tag{11.3}$$

In Fig. 11.1 the values of those two statistics have been calculated for the words of the set $R(2,1,1)$, the rearrangements of the word $1,1,2,3$. In the first, second, third table the words ending by 1, 2, 3 are respectively listed. We can observe that the distribution of "maj" and "inv" in each table is the same.

w	maj	inv
$1,2,3,1$	3	2
$1,3,2,1$	5	3
$2,1,3,1$	4	3
$2,3,1,1$	2	4
$3,1,2,1$	4	4
$3,2,1,1$	3	5

w	maj	inv
$1,1,3,2$	3	1
$1,3,1,2$	2	2
$3,1,1,2$	1	3

w	maj	inv
$1,1,2,3$	0	0
$1,2,1,3$	2	1
$2,1,1,3$	1	2

Fig. 11.1

For each letter x belonging to a linearly ordered alphabet X and each word w let $\mathrm{bot}_x(w)$ (resp. $\mathrm{top}_x(w)$, resp. $|w|_x$) denote the number of letters in w which are less than or equal to (resp. greater than, resp. equal to) x. In particular,

$$\mathrm{bot}_x(w) + \mathrm{top}_x(w) = |w| \quad \text{(length of } w). \tag{11.4}$$

Further, let $R(\mathbf{m})x$ be the set of words wx, where $w \in R(\mathbf{m})$. If w' is a rearrangement of w, the following properties hold:

$$\mathrm{inv}\, wx = \mathrm{inv}\, w + \mathrm{top}_x(w); \tag{11.5}$$

$$\mathrm{maj}\, wx = \begin{cases} \mathrm{maj}\, w, & \text{if } L(w) \le x\ ; \\ \mathrm{maj}\, w + \mathrm{bot}_x(w) + \mathrm{top}_x(w), & \text{if } L(w) > x. \end{cases} \tag{11.6}$$

Suppose that property (11.3) holds for every class $R(\mathbf{m})$. Then there exists a bijection $w \mapsto w'$ of $R(\mathbf{m})$ onto itself such that $\mathrm{maj}\, w = \mathrm{inv}\, w'$ *and* $L(w) = L(w')$. In the same manner, the letter x being given, there also exists a bijection $w \mapsto w''$ such that $\mathrm{maj}\, wx = \mathrm{inv}\, w''x$.

If $L(w) \le x$, we then have:

$$\begin{aligned} L(w') &= L(w); \\ \mathrm{inv}\, w' &= \mathrm{maj}\, w \\ &= \mathrm{maj}\, wx && \text{[by (11.6)]} \\ &= \mathrm{inv}\, w''x \\ &= \mathrm{inv}\, w'' + \mathrm{top}_x(w') && \text{[by (11.5)].} \end{aligned}$$

If $L(w) > x$, we also have:

$$\begin{aligned} L(w') &= L(w); \\ \mathrm{inv}\, w' &= \mathrm{maj}\, w \\ &= \mathrm{maj}\, wx - \mathrm{bot}_x(w) - \mathrm{top}_x(w) && \text{[by (11.6)]} \\ &= \mathrm{inv}\, w''x - \mathrm{bot}_x(w) - \mathrm{top}_x(w) \\ &= \mathrm{inv}\, w'' - \mathrm{bot}_x(w') && \text{[by (11.5)].} \end{aligned}$$

Consequently, if property (11.3) holds, there exists a bijection γ_x of $R(\mathbf{m})$ onto itself, namely $w' \mapsto w''$, having the property

$$\text{inv}\,\gamma_x(w') = \begin{cases} \text{inv}\,w' - \text{top}_x(w'), & \text{if } L(w') \le x\ ; \\ \text{inv}\,w' + \text{bot}_x(w'), & \text{if } L(w') > x. \end{cases} \tag{11.7}$$

Conversely, if there exists a bijection $\gamma_x : w' \mapsto w''$ that sastisfies (11.7), we can define a bijection Φ of each class of rearrangements of words onto itself by letting

$$\Phi(w) := w, \tag{11.8}$$

if w is of length 1, and for each nonempty word w and each letter x by letting

$$\Phi(wx) := \gamma_x(\Phi(w))x. \tag{11.9}$$

Thus by induction we determine the image $\Phi(w)$ of w, then apply the bijection γ_x to $\Phi(w)$, finally the letter x is juxtaposed at the end of the word.

Theorem 11.1. *The following two statements are equivalent:*

(a) *Property* (11.3) *holds for every class.*

(b) *For every letter x there exists a bijection $\gamma_x : w' \mapsto w''$ such that property* (11.7) *holds; moreover, the bijection Φ defined by* (11.8) *and* (11.9) *has the properties:*

$$\text{maj}\,w = \text{inv}\,\Phi(w) \quad \text{and} \quad L(w) = L(\Phi(w)). \tag{11.10}$$

Proof. The implication (b) $\Rightarrow$ (a) is straightforward. To prove the converse it suffices to verify (11.10). First, $L(w) = L(\Phi(w))$ by the definition of Φ given in (11.9). Then, for each nonempty word w and each letter x such that $L(w) \le x$ we have:

$$\begin{aligned} \text{inv}\,\Phi(wx) &= \text{inv}\,\gamma_x(\Phi(w))x \\ &= \text{inv}\,\gamma_x(\Phi(w)) + \text{top}_x(w) && \text{by (11.5)]} \\ &= (\text{inv}\,\Phi(w) - \text{top}_x(\Phi(w))) + \text{top}_x(w) && \text{[by (11.7)]} \\ &= \text{maj}\,w \\ &= \text{maj}\,wx. \end{aligned}$$

If $L(w) > x$, we have:

$$\begin{aligned} \text{inv}\,\Phi(wx) &= \text{inv}\,\gamma_x(\Phi(w)) + \text{top}_x(w) \\ &= (\text{inv}\,\Phi(w) + \text{bot}_x(w)) + \text{top}_x(w) \\ &= \text{maj}\,w + |w| \\ &= \text{maj}\,wx. \quad \square \end{aligned}$$

Thus, the construction of a bijection Φ boils down to the constructions of bijections γ_x having property (11.7).

11.2. *The binary case.* Let a, b be two elements of the alphabet such that $a < b$ and consider a (binary) word w' of the class $R(m_a, m_b)$ with $m_a, m_b \geq 1$. We have

$$\operatorname{bot}_x(w') = \begin{cases} 0, & \text{if } x < a; \\ |w'|_a, & \text{if } a \leq x < b; \\ |w'|, & \text{if } b \leq x; \end{cases}$$

$$\operatorname{top}_x(w') = \begin{cases} |w'|, & \text{if } x < a; \\ |w'|_b, & \text{if } a \leq x < b; \\ 0, & \text{if } b \leq x. \end{cases}$$

Condition (11.7) can be rewritten as

$$\operatorname{inv} \gamma_x(w') = \begin{cases} \operatorname{inv} w', & \text{if } x < a; \\ \operatorname{inv} w' - |w'|_b, & \text{if } L(w') = a \leq x < b; \\ \operatorname{inv} w' + |w'|_a, & \text{if } a \leq x < L(w') = b; \\ \operatorname{inv} w', & \text{if } b = L(w') \leq x. \end{cases} \tag{11.11}$$

We can take the identity map for γ_x when $x < a$ or $b = L(w') \leq x$. Let vy belong to $R(m_a, m_b)$ with y the last *letter* equal to a or b. The *most straightforward* transformation we can think of for γ_x in the remaining two cases is

$$\gamma_x(vy) := yv, \tag{11.12}$$

which is obviously bijective and satisfies (11.11). The next theorem is then a consequence of Theorem 11.1.

Theorem 11.2. *Let $\{a, b\}$ be a two-letter alphabet $(a < b)$ and for each $x = a, b$ let γ_x be defined, for every binary word v in the letters a, b, by*

$$\gamma_a(va) = av \quad \text{and} \quad \gamma_b(vb) = vb. \tag{11.13}$$

Then, the transformation Φ, as defined in (11.8) and (11.9), is a bijection of every rearrangement class onto itself having the property:

$$\operatorname{maj} w = \operatorname{inv} \Phi(w) \quad \text{and} \quad L(w) = L(\Phi(w)).$$

Example. Starting with $w = 0, 0, 1, 0, 1, 1, 0$, so that $\operatorname{maj} w = 3+6 = 9$, we successively have:

v		$\Phi(v)$
0		0
$0,0$	$\gamma_0(0)=0$	$0,0$
$0,0,1$	$\gamma_1(0,0)=0,0$	$0,0,1$
$0,0,1,0$	$\gamma_0(0,0,1)=1,0,0$	$1,0,0,0$
$0,0,1,0,1$	$\gamma_1(1,0,0,0)=1,0,0,0$	$1,0,0,0,1$
$0,0,1,0,1,1$	$\gamma_1(1,0,0,0,1)=1,0,0,0,1$	$1,0,0,0,1,1$
$0,0,1,0,1,1,0$	$\gamma_0(1,0,0,0,1,1,)=1,1,0,0,0,1$	$1,1,0,0,0,1,0$

Thus $\Phi(w)=1,1,0,0,0,1,0$ and $\operatorname{inv}\Phi(w)=9=\operatorname{maj} w$.

11.3. *From the binary to the general case.* The next step is to start with the *natural* bijection γ_x, introduced in (11.13) and see how it can be extended to arbitrary words while keeping property (11.7).

For each $n\ge 1$ let $E_n:=\{0,1,2,\dots,n\}$ and let E_n^* denote the set of all finite words whose letters belong to E_n. For $x, x_i\in E_n$ define

$$\beta_x(x_i):=\begin{cases}0, & \text{if } x_i\le x;\\ 1, & \text{if } x_i>x;\end{cases} \tag{11.14}$$

and for each word $w=x_1x_2\dots x_m\in E_n^*$ let

$$u:=\beta_x(x_1)\beta_x(x_2)\dots\beta_x(x_m).$$

If $i_1<i_2<\dots<i_a$ (resp. $j_1<j_2<\dots<j_b$) is the sequence of the subscripts i such that $x_i\le x$ (resp. subscripts j such that $x_j>x$), let w_0, w_1 be the subwords: $w_0=x_{i_1}x_{i_2}\dots x_{i_a}$ and $w_1=x_{j_1}x_{j_2}\dots x_{j_b}$. Also let

$$B_x(w):=(u,w_0,w_1). \tag{11.15}$$

Proposition 11.3. *For each $x\in E_n$ the mapping B_x defined in* (11.15) *is a bijection of E_n^* onto the set of triples (u,w_0,w_1) such that $|w|=|u|$, $|u|_0=|w_0|$ and $|u|_1=|w_1|$ having the further property that*

$$\operatorname{inv} w=\operatorname{inv} u+\operatorname{inv} w_0+\operatorname{inv} w_1. \tag{11.16}$$

Proof. Relation (11.16) simply indicates a sorting of the inversions within the word w. The bijective property is obvious. ∎

Now, let w be a word in E_n^* and let x be a letter. Form the chain

$$w\xrightarrow{B_x}(u,w_0,w_1)\mapsto(\gamma_x(u),w_0,w_1)\xrightarrow{B_x^{-1}}w', \tag{11.17}$$

where B_x^{-1} designates the inverse of B_x, and define $\gamma_x(w):=w'$.

When w is a binary word, *i. e.* $n=1$, the above chain gives back the value of $\gamma_x(w)$ defined in the previous subsection. Assume $n\ge 2$ and let $w=x_1x_2\dots x_m$ be an arbitrary word. If all its letters are greater than (resp. less than or equal to) x, then $B_w(w)=(1^m,e,w)$ (resp. $B_x(w)=(0^m,w,e)$), so that $\gamma_x(w)=w$. Condition (11.7) holds.

If $L(w)\le x$ and there is at least one letter of w greater than x, then $L(u)=0$ and $\operatorname{inv}\gamma_x(w)=\operatorname{inv}\gamma_x u+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} u-|u|_1+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} u-\operatorname{top}_x(w)+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} w-\operatorname{top}_x(w)$.

If $L(w)>x$ and there is at least one letter of w less than or equal to x, then $L(u)=1$ and $\operatorname{inv}\gamma_x(w)=\operatorname{inv}\gamma_x u+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} u+|u|_0+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} u+\operatorname{bot}_x(w)+\operatorname{inv} w_0+\operatorname{inv} w_1=\operatorname{inv} w+\operatorname{bot}_x(w)$. Thus, condition (11.7) always holds.

Accordingly, our program is fulfilled: by means of the bijections defined in (11.13) for binary words, then, the bijections B_x defined in (11.15) and finally the chain (11.17), we hold a family of bijections γ_x that, when incorporated in the definition of Φ given in (11.9), provide a bijection having properties (11.10).

The three steps (11.13), (11.15), (11.17) can be combined and described more quickly by making use of the *x-factorisation* defined as follows: let x be a letter and w a word. If the last letter $L(w)$ of w is less than or equal to (resp. greater than) x, the word w admits the unique factorization:

$$(v_1y_1,v_2y_2,\dots,v_py_p),$$

called its *x-factorisation* having the following properties:

(i) each y_i $(1\le i\le p)$ is a *letter* verifying $y_i\le x$ (resp. $y_i>x$) ;

(ii) each v_i $(1\le i\le p)$ is a factor which is either empty or has all its letters greater than (resp. smaller than or equal to) x.

We then let:

$$\gamma_x(w):=y_1v_1y_2v_2\dots y_pv_p. \tag{11.18}$$

We see again that γ_x is a bijection of each class $R(\mathbf{m})$ onto itself, since it maps each x-factorisation $(v_1y_1,v_2y_2,\dots,v_py_p)$, obtained by cutting the word *after* every letter less than or equal to (resp. greater than) x onto the factorisation $(y_1v_1,y_2v_2,\dots,y_pv_p)$ obtained by cutting the word *before* every letter less than or equal to (resp. greater than) x.

The transformation Φ itself, as defined in (11.9), can also be described in the following algorithmic manner:

Algorithm for Φ. *Let $w = x_1x_2\dots x_m$;*

1. *Put $i := 1$, $w'_i := x_1$;*
2. *If $i = m$, let $\Phi(w) := w'_i$ and stop; else continue;*
3. *If the last letter of w'_i is less than or equal to (resp. greater than) x_{i+1}, cut w'_i after every letter less than or equal to (resp. greater than) x_{i+1} ;*
4. *In each compartment of w'_i determined by the previous cuttings, move the last letter in the compartment to the beginning of it; let v' be the word obtained after all those moves; put $w'_{i+1} := v'x_{i+1}$; replace i by $i+1$ and go to* 2.

For example, the image of $w = 4,3,5,1,1,3,4,2,3$ under Φ is obtained as follows:

$$\begin{aligned}
w'_1 &= 4 \mid \\
w'_2 &= 4 \mid 3 \mid \\
w'_3 &= 4 \mid 3 \mid 5 \mid \\
w'_4 &= 4\,3\,5\,1 \mid \\
w'_5 &= 1 \mid 4\,3 \mid 5\,1 \mid \\
w'_6 &= 1 \mid 3 \mid 4 \mid 1 \mid 5\,3 \mid \\
w'_7 &= 1\,3 \mid 4 \mid 1\,3 \mid 5 \mid 4 \mid \\
w'_8 &= 3 \mid 1 \mid 4\,3 \mid 1 \mid 5\,4\,2 \mid \\
\Phi(w) = w'_9 &= 3,1,3,4,1,2,5,4,3.
\end{aligned}$$

The descents in the word $w = 4,3,5,1,1,3,4,2,3$ are in position 1, 3 and 7, so that $\operatorname{maj} w = 11$. But $\Phi(w) = 3,1,3,4,1,2,5,4,3$ has $\operatorname{inv}\Phi(w) = 11$ inversions. Finally, w and $\Phi(w)$ end with the same letter, namely 3.

11.4. *Further properties of the transformation.* As we now see, the transformation Φ just defined preserves other statistics, first, the set-valued statistic *inverse ligne of route* and also the subword-valued statistics *right-to-left minimum letter subword* and *right-to-left maximum letter subword* we now define.

If a multiplicity $\mathbf{m}$ reads $\mathbf{m} = (m_1,\dots,m_i,0,\dots,0,m_j,\dots,m_r)$ with $1 \le i < j \le r$ and $m_i, m_j \ge 1$, we say that j is the *succcessor* of i in $\mathbf{m}$ and we write $j = \operatorname{succ} i$. Of course, if all the components of $\mathbf{m}$ are positive, the successor of each i $(1 \le i \le r-1)$ is $(i+1)$. Let w be a word in the class $R(\mathbf{m})$.

Definition. The *inverse ligne of route* of w is defined to be the set, denoted by $\operatorname{Iligne} w$, of all the letters i such that the rightmost occurrence of $\operatorname{succ}(i)$ lies *to the left* of the rightmost occurrence of i. In an equivalent manner, the letter i is said to belong to the *inverse ligne of route* of w, if w can be written $v\,\mathrm{succ}(i)\,v'\,i\,v''$, where the factor v'' contains no letter equal to $\mathrm{succ}(i)$.

For example, consider the rearrangement $w = 4\,3\,6\,1\,1\,3\,4\,2\,3$ of the nondecreasing word $1^2 2 3^3 4^2 6$. The inverse ligne of route of w is $\mathrm{Iligne}\,w = \{3,4\}$. Notice that $4 \in \mathrm{Iligne}\,w$, since $\mathrm{succ}(4) = 6$ is located *on the left* of the rightmost occurrence of 4. Also $1 \notin \mathrm{Iligne}\,w$, since there is a letter 2 *on the right* of the rightmost occurrence of 1.

Remark. The expression "line of route" is classical; we have added the letter "g" making up "ligne of route," thus bringing a slight touch of French. The *ligne of route* of a word $w = x_1 x_2 \ldots x_m$ is defined to be the set, denoted by $\mathrm{Ligne}\,w$, of all the i's such that $1 \le i \le m-1$ and $x_i > x_{i+1}$.

Definition. Let $w = x_1 x_2 \ldots x_m$ be a word of length m; denote by $1 \le i_1 < i_2 < \cdots < i_a$ the sequence of all the i's such that $i_a = m$ and $x_i \le x_k$ for all $k \ge i$. The *right-to-left minimum* **place** *subword* and the *right-to-left minimum* **letter** *subword* of w are respectively defined by:

$$\begin{aligned} \mathrm{Rmip}(w) &:= i_1 i_2 \ldots i_a;\\ \mathrm{Rmil}(w) &:= x_{i_1} x_{i_2} \ldots x_{i_a}. \end{aligned}$$

In the same manner, let $1 \le j_1 < j_2 < \cdots < j_b$ be the sequence of the integers j's such that $j_b = m$ and $x_j > x_k$ for all $k > j$. The *right-to-left maximum* **place** *subword* and *right-to-left maximum* **letter** *subword* of w are defined by

$$\begin{aligned} \mathrm{Rmap}(w) &:= j_1 j_2 \ldots j_b;\\ \mathrm{Rmals}(w) &:= x_{j_1} x_{j_2} \ldots x_{j_b}. \end{aligned}$$

For example,

$$\begin{array}{r|cccccccccc} & 1&2&3&4&5&6&7&8&9&10\\ \mathrm{Rmap}(w) = & & &3& & & &7& & &10\\ \mathrm{Rmip}(w) = & & & &4&5& & &8&9&10\\ w = &4&3&5&1&1&3&4&2&3&3\\ \mathrm{Rmil}(w) = & & & &1&1& & &2&3&3\\ \mathrm{Rmals}(w) = & & &5& & & &4& & &3 \end{array}$$

Notice that the words $\mathrm{Rmip}(w)$ and $\mathrm{Rmap}(w)$ are always *strictly increasing*, as they are subwords of the increasing word $1\,2\,\ldots\,m$; also observe the discrepancy between $\mathrm{Rmals}(w)$, which is *strictly decreasing* and $\mathrm{Rmil}(w)$, which is increasing *in the large sense*. This explains the presence of the "s" in our notation for "Rmals." Finally, let $y := x_{i_1}$, then $\operatorname{Rmil} w = y^{m_y} v$, where v has no letter equal to y.

Four other analogous definitions "Lmip," "Lmap," "Lmil" and "Lmals" can be introduced by considering the subwords *from left to right*, instead of right to left. We will do it later on for permutations only.

Remark. For a permutation w the numerical statistics $\# \operatorname{Lmil} w$ and $\# \operatorname{Lmals} w$ have been considered long ago by probabilists under the names of *lower records* and *upper records*.

Theorem 11.3. *Let Φ be the transformation constructed in subsection* 11.3. *Then, the following properties hold for every word w:*

(a) $\operatorname{maj} w = \operatorname{inv} \Phi(w)$;

(b) $\Phi(w)$ *is a rearrangement of w and the restriction of Φ to each rearrangement class $R(\mathbf{m})$ is a bijection of $R(\mathbf{m})$ onto itself;*

(c) $\operatorname{Iligne} w = \operatorname{Iligne} \Phi(w)$;

(d) $\operatorname{Rmil} w = \operatorname{Rmil} \Phi(w)$;

(e) $\operatorname{Rmals} w = \operatorname{Rmals} \Phi(w)$.

The next corollary is an immediate consequence of Theorem 1.2.

Corollary 11.4. *Let A be a finite set of integers, u, v be two words and $R(\mathbf{m})$ be a rearrangement class. Then, the statistics "maj" and "inv" are equidistributed on the subclass*

$$\{w \in R(\mathbf{m}) : (\operatorname{Iligne}, \operatorname{Rmil}, \operatorname{Rmals})\, w = (A, u, v)\}.$$

In Fig. 11.2 the values of these statistics have been calculated for the words belonging to the class $R(2,1,1)$, the rearrangements of the word $1,1,2,3$. There are five tables corresponding to five subclasses characterized by a value (A, u, v) of the triple $(\operatorname{Iligne}, \operatorname{Rmil}, \operatorname{Rmals})$. Notice that within each subclass the statistics "inv" and "maj" are equidistributed.

When Φ is restricted to $\mathfrak{S}_n$, the group of permutations of $1, 2, \ldots, n$, i.e., the rearrangement class $R(1^n)$, several properties can also be derived. For each permutation w let

$$\operatorname{imaj} w := \sum_i i\, \chi(i \in \operatorname{Iligne} w).$$

It is immediate to verify

$$\operatorname{imaj} w = \operatorname{maj} w^{-1},$$

where w^{-1} denotes the inverse permutation of w. We then deduce the following corollary that is proved in section 5.

w	maj	inv	Iligne	Rmil	Rmals
$1,2,3,1$	3	2	1	$1,1$	$3,1$
$2,1,3,1$	4	3	1	$1,1$	$3,1$
$2,3,1,1$	2	4	1	$1,1$	$3,1$

w	maj	inv	Iligne	Rmil	Rmals
$1,1,3,2$	3	1	2	$1,1,2$	$3,2$
$1,3,1,2$	2	2	2	$1,1,2$	$3,2$
$3,1,1,2$	1	3	2	$1,1,2$	$3,2$

w	maj	inv	Iligne	Rmil	Rmals
$1,3,2,1$	5	3	$1,2$	$1,1$	$3,2,1$
$3,1,2,1$	4	4	$1,2$	$1,1$	$3,2,1$
$3,2,1,1$	3	5	$1,2$	$1,1$	$3,2,1$

w	maj	inv	Iligne	Rmil	Rmals
$1,1,2,3$	0	0	$\emptyset$	$1,1,2,3$	3

w	maj	inv	Iligne	Rmil	Rmals
$1,2,1,3$	2	1	1	$1,1,3$	3
$2,1,1,3$	1	2	1	$1,1,3$	3

Fig. 11.2

Corollary 11.4. *The two statistics* "inv" *and* "imaj" *are equally distributed on each set of permutations having a given ligne of route A, a given left-to-right maximum place subword C and a given right-to-left maximum place subword D. In other words, let*

$$S := \{w \in \mathfrak{S}_n : \operatorname{Ligne} w = A, \operatorname{Lmap} w = C, \operatorname{Rmap} w = D\};$$

then

$$\sum_{w\in S} q^{\operatorname{inv} w} = \sum_{w\in S} q^{\operatorname{imaj} w}.$$

The preceding corollary is an extension of the classical result (see, e.g. [Lo02, Theorem 11.4.4]), where no restriction is made neither on "Lmap," nor on "Rmap", but only on "Ligne."

Proof of Theorem 11.3. Properties (a) and (b) have been proved with Theorem 11.1. For the remaining ones we proceed as follows:

Proof of Property (c).

Consider the mapping $wx \mapsto \Phi(wx)$ defined by (11.9), where wx belongs to the rearrangement class $R(\mathbf{m})$. Let z a letter of wx having a successor $z' := \operatorname{succ}(z)$ in $\mathbf{m}$. Suppose $L(w) \le x$ (resp. $L(w) > x$). There are four cases to be considered:

(i) $z < z' < x$; let $\mathcal{P}$ be the property "the rightmost occurrence of z' is to the left of the rightmost occurrence of z." Then, all the following properties are equivalent: (1) $\mathcal{P}$ holds for wx; (2) $\mathcal{P}$ holds for w; (3) $\mathcal{P}$ holds for $\Phi(w)$ [by induction]; (4) $\mathcal{P}$ holds for $y_1y_2\ldots y_p$; (resp. (4) $\mathcal{P}$ holds for $v_1v_2\ldots v_p$;) (5) $\mathcal{P}$ holds for $y_1v_1y_2v_2\ldots y_pv_p = \gamma_x(\Phi(w))$; (6) $\mathcal{P}$ holds for $\Phi(wx) = \gamma_x(\Phi(w))x$.

(ii) $x < z < z'$; same proof, only property (4) is to reverse: (4) $\mathcal{P}$ holds for $v_1v_2\ldots v_p$; (resp. (4) $\mathcal{P}$ holds for $y_1y_2\ldots y_p$;)

(iii) $x = z < z'$; then $x \in \mathrm{Iligne}\, wx$ and $x \in \mathrm{Iligne}\, \Phi(wx)$, since $L(\Phi(w)) = L(w) = x$.

(iv) $z < z' = x$; then $x \notin \mathrm{Iligne}\, wx$ and $x \notin \mathrm{Iligne}\, \Phi(wx)$. ∎

Proof of Property (d).

Assume that (d) holds for a nonempty word w and let x be a letter. If all the letters of wx are equal, the result is banal. Otherwise, let $\mathrm{Rmil}(w) := x_{i_1} x_{i_2} \dots x_{i_a}$, so that, by induction $\mathrm{Rmil}\, \Phi(w) = x_{i_1} x_{i_2} \dots x_{i_a}$ Notice that $L(w) = L(\Phi(w)) = x_m = x_{i_a}$ and the smallest letter of w is equal to x_{i_1}.

If all the letters of w are less than x, then $\mathrm{Rmil}(wx) = x = \mathrm{Rmil}\, \Phi(wx)$. If it is not the case, but still if $L(w) \le x$, then

$$\begin{aligned}
x_{i_1} x_{i_2} \dots x_{i_a} x &= \mathrm{Rmil}(wx) \\
&= \mathrm{Rmil}(\Phi(w)x) && \text{[by induction]} \\
&= \mathrm{Rmil}(y_1 y_2 \dots y_p x) && \text{[by definition of the } x\text{-factorization]} \\
&= \mathrm{Rmil}(y_1 v_1 y_2 v_2 \dots y_p v_p x) \\
&= \mathrm{Rmil}\, \Phi(wx). && \text{[by definition of } \Phi]
\end{aligned}$$

If $L(w) > x$, there is a unique integer k such that $1 \le k \le a-1$ and $x_{i_1} \le \dots \le x_{i_k} \le x < x_{i_{k+1}} \le \dots \le x_{i_a}$. Then

$$\begin{aligned}
x_{i_1} \dots x_{i_k} x &= \mathrm{Rmil}(wx) \\
&= \mathrm{Rmil}(\Phi(w)x) && \text{[by induction]} \\
&= \mathrm{Rmil}(v_1 v_2 \dots v_p x) && \text{[by definition of the } x\text{-factorization]} \\
&= \mathrm{Rmil}(y_1 v_1 y_2 v_2 \dots y_p v_p x) \\
&= \mathrm{Rmil}\, \Phi(wx). \quad \blacksquare && \text{[by definition of } \Phi]
\end{aligned}$$

Proof of Property (e).

Again, the result is banal if all the letters of wx are identical. Otherwise, let $\mathrm{Rmals}(w) := x_{j_1} x_{j_2} \dots x_{j_b}$, so that $\mathrm{Rmals}\, \Phi(w) = x_{j_1} x_{j_2} \dots x_{j_b}$. If all the letters of w are less than or equal to x, then $\mathrm{Rmals}\, wx = x = \mathrm{Rmals}\, \Phi(wx)$. If it is not the case, but still $L(w) \le x$, there is a unique integer k such that $1 \le k \le b-1$ and $x_{j_1} > \dots > x_{j_k} > x \ge x_{j_{k+1}} > \dots > x_{j_b}$. Then

$$\begin{aligned}
x_{j_1} \dots x_{j_k} x &= \mathrm{Rmals}(wx) \\
&= \mathrm{Rmals}(\Phi(w)x) && \text{[by induction]} \\
&= \mathrm{Rmals}(v_1 v_2 \dots v_p x) && \text{[by definition of the } x\text{-factorization]} \\
&= \mathrm{Rmals}(y_1 v_1 y_2 v_2 \dots y_p v_p x) \\
&= \mathrm{Rmals}\, \Phi(wx). && \text{[by definition of } \Phi]
\end{aligned}$$

If $L(w) > x$, then

$$\begin{aligned}
x_{j_1}\dots x_{j_k}x &= \mathrm{Rmals}(wx)\\
&= \mathrm{Rmals}(\Phi(w)x) && \text{[by induction]}\\
&= \mathrm{Rmals}(v_1v_2\dots v_px) && \text{[by definition of the x-factorization]}\\
&= \mathrm{Rmals}(y_1v_1y_2v_2\dots y_pv_px)\\
&= \mathrm{Rmals}\,\Phi(wx). \quad \Box && \text{[by definition of Φ]}
\end{aligned}$$

This achieves the proof of Theorem 11.3.

11.5. *Application to permutations.* Consider the classes $R(\mathbf{m})$ of words without repeated letters (all the m_i's are equal to 1), i.e., the *permutations.* The *inverse ligne of route* of a permutation $w = x_1x_2\dots x_r$ (of the word $12\dots r$) is then the set of all the integers j such that $1 \le j \le r-1$ and $(j+1)$ is *to the left of* j within the word $x_1x_2\dots x_r$.

The *ligne of route* of a permutation w is the set, $\mathrm{Ligne}\,w$, of the integers j such that $x_j > x_{j+1}$, so that

$$\mathrm{des}\,w = |\,\mathrm{Ligne}\,w| \quad \text{and} \quad \mathrm{maj}\,w = \sum_j j \quad (j \in \mathrm{Ligne}\,w).$$

It is readily seen that

$$\mathrm{Iligne}\,w = \mathrm{Ligne}\,w^{-1}, \tag{11.19}$$

where w^{-1} designates the inverse of the permutation w. Also let

$$\mathrm{ides}\,w := |\,\mathrm{Iligne}\,w| \quad \text{and} \quad \mathrm{imaj}\,w := \sum_j j \quad (j \in \mathrm{Iligne}\,w).$$

The statistic "imaj" is called the *Inverse Major Index* of w. Also let $\mathbf{i}(w)$ denote the inverse w^{-1} of the permutation w. As $\mathrm{inv}\,\mathbf{i}(w) = \mathrm{inv}\,w$, it follows from the property (c) that the chain

$$w \overset{\mathbf{i}}{\mapsto} w_1 \overset{\Phi^{-1}}{\mapsto} w_2 \overset{\mathbf{i}}{\mapsto} w_3 \overset{\Phi}{\mapsto} w_4 \overset{\mathbf{i}}{\mapsto} w_5$$

has the properties:

$$\begin{gathered}
\mathrm{Ligne}\,w = \mathrm{Iligne}\,w_1 = \mathrm{Iligne}\,w_2 = \mathrm{Ligne}\,w_3;\\
\mathrm{des}\,w = \mathrm{ides}\,w_1 = \mathrm{ides}\,w_2 = \mathrm{des}\,w_3;\\
\mathrm{maj}\,w = \mathrm{imaj}\,w_1 = \mathrm{imaj}\,w_2 = \mathrm{maj}\,w_3 = \mathrm{inv}\,w_4 = \mathrm{inv}\,w_5;\\
\mathrm{inv}\,w = \mathrm{inv}\,w_1 = \mathrm{maj}\,w_2 = \mathrm{imaj}\,w_3 = \mathrm{imaj}\,w_4 = \mathrm{maj}\,w_5.
\end{gathered}$$

This result has several consequences that are now stated.

Corollary 11.5. *The six pairs* (maj, inv), (imaj, inv), (imaj, maj), (maj, imaj), (inv, imaj), (inv, maj) *are equally distributed on each symmetric group* $\mathfrak{S}_n$*, i.e., with* w *running over* $\mathfrak{S}_n$ *we have:*

$$\sum_w q_1^{\operatorname{maj} w} q_2^{\operatorname{inv} w} = \sum_w q_1^{\operatorname{maj} w} q_2^{\operatorname{inv} w} = \sum_w q_1^{\operatorname{imaj} w} q_2^{\operatorname{maj} w}$$
$$= \sum_w q_1^{\operatorname{maj} w} q_2^{\operatorname{imaj} w} = \sum_w q_1^{\operatorname{inv} w} q_2^{\operatorname{imaj} w} = \sum_w q_1^{\operatorname{inv} w} q_2^{\operatorname{maj} w}. \tag{11.20}$$

Corollary 11.6. *With* w *running over* $\mathfrak{S}_n$ *we have:*

$$^{\operatorname{inv}}A_n(t,q) = \sum_w t^{\operatorname{des} w} q^{\operatorname{inv} w} = \sum_w t^{\operatorname{des} w} q^{\operatorname{imaj} w}. \tag{11.21}$$

Proof. Consider the bijection $w \mapsto w_3$. ∎

Corollary 11.7. *The two statistics* "inv" *and* "imaj" *have the same distribution on each set of permutations having a given ligne of route. In other words, for every subset* $A \subset [n-1]$ *the following identity holds:*

$$\sum_w q^{\operatorname{inv} w} = \sum_w q^{\operatorname{imaj} w} \quad (w \in \mathfrak{S}_n,\ \operatorname{Ligne} w = A).$$

Proof. Again consider the bijection $w \mapsto w_3$. ∎

We make a further use of the transformations, $\mathbf{i}$, $\mathbf{c}$, $\mathbf{r}$ of the dihedral group. Recall that $\mathbf{c}$ is the *complement to* $(n+1)$ and $\mathbf{r}$ the *reverse image* that map the permutation w, written as a linear word $w = x_1 \dots x_n$, onto

$$\mathbf{c}\, w := (n+1-x_1)(n+1-x_2)\dots(n+1-x_n);$$
$$\mathbf{r}\, w := x_n\ \dots\ x_2 x_1.$$

As $\operatorname{inv} \mathbf{i}\, w = \operatorname{inv} w$ and also $\operatorname{Rmap} \mathbf{i}\, w = \operatorname{Rmals} w$, $\operatorname{Lmap} \mathbf{i}\, w = \operatorname{Rmil} w$ (easy to verify), it follows from Theorem 11.3 that the chain

$$w \overset{\mathbf{i}}{\mapsto} w_1 \overset{\Phi}{\mapsto} w_2 \overset{\mathbf{i}}{\mapsto} w_3$$

has the properties:

$$\operatorname{Ligne} w = \operatorname{Iligne} w_1 = \operatorname{Iligne} w_2 = \operatorname{Ligne} w_3;$$
$$\operatorname{Lmap} w = \operatorname{Rmil} w_1 = \operatorname{Rmil} w_2 = \operatorname{Lmap} w_3;$$
$$\operatorname{Rmap} w = \operatorname{Rmals} w_1 = \operatorname{Rmals} w_2 = \operatorname{Rmap} w_3;$$
$$\operatorname{imaj} w = \operatorname{maj} w_1 = \operatorname{inv} w_2 = \operatorname{inv} w_3.$$

Hence, for each triple (A, B, C) the bijection $w \mapsto w_3$ maps each set of permutations w such that $(\operatorname{Ligne}, \operatorname{Lmap}, \operatorname{Rmap})\, w = (A, B, C)$ onto itself with the property that $\operatorname{imaj} w = \operatorname{inv} w_3$. This proves the following corollary.

Corollary 11.8. *The two statistics* "inv" *and* "imaj" *are equally distributed on each set of permutations having a given ligne of route A, a given left-to-right maximum place subword C and a given right-to-left maximum place subword D. In other words, let*

$$S := \{w \in \mathfrak{S}_n : \operatorname{Ligne} w = A, \operatorname{Rmap} w = C, \operatorname{Lmap} w = D\};$$

then

$$\sum_{w\in S} q^{\operatorname{inv} w} = \sum_{w \in S} q^{\operatorname{imaj} w}.$$

There are other consequences we can deduce from Theorem 11.3 by taking the composition product of Φ with other operations of the dihedral group. First, we can verify that $\operatorname{inv} \mathbf{r}\,\mathbf{c}\, w = \operatorname{inv} w$ and $\operatorname{Ligne} \mathbf{r}\,\mathbf{c}\, w = n - \operatorname{Ligne} w := \{n - i : i \in \operatorname{Ligne} w\}$, so that

$$\operatorname{maj} \mathbf{r}\,\mathbf{c}\, w = \sum_i (n-i)\chi(x_i > x_{i+1}),$$

a statistic that will be denoted by $\operatorname{comaj} w$. Let $\operatorname{Lmil} w$ (resp. $\operatorname{Lmals} w$) denote the *left-to-right* minimum (resp. maximum) letter subword of the permutation w. Again, it is easy to verify that

$$\operatorname{Lmil} \mathbf{r}\,\mathbf{c}\, w = n + 1 - \operatorname{Rmals} w, \quad \operatorname{Lmals} \mathbf{r}\,\mathbf{c}\, w = n + 1 - \operatorname{Rmil} w.$$

Consider the sequence:

$$w \overset{\mathbf{r}\,\mathbf{c}}{\mapsto} w_1 \overset{\Phi}{\mapsto} w_2 \overset{\mathbf{r}\,\mathbf{c}}{\mapsto} w_3.$$

Then

$$\begin{gathered}
\operatorname{Iligne} w = n - \operatorname{Iligne} w_1 = n - \operatorname{Iligne} w_2 = \operatorname{Iligne} w_3;\\
\operatorname{Lmil} w = n + 1 - \operatorname{Rmals} w_1 = n + 1 - \operatorname{Rmals} w_2 = \operatorname{Lmil} w_3;\\
\operatorname{Lmals} w = n + 1 - \operatorname{Rmil} w_1 = n + 1 - \operatorname{Rmil} w_2 = \operatorname{Lmals} w_3;\\
\operatorname{comaj} w = \operatorname{maj} w_1 = \operatorname{inv} w_2 = \operatorname{inv} w_3.
\end{gathered}$$

This implies the following corollary.

Corollary 11.9. *The two statistics* "inv" *and* "imaj" *are equally distributed on each set of permutations having a given inverse ligne of route A, a given left-to-right minimum letter subword C and a given left-to-right maximum letter subword D. In other words, let*

$$S := \{w \in \mathfrak{S}_n : \operatorname{Iligne} w = A, \operatorname{Lmil} w = C, \operatorname{Lmals} w = D\}.$$

then

$$\sum_{w\in S} q^{\operatorname{inv} w} = \sum_{w \in S} q^{\operatorname{comaj} w};$$

that can also be expressed as:

$$\sum_{w\in S'} q^{\operatorname{inv} w} = \sum_{w\in S'} q^{\operatorname{comaj} w},$$

where $S' := \{w \in \mathfrak{S}_n : \operatorname{Ligne} w^{-1} = A,\ \operatorname{Lmip} w^{-1} = C,\ \operatorname{Lmap} w^{-1} = D\}$.

12. Major Index and Inverse Major Index

For each $n \geq 0$ let $A_n(q_1, q_2)$ be the generating polynomial for $\mathfrak{S}_n$ by the pair $(\operatorname{maj}, \operatorname{imaj})$:

$$A_n(q_1, q_2) := \sum_{\sigma\in\mathfrak{S}_n} q_1^{\operatorname{maj}\sigma} q_2^{\operatorname{imaj}\sigma}. \tag{12.1}$$

Corollary 11.3 shows that there are five other ways of expressing such a polynomial. By analogy with the one-basis q-ascending factorials, introduce the following notations:

$$(u; q_1, q_2)_{r,s} := \begin{cases} 1, & \text{if } r \text{ or } s \text{ is zero;} \\ \displaystyle\prod_{0\leq i\leq r-1}\ \prod_{0\leq j\leq s-1} (1 - uq_1^i q_2^j), & \text{if } r, s \geq 1, \end{cases}$$

$$(u; q_1, q_2)_{\infty,\infty} := \lim_{r,s} (u; q_1, q_2)_{r,s} = \prod_{i\geq 0}\prod_{j\geq 0} (1 - uq_1^i q_2^j). \tag{12.2}$$

The purpose of this section is to prove the following theorem.

Theorem 12.1. *The bibasic generating function for the polynomials $A_n(q_1, q_2)$ is given by:*

$$\sum_{n\geq 0} A_n(q_1, q_2) \frac{u^n}{(q_1; q_1)_n\,(q_2; q_2)_n} = \frac{1}{(u; q_1, q_2)_{\infty,\infty}}. \tag{12.3}$$

The term "bibasic" refers to the normalization $(q_1; q_1)_n\,(q_2; q_2)_n$ of the denominator as a product of two q-ascending factorials. *A priori*, there was no evidence that such a normalization was to be introduced. In fact, the infinite product on the right-hand side preexisted in the literature, ready to be unearthed for combinatorial purposes. It could also be regarded as a specialization of the celebrated Cauchy infinite product

$$\prod_{i\geq 1, j\geq 1} \frac{1}{1 - ux_i y_j}$$

with the substitutions $x_i \leftarrow q_1^{i-1}$, $y_j \leftarrow q_j^{j-1}$, for which several expansions are known, in particular in terms of Schur functions. That method will be exploited more in detail in subsequent chapters.

In the proof of Theorem 12.1 the starting point is the infinite product $1/(u;q_1,q_2)_{\infty,\infty}$ that is first expanded as an infinite series and shown to be the generating function for pairs of finite sequences, called *biwords.* This is the "manipulatorics" part of the proof. The next step, the "combinatorics" part, consists of mapping each such a biword onto a triple (σ, b', c'), where σ is a permutation and where (b', c') is another biword that is precisely counted by the product $1/(q_1;q_1)_n\,(q_2;q_2)_n$. The construction of that mapping may be regarded as another application of the MacMahon *Verfahren.* Accordingly, let us decompose the proof into those two parts.

12.1. *The biword expansion.* On the right-hand side of (12.3) each fraction $1/(1-uq_1^i\,q_2^j)$ expands into a geometric series $\sum_{a_{ij}\ge 0}(uq_1^i\,q_2^j)^{a_{ij}}$, whose first nonconstant term is the monomial $uq_1^i\,q_2^j$. Hence, the coefficient of the monomial $u^\alpha q_1^\beta q_2^\gamma$ in the expansion of the infinite product $1/(u;q_1,q_2)_{\infty,\infty}$ is equal to the coefficient of the same monomial in the *finite* product $\prod\limits_{i\le\alpha,j\le\beta} 1/(1-uq_1^i\,q_2^j)$. We can then write $1/(u;q_1,q_2)_{\infty,\infty}$ as the series

$$\sum_A \prod_{i,j}(uq_1^i\,q_2^j)^{a_{ij}} = \sum_A u^{\Sigma a_{ij}} q_1^{\Sigma i\, a_{ij}}\, q_2^{\Sigma j\, a_{ij}}, \tag{12.4}$$

where A runs over the set of all matrices of the form $A=(a_{ij})$ $(i\ge 0, j\ge 0)$, whose entries a_{ij} are integers which are all zero except finitely many of them. Beside the null matrix we can then express each such matrix as a bounded matrix having at least one nonzero entry on its rightmost column and its lowest row.

For example,

$$A = \begin{array}{c@{}c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0\\1\\2\\3 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 & 2\\ 1 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 2 \end{pmatrix} \end{array}$$

is such a matrix.

Now with each matrix A we associate a two-row matrix or *biword* $\mathbf{b} = \begin{pmatrix} b\\ c\end{pmatrix} = \begin{pmatrix} b_1 \ldots b_n\\ c_1 \ldots c_n\end{pmatrix}$, with integral entries such that

$$\sum_{i,j} a_{ij} = n,\ \sum_{i,j} i\, a_{ij} = b_1+\cdots+b_n,\ \sum_{i,j} j\, a_{ij} = c_1+\cdots+c_n; \tag{12.5}$$

$$(b_1,c_1)\le(b_2,c_2)\le\cdots\le(b_n,c_n) \tag{12.6}$$

with respect to the lexicographic order. We then say that the biword **b** is *nondecreasing.*

The sum in (12.4) is then replaced by:

$$\frac{1}{(u;q_1,q_2)_{\infty,\infty}}=\sum_{n\ge 0}u^n\sum_{\mathbf{b}}q_1^{\operatorname{tot}b}\,q_2^{\operatorname{tot}c},\tag{12.7}$$

where **b** runs over all nondecreasing biwords (whose biletters are pairs of nonnegative integers) of length n. To fulfill the relations (12.5) start with matrix A, read its rows from left to right and top to bottom and, for each *positive* entry a_{ij} write down a_{ij} biletters $\begin{pmatrix} i\\ j\end{pmatrix}$ one after another. The number of biletters written in this way is then equal to $\sum_{i,j}a_{ij}$. Moreover, on the top (resp. bottom) row of **b** each number i (resp. j) is repeated a_{ij} times. Consequently, the last two conditions of (12.5) hold.

Finally, as the biletters $\begin{pmatrix} b_i\\ c_i\end{pmatrix}$ were written starting with the first row from left to right, then the second row, … those biletters are in *increasing lexicographic order* when the biword **b** is read from left to right, so that the relations (12.6) hold. Conversely, when starting with a nondecreasing biword, we can reconstruct the matrix A in a unique manner.

For example, to the matrix A above there corresponds the nondecreasing biword

$$\mathbf{b}=\begin{pmatrix} b\\ c\end{pmatrix}=\begin{pmatrix} 0\,0\,0\,1\,1\,1\,1\,3\,3\,3\,3\,3\\ 2\,4\,4\,0\,3\,3\,3\,1\,2\,2\,4\,4\end{pmatrix},$$

a biword of length $n=12$ and such that $\operatorname{tot}b=19$, $\operatorname{tot}c=32$.

12.2. *Another application of the MacMahon Verfahren.* For convenience define the *Comajor Index* of a permutation $\sigma\in\mathfrak{S}_n$ by

$$\operatorname{comaj}\sigma:=\sum_{1\le i\le n-1}(n-i)\,\chi\big(\sigma(i)>\sigma(i+1)\big).\tag{12.8}$$

We now construct a bijection $\mathbf{b}\mapsto(\sigma,b',c')$ that maps each nondecreasing biword $\mathbf{b}=\begin{pmatrix} b\\ c\end{pmatrix}$ of length n onto a triple (σ,b',c'), where $\sigma\in\mathfrak{S}_n$ and $b',c'\in\mathrm{NDS}(n)$, that is, σ is a permutation of order n and b', c' are both nondecreasing words of length n whose letters are nonnegative integers. Moreover, the bijection has the properties:

$$\operatorname{tot}b=\operatorname{comaj}\sigma+\operatorname{tot}b',\quad \operatorname{tot}c=\operatorname{comaj}\sigma^{-1}+\operatorname{tot}c'.\tag{12.9}$$

To do so start with a nondecreasing biword $\mathbf{b} = \begin{pmatrix} b \\ c \end{pmatrix}$. Since the biletters $\begin{pmatrix} b_i \\ c_i \end{pmatrix}$ are written in nondecreasing order with respect to the lexicographic order, the following property holds:

$$c_i > c_{i+1} \Rightarrow b_i < b_{i+1}. \tag{12.10}$$

For each $i = 1, 2, \ldots, n$ let

$$\sigma(i) = |\{j : 1 \le j \le n,\, c_j < c_i\}| + |\{j : 1 \le j \le i,\, c_j = c_i\}|. \tag{12.11}$$

In other words, $\sigma(i)$ is equal to the number of letters in c which are less than c_i, plus the number of letters equal to c_i, but lie to the left of c_i, with c_i included. This defines a permutation σ of order n.

By construction $c_i > c_{i+1}$ if and only if $\sigma(i) > \sigma(i+1)$. For $1 \le i \le n$ let y_i be the number of integers j such that $1 \le j \le i-1$ and $\sigma(j) > \sigma(j+1)$, which is also the number of integers j such that $1 \le j \le i-1$ and $c_j > c_{j+1}$. The word $y = y_1 \ldots y_n$ is nondecreasing. Moreover,

$$\operatorname{comaj} \sigma = \operatorname{tot} y = y_1 + \cdots + y_n \tag{12.12}$$

and (12.10) implies

$$y_i < y_{i+1} \Leftrightarrow \sigma(i) > \sigma(i+1) \Leftrightarrow c_i > c_{i+1} \Rightarrow b_i < b_{i+1}. \tag{12.13}$$

As $y_1 = 0$, the relations (12.13) imply that

$$y_i \le b_i \quad (1 \le i \le n). \tag{12.14}$$

We then define a *nondecreasing* word $b' = b'_1 \ldots b'_n$ by

$$b'_i := b_i - y_i \quad (1 \le i \le n). \tag{12.15}$$

Finally, because of (12.12) the first of the relations (12.9) holds.

Again use the previous example. Under the nondecreasing word $\begin{pmatrix} b \\ c \end{pmatrix}$ we have written the values of σ, of y and of $b' = b - y$.

$$\begin{array}{rcccccccccccc}
b = & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\
c = & 2 & 4 & 4 & 0 & 3 & 3 & 3 & 1 & 2 & 2 & 4 & 4 \\
\sigma = & 3 & 9 & 10 & 1 & 6 & 7 & 8 & 2 & 4 & 5 & 11 & 12 \\
y = & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
b' = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}$$

We can also start with the biword $\begin{pmatrix} c \\ b \end{pmatrix}$ (and not $\mathbf{b} = \begin{pmatrix} b \\ c \end{pmatrix}$) and rearrange its biletters $\begin{pmatrix} c_i \\ b_i \end{pmatrix}$ in increasing order. We obtain a biword $\widetilde{\mathbf{b}}$ whose top row is the nondecreasing rearrangement of the word c.

With the running example we get:

$$\widetilde{\mathbf{b}} = \begin{pmatrix} 0\,1\,2\,2\,2\,3\,3\,3\,4\,4\,4\,4 \\ 1\,3\,0\,3\,3\,1\,1\,1\,0\,0\,3\,3 \end{pmatrix}.$$

Let us go back to the general case. By definition of σ given in (12.11), there are exactly $\sigma(i)$ biletters $\begin{pmatrix} b_j \\ c_j \end{pmatrix}$ such that $1 \le j \le n$ and $c_j < c_i$ or such that $1 \le j \le i$ and $c_j = c_i$. As in the biword $\widetilde{\mathbf{b}}$ we have sorted the biletter $\begin{pmatrix} c_j \\ b_j \end{pmatrix}$ in increasing order, the biletter $\begin{pmatrix} c_i \\ b_i \end{pmatrix}$ will be found in $\widetilde{\mathbf{b}}$ in the $\sigma(i)$-th position. Hence $\widetilde{\mathbf{b}} = \begin{pmatrix} c_{\sigma^{-1}(1)} & \cdots & c_{\sigma^{-1}(n)} \\ b_{\sigma^{-1}(1)} & \cdots & b_{\sigma^{-1}(n)} \end{pmatrix}$. Let τ be the permutation defined as in (12.11) when the counting is applied to the nondecreasing biword $\widetilde{\mathbf{b}}$, i.e.,

$$\begin{aligned} \tau(i) = {} & |\{j : 1 \le j \le n,\, b_{\sigma^{-1}(j)} < b_{\sigma^{-1}(i)}\}| \\ & + |\{j : 1 \le j \le i,\, b_{\sigma^{-1}(j)} = b_{\sigma^{-1}(i)}\}|. \end{aligned}$$

We also have:

$$\tau\sigma(i) = |\{j : 1 \le j \le n,\, b_j < b_i\}| + |\{j : 1 \le j \le \sigma(i),\, b_{\sigma^{-1}(j)} = b_i\}|.$$

But by definition of σ, if $b_k = b_l$, we have $\sigma(k) < \sigma(l) \Leftrightarrow k < l$. Therefore, if $b_{\sigma^{-1}(j)} = b_i$, we have $j < \sigma(i) \Leftrightarrow \sigma^{-1}(j) < i$. Since b is nondecreasing, we obtain:

$$\begin{aligned} \tau\sigma(i) &= |\{j : 1 \le j \le n,\, b_j < b_i\}| + |\{j : 1 \le \sigma^{-1}(j) \le i,\, b_{\sigma^{-1}(j)} = b_i\}| \\ &= |\{j : 1 \le j \le i,\, b_j < b_i\}| + |\{j : 1 \le j \le i,\, b_j = b_i\}| \\ &= i. \end{aligned}$$

Therefore, $\tau = \sigma^{-1}$.

Under the same procedure, to the nondecreasing biword $\widetilde{\mathbf{b}}$ there corresponds a pair (τ, c'), in a one-to-one manner, where $\tau = \sigma^{-1}$ and where c' is a nondecreasing word. Moreover, the second relation of (12.9) holds.

Keeping the same example we determine σ^{-1}, y and c', where this time the word y serves to calculate $\operatorname{comaj}\sigma^{-1}$:

$$\begin{array}{rcccccccccccc} \tilde{c} = & 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ \tilde{b} = & 1 & 3 & 0 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 3 & 3 \\ \sigma^{-1} = & 4 & 8 & 1 & 9 & 10 & 5 & 6 & 7 & 2 & 3 & 11 & 12 \\ y = & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ c' = & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Now by using the bijection $\mathbf{b} \mapsto (\sigma, b', c')$ just obtained we can derive:

$$\sum_{n\ge 0} u^n \sum_{\mathbf{b}} q_1^{\operatorname{tot} b}\, q_2^{\operatorname{tot} c} = \sum_{n\ge 0} u^n \sum_{\substack{\sigma\in\mathfrak{S}_n \\ b',c'\in \mathrm{NDS}(n)}} q_1^{\operatorname{comaj}\sigma + \operatorname{tot} b'}\, q_2^{\operatorname{comaj}\sigma^{-1} + \operatorname{tot} c'};$$

and then by (6.1):

$$\sum_{n\ge 0} u^n \sum_{\mathbf{b}} q_1^{\operatorname{tot} b}\, q_2^{\operatorname{tot} c} = \sum_{n\ge 0} u^n \frac{1}{(q_1;q_1)_n\,(q_2;q_2)_n} \sum_{\sigma\in\mathfrak{S}_n} q_1^{\operatorname{comaj}\sigma} q_2^{\operatorname{comaj}\sigma^{-1}}.$$

There remains to show that the polynomial $A_n(q_1,q_2)$, defined in (12.1), is also equal to $\sum\limits_{\sigma\in\mathfrak{S}_n} q_1^{\operatorname{comaj}\sigma} q_2^{\operatorname{comaj}\sigma^{-1}}$. This can be proved by means of the bijection $\mathbf{rc}$ that maps the permutation σ onto the permutation $\mathbf{rc}\,\sigma$ defined by

$$\mathbf{rc}\,\sigma(i) := n+1-\sigma(n+1-i) \quad (1\le i\le n).$$

It is easily seen that

$$\operatorname{comaj}\mathbf{rc}\,\sigma = \operatorname{maj}\sigma.$$

This completes the proof of Theorem 12.1.

13. A four-variable distribution

In section 7 we have introduced the Euler-Mahonian polynomial $A_{\mathbf{m}}(t,q)$ as a *t-extension* of the polynomial $A_{\mathbf{m}}(q)$ by noticing that the combinatorial correspondence used in the calculation of the generating function for the $A_{\mathbf{m}}(q)$'s had a further property. We will do the same for the bijection constructed in the previous section and derive what could be called a t_1,t_2*-extension* of formula (12.3).

Consider the inverse bijection $(\sigma,b',c')\mapsto \mathbf{b}$ described in 12.2. The nondecreasing word y defined just before (12.12), such that $\operatorname{tot} y = \operatorname{comaj}\sigma$, has the further property

$$y_n = \operatorname{des}\sigma. \tag{13.1}$$

Now consider the *finite* product

$$\frac{1}{(u;q_1,q_2)_{r+1,s+1}} = \prod_{0\le i\le r}\prod_{0\le j\le s}\frac{1}{1-uq_1^i q_2^j}.$$

It can be expanded into the series

$$\sum_A \prod_{i,j}(uq_1^i\, q_2^j)^{a_{ij}} = \sum_A u^{\Sigma a_{ij}} q_1^{\Sigma i\, a_{ij}}\, q_2^{\Sigma j\, a_{ij}},$$

but this time the matrices A are $(r+1)\times(s+1)$-matrices. The nondecreasing biword $\mathbf{b} = \begin{pmatrix} b\\ c\end{pmatrix} = \begin{pmatrix} b_1 \dots b_n\\ c_1 \dots c_n\end{pmatrix}$ that corresponds to such a matrix has the further property:

$$\max b_i = b_n \le r \quad\text{and}\quad \max c_i \le s.$$

Properties (12.15) and (13.1) imply: $b'_n = b_n - y_n \le r - \operatorname{des}\sigma$, with an analogous property for the c'_i's, that is $c'_n \le s - \operatorname{des}\sigma^{-1}$.

Start with a quintuple (σ, r', s', b', c'), where

$$\sigma \in \mathfrak{S}_n, \quad r', s' \ge 0, \quad b' \in \mathrm{NDS}(n, r'), \quad c' \in \mathrm{NDS}(n, s'). \tag{13.2}$$

Under the inverse bijection $(\sigma, b', c') \mapsto \mathbf{b}$ the quintuple is mapped onto a triple $(r, s, \mathbf{b})$ such that $r = r' + \operatorname{des}\sigma$, $s = s' + \operatorname{des}\sigma^{-1}$, $\mathbf{b} = \begin{pmatrix} b \\ c \end{pmatrix}$ with $b \in \mathrm{NDS}(n, r)$ and $\tilde{c}$ (the nondecreasing rearrangement of c) in $\mathrm{NDS}(n, s)$. We can then write

$$\begin{aligned} \sum_{r,s\ge 0} \frac{t_1^r\, t_2^s}{(u; q_1, q_2)_{r+1,s+1}} &= \sum_{r,s\ge 0} t_1^r\, t_2^s \sum_{n\ge 0} u^n \sum_{\substack{b\in \mathrm{NDS}(n,r),\\ c\in \mathrm{NDS}(n,s)}} q_1^{\operatorname{tot} b} q_2^{\operatorname{tot} c} \\ &= \sum_{r,s\ge 0} \sum_{n\ge 0} u^n \sum_{(\sigma, r', s', b', c')} t_1^{r'+\operatorname{des}\sigma} t_2^{s'+\operatorname{des}\sigma^{-1}} q_1^{\operatorname{comaj}\sigma + \operatorname{tot} b'} q_2^{\operatorname{comaj}\sigma^{-1} + \operatorname{tot} c'}, \end{aligned}$$

where the relations (13.2) hold and also $r = r' + \operatorname{des}\sigma$, $s = s' + \operatorname{des}\sigma^{-1}$. Let

$$A_n(t_1, t_2, q_1, q_2) := \sum_{\sigma\in\mathfrak{S}_n} t_1^{\operatorname{des}\sigma} t_2^{\operatorname{des}\sigma^{-1}} q_1^{\operatorname{comaj}\sigma} q_2^{\operatorname{comaj}\sigma^{-1}}. \tag{13.3}$$

It follows that

$$\begin{aligned} \sum_{r,s\ge 0} \frac{t_1^r\, t_2^s}{(u; q_1, q_2)_{r+1,s+1}} &= \sum_{n\ge 0} u^n\, A_n(t_1, t_2, q_1, q_2) \sum_{\substack{r'\ge 0\\ b'\in \mathrm{NDS}(n,r')}} t_1^{r'} q_1^{\operatorname{tot} b'} \sum_{\substack{s'\ge 0\\ c'\in \mathrm{NDS}(n,s')}} t_2^{s'} q_2^{\operatorname{tot} c'} \\ &= \sum_{n\ge 0} u^n\, A_n(t_1, t_2, q_1, q_2) \frac{1}{(t_1; q_1)_{n+1}\, (t_2; q_2)_{n+1}}, \end{aligned} \tag{13.4}$$

by (3.9) and (4.5).

Formula (13.4) provides an expression for the generating function for the polynomials $A_n(t_1, t_2, q_1, q_2)$. Observe the nature of the denominators. They are products of a q_1-ascending factorial by q_2-ascending factorial. There remains to verify that

$$A_n(t_1, t_2, q_1, q_2) = \sum_{\sigma\in\mathfrak{S}_n} t_1^{\operatorname{des}\sigma} t_2^{\operatorname{ides}\sigma} q_1^{\operatorname{maj}\sigma} q_2^{\operatorname{imaj}\sigma}, \tag{13.5}$$

where, by analogy with “imaj”, the symbol “ides” means

$$\operatorname{ides}\sigma := \operatorname{des}\sigma^{-1}. \tag{13.6}$$

Again this is proved by means of the bijection **rc** that maps each permutation σ onto the permutation $\mathbf{rc}\,\sigma$ defined by

$$\mathbf{rc}\,\sigma(i) := n+1-\sigma(n+1-i) \quad (1 \le i \le n).$$

Clearly, we have

$$\operatorname{comaj} \mathbf{rc}\,\sigma = \operatorname{maj} \sigma, \quad \operatorname{des} \mathbf{rc}\,\sigma = \operatorname{des} \sigma.$$

We have then proved the following result.

Theorem 13.1. *The bibasic generating function for the polynomials* $A_n(t_1,t_2,q_1,q_2)$, *as defined in* (13.5), *is given by*

$$\text{(13.7)} \quad \sum_{n\ge 0} A_n(t_1,t_2,q_1,q_2) \frac{u^n}{(t_1;q_1)_{n+1}\,(t_2;q_2)_{n+1}} = \sum_{r,s\ge 0} \frac{t_1^r\,t_2^s}{(u;q_1,q_2)_{r+1,s+1}}.$$

Specializations. The right-hand side of (13.7) is symmetric in the pairs (t_1,q_1), (t_2,q_2), so that the polynomial $A_n(t_1,q_1,t_2,q_2)$ is also symmetric in (t_1,q_1), (t_2,q_2). This can also be seen by using the bijection $\sigma \mapsto \sigma^{-1}$ of $\mathfrak{S}_n$ on itself. In particular, both specializations $A_n(t,1,q,1)$ and $A_n(1,t,1,q)$ are equal. Moreover,

$$\text{(13.8)} \qquad A_n(t,1,q,1) = A_n(1,t,1,q) = {}^{\operatorname{maj}}A(t,q),$$

where ${}^{\operatorname{maj}}A(t,q)$ is the *q-maj-Eulerian polynomial* defined in section 10. On the other hand, it follows from Corollary 11.4 that

$$\text{(13.9)} \qquad t\,A_n(t,1,1,q) = t\,A_n(1,t,1,q) = {}^{\operatorname{inv}}A(t,q),$$

where ${}^{\operatorname{inv}}A(t,q)$ is the *q-inv-Eulerian polynomial*, also defined in section 10.

Tables of the polynomials $A_n(t_1,q_1,t_2,q_2)$ for $n=3,4,5$ are shown in Fig. 13.1. Keeping in mind (13.8), the following notations have been used: ${}^{\operatorname{maj}}A(t,q) = \sum_k A_{n,k}(q)$ and $A_{n,k} := A_{n,k}(1)$ (the *Eulerian coefficient*). Notice the numerous symmetries within the tables.

	ides →	0	1	2	$n = 3$	
	imaj →	0	1 2	3	$A_{3,k}(q)$	$A_{3,k}$
des ↓	maj ↓					
0	0	1			1	1
1	1		1 1		2	
	2		1 1		2	4
2	3			1	1	1

	ides →	0	1	2	3	$n = 4$	
	imaj →	0	1 2 3	3 4 5	6	$A_{4,k}(q)$	$A_{4,k}$
des ↓	maj ↓						
0	0	1				1	1
	1		1 1 1			3	
1	2		1 2 1	1		5	11
	3		1 1 1			3	
	3			1 1 1		3	
2	4		1	1 2 1		5	11
	5			1 1 1		3	
3	6				1	1	1

	ides →	0	1	2	3	4	$n = 5$	
	imaj →	0	1 2 3 4	3 4 5 6 7	6 7 8 9	10	$A_{5,k}(q)$	$A_{5,k}$
des ↓	maj ↓							
0	0	1					1	1
1	1		1 1 1 1				4	26
	2		1 2 2 1	1 1 1			9	
	3		1 2 2 1	1 1 1			9	
	4		1 1 1 1				4	
2	3			1 1 2 1 1			6	66
	4		1 1	1 3 4 3 1	1 1		16	
	5		1 1	2 4 6 4 2	1 1		22	
	6		1 1	1 3 4 3 1	1 1		16	
	7			1 1 2 1 1			6	
3	6				1 1 1 1		4	26
	7			1 1 1	1 2 2 1		9	
	8			1 1 1	1 2 2 1		9	
	9				1 1 1 1		4	
4	10					1	1	1

Fig. 13.1

14. Symmetric Functions

The Cauchy identity for Schur functions will be an essential tool for deriving several combinatorial formulas for symmetric group statistics. It matters to have a brief account for the algebra of symmetric function and a complete description of the combinatorial properties of the Schur functions. This is the content of the next three sections.

14.1. *Partitions of integers.* Those structures remain the privileged objects in the theory of symmetric functions. As already discussed in section 4, by *partition* of an integer $n \geq 1$ it is meant a nondecreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers, where finitely many of them are nonzero. The nonzero elements of λ are the *parts* of the partition. The *number of parts* is denoted by $l(\lambda)$ and the *weight*, denoted by $|\lambda|$, of the partition λ is defined by

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots$$

When $|\lambda| = n$, we say that λ is a *partition* of the integer n. The symbol $\mathcal{P}_n$ will designate the set of partitions of n.

The *multiplicative notation* of the partition λ is reads as $\lambda = 1^{m_1}2^{m_2}\ldots$, where for each $i = 1, 2, \ldots$ the exponent m_i is equal to the number of parts of λ equal to i. The integer $m_i = m_i(\lambda)$ is called the *multiplicity* of i in λ.

For example, $\lambda = (5, 4, 4, 2, 1, 1)$ is a partition of $n = 17$, whose multiplicative notation reads $1^2 2^1 3^0 4^2 5^1$.

The *shape* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ $(\lambda_r \geq 1)$ is the set of all the $|\lambda|$ points $(1, 1)$, $(1, 2)$, ... , $(1, \lambda_1)$, $(2, 1)$, $(2, 2)$, ... , $(2, \lambda_2)$, ... , $(r, 1)$, $(r, 2)$, ... , (r, λ_r) located in the north-eastern quadrant $\mathbb{N}^2$ of $\mathbb{Z}^2$. Each shape if also represented by a set of squared boxes left justified, where every point of the previous sequence is the center of a box. For example, the partition $\lambda = (5, 4, 4, 2, 1, 1)$ is represented by the shape drawn in Fig. 14.1

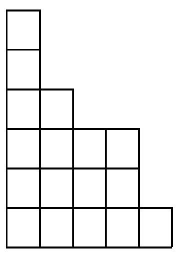


Fig. 14.1

This geometric representation is also called *Ferrers diagram* and denoted by the same symbol λ.

The *conjugate* partition of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is the partition $\lambda' = 1^{\lambda_1 - \lambda_2} 2^{\lambda_2 - \lambda_3} \ldots$ (written in multiciplicative notation) or $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$, where $\lambda'_i := |\{j : \lambda_j \geq i\}|$ $(i = 1, \dots, \lambda_1)$. In particular, $\lambda'_1 = l(\lambda)$. The Ferrers diagram of λ' is obtained by taking the symmetry of the Ferrers diagram of λ with respect of the line $y = x$ of the plane. With the previous example we have $\lambda' = 1^1 2^0 3^2 4^1 5^0 6^1 = (6, 4, 3, 3, 1)$.

14.2. *The algebra of symmetric functions.* Let $\mathbb{Z}[x_1, \dots, x_n]$ be the ring of polynomials in n variables with integral coefficients. A polynomial in that ring is said to be *symmetric*, if it is invariant by permutation of its variables. Let Λ_n be the subring of $\mathbb{Z}[x_1, \dots, x_n]$ of all symmetric polynomials. For every $k \geq 0$ let Λ_n^k be the set of all homogeneous symmetric polynomials of degree k, the zero polynomial included. Then,

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k.$$

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ let x^α be the monomial $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and for each partition λ of length $l(\lambda) \leq n$ let $m_\lambda(x_1, \dots, x_n) := \sum x^\alpha$ denote the sum of all (distinct) monomials x^α, where α is a permutation of $\lambda = (\lambda_1, \dots, \lambda_n)$. The polynomial m_λ is called *monomial symmetric polynomial*. We also use the notation $\sum x_1^{\lambda_1} x_2^{\lambda_2} \dots$ in place of m_λ.

For example, with $n = 4$ variables,
$m_{(1)} = \sum x_1 = x_1 + x_2 + x_3 + x_4$;
$m_{(1,1)} = \sum x_1 x_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$;
$m_{(2,1)} = \sum x_1^2 x_2 = x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_1 + x_2^2x_3 + x_2^2x_4 + x_3^2x_1 + x_3^2x_2 + x_3^2x_4 + x_4^2x_1 + x_4^2x_2 + x_4^2x_3$.

The monomial symmetric polynomials $m_\lambda(x_1, \dots, x_n)$ form a $\mathbb{Z}$-basis for Λ_n, when λ is restricted to the set of all partitions of length $l(\lambda) \leq n$. On the other hand, the polynomials $m_\lambda(x_1, \dots, x_n)$ ($l(\lambda) \leq n$; $|\lambda| = k$) form a $\mathbb{Z}$-basis for Λ_n^k. When $n \geq k$, that is to say, when the number of variables is *large*, the set of *all* the polynomials m_λ such that $|\lambda| = k$ form a $\mathbb{Z}$-basis for Λ_n^k. Hence, $\dim \Lambda_n^k = p(k)$, the number of partitions of k.

In the theory of symmetric functions we assume that the number of variables is finite, but *large*; some authors prefer to deal with infinitely many of them, but the statements of certain properties are less intuitive. To make the notion of largeness more precise, we deal, not with polynomials, but with sequences of polynomials $f = (f_n)$ ($n \geq 0$), where each term f_n belongs to Λ_n^k and where for each pair $m \geq n$ the polynomials f_m and f_n satisfy the compatibility property:

$$f_m(x_1, \dots, x_n, 0, \dots, 0) = f_n(x_1, \dots, x_n).$$

Let Λ^k be the set of the sequences $f = (f_n)$, where each term f_n is a symmetric polynomial of degree k. We can show that for $n \geq k$ the mapping ρ_n^k of Λ^k into Λ_n^k that sends $f = (f_n)$ onto f_n is an *isomorphism.* Consequently, Λ^k is of dimension $p(k)$. The set of the sequences $m_\lambda = (m_\lambda(x_1, \dots, x_n))$ ($n \geq 0$) such that $|\lambda| = k$ is a $\mathbb{Z}$-basis for Λ^k. We then let

$$\Lambda = \bigoplus_{\lambda \geq 0} \Lambda^k.$$

The elements of Λ are formal series with integral coefficients in the functions m_λ. The ring Λ is called the *ring of the symmetric functions.* For every $n \ge 0$ the mapping of Λ into Λ_n that sends the formal series $\sum_\lambda a_\lambda m_\lambda$ onto the symmetric function $\sum_\lambda a_\lambda m_\lambda(x_1,\ldots,x_n)$ is a surjective homomorphism for $n \ge 0$ and maps the formal series $\sum_\lambda a_\lambda m_\lambda$ such that $|\lambda| \le n$ onto the corresponding series, in an injective manner.

14.3. *The classical bases.* For each $r \ge 1$ the polynomial

$$e_r = m_{1^r} = \sum x_1 x_2 \ldots x_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r}$$

is called the r-th *elementary symmetric function.* By convention, $e_0 = 1$. The generating function for the e_r's is obviously:

$$E(u) = \sum_{r \ge 0} e_r u^r = \prod_{i \ge 1} (1 + x_i u). \tag{14.1}$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we define:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

Notice that if λ is written $\lambda = 1^{m_1} 2^{m_2} \ldots$ (multiplicative notation), then $e_\lambda = e_1^{m_1} e_2^{m_2} \ldots$

For each $r \ge 0$ define the *homogeneous symmetric function* of degree r by

$$h_r := \sum_\lambda m_\lambda \qquad (|\lambda| = r).$$

In particular, $h_0 = 1$, $h_1 = e_1$, $h_2 = m_{(2)} + m_{(1,1)}$, $h_3 = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$. Now

$$\begin{aligned}
\prod_{i \ge 1} (1 - x_i u)^{-1} &= \prod_{i \ge 1} \sum_{k_i \ge 0} (x_i u)^{k_i} \\
&= \sum_{r \ge 0} u^r \sum_{1 \le i_1 < \cdots < i_m} \sum_{\substack{(k_1,\ldots,k_m) \\ \Sigma\, k_i = r}} x_{i_1}^{k_1} \ldots x_{i_m}^{k_m} \\
&= \sum_{r \ge 0} u^r \sum_{|\lambda| = r} \sum_{(k_1,\ldots,k_m)} \sum_{1 \le i_1 < \cdots < i_m} x_{i_1}^{k_1} \ldots x_{i_m}^{k_m}
\end{aligned}$$

[where $(k_1,\ldots,k_m)$ is a rearrangement of $(\lambda_1,\ldots,\lambda_m)$]

$$= \sum_{r \ge 0} u^r \sum_{|\lambda| = r} m_\lambda = \sum_{r \ge 0} u^r h_r.$$

Thus, the generating function for the h_r's is

$$H(u) = \sum_{r\geq 0} h_r u^r = \prod_{i\geq 1}(1 - x_i u)^{-1}. \tag{14.2}$$

For every partition $\lambda = (\lambda_1, \lambda_2, \dots)$ define

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots$$

The *power sums* p_r are defined by

$$p_r := m_{(r)} = \sum_i x_i^r;$$

furthermore, let

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots$$

The generating function for the p_r's, defined by $P(u) := \sum_{r\geq 1} p_r u^{r-1}$, can also be expressed as

$$\begin{aligned} P(u) &= \sum_{i\geq 1}\sum_{r\geq 1} x_i^r u^{r-1} = \sum_{i\geq 1} \frac{x_i}{1 - x_i u} = \sum_{i\geq 1} \frac{d}{du} \log \frac{1}{1 - x_i u} \\ &= \frac{d}{du} \log \prod_i \frac{1}{1 - x_i u} = \frac{d}{du} \log H(u) = \frac{H'(u)}{H(u)}. \end{aligned}$$

The proof of the following theorem can be found in Macdonald [Ma95].

Theorem 14.1. *The functions e_λ (resp. h_λ) form a $\mathbb{Z}$-basis for Λ. The functions p_λ form a $\mathbb{Q}$-basis for Λ.*

Theorem 14.2. *We have the relations:*

(i)
$$\prod_{i\geq 1}(1 + x_i u) = \sum_{r\geq 0} e_r u^r = E(u)\,;$$

(ii)
$$\prod_{i\geq 1}(1 - x_i u)^{-1} = \sum_{r\geq 0} h_r u^r = H(u)\,;$$

(iii)
$$H(u)E(-u) = 1\,;$$

(iv)
$$\sum_{0\leq r\leq n} (-1)^r e_r h_{n-r} = 0 \qquad (n \geq 1)$$

(v)
$$\frac{H'(u)}{H(u)} = \sum_{r\geq 1} p_r u^{r-1} = P(u)\,;$$

(vi) $$\frac{E'(u)}{E(u)} = P(-u)\,;$$

(vii) $$nh_n = \sum_{1\le r\le n} p_r h_{n-r}\,;$$

(viii) $$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{n\ge 0}\sum_{|\lambda|=n} m_\lambda(x) h_\lambda(y)\,;$$

(ix) $$\prod_i \frac{1}{1-x_i u} = \exp \sum_{n\ge 1} p_n(x)\frac{u^n}{n}\,;$$

(x) $$\prod_{i,j} \frac{1}{1-x_i y_j} = \exp \sum_{n\ge 1} \frac{p_n(x)p_n(y)}{n}\,;$$

Proof. Relations (i), (ii) have already been proved; (iii) is a straightforward consequence of (i) and (ii). Identity (iv) is derived by considering the coefficient of t^r on the two sides of (iii). Identity (v) has been proved and (vi) follows from (iii) and (v). When (v) is written in the form $H'(u) = H(u)P(u)$, we obtain (vii).

For the proof of (viii) write

$$\begin{aligned}
\prod_i \prod_j \frac{1}{1-x_i y_j} &= \prod_i \sum_{r\ge 0} x_i^r h_r(y) \qquad \text{[by (ii)]}\\
&= \sum_{n\ge 0} \sum_{(r_1,\dots,r_n)} h_{r_1}(y)\dots h_{r_n}(y) \sum_{1\le i_1<\dots<i_n} x_{i_1}^{r_1}\dots x_{i_n}^{r_n}\\
&= \sum_{n\ge 0}\sum_{|y|=n} h_\lambda(y) m_\lambda(x) = \sum_{n\ge 0}\sum_{|\lambda|=n} h_\lambda(x) m_\lambda(y).
\end{aligned}$$

For (ix) write:

$$\begin{aligned}
\log \prod_i \frac{1}{1-x_i u} &= \sum_i \log\frac{1}{1-x_i u} = \sum_i \sum_{n\ge 1} \frac{x_i^n}{n} u^n\\
&= \sum_{n\ge 1} \frac{u^n}{n} \sum_i x_i^n = \sum_{n\ge 1} \frac{u^n}{n} p_n(x).
\end{aligned}$$

Finally,

$$\log \prod_i \prod_j \frac{1}{1-x_i y_j} = \sum_i \sum_{n\ge 1} \frac{x_i^n}{n} p_n(y) = \sum_n p_n(x)p_n(y)\frac{1}{n}. \quad \square$$

Let us give further relations between h_n, e_n and p_λ. If $\lambda = 1^{m_1}2^{m_2}\dots$ is a partition, define

$$z_\lambda := 1^{m_1} m_1!\, 2^{m_2} m_2!\,\dots$$

Notice that if C_λ designates the set of the permutations of order n whose cycle structure is given by $1^{m_1}2^{m_2}\dots n^{m_n}$, in other words, if C_λ is the set of all the permutations having m_1 cycles of length 1, m_2 cycles of length 2, $\dots$, then $z_\lambda = n!/\,|C_\lambda|$.

Theorem 14.3. *We have:*

$$H(u)=\sum_\lambda \frac{1}{z_\lambda}p_\lambda u^{|\lambda|}\,; \qquad E(u)=\sum_\lambda (-1)^{|\lambda|-l(\lambda)}\frac{1}{z_\lambda}p_\lambda u^{|\lambda|};$$
$$h_n=\sum_{|\lambda|=n}\frac{1}{z_\lambda}p_\lambda\,; \qquad e_n=\sum_{|\lambda|=n}(-1)^{|\lambda|-l(\lambda)}\frac{1}{z_\lambda}p_\lambda.$$

Proof. It suffices to prove the first identity. But $P(u)=(d/du)\log H(u)$ implies

$$H(u)=\exp\sum_{r\ge1}p_r\frac{u^r}{r}=\prod_{r\ge1}\exp\Bigl(p_r\frac{u^r}{r}\Bigr)=\prod_{r\ge1}\sum_{m_r\ge0}\frac{(p_ru^r)^{m_r}}{r^{m_r}}\frac{1}{m_r!}$$
$$=\sum_\lambda u^{\Sigma r m_r}\frac{\prod p_r^{m_r}}{\prod r^{m_r}m_r!}=\sum_\lambda u^{|\lambda|}p_\lambda\frac{1}{z_\lambda}. \quad \square$$

15. Schur Functions

Let $\varepsilon(\sigma)$ designate the *signature* of the permutation σ. A polynomial $P(x)=P(x_1,\dots,x_n)$ in n variables is said to be *alternant* or *antisymmetric*, if for every permutation σ of $(1,2,\dots,n)$ the following relation holds:

$$P(x_{\sigma1},x_{\sigma2},\dots,x_{\sigma n})=\varepsilon(\sigma)\,P(x_1,x_2,\dots,x_n).$$

Let A_n (resp. A_n^k) be the space of all the alternants in n variables (resp. in n variables and homogeneous of degree k, the zero polynomial included). Then every alternant $P(x)$ that belongs to A_n^k can be written

$$P(x)=\sum_{\substack{\alpha_1>\alpha_2>\dots>\alpha_n\\ |\alpha|=\alpha_1+\dots+\alpha_n=k}} c(\alpha)\,\det\bigl(x_i^{\alpha_j}\bigr)_{(1\le i,j\le n)},$$

since, if $P(x)$ contains the term $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ with coefficient $c(\alpha)$, it also contains the term $x_{\sigma1}^{\alpha_1}\dots x_{\sigma n}^{\alpha_n}$ with the coefficient $\varepsilon(\alpha)c(\alpha)$. On the other hand, the α_i's are all distinct, for if it were not the case, every determinant $\det\bigl(x_i^{\alpha_j}\bigr)$ would be zero. Also notice that $c(\alpha)$ is the coefficient of x^α in $P(x)$.

As $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, we can write $\alpha_i = \lambda_i + n - i$ $(i = 1, 2, \ldots)$, so that $\lambda_1 \ge \lambda_2 \ge \cdots > \lambda_n$ and $k = |\alpha| = \lambda_1 + \lambda_2 + \cdots + n(n-1)/2$. Thus,

$$P(x) = \sum_{|x|=k=n(n-1)/2} c(\lambda)\, \det\bigl(x_i^{\lambda_j+n-j}\bigr)_{(1\le i,j\le n)}.$$

Consequently, there exists no alternant in n variables of degree (strictly) less than $n(n-1)/2$.

The determinant $a_\delta = \det\bigl(x_i^{n-j}\bigr)$ $(1 \le i, j \le n)$, where $\delta = (n-1, n-2, \ldots, 1, 0)$ is the Vandermonde determinant, equal to $\prod_{i<j}(x_i - x_j)$. The determinant $a_\alpha = \det\bigl(x_i^{\alpha_j}\bigr)$ can be written $a_\alpha = a_{\lambda+\delta} = \det\bigl(x_i^{\lambda_j+n-j}\bigr)$. But if two α_i are equal, the determinant is zero. Therefore, it is *divisible* by $(x_i - x_j)$ $(i \ne j)$ and then by the product $a_\delta = \prod_{i<j}(x_i - x_j)$, that is, by a_δ.

Definition. The quotient $s_\lambda(x_1, \ldots, x_n) := a_{\lambda+\delta}/a_\delta$ is called the *Schur function* in the variables $x_1, \ldots, x_n$ associated with the partition λ.

The Schur function $a_{\lambda+\delta}$ is *symmetric* and *homogeneous* of degree k. This follows from the fact that it is the ratio of two alternants. On the other hand, the alternants $a_{\lambda+\delta}$ $(|\lambda| = k, l(\lambda) \le n)$ form a basis for $A_n^{k+n(n-1)/2}$. The mapping $A \mapsto a_\delta Q$ is an isomorphism of Λ_n^k onto $A_n^{k+n(n-1)/2}$, the kernel being zero, since $a_\delta Q = 0 \Rightarrow Q = 0$. The following theorem has then be proved.

Theorem 15.1. *The Schur functions $s_\lambda(x_1, \ldots, x_n)$ $(|\lambda| = k, l(\lambda) \le n)$ form a $\mathbb{Z}$-basis for Λ_n^k and the Schur functions $s_\lambda(x_1, \ldots, x_n)$ $(l(\lambda) \le n)$ form a $\mathbb{Z}$-basis for Λ_n.*

There is a compatibility relation that holds for the Schur functions, as shown in the next proposition.

Proposition 15.2. *Let $l(\lambda) = l$ and p, q be two integers such that $l \le p < q$. Then,*

$$s_\lambda(x_1, \ldots, x_p) = s_\lambda(x_1, \ldots, x_p, x_{p+1}, \ldots, x_q)\,\Bigr|_{x_{p+1} = \cdots = x_q = 0}.$$

Proof. It suffices to verify the proposition for $q = p + 1$. First, $a_{\lambda+\delta}(x_1, \ldots, x_{p+1})$ is equal to

$$\begin{vmatrix} x_1^{\lambda_1+p} & \ldots & x_1^{\lambda_p+1} & x_1^{\lambda_{p+1}} \\ \ldots & \ldots & \ldots & \ldots \\ x_p^{\lambda_1+p} & \ldots & x_p^{\lambda_p+1} & x_p^{\lambda_{p+1}} \\ x_{p+1}^{\lambda_1+p} & \ldots & x_{p+1}^{\lambda_p+1} & x_{p+1}^{\lambda_{p+1}} \end{vmatrix} = \begin{vmatrix} x_1^{\lambda_1+p} & \ldots & x_1^{\lambda_p+1} & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_p^{\lambda_1+p} & \ldots & x_p^{\lambda_p+1} & 1 \\ x_{p+1}^{\lambda_1+p} & \ldots & x_{p+1}^{\lambda_p+1} & 1 \end{vmatrix},$$

since $\lambda_{p+1} = 0$. Hence, $a_{\lambda+\delta}(x_1, \ldots, x_p, 0)$ is equal to

$$\begin{vmatrix} x_1^{\lambda_1+p} & \ldots & x_1^{\lambda_p+1} & 1 \\ \multicolumn{4}{c}{\dotfill} \\ x_p^{\lambda_1+p} & \ldots & x_p^{\lambda_p+1} & 1 \\ 0 & \ldots & 0 & 1 \end{vmatrix} = x_1 \ldots x_p \begin{vmatrix} x_1^{\lambda_1+p-1} & \ldots & x_1^{\lambda_p} \\ \multicolumn{3}{c}{\dotfill} \\ x_p^{\lambda_1+p-1} & \ldots & x_p^{\lambda_p} \end{vmatrix}$$

$$= x_1 \ldots x_p \, a_{\lambda+\delta}(x_1, \ldots, x_p).$$

Thus,

$$s_\lambda(x_1, \ldots, x_p, 0) = \frac{x_1 \ldots x_p \, a_{\lambda+\delta}(x_1, \ldots, x_p)}{x_1 \ldots x_p \, a_\delta(x_1, \ldots, x_p)} = s_\lambda(x_1, \ldots, x_p). \quad \square$$

We can then define the Schur functions as infinite sequences $s_\lambda = \big(s_\lambda(x_1, \ldots, x_n)\big)$ $(n \geq 0)$. Thus the *s_λ's form a $\mathbb{Z}$-basis for Λ and the s_λ's $(|\lambda| = k)$ for a $\mathbb{Z}$-basis for Λ^k.*

The Schur functions can also be expressed as plain determinants in the h_k's and also the e_k's, as shown in the next proposition. Remember that λ' designates the conjugate of the partition λ.

Proposition 15.3. *We have*

$$s_\lambda = \det\big(h_{\lambda_i - i + j}\big)_{(1 \leq i,j \leq n)} \quad (n \geq l(\lambda))\,;$$
$$s_\lambda = \det\big(e_{\lambda'_i - i + j}\big)_{(1 \leq i,j \leq m)} \quad (m \geq l(\lambda'))\,;$$

where, by convention, the coefficients are zero when the subscripts of the h_k's or the e_k's are strictly negative.

Proof. We only give the proof of the first identity. Start with the formula (iii) of Theorem 14.2, that is, $H(u)E(-u) = 1$ and let $e_i^{(k)}$ be the elementary symmetric function of $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$, with $E^{(k)}(u)$ being the generating function $\sum_{r=0}^{n-1} e_r^{(k)} u^r$. Obviously,

$$\sum_{p \geq 0} h_p u^p \sum_{r=0}^{n-1} e_r^{(k)} (-u)^r = (1 - x_k u)^{-1}.$$

Now consider a sequence $(b_1, \ldots, b_n)$ of nonnegative integers and determine the coefficient of u^{b_m} on the two sides of the previous equation. We get

$$\sum_{r=0}^{n-1} h_{b_m - r} (-1)^r e_r^{(k)} = x_k^{b_m},$$

where with $r + j = n$

$$\sum_{r=1}^{n} h_{b_m-n+j}(-1)^{n-j}e_{n-j}^{(k)} = x_k^{b_m}. \tag{15.1}$$

Let $\mathcal{E}$ be the $n \times n$-matrix whose k-th row is given by

$$\big((-1)^{n-1}e_{n-1}^{(k)}, \ldots, (-1)e_1^{(k)}, e_0^{(k)}\big) \tag{15.2}$$

and let $\mathcal{H}$ be the matrix whose m-th column $\mathcal{H}_{\cdot,m}$ reads

$$\mathcal{H}_{\cdot,m}^t = \big(h_{b_m-n+1}, h_{b_m-n+2}, \ldots, h_{b_m}\big). \tag{15.3}$$

The left-hand side of identity (15.1) can be rewritten as a matrix product, so that

$$\mathcal{H}\mathcal{E} = \big(x_k^{b_m}\big).$$

Now take the determinant of each side: $\det\mathcal{H}.\det\mathcal{E} = \det\big(x_k^{b_m}\big)$. When $b = \delta$, we get $1 \times \det\mathcal{E} = \det\big(x_k^{n-j}\big) = a_\delta$, so that $\det\big(h_{b_j-n+i}\big)a_\delta = \det\big(x_i^{b_j}\big)$. With $b_i := \lambda_i + n - i$, this can be rewritten as

$$\det\big(h_{\lambda_j+n-j-n+i}\big)a_\delta = \det\big(x_i^{\lambda_j+n-j}\big),$$

i.e.,

$$\det\big(h_{\lambda_j-j+i}\big) = \det\big(h_{\lambda_i-i+j}\big) = \det\big(x_i^{\lambda_j+n-j}\big)/\det\big(x_i^{n-j}\big) = s_\lambda. \quad \square$$

Let ν, θ be two Ferrers diagrams such that $\nu \supset \theta$. The set difference $\nu \setminus \theta$, usually denoted by ν/θ, is called a *skew diagram.* When $n \ge l(\nu)$, we can define the *skew Schur function* $s_{\nu/\theta}(x) = s_{\nu/\theta}(x_1, \ldots, x_n)$ by the determinantal expression

$$s_{\nu/\theta}(x) = \det(h_{\nu_i-\theta_j-i+j})\ (1 \le i, j, \le n). \tag{15.4}$$

As above, it can be shown that also

$$s_{\nu/\theta}(x) = \det(e_{\nu'_i-\theta'_j-i+j})\ (1 \le i, j, \le n). \tag{15.5}$$

when $n \ge l(\nu')$. The skew Schur Function $s_{\nu/\theta}(x)$ is a symmetric function that reduces to $s_\nu(x)$ when θ is the zero partition.

16. The Cauchy identity

We have already found an expression for the expansion of the product $\prod(1 - x_iy_j)^{-1}$, namely

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_\lambda m_\lambda(x)h_\lambda(y). \tag{16.1}$$

Two other expressions can be obtained:

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_\lambda z_\lambda^{-1}p_\lambda(x)p_\lambda(y)\,; \tag{16.2}$$

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x)s_\lambda(y). \tag{16.3}$$

In those expansions λ runs over all partitions of integers. First, (16.2) follows from

$$\prod_{i,j}(1-x_iy_j)^{-1}=\exp\sum_{n\geq 1}p_n(x)p_n(y)\frac{1}{n},$$

since the right-hand side can be written

$$\begin{aligned}\prod_{n\geq 1}\sum_{k_n\geq 0}p_n(x)^{k_n}p_n(y)^{k_n}\frac{1}{n^{k_n}k_n!}&=\sum_{(n_i)}\sum_{(k_{n_i})}\prod_i\frac{p_{n_i}(x)^{k_{n_i}}\,p_{n_i}(y)^{k_{n_i}}}{n_i^{k_{n_i}}\,k_{n_i}!}\\&=\sum_\lambda\frac{1}{z_\lambda}p_\lambda(x)p_\lambda(y).\end{aligned}$$

This proves (16.2). The third identity follows from the Binet-Cauchy identity that reads:

$$\det\bigl((1-x_iy_j)^{-1}\bigr)_{(1\leq i,j\leq n)}=a_\delta(x)a_\delta(y)\prod_{i,j=1}^{n}(1-x_iy_j)^{-1}.$$

The proof of the Binet-Cauchy identity can be made as follows. Multiply the i-th row of the determinant $\det\bigl((1-x_iy_j)^{-1}\bigr)$ by the product $\prod_{k=1}^n(1-x_iy_k)$ and do it for each $i=1,\dots,n$. The entry in (i,j) becomes

$$\begin{aligned}\prod_{k\neq j}(1-x_iy_k)&=\sum_{r=0}^{n-1}x_i^r(-1)^re_r^{(j)}(y)\qquad[e_r^{(j)}(y)=e_r(y_1,\dots,\check{y}_j,\dots,y_n)]\\&=\sum_{r=1}^{n}x_i^{n-r}(-1)^{n-r}e_{n-r}^{(j)}(y).\end{aligned}$$

As product of matrices this can be read as

$$\Bigl(\prod_{k\neq j}(1-x_iy_k)\Bigr)_{(i,j)}=\bigl(x_i^{n-j}\bigr)_{(i,j)}\bigl((-1)^{n-i}e_{n-i}^{(j)}(y)\bigr)_{(i,j)}\,;$$

and as product of determinants as

$$\begin{aligned}\det\Bigl(\prod_{k\neq j}(1-x_iy_k)\Bigr)\Bigl(=\prod_{i,j}(1-x_iy_j)\det\bigl((1-x_iy_j)^{-1}\bigr)\Bigr)&\\=\det\bigl(x_i^{n-j}\bigr)\det\bigl((-1)^{n-i}e_{n-i}^{(j)}(y)\bigr)&\\=a_\delta(x)a_\delta(y),&\end{aligned}$$

remembering that $a_\delta(x) = \det\bigl(x_i^{n-j}\bigr)$ and $\det\bigl((-1)^{n-i}e^{(j)}_{n-i}(y)\bigr) = \det\mathcal{E} = a_\delta(y)$, using the notations of the previous section.

Now to derive (16.3) we expand each term $(1-x_iy_j)^{-1}$ in the determinant:

$$\begin{aligned}
\det\bigl((1-x_iy_j)^{-1}\bigr) &= \det\Bigl(\Bigl(\sum_{m\ge 0} x_i^m y_j^m\Bigr)\Bigr)\\
&= \det\Bigl(\sum_{\alpha_1}\begin{pmatrix} x_1^{\alpha_1}y_1^{\alpha_1}\\ \vdots\\ x_n^{\alpha_1}y_1^{\alpha_1}\end{pmatrix},\ldots,\sum_{\alpha_n}\begin{pmatrix} x_1^{\alpha_n}y_n^{\alpha_n}\\ \vdots\\ x_n^{\alpha_n}y_n^{\alpha_n}\end{pmatrix}\Bigr)\\
&= \sum_\alpha \det\bigl(x_i^{\alpha_j}y_j^{\alpha_j}\bigr) \qquad [\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n]\\
&= \sum_\alpha \det\Bigl(y_1^{\alpha_1}\begin{pmatrix} x_1^{\alpha_1}\\ \vdots\\ x_n^{\alpha_1}\end{pmatrix},\ldots,y_n^{\alpha_n}\begin{pmatrix} x_1^{\alpha_n}\\ \vdots\\ x_n^{\alpha_n}\end{pmatrix}\Bigr)\\
&= \sum_\alpha y^\alpha \det\bigl(x_i^{\alpha_j}\bigr) = \sum_\alpha y^\alpha a_\alpha(x).
\end{aligned}$$

As $a_\alpha(x)=0$ if the α_i's are not all distinct, this yields:

$$\begin{aligned}
\det\bigl((1-x_iy_j)^{-1}\bigr) &= \sum_{\beta_1>\cdots>\beta_n\ge 0}\ \sum_{\sigma\in\mathfrak{S}_n} y^{\sigma\beta}a_{\sigma\beta}(x)\\
&= \sum_\beta \sum_{\sigma\in S_n} y^{\sigma\beta}\varepsilon(\sigma)a_\beta(x)\\
&= \sum_\beta a_\beta(x)a_\beta(y) = \sum_\lambda a_{\lambda+\delta}(x)a_{\lambda+\delta}(y),
\end{aligned}$$

where $l(\lambda)\le n$. Hence,

$$\prod_{i,j=1}^n (1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x_1,\ldots,x_n)s_\lambda(y_1,\ldots,y_n).$$

The identity also holds for infinitely many variables (x_i, y_j), for the coefficient of $x_{i_1}^{r_1}\ldots x_{i_n}^{r_n}y_{j_1}^{r_1}\ldots y_{j_n}^{r_n}$ $(i_1<\cdots<i_n\,;\ j_1<\cdots<j_n)$ in the product $\prod_{i,j\ge 1}(1-x_iy_j)^{-1}$ is equal to the coefficient of the same monomial in the *finite* product $\prod_{1\le i,j\le N}(1-x_iy_j)^{-1}$, where $i_n, j_n\le N$. The identity holding in the finite case, the previous coefficient is equal to the coefficient of the same monomial in $s_\lambda(x_1,\ldots,x_N)s_\lambda(y_1,\ldots,y_N)$, and also in $s_\lambda(x_1,\ldots,x_M)s_\lambda(y_1,\ldots,y_M)$ for every $M\ge N$, because of the compatibility property of the s_λ's: $s_\lambda(x_1,\ldots,x_n,0,\ldots,0)=s_\lambda(x_1,\ldots,x_n)$. $\square$

Identity (16.3) is referred to as the *Cauchy identity* and is used in many combinatorial contexts, especially under the form

$$\sum_{n\geq 0} u^n \sum_{|\lambda|=n} s_\lambda(x)s_\lambda(y) = \prod_{i,j} \frac{1}{1-ux_iy_j}. \tag{16.4}$$

17. The combinatorial definition of the Schur functions

The theory of symmetric functions would not have such an impact in Combinatorics, if there was no interpretation of the Schur functions in terms of sums of tableau evaluations. The tableaux in questions are called semi-standard and can be introduced as follows.

First, recall that the *Ferrers diagram* associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the set of ordered pairs (i,j) of the Euclidean plane with the property that $1 \leq i \leq \lambda_j$, $1 \leq j \leq r$. Each Ferrers diagram is usually identified with its corresponding partition. For instance, the following Ferrers diagram

$$\begin{array}{l} \times\,\times \\ \times\,\times\,\times \\ \times\,\times\,\times \\ \times\,\times\,\times\,\times\,\times \end{array}$$

corresponds to the partition $\lambda = (5,3,3,2)$.

Let λ be a Ferrers diagram with n points and let $i_1 i_2 \ldots i_n$ be a nonincreasing word of length n whose letters are integers. Suppose that those n letters are written on the n points of λ in such a way that every column (resp. every row) is (strictly increasing) from bottom to top (resp. nondecreasing from left to right). The configuration τ thereby obtained is called a *semi-standard tableau*, of *content* $\{i_1, i_2, \ldots, i_n\}$ and *of shape* λ. If $i_1 i_2 \ldots i_n$ is the word $1, 2, \ldots, n$ the semi-standard tableau is called *standard* of order n.

For instance,

$$\tau = \begin{array}{l} 6\;7 \\ 4\;4\;8 \\ 3\;3\;5 \\ 1\;2\;2\;4\;5 \end{array}$$

is a semi-standard tableau of shape $(5,3,3,2)$ and of content $12^2 3^2 4^3 5^2 678$.

Let $x = \{x_1, x_2, \ldots\}$ be a set of variables (finite or infinite) and τ be a semi-standard tableau of shape λ and of content $\{i_1 \leq i_2 \leq \cdots \leq i_n\}$. If the cardinality of x is greater than or equal to i_n, the *x-evaluation* of τ is defined to be the monomial $x(\tau) := x_{i_1} x_{i_2} \cdots x_{i_n}$, i.e., the product

$$\prod x_{\tau(i,j)},$$

where (i,j) runs over all points (i,j) of the Ferrers diagram λ.

For each alphabet $x = \{x_1, x_2, \ldots, x_n\}$ with $n \geq 7$ the semi-standard tableau of the running example has the x-evaluation $x_1x_2^2x_3^2x_4^3x_5^2x_6x_7x_8$.

Theorem 17.1. *The Schur function $s_\lambda(x)$ associated with the partition λ in the alphabet x is equal to*

$$s_\lambda(x) = \sum_\tau x(\tau), \tag{17.1}$$

where the summation is over all semi-standard tableaux τ of shape λ.

Thus Theorem 17.1 provides a *combinatorial definition* of the Schur functions. Notice that with that definition it is not at all obvious that each Schur function is symmetric in the x_i's. This follows from the algebraic definition given in section 15, although there are several ways of proving the symmetry property using (17.1) only. To obtain the expansion of $s_\lambda(x)$ it suffices to list the semi-standard tableaux of contents $\{1^{c_1}, 2^{c_2}, \ldots\}$ with $c_1 \geq c_2 \geq \cdots \geq 1$. As the Schur function is symmetric, the expansion will also include the semi-standard tableaux of contents $\{1^{c_{\sigma 1}}, 2^{c_{\sigma 2}}, \ldots\}$ for each permutation σ of the subscripts.

For example, consider the partition $\lambda = (3,1)$ and the alphabet $x = \{x_1, x_2, x_3, x_4\}$. The semi-standard tableaux τ of shape λ whose contents are of the form $\{1^{c_1}, 2^{c_2}, \ldots\}$ with $c_1 \geq c_2 \geq \cdots \geq 1$ are the following

$$\begin{array}{lllllll} 2 & 2 & 3 & 2 & 4 & 3 & 2 \\ 1\,1\,1; & 1\,1\,2; & 1\,1\,2; & 1\,1\,3; & 1\,2\,3; & 1\,2\,4; & 1\,3\,4. \end{array}$$

According to Theorem 17.1 we have:

$$s_{(3,1)}(x_1, x_2, x_3, x_4) = \sum x_1^3x_2 + \sum x_1^2x_2^2 + 2\sum x_1^2x_2x_3 + 3\,x_1x_2x_3x_4.$$

Notice that the first summation involves twelve monomials. For $1 \leq i < j \leq 4$, the monomial $x_i^3x_j$ (resp. $x_ix_j^3$) corresponds to the semi-standard tableau $\begin{smallmatrix} j & & \\ i & i & i \end{smallmatrix}$ (resp. $\begin{smallmatrix} j & & \\ i & j & j \end{smallmatrix}$).

To prove Theorem 17.1 we start with the very first definition of the Schur function as a ratio of two determinants, as given in Section 15. An alternate proof consists of using Proposition 15.3, but this involves other combinatorial techniques

Let λ and μ be two partitions (or Ferrers diagrams) such that μ is contained in λ. The set difference $\lambda \setminus \mu$ is called a *skew tableau* and denoted by λ/μ. A skew tableau is called a *horizontal strip*, if each column contains at most one box (or one cross) of the skew tableau. Let λ_i and μ_i denote

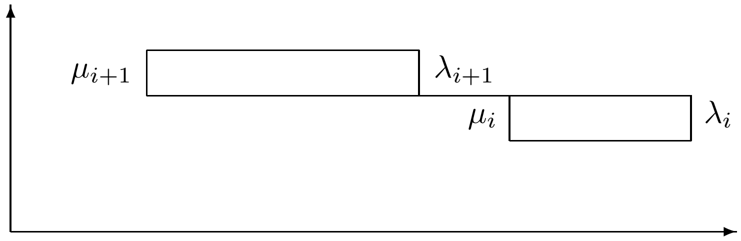


Fig. 17.1

the parts of the partitions λ and μ, respectively. As shown in Fig. 17.1, the skew tableau λ/μ is a horizontal strip if and only if for all i we have $\lambda_{i+1} \le \mu_i$, i.e.,

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \lambda_3 \ge \cdots$$

Let τ be a semi-standard tableau of shape λ and of content $1^{c_1}2^{c_2}\dots n^{c_n}$. Let $\lambda_0 := \emptyset$ and $\lambda_n := \lambda$. For each $i = 1, 2, \dots, n$ the subset of the boxes of τ containing an integer at most equal to i (resp. an integer equal to i) is a semi-standard tableau (resp. a horizontal strip) we denote by $\lambda^{(i)}$ (resp. $\lambda^{(i)}/\lambda^{(i-1)}$). Let $\lambda_j^{(i)}$ denote the parts of $\lambda^{(i)}$. Hence, for each $i = 1, 2, \dots, n$ the following inequalities holds

$$\lambda_1^{(i)} \ge \lambda_1^{(i-1)} \ge \lambda_2^{(i)} \ge \lambda_2^{(i-1)} \ge \lambda_3^{(i)} \ge \lambda_3^{(i-1)} \ge \cdots$$

With those notations the x-evaluation of τ is given by:

$$x(\tau) = x_1^{|\lambda^{(1)}|-|\lambda^{(0)}|} x_2^{|\lambda^{(2)}|-|\lambda^{(1)}|} \dots\ x_n^{|\lambda^{(n)}|-|\lambda^{(n-1)}|}.$$

Theorem 17.1 can then be rephrased as follows.

Theorem 17.1′. *We have:*

$$s_\lambda(x_1, \dots, x_n) = \sum_{\substack{\emptyset=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(n)}=\lambda \\ \lambda^{(i)}/\lambda^{(i-1)} \text{ horizontal strip}}} x_1^{|\lambda^{(1)}|-|\lambda^{(0)}|} x_2^{|\lambda^{(2)}|-|\lambda^{(1)}|} \dots\ x_n^{|\lambda^{(n)}|-|\lambda^{(n-1)}|}.$$

The formula holds for $n = 1$, since the only Schur functions $s_\lambda(x_1)$ in one variable x_1 are the monomials $x_1^{|\lambda|}$, where $\lambda = \lambda^{(1)}$ is reduced to a single part. The formula can be rewritten

$$s_\lambda(x_1, \dots, x_n) = \sum_{\lambda^{(n-1)}\subset\lambda} x_n^{|\lambda|-|\lambda^{(n-1)}|} \sum_{\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(n-1)}} x_1^{|\lambda^{(1)}|-|\lambda^{(0)}|} \dots\ x_{n-1}^{|\lambda^{(n-1)}|-|\lambda^{(n-2)}|},$$

keeping in mind that each skew tableau $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. The second summation is nothing but the formula of the theorem for

$(n-1)$ variables. Let $\mu := \lambda^{(n-1)}$ and use the induction hypothesis. We are left to prove the identity

$$(\star) \qquad s_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\mu \\ \lambda/\mu \text{ horizontal strip}}} x_n^{|\lambda|-|\mu|}\, s_\mu(x_1,x_2,\ldots,x_{n-1}).$$

To prove $(\star)$ we proceed as follows. Consider the determinant

$$a_\lambda(x_1,\ldots,x_n) := \det(x_i^{\lambda_j+n-j})_{(1\le i,j\le n)}$$

and evaluate

$$a_\lambda(x_1,\ldots,x_{n-1},1) = \begin{vmatrix} x_1^{\lambda_1+n-1} & \ldots & x_1^{\lambda_n+n-n} \\ \vdots & \ddots & \vdots \\ x_{n-1}^{\lambda_1+n-1} & \ldots & x_{n-1}^{\lambda_n+n-n} \\ 1 & \ldots & 1 \end{vmatrix}.$$

Subtract the $(j+1)$-st column from the j-th column for each $j = 1,2,\ldots,n-1$. We get:

$$\begin{vmatrix} & & & x_1^{\lambda_n} \\ & E & & \\ & & & x_{n-1}^{\lambda_n} \\ 0 & \ldots & 0 & 1 \end{vmatrix},$$

where E is the matrix $(x_i^{\lambda_j+n-j} - x_i^{\lambda_{j+1}+n-j-1})_{(1\le i,j\le n-1)}$. But

$$\begin{aligned} x_i^{\lambda_j+n-j} - x_i^{\lambda_{j+1}+n-j-1} &= (x_i-1)\sum_{\lambda_{j+1}+n-j-1\le t\le \lambda_j+n-j-1} x_i^t \\ &= (x_i-1)\sum_{\lambda_{j+1}\le \mu_j\le \lambda_j} x_i^{\mu_j+n-j-1}. \end{aligned}$$

Hence

$$\begin{aligned} \det E &= \det\Bigl((x_i-1)\sum_{\lambda_{j+1}\le\mu_j\le\lambda_j} x_i^{\mu_j+n-j-1}\Bigr)_{(1\le i,j\le n-1)} \\ &= \prod_{1\le i\le n}(x_i-1)\times \sum_{\substack{(\lambda_{j+1}\le\mu_j\le\lambda_j)\\(1\le j\le n-1)}} \det(x_i^{\mu_j+n-j-1})_{(1\le i,j\le n-1)}. \end{aligned}$$

Now if we divide $\det E$ by

$$\Delta(x_1,\ldots,x_{n-1},1) = \prod_{1\le i\le n}(x_i-1)\cdot\Delta(x_1,\ldots,x_{n-1}),$$

we obtain the Schur function $s_\lambda(x_1,\dots,x_{n-1},1)$. Consequently,

$$\begin{aligned} s_\lambda(x_1,\dots,x_{n-1},1) &= \sum_{\substack{(\lambda_{j+1}\le\mu_j\le\lambda_j)\\(1\le j\le n-1)}} \frac{\det(x_i^{\mu_j+n-1-j})_{(1\le i,j\le n-1)}}{\Delta(x_1,\dots,x_{n-1})} \\ &= \sum_{\substack{\mu\\ \lambda/\mu \text{ horizontal strip}}} s_\mu(x_1,\dots,x_{n-1}). \end{aligned}$$

As $s_\lambda(x_1,\dots,x_{n-1},x_n)$ is of degree $|\lambda|$ with respect to the set of all variables and as $s_\mu(x_1,\dots,x_{n-1})$ is of degree $|\mu|$, we get:

$$s_\lambda(x_1,\dots,x_{n-1},x_n) = \sum_{\substack{\mu\\ \lambda/\mu \text{ horizontal strip}}} x_n^{|\lambda|-|\mu|}\, s_\mu(x_1,\dots,x_{n-1}),$$

which is formula $(\star)$. $\square$

There is also a combinatorial definition for the skew Schur function that reads

$$s_{\nu/\theta}(x) = \sum_\tau x(\tau), \tag{17.2}$$

where the summation is over all semi-standard tableaux τ of shape ν/θ. Those semi-standard tableaux obey the same rules as the semi-standard tableaux occurring in (17.1): every column (resp. every row) is strictly increasing from bottom to top (resp. nondecreasing from left to right). However their shapes are skew diagrams.

For instance,

$$\tau = \begin{array}{llllll} 6 & 8 & & & & \\ 1 & 5 & 5 & & & \\ & & & 2 & & \\ & & & 1 & 2 & 2 \end{array}$$

is such a semi-standard tableau of shape is ν/θ with $\nu = (6,4,3,2)$ and $\theta = (3,3)$.

To derive (17.2) we have to use another combinatorial technique, since we have to start from a determinantal definition such as (15.4) and not a ratio of two determinants. The proof of (17.2) is not given in this memoir.

18. The inverse ligne of route of a standard tableau

In this section we consider standard tableaux whose shapes can be skew diagrams ν/θ, as introduced in section 15. If the skew diagram has n

boxes (or crosses), the entries written on those crosses are the n integers $1, 2, \ldots, n$, the rows (resp. columns) being increasing when read from left to right (resp. from bottom to top). For instance,

$$T = \begin{array}{llllll} 6 & 8 & & & & \\ 4 & 5 & 9 & & & \\ & & & 3 & & \\ & & & 1 & 2 & 7 \end{array}$$

is a standard tableau of shape ν/θ with $\nu = 6, 4, 3, 2$ and $\theta = 3, 3$.

By analogy with the permutations, we can define the *inverse ligne of route* of the standard tableau T as the set, $\operatorname{Iligne} T$, of all k such that $(k+1)$ is *above* k in T (or, equivalently, to its left). With the above example we have

$$\operatorname{Iligne} T = \{2, 3, 5, 7\}.$$

Notice that for a standard tableau no ligne of route is introduced. The word "inverse" has been here added for convenience. We further define:

$$\text{(18.1)} \qquad \operatorname{ides} T := |\operatorname{Iligne} T| \quad \text{and} \quad \operatorname{imaj} T := \sum_j j \quad (j \in \operatorname{Iligne} T).$$

A more refined statistic is the *y-vector* of T defined as the word $y(T) = y(1)y(2)\ldots y(n)$, where

$$\text{(18.2)} \qquad y(i) := \#\{j \geq i : j \in \operatorname{Iligne} T\}.$$

Using the same example, $\operatorname{ides} T = 4$, $\operatorname{imaj} T = 2 + 3 + 5 + 7 = 17$ and

$$\begin{array}{rccccccccc} & 1 & \mathbf{2} & \mathbf{3} & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 \\ y(T) = & 4 & 4 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \end{array}$$

The next proposition is straightforward and given without proof.

Proposition 18.1. *For each standard tableau T whose y-vector reads $y(T) = y(1)y(2)\ldots y(n)$ the following holds:*

$$\operatorname{ides} T = y(1), \quad \operatorname{imaj} T = \operatorname{tot} y(T) = y(1) + y(2) + \cdots + y(n).$$

Now let $\{x_1, x_2, \ldots\}$ and $\{q_1, q_2, \ldots\}$ be two sets of independent variables. If $x_{c(1)}x_{c(2)}\ldots x_{c(n)}$ is a monomial written in such a way that $c(1) \geq c(2) \geq \cdots \geq c(n)$, define

$$\phi_{\mathbf{q}}(x_{c(1)}x_{c(2)}\ldots x_{c(n)}) := q_1^{c(1)-1} q_2^{c(2)-1} \ldots q_n^{c(n)-1}$$

and let $\phi_{\mathbf{q}}$ act linearly on the algebra of the polynomials in the x_i's. If the variables q_i's are distinct, then $\phi_{\mathbf{q}}$ cannot be a ring homomorphism. For a standard tableau T of order n it will be convenient to use the notation: $\mathbf{q}^{y(T)} := q_1^{y(1)} q_2^{y(2)} \dots q_n^{y(n)}$. Also, for each variable t introduce

$$[t;\mathbf{q}]_n := \begin{cases} 1-t, & \text{if } n=0; \\ (1-t)(1-tq_1)(1-tq_1q_2)\cdots(1-tq_1\cdots q_n), & \text{if } n \geq 1. \end{cases}$$

Theorem 18.2. *Let ν/θ be a skew diagram with n points. Then, the following identities hold:*

$$\phi_{\mathbf{q}}(s_{\nu/\theta}(x_1,x_2,\dots)) = \frac{1}{\prod_{i=1}^{n}(1-q_1\cdots q_i)} \sum_{T,T \text{ of shape } \nu/\theta} \mathbf{q}^{y(T)}; \tag{18.3}$$

$$\sum_{r\geq 0} t^r\, \phi_{\mathbf{q}}(s_{\nu/\theta}(x_1,\dots,x_{r+1})) = \frac{1}{[t;\mathbf{q}]_{n+1}} \sum_{T,T \text{ of shape } \nu/\theta} t^{\operatorname{ides} T}\mathbf{q}^{y(T)}; \tag{18.4}$$

the summations on the two right-hand sides being over all standard tableaux of shape ν/θ.

When all the variables q_i are equal to a single variable q, the homomorphism $\phi_{\mathbf{q}}$, that will then be denoted by ϕ_q, becomes a *ring homomorphism.* We then have

$$\phi_q(s_{\nu/\theta}(x_1,x_2,\dots)) = s_{\nu/\theta}(1,q,q^2,\dots)); \tag{18.5}$$

$$\phi_q(s_{\nu/\theta}(x_1,\dots,x_{r+1})) = s_{\nu/\theta}(1,q,\dots,q^r)). \tag{18.6}$$

In view of Proposition 18.1 this implies the subsequent corollary.

Corollary 18.3. *Let ν/θ be a skew diagram with n points. Then, the following identities hold:*

$$s_{\nu/\theta}(1,q,q^2,\dots) = \frac{1}{(q;q)_n} \sum_{T,T \text{ of shape } \nu/\theta} q^{\operatorname{imaj} T}; \tag{18.7}$$

$$\sum_{r\geq 0} t^r\, s_{\nu/\theta}(1,q,\dots,q^r) = \frac{1}{(t;q)_{n+1}} \sum_{T,T \text{ of shape } \nu/\theta} t^{\operatorname{ides} T} q^{\operatorname{imaj} T}; \tag{18.8}$$

the summations on the two right-hand sides being over all standard tableaux of shape ν/θ.

The proof of Theorem 18.2 is based upon the *combinatorial* definition of Schur functions (or skew Schur functions), as discussed in the previous section, and a bijection

$$\tau \mapsto (T,d)$$

of the set of semi-standard tableaux onto pairs (T,d), where T is a standard tableau having the same shape as τ and d a nonincreasing sequence of integers.

The *semi-standard* tableaux we will consider this time are *non-increasing* along the rows (from left to right) and (strictly) decreasing (from bottom to top) along the columns, as opposed to the definition given in the previous section. This convention will facilitate the calculations. For instance

$$\tau = \begin{array}{cccccc} 2 & 1 & & & & \\ 3 & 3 & 1 & & & \\ & & & 5 & & \\ & & & 8 & 6 & 2 \end{array}$$

is such a semi-standard tableau, whose shape is ν/θ with $\nu=(6,4,3,2)$ and $\theta=(3,3)$.

Each such semi-standard tableau τ determines a total ordering on the points (i,j) of its diagram ν/θ in the following manner: (i,j) is said to be less than (i',j'), if the integer $\tau(i,j)$ written on (i,j) is greater than $\tau(i',j')$, or if $\tau(i,j)=\tau(i',j')$ and (i,j) is to the left of (i',j'), that is, $i<i'$.

Suppose that the diagram ν/θ has n elements; write k on the point (i,j) if (i,j) is the smallest k-th element using the previous ordering. This produces a *standard tableau* T, of order n, of shape ν/θ. Reading the elements $\tau(i,j)$ from smallest to greatest yields a non-increasing sequence $c=c(1)c(2)\dots c(n)$, called the *content* of τ. Moreover, we have

$$k\in \operatorname{Iligne} T \Rightarrow c(k)>c(k+1). \tag{18.9}$$

As $i\in \operatorname{Iligne} T \Rightarrow y(i)=y(i+1)+1$ and $i\notin \operatorname{Iligne} T \Rightarrow y(i)=y(i+1)$, we see that the sequence $d:=d(1)d(2)\dots d(n)$, defined by $d(i):=c(i)-1-y(i)$ for each $i=1,2,\dots,n$, satisfies $d(1)\ge d(2)\ge\cdots\ge d(n)\ge 0$. Moreover, the mapping $\tau\mapsto(T,d)$ is *bijective.*

With the previous semi-standard tableau τ the standard tableau T is the tableau given in the beginning of this section. We further have:

$$\begin{array}{rccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ c= & 8 & 6 & 5 & 3 & 3 & 2 & 2 & 1 & 1\\ y(T)= & 4 & 4 & 3 & 2 & 2 & 1 & 1 & 0 & 0\\ d= & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Next, for each semi-standard tableau τ of content $c=c(1)\dots c(n)$ define

$$x(\tau):=x_{c(1)}\cdots x_{c(n)},$$

so that

$$\phi_{\mathbf{q}}(x(\tau)):=q_1^{c(1)-1}q_2^{c(2)-1}\dots q_n^{c(n)-1}.$$

Using the above bijection $\tau \mapsto (T, d)$, we get

$$\begin{aligned}\phi_{\mathbf{q}}(x(\tau)) &= q_1^{y(1)} q_2^{y(2)} \dots q_n^{y(n)} q_1^{d(1)} q_2^{d(2)} \dots q_n^{d(n)} \\ &= \mathbf{q}^{y(T)} \mathbf{q}^d.\end{aligned}$$

Accordingly,

$$\sum_{\tau, \tau \text{ of shape } \nu/\theta} \phi_{\mathbf{q}}(x(\tau)) = \sum_{T, T \text{ of shape } \nu/\theta} \mathbf{q}^{y(T)} \sum_d \mathbf{q}^d.$$

As the last summation is over all non-increasing sequences $d = d(1) \geq \cdots \geq d(n) \geq 0$, we obtain

$$\sum_{\tau, \tau \text{ of shape } \nu/\theta} \phi_{\mathbf{q}}(x(\tau)) = \frac{1}{\prod_{i=1}^{n}(1 - q_1 \cdots q_i)} \sum_{T, T \text{ of shape } \nu/\theta} \mathbf{q}^{y(T)}. \tag{18.10}$$

Now make use of the combinatorial definition for the skew Schur function

$$s_{\nu/\theta}(x_1, x_2, \dots) = \sum_{\tau, \tau \text{ of shape } \nu/\theta} x(\tau),$$

and take the homomorphic image of identity (18.10) under $\phi_{\mathbf{q}}$. This yields (18.3).

For the proof of (18.4) proceed as follows. In the expansion of the skew Schur function $s_{\nu/\theta}(x_1, x_2, \dots, x_{r+1})$ in the finite alphabet $\{x_1, x_2, \dots, x_{r+1}\}$ the semi-standard tableaux τ of content c such that $c(1) \geq r+2$ bring no contribution. Consequently,

$$\sum_{c(1)=r+1} x(c) = s_{\nu/\theta}(x_1, x_2, \dots, x_{r+1}) - s_{\nu/\theta}(x_1, x_2, \dots, x_r);$$

hence,

$$\begin{aligned}\sum_r t^r \sum_{c(1)=r+1} x(c) &= \sum_r t^r (s_{\nu/\theta}(x_1, x_2, \dots, x_{r+1}) - s_{\nu/\theta}(x_1, x_2, \dots, x_r)) \\ &= (1-t) \sum_r t^r s_{\nu/\theta}(x_1, x_2, \dots, x_{r+1}).\end{aligned}$$

On the other hand, by using (18.10) we get

$$\begin{aligned}\phi_{\mathbf{q}}\Bigl(\sum_r t^r \sum_{c(1)=r+1} x(c)\Bigr) &= \sum_{\tau, \tau \text{ of shape } \nu/\theta} (tq_1)^{c(1)-1} q_1^{c(2)-1} \cdots q_n^{c(n)-1} \\ &= \frac{1}{\prod_{i=1}^{n}(1 - tq_1 \cdots q_i)} \sum_{T, T \text{ of shape } \nu/\theta} t^{\operatorname{ides} T} \mathbf{q}^{y(T)},\end{aligned}$$

since $y(1) = \operatorname{ides} T$. Thus identity (18.4) is proved.

19. The Robinson-Schensted Correspondence

This is one of the most celebrated correspondences. Its main interest in Combinatorics lies in the fact that geometric properties held by permutations can be transferred onto tableaux. Remember that we have introduced inverse lignes of route for both permutations and standard tableaux. One of our problems will be to relate those two lignes of routes through the Robinson-Schensted correspondence.

19.1. *The Schensted-Knuth algorithm.* In section 12.2 each nondecreasing biword $\mathbf{b}$ has been mapped onto a triple (σ, b', c'), where σ is a permutation and b', c' two nondecreasing words. The Schensted-Knuth algorithm will map such a biword onto a pair of two semi-standard tableaux *of the same shape*, that shape being a (right) Ferrers diagram. Again take up the example of section 12.1

$$\mathbf{b} = \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0\,0\,0\,1\,1\,1\,1\,3\,3\,3\,3\,3 \\ 2\,4\,4\,0\,3\,3\,3\,1\,2\,2\,4\,4 \end{pmatrix},$$

a biword of length $n = 12$, which is nondecreasing in the sense that the biletters $(0,2)$, $(0,4)$, ..., are in nondreasing order with respect to the lexicographic order. The Schensted-Knuth algorithm applied to $\mathbf{b}$ can be described as follows:

start with two empty tableaux $P = \begin{array}{|c}\ \ \\ \hline\end{array}$, $Q = \begin{array}{|c}\ \ \\ \hline\end{array}$. Next, insert the first entry 0 on the top row b into Q and the first entry 2 on the bottom row c into P, to obtain $P = \begin{array}{|c}2\\ \hline\end{array}$, $Q = \begin{array}{|c}0\\ \hline\end{array}$. As $2 \le 4 \le 4$ (on the bottom row c), insert $4\,4$ next to 2 onto the same row in P and record the positions by inserting the corresponding entries $0\,0$ (on the top row b) onto analogous places in Q:

$$P = \begin{array}{|ccc}2&4&4\\ \hline\end{array}, \quad Q = \begin{array}{|ccc}0&0&0\\ \hline\end{array}.$$

Next, the fourth letter 0 of c will *bump up the smallest entry in P that is strictly larger than itself*. This is 2. We also record the new position occupied by 2 on the second row in P by writing 1 (the fourth entry on the top row) into Q:

$$P = \begin{array}{|ccc}2&&\\0&4&4\\ \hline\end{array}, \quad Q = \begin{array}{|ccc}1&&\\0&0&0\\ \hline\end{array}.$$

Then, the fifth letter 3 of c will *bump up the smallest leftmost entry on the first row in P that is strictly larger than itself*. This is the leftmost 4. There is no entry on the second row of P smaller than 4, so that 4 is inserted next to the right of 2. We then record the new position occupied by 4 on the second row in P by writing 1 (the fifth letter of b) into Q:

$$P = \begin{array}{|l} 2\ 4 \\ 0\ 3\ 4 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 1\ 1 \\ 0\ 0\ 0 \\ \hline \end{array}\,.$$

Again, the sixth letter 3 of c will *bump up the smallest leftmost entry on the first row in P that is strictly larger than itself*. This is 4. There is no entry on the second row of P smaller than 4, so that 4 is inserted to the right of the second row. We then record the new position occupied by 4 on the second row in P by writing 1 (the sixth letter of b) into Q:

$$P = \begin{array}{|l} 2\ 4\ 4 \\ 0\ 3\ 3 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 1\ 1\ 1 \\ 0\ 0\ 0 \\ \hline \end{array}\,.$$

The seventh letter of c is 3. There is no entry on the first row in P that is greater than 3, so that 3 is inserted to the right of the first row and the seventh letter 1 of b is written to the right of the first row in Q:

$$P = \begin{array}{|l} 2\ 4\ 4 \\ 0\ 3\ 3\ 3 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 1\ 1\ 1 \\ 0\ 0\ 0\ 1 \\ \hline \end{array}\,.$$

The eighth letter of c is 1. It will *bump up the smallest leftmost entry on the first row in P that is strictly larger than itself*. This is the leftmost 3. In its turn that entry 3 will *bump up the smallest leftmost entry on the second row in P that is strictly larger than itself*. This is the leftmost 4. The third row is empty, so that 4 will form a new one-entry row for P. We then record the new position occupied by 4 on the third row in P by writing 3 (the eighth letter of b) into Q:

$$P = \begin{array}{|l} 4 \\ 2\ 3\ 4 \\ 0\ 1\ 3\ 3 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 3 \\ 1\ 1\ 1 \\ 0\ 0\ 0\ 1 \\ \hline \end{array}\,.$$

Using the same insertion technique we successively get

$$P = \begin{array}{|l} 4\ 4 \\ 2\ 3\ 3 \\ 0\ 1\ 2\ 3 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 3\ 3 \\ 1\ 1\ 1 \\ 0\ 0\ 0\ 1 \\ \hline \end{array}\,; \qquad P = \begin{array}{|l} 4\ 4 \\ 2\ 3\ 3\ 3 \\ 0\ 1\ 2\ 2 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 3\ 3 \\ 1\ 1\ 1\ 3 \\ 0\ 0\ 0\ 1 \\ \hline \end{array}\,;$$

and finally

$$P = \begin{array}{|l} 4\ 4 \\ 2\ 3\ 3\ 3 \\ 0\ 1\ 2\ 3\ 4\ 4 \\ \hline \end{array}\,, \quad Q = \begin{array}{|l} 3\ 3 \\ 1\ 1\ 1\ 3 \\ 0\ 0\ 0\ 1\ 3\ 3 \\ \hline \end{array}\,.$$

Theorem 19.1. *The insertion algorithm*

$$\mathbf{b} = \binom{b}{c} = \begin{pmatrix} b_1\ b_2\ \ldots\ b_n \\ c_1\ c_2\ \ldots\ c_n \end{pmatrix} \mapsto P\,Q$$

provides a bijection of the set of all nondecreasing biwords onto the set of ordered pairs $P\,Q$ of semi-standard tableaux, of the same shape, the content of P (resp. of Q) being $\{c_1, c_2, \ldots, c_n\}$ (resp. $\{b_1, b_2, \ldots, b_n\}$).

It is not our intention to give a formal proof of the theorem. The only difficult part is to see that the algorithm provides two *tableaux* which are both *semi-standard*. The fact that both P and Q have the same shape is then evident. Also, the algorithm for the reverse construction is easy to formulate.

In the sequel, *standard Young tableau* of order n means standard tableau whose shape is a (right) Ferrers diagram corresponding to a partition of n. Let $\mathfrak{T}_n$ (resp. $\mathfrak{T}_n^{(2)}$) be the set of all standard Young tableaux of order n (resp. of all ordered pairs $P\,Q$ of standard Young tableaux of order n *of the same shape*).

Corollary 19.2. *When restricted to permutations σ of order n written as increasing biwords, the insertion algorithm*

$$\sigma \mapsto P\,Q$$

establishes a bijection of the group $\mathfrak{S}_n$ of $n!$ elements onto $\mathfrak{T}_n^{(2)}$. The bijection is called the Robinson-Schensted correspondence.

19.2. *A combinatorial proof of the Cauchy identity.* Let

$$A = (a_{ij})_{(i,j\ge 0)} \mapsto \mathbf{b} = \binom{b}{c} = \begin{pmatrix} b_1\ b_2\ \ldots\ b_n \\ c_1\ c_2\ \ldots\ c_n \end{pmatrix}$$

be the bijection constructed in section 12.1 that maps each matrix $A = (a_{ij})$ $(i, j \ge 0)$ with integral entries such that $\sum_{i,j} a_{ij} = n$ onto a nondecreasing biword of length n. By construction the biletter $\binom{i}{j}$ occurs a_{ij} times in $\mathbf{b}$. Accordingly, if $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ are two alphabets of variables, we have

$$x_{b_1} x_{b_2} \ldots x_{b_n} y_{c_1} y_{c_2} \ldots y_{c_n} = \prod_{i,j} (x_i y_j)^{a_{ij}}. \tag{19.1}$$

Now consider the Schensted-Knuth bijection $\mathbf{b} \mapsto P\,Q$. The product (19.1) is equal to the product of the x-evaluation $x(Q)$ by the y-evaluation $y(P)$.

When the matrix A runs over all matrices with integral entries whose finitely many of them are nonzero, the pair $P\,Q$ runs over all pairs of semi-standard tableaux, of the same shape. For short, designate the shape of a tableau P by $|P|$. We can then write:

$$\begin{aligned}\prod_{i,j}(1-x_iy_j)^{-1} &= \prod_{i,j}\sum (x_iy_j)^{a_{ij}}\\ &= \sum_A \prod_{i,j}(x_iy_j)^{a_{ij}}\\ &= \sum_{P\,Q} x(Q)y(P)\\ &= \sum_\lambda \sum_{|P|=|Q|=\lambda} x(Q)y(P)\\ &= \sum_\lambda \Bigl(\sum_{|Q|=\lambda} x(Q)\Bigr)\Bigl(\sum_{|P|=\lambda} y(P)\Bigr)\\ &= \sum_\lambda s_\lambda(y)s_\lambda(x),\end{aligned}$$

by the very combinatorial definition for the Schur functions given in section 17.

In the same vein we can prove the dual Cauchy identity, i.e.,

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda s_{\lambda'}(x)s_\lambda(y), \tag{19.2}$$

where λ' still designates the conjugate partition. The product on the left-hand side is equal to

$$\sum_A \prod_{i,j}(x_iy_j)^{a_{ij}},$$

but this time the entries of the matrices are only 0 or 1. The corresponding biwords $\mathbf{b}$ have *distinct* biletters, so that we can derive an inserting process in which each inserted element bumps up the entry that is *larger than or equal to* instead of *the smallest leftmost entry strictly larger than* itself. Recording the tableau Q as before, we obtain a pair $P\,Q$, of tableaux of the same shape, such that the transpose P^T of P and Q itself are semi-standard. An analogous calculation leads to identity (19.2).

19.3. *Geometric properties of correspondence.* The ligne of route "Ligne" and the inverse ligne of route "Iligne" of a permutation have been defined in section 11 and the inverse ligne of route "Iligne" of a tableau in section 18.

Theorem 19.3. *Let $\sigma \mapsto P\,Q$ be the Robinson-Schensted correspondence. Then*

$$\text{Ligne}\,\sigma = \text{Iligne}\,Q; \quad \text{Iligne}\,\sigma = \text{Iligne}\,P. \tag{19.3}$$

This property can be verified by a careful study of the various steps in the construction of the Schensted-Knuth algorithm. We omit the detailed proof. It is essential to notice that the Q-tableau retains the geometric information on the permutation σ, on the one hand, and the P-tableau on the inverse permutation σ^{-1}, on the other hand.

For instance, with

$$\sigma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 3\ 1\ 4\ 2\ 5 \end{pmatrix} \mapsto P\,Q = \begin{matrix} 3\ 4 & \\ 1\ 2\ 5 & \end{matrix}, \begin{matrix} 2\ 4 & \\ 1\ 3\ 5 & \end{matrix}$$

we observe the equalities: $\text{Ligne}\,\sigma = \{1,3\} = \text{Iligne}\,Q$ and $\text{Iligne}\,\sigma = \{2\} = \text{Iligne}\,P$.

The Robinson-Schensted correspondence is a useful ingredient in the construction of bijections of the symmetric group onto itself. For instance, suppose that we have a a bijection $Q \mapsto Q^J$ of $\mathfrak{T}_n$ onto itself that preserves the shape and has the property: $\text{Iligne}\,Q^J = n - \text{Iligne}\,Q = \{n - i : i \in \text{Iligne}\,Q\}$. Then the chain

$$\sigma \mapsto P\,Q \mapsto P\,Q^J \mapsto \mathbf{j}\,\sigma,$$

where the rightmost arrow stands for the inverse of the Robinson-correspondence, is a bijection of $\mathfrak{S}_n$ onto itself such that

$$\text{Iligne}\,\mathbf{j}\,\sigma = \text{Iligne}\,\sigma, \quad \text{Ligne}\,\mathbf{j}\,\sigma = n - \text{Ligne}\,\sigma.$$

The other important property of the correspondence (for which there exist many proofs) can be stated as follows:

$$[\sigma \mapsto P\,Q] \Leftrightarrow [\sigma^{-1} \mapsto Q\,P]. \tag{19.4}$$

19.4. *A permutation statistic distribution.* As was done in section 18 for standard tableaux, we can introduce the *y-vector* of a permutation $\sigma = \sigma(1)\ldots\sigma(n)$ and define it as the word $y(\sigma) = y_\sigma(1)\ldots y_\sigma(n)$, where

$$y_\sigma(i) := \#\{j \geq i : j \in \text{Ligne}\,\sigma\}.$$

Hence, $\operatorname{des}\sigma = y_\sigma(1)$, $\operatorname{maj}\sigma = \operatorname{tot} y(\sigma) = y_\sigma(1) + \cdots + y_\sigma(n)$. We can also consider the y-vector for the inverse σ^{-1} of σ. Taking two sequences $(p_1, p_2, \ldots)$ and $(q_1, q_2, \ldots)$ of variables define the monomials

$$\mathbf{q}^{y(\sigma)} := q_1^{y_\sigma(1)} \cdots q_n^{y_\sigma(n)}, \quad \mathbf{p}^{y(\sigma^{-1})} := p_1^{y_{\sigma^{-1}}(1)} \cdots p_n^{y_{\sigma^{-1}}(n)}$$

and consider the polynomial

$$A_n(t, s, \mathbf{q}, \mathbf{p}) := \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}\sigma} s^{\operatorname{des}\sigma^{-1}} \mathbf{q}^{y(\sigma)} \mathbf{p}^{y(\sigma^{-1})}.$$

The homomorphism $\phi_{\mathbf{q}}$ that has been introduced just after Proposition 18.1, keeps the same meaning. Also let $\phi_{\mathbf{p}}$ be the analogous homomorphism defined by means of the variables p_i's.

Theorem 19.4. *The bi-factorial generating function for the polynomials* $A_n(t, s, \mathbf{q}, \mathbf{p})$ *is given by:*

$$\sum_{n \geq 0} \frac{u^n}{[t; \mathbf{q}]_{n+1} \, [s; \mathbf{p}]_{n+1}} A_n(t, s, \mathbf{q}, \mathbf{p}) = \phi_{\mathbf{q}} \phi_{\mathbf{p}} \sum_{l,k} t^l s^k \prod_{\substack{1 \leq j \leq l+1 \\ 1 \leq i \leq k+1}} \frac{1}{1 - u x_j y_i}.$$

Again, we emphasize the fact that $\phi_{\mathbf{q}}$ and $\phi_{\mathbf{p}}$ are (non-ring) homomorphisms acting on the x_j's and the y_i's, respectively. The infinite product on the right-hand is to be extended first in monomials in the x_j's and the y_i's and the mappings $\phi_{\mathbf{p}}$, $\phi_{\mathbf{q}}$ are to be applied next to each monomial.

The proof of Theorem 19.4 is a consequence of the geometric properties of the Robinson-Schensted correspondence, as stated in Theorem 19.3, and of the Cauchy identity. From (19.3) we can write:

$$A_n(t, s, \mathbf{q}, \mathbf{p}) = \sum_{|\lambda| = n} \sum_{|P| = |Q| = \lambda} s^{\operatorname{ides} Q} t^{\operatorname{ides} P} \mathbf{p}^{y(Q)} \mathbf{q}^{y(P)}$$

Hence,

$$\begin{aligned}
&\frac{1}{[t; \mathbf{q}]_{n+1} \, [s; \mathbf{p}]_{n+1}} A_n(t, s, \mathbf{q}, \mathbf{p}) \\
&\qquad = \sum_{|\lambda| = n} \Bigl(\frac{1}{[t; \mathbf{q}]_{n+1}} \sum_{|Q| = \lambda} t^{\operatorname{ides} Q} \mathbf{q}^{y(Q)} \Bigr) \Bigl(\frac{1}{[s; \mathbf{p}]_{n+1}} \sum_{|P| = \lambda} s^{\operatorname{ides} P} \mathbf{p}^{y(P)} \Bigr) \\
&\qquad = \sum_{|\lambda| = n} \Bigl(\sum_{l \geq 0} t^l \phi_{\mathbf{q}}(s_\lambda(x_1, \ldots, x_{l+1}) \Bigr) \Bigl(\sum_{k \geq 0} s^k \phi_{\mathbf{p}}(s_\lambda(y_1, \ldots, y_{k+1}) \Bigr)
\end{aligned}$$

[by (18.4).]

Thus

$$\sum_{n\ge 0}\frac{u^n}{[t;\mathbf{q}]_{n+1}\,[s;\mathbf{p}]_{n+1}}A_n(t,s,\mathbf{q},\mathbf{p})$$
$$=\phi_{\mathbf{q}}\phi_{\mathbf{p}}\sum_{l,k}t^l s^k\sum_{n\ge 0}u^n\sum_{|\lambda|=n}s_\lambda(x_1,\ldots,x_{l+1})s_\lambda(y_1,\ldots,y_{k+1})$$
$$=\phi_{\mathbf{q}}\phi_{\mathbf{p}}\sum_{l,k}t^l s^k\prod_{\substack{1\le j\le l+1\\ 1\le i\le k+1}}\frac{1}{1-ux_jy_i}\quad\text{[by the Cauchy formula (16.4).]}$$

When all the variables q_j (resp. the variables p_i) are equal to a single variable q (resp. p), the homomorphisms become ring homomorphisms. We recover formula (13.7) that gives the bi-basic generating functions by the four-statistic $(\mathrm{des},\mathrm{ides},\mathrm{maj},\mathrm{imaj})$.

20. Eulerian Calculus; the first extensions

The next three sections will be devoted to deriving the various extensions of the fundamental identity for the Eulerian polynomials $(A_n(t))$ $(n\ge 0)$ (see (10.9)) that reads

$$\frac{1-t}{-t+\exp(u(t-1))}=\sum_{n\ge 0}\frac{u^n}{n!}A_n(t),\tag{20.1}$$

when the exponential occurring in the denominator of the rational fraction is replaced by the q-exponential (as was done in section 10), then by a Bessel function J or by a q-Bessel function $\mathbf{J}$, in its infinite or finite form, and also when the fraction itself is replaced by

$$\frac{(1-t)\exp(u(t-1)X)}{-t+\exp(u(t-1)(X+Y))}=\sum_{n\ge 0}\frac{u^n}{n!}B_n(X,Y,t),\tag{20.2}$$

that stands for the natural extension of the fraction, when going from the group A_n of the *permutations* to the group B_n of the *signed permutations.*

Those various extensions are symbolized by the diagram of Fig. 20.1, where the identities (20.1) and (20.2) sit on the vertices $(\cdot,\cdot,\cdot)$ and $(\mathrm{sgn},\cdot,\cdot)$, respectively, and where the horizontal arrow $\rightarrow$ stands for the passage from the permutations to the signed permutations, the oblique arrow $\nearrow$ stands for the q-extension, the uparrow $\uparrow$ for the extension by means of a Bessel function, in its infinite form (J) for the first level and in its finite form (J_k) for the second one. All those terms will be fully explained in the sequel.

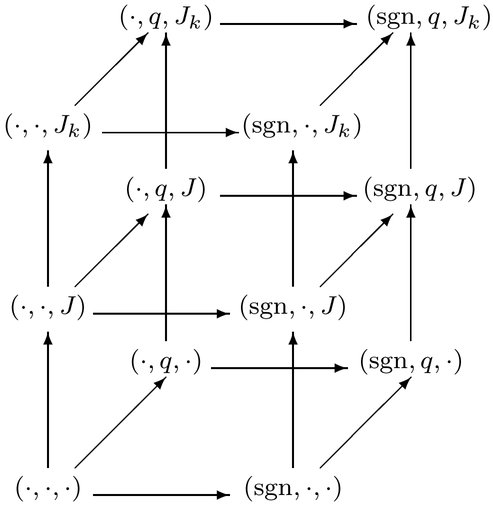


Fig. 20.1

Of course, we could start from the most general extension that sits on (sgn, q, J_k) and get the other results by successive specializations. However proceeding in such a way would overlook the local techniques that have their own interest. We then proceed from the particular to the general, at least in the beginning. When we have enough material, we will attack (sgn, q, J_k) itself and obtain the remaining identities with their combinatorial interpretations.

20.1. *The signed permutations.* A *signed permutation* of order n can be defined as a pair (σ, ε), where σ is a permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ and $\varepsilon = \varepsilon(1)\varepsilon(2)\ldots\varepsilon(n)$ is a word of length n in the alphabet $\{x, y\}$. The word ε is called a *xy-word*; the number of letters in ε equal to x (resp. equal to y) is denoted by $\ell(\varepsilon|x)$ (resp. $\ell(\varepsilon|y)$). Also $\sigma_{\varepsilon|x}$ (resp. $\sigma_{\varepsilon|y}$) designates the *subword* of σ made of all letters $\sigma(i)$ such that $\varepsilon(i) = x$ (resp. $\varepsilon(i) = y$).

Definition. The integer i is said to be a *descent* of the signed permutation (σ, ε), if one of the following conditions is verified

(i) $i = n$ and $\varepsilon(n) = x$;
(ii) $1 \le i \le n-1$, $\varepsilon(i) = x$, $\varepsilon(i+1) = y$;
(iii) $1 \le i \le n-1$, $\varepsilon(i) = \varepsilon(i+1)$ and $\sigma(i) > \sigma(i+1)$.

The *number of descents* of (σ, ε) is denoted by $\mathrm{des}(\sigma, \varepsilon)$.

Example. Let

$$\begin{array}{rccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \sigma = & 6 & 7 & 2 & 4 & 3 & 1 & 8 & 5 & 9 \\ \varepsilon = & x & y & y & y & x & x & y & x & x \end{array}$$

be a signed permutation. Then $\sigma_{\varepsilon|x}=63159$, $\sigma_{\varepsilon|y}=7248$ and $\operatorname{des}(\sigma,\varepsilon)=\operatorname{Card}\{1,2,5,6,9\}=5$.

Another way of introducing the descents is to start with the linear order $(n,x)>\cdots>(1,x)>(n,y)>\cdots>(1,y)$, keep item (i) and replace conditions (ii) and (iii) above by

(ii′) $1\le i\le n-1$ and $(\sigma(i),\varepsilon(i))>(\sigma(i+1),\varepsilon(i+1))$.

Now let

$$B_n(X,Y,t)=\sum_{(\sigma,\varepsilon)} X^{\ell(\varepsilon|x)}Y^{\ell(\varepsilon|y)}t^{\operatorname{des}(\sigma,\varepsilon)} \tag{20.3}$$

be the generating polynomial for the set of signed permutations of order n by the number of descents.

Theorem 20.1. *The following identity holds:*

$$\frac{(1-t)\,\exp(u(t-1)X)}{-t+\exp(u(t-1)(X+Y))}=\sum_{n\ge0}\frac{u^n}{n!}\,B_n(X,Y,t). \tag{20.4}$$

Notice that when $X=0$ and $Yu=u$ in (20.4) we revover identity (20.1) with their combinatorial interpretation.

For the proof of Theorem 20.1 we proceed as follows. Define the statistic $\operatorname{des}_k\sigma$ of a permutation (not a signed permutation) $\sigma=\sigma(1)\ldots\sigma(n)$ to be

$$\operatorname{des}_k\sigma:=\begin{cases}\operatorname{des}\sigma, & \text{if } 1\le\sigma(n)\le n-k;\\ 1+\operatorname{des}\sigma, & \text{if } n-k+1\le\sigma(n)\le n;\end{cases}$$

and let

$$A_n^k(t):=\sum_\sigma t^{\operatorname{des}_k\sigma}\quad(\sigma\in\mathfrak{S}_n).$$

Because of item (i) in the definition of a descent in a signed permutation we can write:

$$B_n(X,Y,t)=\sum_{k=0}^{n}\binom{n}{k}X^kY^{n-k}A_n^k(t). \tag{20.5}$$

Now the "des_k" interpretation given for $A_n^k(t)$ provides the following recurrence relation

$$A_n^k(t)=A_n^{k-1}(t)+(t-1)A_{r-1}^{k-1}(t)\qquad(1\le k\le r), \tag{20.6}$$

since $t\,A_{n-1}^{k-1}(t)$ is the generating polynomial for the permutations ending with $(n-k+1)$ by "des_k," while $A_n^{k-1}(t) - A_{n-1}^{k-1}(t)$ is the generating polynomials for the other permutations. By iteration,

$$\text{(20.7)}\quad A_n^k(t) = A_n^0(t) + \binom{k}{1}(t-1)A_{n-1}^0(t) + \binom{k}{2}(t-1)^2A_{n-2}^0(t) + \cdots + \binom{k}{k}(t-1)^kA_{n-k}^0(t).$$

Using Umbral Calculus (dear to the late professor Rota, who had made an extensive usage of that approach) $A^n \equiv A_n \equiv A_n^0(t)$ and $A^0 \equiv 1$, formula (20.7) may be rewritten as

$$\text{(20.8)}\qquad A_n^k(t) = A^{n-k}\big((t-1)+A\big)^k.$$

In view of (20.5) and (20.8) we then have

$$\text{(20.9)}\qquad B_n(X,Y,t) = \big(X(t-1)+(X+Y)A\big)^r, \qquad A^n \equiv A_n.$$

Now exponential generating function (20.1) for the Eulerian polynomials may be rewritten as

$$\exp(uA) = \frac{1-t}{-t+\exp((t-1)u)},$$

so that

$$\begin{aligned}\sum_{n\ge 0}\frac{u^n}{n!}B_n(X,Y,t) &= \sum_{n\ge 0}\frac{u^n}{n!}\big(X(t-1)+(X+Y)A\big)^n\\ &= \exp\big(uX(t-1)\big)\exp\big(u(X+Y)A\big)\\ &= \frac{(1-t)\exp(uX(t-1))}{-t+\exp(u(X+Y)(t-1))}. \quad\Box\end{aligned}$$

20.2. *Pairs of permutations.* Let $\pi = \pi(1)\pi(2)\ldots\pi(n)$ and $\rho = \rho(1)\rho(2)\ldots\rho(n)$ be two permutations of order n. The *number of common descents* of π and ρ, denoted by $\mathrm{ddes}(\pi,\rho)$, is defined to be the number of integers k such that $1 \le k \le n-1$ and $\pi(k) > \pi(k+1)$, $\rho(k) > \rho(k+1)$. The generating polynomial for the Cartesian product $\mathfrak{S}_n \times \mathfrak{S}_n$ by "ddes" is denoted by

$$\text{(20.10)}\qquad A_n^{\mathrm{ddes}}(t) := \sum_{(\pi,\rho)} t^{\mathrm{ddes}(\pi,\rho)} \quad ((\pi,\rho)\in\mathfrak{S}_n\times\mathfrak{S}_n).$$

Now, let

$$J_0(u)=\sum_{n\ge 0}(-1)^n\frac{(u/2)^{2n}}{n!\,n!}$$

be the *Bessel function of the first kind, of order zero* and let

$$J(u):=J_0(2\sqrt{u})=\sum_{n\ge 0}(-1)^n\frac{u^n}{n!\,n!}. \tag{20.10}$$

Theorem 20.3. *The bi-exponential generating function for the polynomials $A_n^{\rm ddes}(t)$ $(n\ge 0)$ is given by:*

$$\frac{1-t}{-t+J(u(1-t))}=1+\sum_{n\ge 1}\frac{u^n}{n!\,n!}A_n^{\rm ddes}(t). \tag{20.11}$$

Notice that we go from the fraction in (20.1) to the fraction in (20.11) by merely replacing "exp" by "J." The proof of Theorem 20.3 is quite similar to the proof of Theorem 10.1 that gives the generating function for the polynomials ${}^{\rm inv}A_n(t,q)$. The fraction in identity (20.11) can be rewritten as

$$\Bigl(1+\sum_{n\ge 1}(-1)^n(1-t)^{n-1}\frac{u^n}{n!\,n!}\Bigr)^{-1},$$

so that (20.11) is equivalent to

$$\Bigl(1-\sum_{n\ge 1}(-1)^{n-1}(1-t)^{n-1}\frac{u^n}{n!\,n!}\Bigr)\cdot\sum_{n\ge 0}A_n^{\rm ddes}(t)\frac{u^n}{n!\,n!}=1$$

and provides the recurrence $A_0^{\rm ddes}(t)=1$ and

$$A_n^{\rm ddes}(t)=\sum_{1\le k\le n}\binom{n}{k}\binom{n}{k}(t-1)^{k-1}A_{n-k}^{\rm ddes}(t)\quad (n\ge 1). \tag{20.12}$$

Let F_k be the generating polynomial for the pairs of permutations (π,ρ) of order n, such that both π and ρ have their *longest decreasing rightmost factors (l.d.r.f.)* equal to k by "ddes." Also, let $G_k:=F_k+F_{k+1}+\cdots+F_n$. Under those conditions we have

$$\binom{n}{k}\binom{n}{k}t^kA_{n-k}^{\rm ddes}(t)=tF_k+G_{k+1},$$

so that

$$
\begin{aligned}
\sum_{1\le k\le n}\binom{n}{k}\binom{n}{k}(t-1)^{k-1}A_{n-k}^{\mathrm{ddes}}(t) &= \sum_{k=1}^{n}\Bigl(\frac{F_k}{t^{k-1}}+\frac{G_{k+1}}{t^k}\Bigr)(t-1)^{k-1}\\
&=\sum_{k=1}^{n}\frac{1}{t^k}\Bigl(t(G_k-G_{k+1})+G_{k+1}\Bigr)(t-1)^{k-1}\\
&=\sum_{k=1}^{n}\frac{1}{t^k}\Bigl(tG_k-(t-1)G_{k+1}\Bigr)(t-1)^{k-1}\\
&=\sum_{k=1}^{n}G_k\frac{(t-1)^{k-1}}{t^{k-1}}-G_{k+1}\frac{(t-1)^k}{t^k}\\
&=G_1=A_n^{\mathrm{ddes}}(t),
\end{aligned}
$$

by definition of the G_k's. $\square$

Remark. The proof of the previous theorem shows that if instead of a pair of permutations we consider a finite sequence of permutations, we can also calculate the generating function for those finite sequences by their common descent. This is one of the extensions that will be proposed in the next section.

Let $A_{n,k}^{\mathrm{ddes}} := |\{(\pi,\rho)\in\mathfrak{S}_n\times\mathfrak{S}_n : \mathrm{ddes}(\pi,\rho)=k\}|$. The first values of the coefficients $A_{n,k}^{\mathrm{ddes}}$, calculated by means of the recurrence (20.12), are shown in Table 20.2.

$k=$	0	1	2	3	4
$n=1$	1				
2	3	1			
3	19	16	1		
4	211	299	65	1	
5	3651	7346	3156	246	1

Table 20.2 (distribution of ddes on $\mathcal{S}_n\times\mathcal{S}_n$)

20.3. *The q-extension.* The q-extension of (20.1) was already done in Theorem 10.1. Recall that when the exponential is replaced by the q-exponential E_q, the expansion of the fraction involves the generating polynomials for $\mathfrak{S}_n$ by the bi-statistic $(1+\mathrm{des},\mathrm{inv})$. We will also consider the replacement of the exponential by the other q-exponential e_q. The formulas obtained together with their combinatorial interpretations will be discussed in the next section as specializations of more general results.

Referring to the diagram represented in Fig. 20.1 the three extensions $(\mathrm{sgn},\cdot,\cdot)$ [signed permutations], $(.,.,J)$ [pairs of permutations] and $(\cdot,q,\cdot)$ [the q-extension] have been described.

20.4. *The t,q-maj extension for signed permutations.* In § 20.1 the notion of *descent* for a signed permutation has been introduced, so that the *major index* can also be defined by adding the *positions* at which a descent occurs. In the example shown in § 20.1 the descents occur at positions $1,2,5,6,9$ so that $\operatorname{maj}(\sigma,\varepsilon)=1+2+5+6+9=23$.

Now form the generating polynomial for the set of signed permutations of order n by the pair $(\operatorname{des},\operatorname{maj})$, that is,

$$B_n(X,Y,t,q)=\sum_{(\sigma,\varepsilon)} X^{\ell(\varepsilon|x)}Y^{\ell(\varepsilon|y)}t^{\operatorname{des}(\sigma,\varepsilon)}q^{\operatorname{maj}(\sigma,\varepsilon)}. \tag{20.13}$$

As we now see, the generating function for those polynomials $B_n(X,Y,t,q)$ $(n\ge 0)$ can be derived, by *q-analogizing* the calculation made in § 20.1 for the polynomial $B_n(X,Y,t)$. For $0\le k\le n$ and each (ordinary) permutation σ let

$$\operatorname{des}_k\sigma := \begin{cases} \operatorname{des}\sigma, & \text{if } 1\le\sigma(n)\le n-k;\\ 1+\operatorname{des}\sigma, & \text{if } n-k+1\le\sigma(n)\le n;\end{cases}$$

$$\operatorname{maj}_k\sigma := \operatorname{maj}\sigma + n\,\chi(n-k+1\le\sigma(n)\le n); \tag{20.14}$$

and let

$$A_n^k(t,q):=\sum_\sigma t^{\operatorname{des}_k\sigma}q^{\operatorname{maj}_k\sigma}\quad(\sigma\in\mathfrak{S}_n).$$

If I is a subset of $[\,n\,]=\{1,2,\dots,n\}$ let $B_I(t,q)$ denote the generating polynomial for the signed permutations $(\sigma,\varepsilon)=(\sigma(1),\varepsilon(1))\dots(\sigma(n),\varepsilon(n))$ containing all the letters (i,x) when $i\in I$ and all the letters (j,y) when $j\in[\,n\,]\setminus I$. Because of the definition of a descent for a signed permutation, we have

$$B_I(t,q)=A_n^k(t,q)\quad\text{if}\quad \#I=k. \tag{20.15}$$

Hence

$$B_n(X,Y,t,q)=\sum_{k=0}^{n}\binom{n}{k}X^kY^{n-k}A_n^k(t,q).$$

Mimicking (20.6) we also have:

$$A_n^k(t,q)=A_n^{k-1}(t,q)+(tq^n-1)A_{n-1}^{k-1}(t,q)\qquad(1\le k\le n).$$

By iteration,

$$A_n^k(t,q)=\sum_{s=0}^{k}\binom{k}{s}(tq^n-1)(tq^{n-1}-1)\dots(tq^{n-s+1}-1)B_{n-s}^0(t,q).$$

Hence

$$\begin{aligned}A_n(X,Y;t,q)=\sum_{0\le s\le k\le n}&\frac{n!}{(n-k)!\,s!\,(k-s)!}X^kY^{n-k}\\ &\times(tq^n-1)(tq^{n-1}-1)\dots(tq^{n-s+1}-1)B_{n-s}^0(t,q)\end{aligned}$$

and

$$\sum_{n\ge 0}\frac{u^n}{n!}\frac{B_n(X,Y;t,q)}{(t;q)_{n+1}} = \sum_{s,n,m\ge 0}\frac{u^{s+n+m}X^{s+n}Y^m}{s!\,n!\,m!}\frac{(-1)^s}{(t;q)_{n+m+1}}B^0_{n+m}(t,q)$$
$$= \exp(-uX)\sum_{r\ge 0}\frac{(u(X+Y))^r}{r!}\frac{A^0_r(t,q)}{(t;q)_{r+1}}.$$

But $A^0_s(t,q)$ is the generating polynomial by $(\mathrm{des}_0,\mathrm{maj}_0)$, that is, the *q-maj Eulerian polynomial* ${}^{\mathrm{maj}}A_s(t,q)$, introduced in Definition 10.1. As, by (10.2*b*)

$$\sum_{r\ge 0}\frac{u^r}{r!}\frac{A^0_r(t,q)}{(t;q)_{r+1}} = \sum_{s\ge 0} t^s\ \exp(u[s+1]_q),$$

we conclude that

$$\sum_{n\ge 0}\frac{u^n}{n!}\frac{B_n(X,Y;t,q)}{(t;q)_{n+1}} = \exp(-uX)\sum_{s\ge 0}t^s\ \exp(u(X+Y)[s+1]_q)$$
$$= \sum_{n\ge 0}t^s\exp\bigl(u(Xq[s]_q+Y[s+1]_q)\bigr).$$

Proposition 20.4. *Let $B_n(X,Y;t,q)$ be the generating polynomial for the signed permutations by* (des, maj)*, as defined in* (20.13)*. Then*

(20.16) $$\frac{B_n(X,Y;t,q)}{(t;q)_{n+1}} = \sum_{s\ge 0}t^s\ (Xq[s]_q+Y[s+1]_q)^n;$$

(20.17) $$\sum_{n\ge 0}u^n\frac{B_n(X,Y;t,q)}{(t;q)_{n+1}} = \sum_{n\ge 0}t^s\frac{1}{1-u(Xq[s]_q+Y[s+1]_q)};$$

(20.18) $$\sum_{n\ge 0}\frac{u^n}{n!}\frac{B_n(X,Y;t,q)}{(t;q)_{n+1}} = \sum_{n\ge 0}t^s\exp\bigl(u(Xq[s]_q+Y[s+1]_q)\bigr).$$

For each subset $I\subset[\,n\,]$ let $B_I(t,q)$ be the generating polynomial for the signed permutations containing all the letters (i,x) with $i\in I$ and all the letters (j,y) with $j\in[\,n\,]\subset I$ by (des, maj)*. If $\#I=k$, we have:*

(20.19) $$\frac{B_I(t,q)}{(t;q)_{n+1}} = \sum_{s\ge 0}t^s\ q^k\,[s]_q^k\,[s+1]_q^{n-k}.$$

Proof. Identity (20.18) has just been proved. The other formulas can then be easily deduced. ∎

20.5. *A first inversion number for signed permutations.* As explained in the next section, the inversion number for signed permutations that will be further studied will be motivated by analytical reasons. However, we can also look for an inversion number, that reduces to the usual "inv" for the ordinary permutations, that has the further property of being equidistributed with "maj" on the set of signed permutations.

Keep the notations of Proposition 20.4 and let SP_I be the set of signed permutations of order n containing all the letters (i,x) with $i \in I$ and all the letters (j,y) with $j \in [\,n\,] \setminus I$. By letting $t=1$ in (20.19) we get

$$B_I(1,q) = q^k \,\frac{(q;q)_n}{(1-q)^n}.$$

The most (?) natural inversion number, say, "Inv", whose distribution on the set SP_I is given by $B_I(1,q)$, can be defined by mapping each $(\sigma,\varepsilon) \in SP_I$ onto

$$\mathrm{Inv}(\sigma,\varepsilon) := \mathrm{inv}\,\sigma + \#\,I.$$

In the notations of § 20.1

$$\mathrm{Inv}(\sigma,\varepsilon) := \mathrm{inv}\,\sigma + \ell(\varepsilon|x). \tag{20.20}$$

To avoid any confusion, denote the major index of a signed permutation w by $\mathrm{Maj}\,w$. To construct a bijection Ψ having the property that

$$\mathrm{Maj}(\sigma,\varepsilon) = \mathrm{Inv}\,\Psi(\sigma,\varepsilon), \tag{20.21}$$

proceed as follows. It suffices to do so for each subset SP_I such that $I = \{n-k+1, n-k+2, \dots, n\}$ $(0 \le k \le n)$. Each signed permutation in such an SP_I is then a permutation of the *increasing* word

$$(1,y)(2,y)\dots(n-k,y)(n-k+1,x)\dots(n,x),$$

Introduce a new letter $*$ and impose the ordering

$$(n-k,y) < * < (n-k+1,x).$$

Consider the alphabet $\{(1,y),\dots,(n-k,y),*,(n-k+1,x),\dots,(n,x)\}$ and let w belong to SP_I. The transformation Φ, introduced in Theorem 11.3, maps $w\,*$ (the signed permutation w, linearly written as a permutation, with the juxtaposition of the letter $*$ at the end) onto

$$\Phi(w\,*) = w'\,*, \tag{20.22}$$

where w' is a rearrangement of w. Furthermore,

$$\mathrm{maj}(w\,*) = \mathrm{inv}\,\Phi(w\,*) = \mathrm{inv}(w'\,*). \tag{20.23}$$

The bijection Ψ of SP_I onto itself is then defined by

$$\Psi(w)*:=\Phi(w*)=w'\ *\,. \tag{20.24}$$

Theorem 20.5. *The mapping Ψ defined in (20.24) is a bijection of each class SP_I onto itself such that $\operatorname{Maj} w=\operatorname{Inv}\Psi(w)$ holds for every $w\in SP_I$.*

Proof. As Φ is a bijection of each rearrangement class onto itself and because of (20.22), the mapping Ψ is itself a bijection. Furthermore, let $w=x_1x_2\dots x_n\in SP_I$. Then $\operatorname{Maj} w=\operatorname{maj} w+n\chi(x_n\in I)=\operatorname{maj}(w*)=\operatorname{inv}\Phi(w*)=\operatorname{inv}(\Psi(w)*)=\operatorname{Inv}\Psi(w)$. ∎

Example. Let $n=6$, $I=\{4,5,6\}$ and consider the signed permutation $w=(3,y)(1,y)(6,x)(5,x)(2,y)(4,x)$, that will be rewritten as $w=3,1,\mathbf{6},\mathbf{5},2,\mathbf{4}$. With $3<*<\mathbf{4}$ the image of $w*$ under Φ (defined in § 11.3) is $\Phi(w*)=\mathbf{6},3,\mathbf{5},1,\mathbf{4},2,*$, so that $\Psi(w)=\mathbf{6},3,\mathbf{5},1,\mathbf{4},2$. We verify that $\operatorname{Maj} w=1+3+4+6=14$. Moreover $\Psi(w)$ has 11 inversions and $\#\,I=3$, so that $\operatorname{Inv}\Psi(w)=11+3=14$.

21. Eulerian Calculus; the analytic choice

When the symmetric group $\mathfrak{S}_n$ is regarded as a Coxeter group generated by the transpositions $(i,i+1)$ $(1\le i\le n-1)$, the *length* $l_{\mathrm{Cox}}(\sigma)$ of a permutation σ is simply the *number of inversions* $\operatorname{inv}\sigma$. As recalled in the previous subsection 20.3, the generating function for the bi-statistic $(1+\operatorname{des},l_{\mathrm{Cox}})=(1+\operatorname{des},\operatorname{inv})$ on the symmetric groups $A_n=\mathfrak{S}_n$ has been derived by taking a suitable q-analog of the generating function for the single statistic "des."

As we already have an expression for the generating function for "des" on the groups B_n, namely (20.4), the natural question arises: can we find an analogous approach to obtain the generating function for $(\operatorname{des},l_{\mathrm{Cox}})$ on the groups B_n? The answer is yes (see Exercise 30), but the formula we can derive does not fit any longer in the set-up of the q-series; its analytic manipulation is cumbersome.

Accordingly, if we wish to remain in the algebra of the q-series and obtain the desired extension for the signed permutations, identity (20.2) must play the role played by identity (20.1) for the plain permutations. In the first place it is necessary to find a q-analog for (20.2), with an interesting combinatorial interpretation.

We definitely adopt that approach of *analytical* nature. The statistic "inv" for the signed permutations will directly come out of that interpretation.

21.1. *Inversions for signed permutations.* A good choice for a q-analog of formula (20.2) is to replace the product of the two exponentials in the denominator by a product of two q-exponentials (and not by $e_q((t-1)(X+Y))$ that would give a q-expansion with some negative terms.) In the q-series expansion

$$\text{(21.1)} \qquad \frac{(1-t)\,e_q((t-1)X)}{-t+e_q((t-1)X)\,e_q((t-1)Y)} = \sum_{n\ge 0} \frac{1}{(q;q)_n}\,B_n(X,Y,t,q)$$

the coefficients $B_n(X,Y,t,q)$ are polynomials with *nonnegative integral* coefficients. One of the consequences of the main result proved in this section is to show that each $B_n(X,Y,t,q)$ is the generating polynomial for the signed permutations of order n by a bi-statistic "(des, coinv)":

$$\text{(21.2)} \qquad B_n(X,Y,t,q) = \sum_{(\sigma,\varepsilon)} X^{\ell(\varepsilon|x)} Y^{\ell(\varepsilon|y)} t^{\operatorname{des}(\sigma,\varepsilon)} q^{\operatorname{coinv}(\sigma,\varepsilon)}.$$

Of course, "$\operatorname{des}(\sigma,\varepsilon)$" is the number of descents, as was defined in subsection 20.2; and "$\operatorname{coinv}(\sigma,\varepsilon)$" is the *number of co-inversions* in the signed permutation, defined as follows.

A pair of integers (i,j) is said to be an *inversion* (resp. a *co-inversion*) of the signed permutation (σ,ε) of order n, if one of the following conditions is verified

(i) $\varepsilon(i)=\varepsilon(j)$, $i<j$; and $\sigma(i)>\sigma(j)$ (resp. $\sigma(i)<\sigma(j)$);

(ii) $\varepsilon(i)=y$, $\varepsilon(j)=x$ and $\sigma(i)>\sigma(j)$.

Let $\operatorname{inv}(\sigma,\varepsilon)$ (resp. $\operatorname{coinv}(\sigma,\varepsilon)$) denote the *number of inversions* (resp. the *number of co-inversions*) of (σ,ε).

Take up again the example of section 20.2. We have:

$$\operatorname{inv}\begin{pmatrix} 6&7&2&4&3&1&8&5&9\\ x&y&y&y&x&x&y&x&x\end{pmatrix} = 4+2+11=17\,;$$
$$\operatorname{coinv}\begin{pmatrix} 6&7&2&4&3&1&8&5&9\\ x&y&y&y&x&x&y&x&x\end{pmatrix} = 6+4+11=21.$$

The first values of the polynomials $B_n(X,Y,t,q)$, i.e., the generating polynomials for the signed permutations by the pair (des, coinv), are shown in Table 21.1.

$$B_1 = tX + Y; \quad B_2 = t(t+q)X^2 + 2t(1+q)XY + (t+q)Y^2;$$
$$B_3 = (2tq + 2tq^2 + t^2 + q^3)Y^3 + t(1+q+q^2)(3q+2t+1)XY^2$$
$$+ t(1+q+q^2)(2q+tq+3t)X^2Y + t(2tq+2tq^2+t^2+q^3)X^3.$$

Table 21.1 (distribution of $(\mathrm{des}, \mathrm{coinv})$)

Another consequence of our main result is to show that if in (21.1) each q-exponential $e_q(u)$ is replaced by the *second* Q-exponential $E_Q(u)$, the polynomials $B_n(X, Y, t, Q)$ on the right-hand side are the generating polynomials for the signed permutations by the pair $(\mathrm{des}, \mathrm{inv})$. The first values of those polynomials are shown in Table 21.2.

$$B_1 = Y + tX; \quad B_2 = (1+tQ)Y^2 + 2t(1+Q)XY + t(1+tQ)X^2;$$
$$B_3 = (1 + 2tQ + 2tQ^2 + t^2Q^3)Y^3 + t(1+Q+Q^2)(3+Q+2tQ)XY^2$$
$$+ t(1+Q+Q^2)(3+Q+2tQ)X^2Y + t(1+2tQ+2tQ^2+t^2Q^3)X^3.$$

Table 21.2 (distribution of $(\mathrm{des}, \mathrm{inv})$)

When restricted to the elements of $\mathfrak{S}_n$, the statistic "inv" for B_n is nothing but the usual number of inversions for the permutations. Likewise, formula (21.1) with $X = 0$ specializes into the analog of (10.12) for e_q.

Thus, (21.1) with its combinatorial interpretation (21.2) is our q-extension of (20.4). In our diagram of Fig. 20.1 that extension sits on vertex $(\mathrm{sgn}, q, \cdot)$.

21.2. *Basic Bessel Functions.* The next step is to study the expansion of (21.1), when the q-exponentials are replaced by q-analogs of Bessel functions. Formally, the left-hand side of (21.1) is to be replaced by the fraction

$$\frac{(1-t)\,\mathbf{J}((1-t)X; \mathbf{Q}, \mathbf{q})}{-t + \mathbf{J}((1-t)X; \mathbf{Q}, \mathbf{q})\,\mathbf{J}((1-t)Y; \mathbf{Q}, \mathbf{q})}, \tag{21.3}$$

where $\mathbf{J}$ is a basic Bessel function defined below. Our problem will be to study the expansion of that fraction in the algebra of series in one or several bases.

Let L, l be two fixed nonnegative integers and $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_L)$, $\mathbf{q} = (q_1, q_2, \ldots, q_l)$ two sequences of variables. Define

$$\mathbf{Q}^{\binom{n}{2}} := Q_1^{\binom{n}{2}} \ldots Q_L^{\binom{n}{2}},$$
$$(\mathbf{Q}; \mathbf{Q})_n := (Q_1; Q_1)_n \ldots (Q_L; Q_L)_n,$$
$$(\mathbf{q}; \mathbf{q})_n := (q_1; q_1)_n \ldots (q_l; q_l)_n.$$

Using the H-notation for the *Hadamard product* of two series, that is,

$$\Bigl(\sum_{n\ge 0}\alpha_n u^n\Bigr)\,\mathrm{H}\Bigl(\sum_{n\ge 0}\beta_n u^n\Bigr)=\sum_{n\ge 0}(\alpha_n\beta_n)u^n,$$

we define *the several-basis Bessel function* by

$$\text{(21.4)}\quad \mathbf{J}(u;\mathbf{Q},\mathbf{q}):=\Bigl(\sum_{n\ge 0}(-u)^n\Bigr)\,\mathrm{H}\,E_{Q_1}(u)\,\mathrm{H}\,E_{Q_2}(u)\,\mathrm{H}\cdots$$
$$\mathrm{H}\,e_{q_1}(u)\,\mathrm{H}\,e_{q_2}(u)\,\mathrm{H}\cdots$$

We can also write:

$$\text{(21.5)}\qquad \mathbf{J}(u;\mathbf{Q},\mathbf{q}):=\sum_{n\ge 0}(-1)^n\frac{\mathbf{Q}^{\binom{n}{2}}}{(\mathbf{Q};\mathbf{Q})_n}\frac{1}{(\mathbf{q};\mathbf{q})_n}u^n.$$

If $L=l=1$, then

$$\mathbf{J}(u;Q,q)=\sum_{n\ge 0}(-1)^n\frac{Q^{\binom{n}{2}}}{(Q;Q)_n}\frac{1}{(q;q)_n}u^n.$$

Finally, for $L=0$ and $l=1$, we regain the q-exponential

$$\mathbf{J}(u;\text{-},q)=\sum_{n\ge 0}(-1)^n\frac{1}{(q;q)_n}u^n=e_q(-u),$$

and for $L=1$ and $l=0$, the Q-exponential

$$\mathbf{J}(u;Q,\text{-})=\sum_{n\ge 0}(-1)^n\frac{Q^{\binom{n}{2}}}{(Q;Q)_n}u^n=E_Q(-u).$$

The coefficients in the series expansion of the fraction (21.3) will be shown to be polynomials with positive integral coefficients. They actually are generating polynomials for combinatorial objects, called *signed multipermutations* by a multivariable statistic that includes a single statistic referred to as the *number of descents*.

Definition. Each triple $(\underline{\Sigma},\underline{\sigma},\varepsilon)$ is called a *signed multipermutation* of order n if $\underline{\Sigma}=(\Sigma_1,\dots,\Sigma_L)$ and $\underline{\sigma}=(\sigma_1,\dots,\sigma_l)$ are two sequences of L and l permutations or order n, respectively, and if ε is an xy-word of length n.

Definition. Each integer i is said to be a *descent* of the signed multipermutation $(\underline{\Sigma},\underline{\sigma},\varepsilon)$, if one of the following three conditions is satisfied:

(i) $i=n$ and $\varepsilon(n)=x$;

(ii) $1\le i\le n-1$, $\varepsilon(i)=x$, $\varepsilon(i+1)=y$;

(iii) $1\le i\le n-1$, $\varepsilon(i)=\varepsilon(i+1)$ and $\Sigma_1(i)>\Sigma_1(i+1)$, … , $\Sigma_L(i)>\Sigma_L(i+1)$, and also $\sigma_1(i)>\sigma_1(i+1)$, … , $\sigma_l(i)>\sigma_l(i+1)$;

Let $\operatorname{ddes}(\underline{\Sigma},\underline{\sigma},\varepsilon)$ denote the *number of descents* of $(\underline{\Sigma},\underline{\sigma},\varepsilon)$.

The inversion and co-inversion numbers "inv" and "coinv" of a signed permutation (σ,ε) have been defined in § 21.1. With a signed multipermutation $(\underline{\Sigma},\underline{\sigma},\varepsilon)$ we further use the notations:

$$\mathbf{Q}^{\mathrm{inv}(\underline{\Sigma},\varepsilon)} = Q_1^{\mathrm{inv}(\Sigma_1,\varepsilon)} \cdots Q_L^{\mathrm{inv}(\Sigma_L,\varepsilon)}\,;$$
$$\mathbf{q}^{\mathrm{coinv}(\underline{\sigma},\varepsilon)} = q_1^{\mathrm{coinv}(\sigma_1,\varepsilon)} \cdots q_l^{\mathrm{coinv}(\sigma_l,\varepsilon)}.$$

Let (σ,ε) be a signed permutation, where $\sigma\in\mathcal{S}_n$ and $\varepsilon=\varepsilon(1)\ldots\varepsilon(n)$ is an xy-word. Recall that the word $\sigma_{\varepsilon|x}$ (resp. $\sigma_{\varepsilon|y}$) is defined to be the subword of $\sigma(1)\ldots\sigma(n)$ made of all letters $\sigma(i)$ such that $\varepsilon(i)=x$ (resp. $\varepsilon(i)=y$). Introduce the number of inversions "$\mathrm{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x})$" *between the words* $\sigma_{\varepsilon|y}$ and $\sigma_{\varepsilon|x}$ by

$$\mathrm{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x}) = \#\{(i,j) : \varepsilon(i)=y,\ \varepsilon(j)=x,\ \sigma(i)>\sigma(j)\}.$$

Another way of defining the statistics "inv" and "coinv" is then to let

$$\mathrm{inv}(\sigma,\varepsilon) = \mathrm{inv}\,\sigma_{\varepsilon|x} + \mathrm{inv}\,\sigma_{\varepsilon|y} + \mathrm{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x})\,;$$
$$\mathrm{coinv}(\sigma,\varepsilon) = \mathrm{coinv}\,\sigma_{\varepsilon|x} + \mathrm{coinv}\,\sigma_{\varepsilon|x} + \mathrm{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x}).$$

Definition. Suppose that ε has α letters equal to x and β letters equal to y $(\alpha+\beta=n)$. We say that the permutation σ is *compatible with* ε (also that the signed permutation (σ,ε) is *compatible*), if $\mathrm{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x})=0$, or, in an equivalent manner, if the subword $\sigma_{\varepsilon|y}$ is a rearrangement of the word $12\ldots\beta$ (and then $\sigma_{\varepsilon|x}$ is a rearrangement of $(\beta+1)(\beta+2)\ldots n$ (of length α)).

A signed multipermutation $(\underline{\Sigma},\underline{\sigma},\varepsilon)$ such that $\underline{\Sigma}=(\Sigma_1,\ldots,\Sigma_L)$ and $\underline{\sigma}=(\sigma_1,\ldots,\sigma_l)$ is said to be *compatible*, if the permutations $\Sigma_1,\ \ldots,\ \Sigma_L$, $\sigma_1,\ \ldots,\ \sigma_l$ are all compatible with ε.

Let

$$B_{\alpha,\beta}(t,\mathbf{Q},\mathbf{q}) := \sum_{\substack{(\underline{\Sigma},\underline{\sigma},\varepsilon)\ (\mathrm{comp.})\\ \ell(\varepsilon|x)=\alpha,\ \ell(\varepsilon|y)=\beta}} t^{\mathrm{ddes}(\underline{\Sigma},\underline{\sigma},\varepsilon)}\mathbf{Q}^{\mathrm{inv}(\underline{\Sigma},\varepsilon)}\mathbf{q}^{\mathrm{coinv}(\underline{\sigma},\varepsilon)} \tag{21.6}$$

be the generating polynomial for the *compatible* signed multipermutations $(\underline{\Sigma},\underline{\sigma},\varepsilon)$ such that ε contains α letters equal to x and β letters equal to y, by the statistic $(\mathrm{ddes},\mathrm{inv},\mathrm{coinv})$. Also let

$$\begin{aligned} B_n(X,Y,t,\mathbf{Q},\mathbf{q}) \\ := \sum_{(\underline{\Sigma},\underline{\sigma},\varepsilon)} X^{\ell(\varepsilon|x)}Y^{\ell(\varepsilon|y)}t^{\mathrm{ddes}(\underline{\Sigma},\underline{\sigma},\varepsilon)}\mathbf{Q}^{\mathrm{inv}(\underline{\Sigma},\varepsilon)}\mathbf{q}^{\mathrm{coinv}(\underline{\sigma},\varepsilon)}, \end{aligned} \tag{21.7}$$

the sum being *over all* signed multipermutations of length n.

Proposition 21.1. *We have the identity*

$$B_n(X,Y,t,\mathbf{Q},\mathbf{q}) = \sum_{\alpha+\beta=n} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{\mathbf{Q}} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{\mathbf{q}} X^\alpha\, Y^\beta\, B_{\alpha,\beta}(t,\mathbf{Q},\mathbf{q}) \tag{21.8}$$

Proof. To avoid cumbersome notations assume that $L=l=1$. Let $(\Sigma,\sigma,\varepsilon)$ be a signed bipermutation of length n such that $\ell(\varepsilon|x)=\alpha$. Let I denote the set of the α letters of the word $\sigma_{\varepsilon|x}$. There exists a unique increasing bijection f_Σ (resp. f_σ) mapping the set $\Sigma(I)$ (resp. $\sigma(I)$) onto the interval $[n-\alpha+1,n]$ and the set $\Sigma([\,n\,]\setminus I)$ (resp. $\sigma([n]\setminus I)$) onto the interval $[1,n-\alpha]$. The triple $(f_\Sigma\circ\Sigma, f_\sigma\circ\sigma,\varepsilon)$ is then a compatible signed permutation. The mapping $(\Sigma,\sigma,\varepsilon)\mapsto(f_\Sigma\circ\Sigma, f_\sigma\circ\sigma,\varepsilon)$ is *surjective.* Moreover, if $(\Sigma_0,\sigma_0,\varepsilon)$ is compatible, the number of preimages $(\Sigma,\sigma,\varepsilon)$ mapped over $(\Sigma_0,\sigma_0,\varepsilon)$ by that surjection is equal to $\binom{n}{\alpha}\binom{n}{\alpha}$.

If $(\Sigma_0,\sigma_0,\varepsilon)$ is compatible and if $(f_\Sigma\circ\Sigma, f_\sigma\circ\sigma,\varepsilon)=(\Sigma_0,\sigma_0,\varepsilon)$, then

$$\begin{aligned} \operatorname{ddes}(\Sigma,\sigma,\varepsilon) &= \operatorname{ddes}(\Sigma_0,\sigma_0,\varepsilon);\\ \operatorname{inv}(\sigma,\varepsilon) &= \operatorname{inv}(\sigma_0,\varepsilon)+\operatorname{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x});\\ \operatorname{coinv}(\sigma,\varepsilon) &= \operatorname{coinv}(\sigma_0,\varepsilon)+\operatorname{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x}), \end{aligned}$$

since f_Σ and f_σ are increasing mappings.

Therefore, the sum $\sum\limits_{(\Sigma,\sigma,\varepsilon)} Q^{\operatorname{inv}(\Sigma,\varepsilon)}\, q^{\operatorname{coinv}(\sigma,\varepsilon)}$ over all triples $(\Sigma,\sigma,\varepsilon)$ such that $(f_\Sigma\circ\Sigma, f_\sigma\circ\sigma,\varepsilon)=(\Sigma_0,\sigma_0,\varepsilon)$, is equal to

$$\begin{aligned} Q^{\operatorname{inv}(\Sigma_0,\varepsilon)}\, q^{\operatorname{coinv}(\sigma_0,\varepsilon)} \sum_{(\Sigma,\sigma,\varepsilon)} & Q^{\operatorname{inv}(\Sigma_{\varepsilon|y},\Sigma_{\varepsilon|x})}\, q^{\operatorname{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x})}\\ &= Q^{\operatorname{inv}(\Sigma_0,\varepsilon)}\, q^{\operatorname{coinv}(\sigma_0,\varepsilon)} \begin{bmatrix} n \\ \alpha \end{bmatrix}_Q \begin{bmatrix} n \\ \alpha \end{bmatrix}_q. \end{aligned}$$

Hence,

$$\begin{aligned} &B_n(X,Y,t,Q,q)\\ &\quad= \sum_{\alpha+\beta=n} X^\alpha Y^\beta \sum_{\substack{(\Sigma_0,\sigma_0,\varepsilon)\ (\text{comp.})\\ \ell(\varepsilon|x)=\alpha}} t^{\operatorname{ddes}(\Sigma_0,\sigma_0,\varepsilon)} \sum_{\substack{(\Sigma,\sigma,\varepsilon)\\ (f_\Sigma\circ\Sigma,f_\sigma\circ\sigma,\varepsilon)=(\Sigma_0,\sigma_0,\varepsilon)}} Q^{\operatorname{inv}(\Sigma,\varepsilon)} q^{\operatorname{coinv}(\sigma,\varepsilon)}\\ &\quad= \sum_{0\le\alpha\le n} X^\alpha Y^{n-\alpha} \begin{bmatrix} n \\ \alpha \end{bmatrix}_Q \begin{bmatrix} n \\ \alpha \end{bmatrix}_q B_{\alpha,\beta}(t,Q,q). \qquad \square \end{aligned}$$

The main result of this section that will be proved in the next subsection is the following theorem.

Theorem 21.2. *The following identities hold:*

$$\begin{aligned}(21.9)\quad \frac{(1-t)\,\mathbf{J}((1-t)X;\mathbf{Q},\mathbf{q})}{-t+\mathbf{J}((1-t)X;\mathbf{Q},\mathbf{q})\,\mathbf{J}((1-t)Y;\mathbf{Q},\mathbf{q})}&\\ =\sum_{n\geq 0}\frac{1}{(\mathbf{Q};\mathbf{Q})_n(\mathbf{q};\mathbf{q})_n}\,B_n(X,Y,t,\mathbf{Q},\mathbf{q})&\\ =\sum_{\alpha,\beta\geq 0}\frac{X^\alpha Y^\beta}{(\mathbf{Q};\mathbf{Q})_\alpha(\mathbf{q};\mathbf{q})_\alpha(\mathbf{Q};\mathbf{Q})_\beta(\mathbf{q};\mathbf{q})_\beta}B_{\alpha,\beta}(t,\mathbf{Q},\mathbf{q}).&\end{aligned}$$

The list of the first values of the polynomials $B_{\alpha,\beta}(t,Q,q)$ is shown in Table 21.3. Remember that to obtain the polynomials $B_n(X,Y,t,Q,q)$ (for $L=l=1$) identity (21.8) is to be used.

$$\begin{aligned}&B_{0,1}=1;\quad B_{1,0}=t;\\ &B_{0,2}=Qq+q+tQ+1;\quad B_{1,1}=2\,t;\quad B_{2,0}=t(Qq+q+tQ+1);\\ &B_{0,3}=2\,Qq+2\,Q^2q+2\,q+2\,Q^3tq+2\,Q^3tq^2+2\,Q^2q^3+2\,q^2\\ &\qquad +q^3+1+2\,Qq^3+2\,tQq+2\,tQ^2q+2\,q^2tQ+2\,q^2tQ^2\\ &\qquad +2\,Qq^2+Q^3t^2+2\,tQ+2\,tQ^2+2\,Q^2q^2+Q^3q^3\\ &B_{1,2}=t(3\,Qq+3\,q+2\,tQ+Q+3)\\ &B_{2,1}=t(2\,Qq+tQq+2\,q+tq+3\,tQ+2+t)\\ &B_{3,0}=t\,B_{0,3}\end{aligned}$$

Table 21.3 (distribution of (ddes, inv, coinv)
over the compatible signed bipermutations)

The list of the first values of the polynomials $B_n(X,Y,t,Q,q)$ (for $L=l=1$) is shown in Table 21.3. If in Table 21.3 we let $q=0$ (resp. $Q=0$), we recover Table 21.1 (resp. Table 21.2). On the other hand, the table contains the first values of the Gaussian polynomials $1+Q$, $1+Q+Q^2$ and $1+q$, $1+q+Q^2$. That occurrence will be fully explained in the next subsection.

Back to the diagram in Fig. 20.1 we see that Theorem 21.2 refers to extension (sgn,q,J) and the distribution of (ddes, inv, coinv) on signed multipermutations. When $X=0$ in (21.9), we go from signed multipermutations to the multipermutations themselves. This corresponds to vertex $(\cdot,q,J)$. For $L=0$, $l=1$ we have the specialization $(\mathrm{sgn},q,\cdot)$, corresponding to identity (21.1) with a combinatorial interpretation in terms of number of descents and co-inversions. Finally, vertex $(\mathrm{sgn},\cdot,J)$ may be regarded as the specialization with all the variables Q_i and q_i equal to 1.

Theorem 21.2 provides a combinatorial interpretation in terms of signed multipermutations counted only by their numbers of descents.

21.3. *The iterative method.* To prove Theorem 21.2 we proceed as follows. Let $(Q)_n := (Q;Q)_n$, $(q)_n := (q;q)_n$ and again assume that $L = l = 1$. Identity (21.9) (in its second form) may be rewritten

$$\Bigl(1-\sum_{i+j\ge 1}\frac{(t-1)^{i+j-1}Q^{\binom{i}{2}+\binom{j}{2}}X^iY^j}{(Q)_i\,(q)_i\,(Q)_j\,(q)_j}\Bigr)^{-1}\mathbf{J}((1-t)X;Q,q)$$
$$=\sum_{\alpha,\beta\ge 0}\frac{X^\alpha Y^\beta}{(Q)_\alpha\,(q)_\alpha(Q)_\beta\,(q)_\beta}\,B_{\alpha,\beta}(t,Q,q),$$

an identity of the form $(1-R)^{-1}S = T$. Express it in the form $S = (1-R)T$ and look for the coefficients of $X^\alpha Y^\beta$ on both sides. Then the identity is equivalent to the set of the two recurrence formulas

$$(21.10)\ B_{n,0} = Q^{\binom{n}{2}}(t-1)^n + \sum_{1\le i\le n}\begin{bmatrix}n\\ i\end{bmatrix}_Q\begin{bmatrix}n\\ i\end{bmatrix}_q(t-1)^{i-1}\,Q^{\binom{i}{2}}\,B_{n-i,0};$$

$$(21.11)\ B_{\alpha,\beta} = \sum_{\substack{0\le i\le\alpha\\ 0\le j\le\beta\\ i+j\ge 1}}\begin{bmatrix}\alpha\\ i\end{bmatrix}_Q\begin{bmatrix}\alpha\\ i\end{bmatrix}_q\begin{bmatrix}\beta\\ j\end{bmatrix}_Q\begin{bmatrix}\beta\\ j\end{bmatrix}_q(t-1)^{i+j-1}\,Q^{\binom{i}{2}+\binom{j}{2}}\,B_{\alpha-i,\beta-j};$$

with $n\ge 1$, $\alpha+\beta\ge 1$ and $B_{0,0}=1$.

First, if $(\Sigma,\sigma,\varepsilon)$ is a signed bipermutation such that ε contains α letters x and β letters y, both signed permutations (Σ,ε), (σ,ε) are permutations of the biletters $(1,y)\dots(\beta,y)(1+\beta,x)\dots(\alpha+\beta,x)$. Consequently, the left factor $(\Sigma(1),\sigma(1),\varepsilon(1))\dots(\Sigma(k),\sigma(k),\varepsilon(k))$ $(1\le k\le\alpha+\beta)$ of $(\Sigma,\sigma,\varepsilon)$ is *decreasing*, if we have $\Sigma(1)>\dots>\Sigma(k)$ and $\sigma(1)>\dots>\sigma(k)$. Each compatible signed bipermutation [in short, *c.s.b.*] has a *longest decreasing left factor* [in short, *l.d.l.f.*] of length at least equal to 1. Each letter $(\Sigma(i),\sigma(i),\varepsilon(i))$ of the signed bipermutation $(\Sigma,\sigma,\varepsilon)$ is said to be *large* (resp. *small*), if $\varepsilon(i)=x$ (resp. $\varepsilon(i)=y$).

In the sequel the *c.s.b.* are supposed to have α large letters and β small letters. Let $F(i,j)$ denote the generating function for those *c.s.b.*, whose *l.d.l.f.* is of length $(i+j)$ and start with i large letters followed by j small letters $(0\le i\le\alpha,\ 0\le j\le\beta,\ i+j\ge 1)$. Let $G(\ge i,\ge 0,\ge k)$ (resp. $G(=i,\ge j,\ge k)$) designate the generating function for those *c.s.b.*, whose *l.d.l.f.* is of length at least equal to k and start with at least i large letters (resp.and start with i large letters followed by at least j small letters). Let

$$u_{i,j} = \begin{bmatrix}\alpha\\ i\end{bmatrix}_Q\begin{bmatrix}\alpha\\ i\end{bmatrix}_q\begin{bmatrix}\beta\\ j\end{bmatrix}_Q\begin{bmatrix}\beta\\ j\end{bmatrix}_q Q^{\binom{i}{2}+\binom{j}{2}}\,B_{\alpha-i,\beta-j},$$

so that (21.10) and (21.11) can be rewritten as

$$B_{\alpha,0} = Q^{\binom{\alpha}{2}}(t-1)^{\alpha} + \sum_{1\le i\le \alpha} u_{i,0}(t-1)^{i-1} \quad (\beta = 0); \tag{21.12}$$

$$B_{\alpha,\beta} = \sum_{\substack{0\le i\le \alpha\\ 0\le j\le \beta\\ i+j\ge 1}} u_{i,j}(t-1)^{i+j-1} \quad (\beta \ge 1). \tag{21.13}$$

Lemma 21.3. *The following identities hold:*

$$t^i\, u_{i,0} = t\, F(i,0) + G(\ge i, \ge 0, \ge i+1); \tag{21.14}$$
$$t^{i+j}\, u_{i,j} = t\, F(i,j) + G(= i, \ge j+1, \ge i+j+1), \quad \textit{if } j \ge 1. \tag{21.15}$$

Proof. The product $t^i\, u_{i,0}$ is the generating function for the *c.s.b.* starting with a decreasing factor having i large letters, with the restriction that the initial number of descents is counted i (instead of $i-1$) when the *l.d.l.f.* is exactly of length i. In that generating function are included, besides the *c.s.b.* just described, the *c.s.b.* whose *l.d.l.f.* contains i large letters and at least a small letter, plus all the *c.s.b.* whose *l.d.l.f.* contains at least $i+1$ large letters and no small letter. This proves (21.14).

When $j \ge 1$, the product $t^{i+j}\, u_{i,j}$ is the generating function for the *c.s.b.* starting with a decreasing factor having i large letters and at least j small letters, still with the restriction that the intial number of descents is counted $i+j$ (instead of $i+j-1$) when the *l.d.l.f.* is of length $i+j$. Beside those *c.s.b.* the generating function involves all the *c.s.b.* whose *l.d.l.f.* contains i large letters and at least $j+1$ small letters. $\square$

As $F(i,0) = G(\ge i, \ge 0, \ge i) - G(\ge i, \ge 0, \ge i+1)$, it follows from (21.14) that

$$G(\ge i, \ge 0, \ge i) = t^{i-1}\, u_{i,0} + \frac{t-1}{t} G(\ge i, \ge 0, \ge i+1); \tag{21.16}$$

it also follows from (21.15), when $j \ge 1$, that

$$G(=i, \ge j, \ge i+j) = t^{i+j-1} u_{i,j} + \frac{t-1}{t} G(=i, \ge j+1, \ge i+j+1). \tag{21.17}$$

Moreover, we have the identity:

$$G(\ge i, \ge 0, \ge i+1) = G(= i, \ge 1, \ge i+1) + G(\ge i+1, \ge 0, \ge i+1). \tag{21.18}$$

Since $B_{\alpha,\beta}$ is equal to $G(\geq 1, \geq 0, \geq 1) + G(=0, \geq 1, \geq 1)$, those three formulas furnish the calculation for $B_{\alpha,\beta}$ as we now see. By iteration,

$$\begin{aligned}
(21.19)\quad B_{\alpha,\beta} = u_{1,0} &+ (t-1)u_{2,0} + \cdots + (t-1)^{\alpha-2}u_{\alpha-1,0}\\
&+ \frac{(t-1)^{\alpha-1}}{t^{\alpha-1}} G(\geq \alpha, \geq 0, \geq \alpha)\\
&+ \frac{(t-1)^{\alpha-1}}{t^{\alpha-1}} G(=\alpha-1, \geq 1, \geq \alpha)\\
&+ \cdots + \frac{(t-1)}{t} G(=1, \geq 1, \geq 2) + G(=0, \geq 1, \geq 1).
\end{aligned}$$

If $\beta = 0$, we have $G(\geq \alpha, \geq 0, \geq \alpha) = G(=\alpha, =0, =\alpha) = F(\alpha, 0) = t^\alpha Q^{\binom{\alpha}{2}}$ and all quantities of the form $G(=i, \geq 1, \geq i+1)$ are null. As $u_{\alpha,0} = Q^{\binom{\alpha}{2}}$, we get

$$\begin{aligned}
B_{\alpha,0} &= u_{1,0} + (t-1)u_{2,0} + \cdots + (t-1)^{\alpha-2}u_{\alpha-1,0} + t(t-1)^{\alpha-1}Q^{\binom{\alpha}{2}}\\
&= u_{1,0} + (t-1)u_{2,0} + \cdots + (t-1)^{\alpha-2}u_{\alpha-1,0} + (t-1)^{\alpha-1}u_{\alpha,0}\\
&\qquad + (t-1)^{\alpha}Q^{\binom{\alpha}{2}},
\end{aligned}$$

which is nothing but formula (21.12) to be proved.

When $\beta \geq 1$, we can pursue the iteration in (21.19) and because $G(\geq \alpha+1, \geq 0, \geq \alpha+1) = 0$ obtain

$$\begin{aligned}
B_{\alpha,\beta} = u_{1,0} &+ (t-1)u_{2,0} + \cdots + (t-1)^{\alpha-1}u_{\alpha,0}\\
&+ \frac{(t-1)^{\alpha}}{t^{\alpha}} G(=\alpha, \geq 1, =\alpha+1)\\
&+ \cdots + \frac{(t-1)}{t} G(=1, \geq 1, \geq 2) + G(=0, \geq 1, \geq 1).
\end{aligned}$$

Then each expression $G(=i, \geq 1, \geq i+1)$ is iterated by making use of the relations $G(=i, \geq \beta+1, \geq i+\beta+1) = 0$ $(0 \leq i \leq \alpha)$. We then obtain (21.13). $\square$

22. Eulerian Calculus; finite analogs of Bessel functions

Looking back on the diagram in Fig. 20.1 there remains to show how a further extension of Theorem 21.1 can be made and what meaning be given to the top level of the diagram, especially to vertex (sgn, q, J_k). As was shown in the previous section, the letter "J" without subscript refers to the basic Bessel function. The subscript "k" will refer to a finite analog of that function, that is now introduced.

Rewrite the two identities of Proposition 4.1 as:

$$\begin{bmatrix} n+k \\ n \end{bmatrix}_q = \sum_{0\le b_1\le\cdots\le b_n<k+1} q^{\operatorname{tot} b}, \tag{22.1}$$

$$Q^{\binom{n}{2}} \begin{bmatrix} K+1 \\ n \end{bmatrix}_Q = \sum_{0\le B_1<\cdots<B_n<K+1} Q^{\operatorname{tot} B}. \tag{22.2}$$

They hold for all integers $k \ge 0$ and $K \ge 0$. On the other hand, as we have (see (6.1))

$$\frac{1}{(q;q)_n} = \sum_{0\le b_1\le\cdots\le b_n} q^{\operatorname{tot} b}, \tag{22.3}$$

$$\frac{Q^{\binom{n}{2}}}{(Q;Q)_n} = \sum_{0\le B_1<\cdots<B_n} Q^{\operatorname{tot} B}, \tag{22.4}$$

we can let

$$\begin{bmatrix} n+\infty \\ n \end{bmatrix}_q := \frac{1}{(q;q)_n}, \tag{22.5}$$

$$Q^{\binom{n}{2}} \begin{bmatrix} \infty+1 \\ n \end{bmatrix}_Q := \frac{Q^{\binom{n}{2}}}{(Q;Q)_n}, \tag{22.6}$$

so that (22.1) and (22.2) are valid for all k and K finite or infinite.

Let $e_q^k(u)$ and $E_Q^K(u)$ denote the generating functions for (22.1) and (22.2), i.e.,

$$e_q^k(u) = \sum_{n\ge 0} \begin{bmatrix} n+k \\ n \end{bmatrix}_q u^n, \quad \text{still equal to } \frac{1}{(u;q)_{k+1}}, \tag{22.7}$$

$$E_Q^K(u) = \sum_{n\ge 0} Q^{\binom{n}{2}} \begin{bmatrix} K+1 \\ n \end{bmatrix}_Q u^n. \tag{22.8}$$

In particular, the two q-exponentials may rewrite as

$$e_q(u) = e_q^\infty(u) = \sum_{n\ge 0} \frac{1}{(q;q)_n} u^n,$$

$$E_Q(u) = E_Q^\infty(u) = \sum_{n\ge 0} \frac{Q^{\binom{n}{2}}}{(Q;Q)_n} u^n.$$

Let K, k be two fixed nonnegative integers. Using the Hadamard product (see (21.4)) we define the *K,k-finite analog basic Bessel function* $\mathbf{J}_k^K(u;Q,q)$ by

$$\mathbf{J}_k^K(u;Q,q) := \Bigl(\sum_{n\ge 0}(-u)^n\Bigr) \,\mathrm{H}\, E_Q^K(u) \,\mathrm{H}\, e_q^k(u); \tag{22.9}$$

or, in an equivalent manner, by

$$\mathbf{J}_k^K(u;Q,q) := \sum_{n\ge 0}(-1)^n Q^{\binom{n}{2}} \begin{bmatrix} K+1 \\ n \end{bmatrix}_Q \begin{bmatrix} n+k \\ n \end{bmatrix}_q u^n. \tag{22.10}$$

When K and k are infinite, then

$$\mathbf{J}_\infty^\infty(u;Q,q) = \mathbf{J}(u;Q,q) = \sum_{n\ge 0}(-1)^n \frac{Q^{\binom{n}{2}}}{(Q,Q)_n}\frac{1}{(q;q)_n}u^n, \tag{22.11}$$

which is the basic Bessel function defined and used in the previous section. We could take the Hadamard product of more than two Bessel functions; our results would still be valid, but awkward to state and handle.

Following the same pattern as in the previous two sections consider the fraction

$$F = F\binom{K,M}{k,m} := \frac{(1-t)\,\mathbf{J}_k^K((1-t)X;P,p)}{-t+\mathbf{J}_k^K((1-t)X;P,p)\,\mathbf{J}_m^M((1-t)Y;Q,q)} \tag{22.12}$$

and expand it first as a generating function for *signed biwords*, then for *signed bipermutations*. Notice that the above fraction involves *two* Bessel functions $\mathbf{J}_k^K((1-t)X;P,p)$ and $\mathbf{J}_m^M((1-t)Y;Q,q)$.

22.1. *Signed biwords.* The next step is to express F as a homomorphic image of a generating function for the so-called *signed biwords*. We just make use of the combinatorial expressions given in (22.1) and (22.2). Let x, X, y, Y, P, p, Q, q designate independent variables and K, k, M, m four fixed nonnegative integers. Consider the alphabet $\mathfrak{A} = \mathfrak{A}\binom{K,M}{k,m}$ whose elements are the triletters $\begin{pmatrix} C \\ c \\ z \end{pmatrix}$, where $z = x$ or y, and C, c are integers such that

$$\begin{cases} 0\le C\le K,\ 0\le c\le k, & \text{if } z = x;\\ 0\le C\le M,\ 0\le c\le m, & \text{if } z = y.\end{cases}$$

Let $\Phi(\cdot) = \Phi(\cdot;P,p,Q,q,X,Y)$ be the map that sends each triletter $\begin{pmatrix} C \\ c \\ z \end{pmatrix}$ onto

$$\Phi\Bigl(\begin{pmatrix} C \\ c \\ z \end{pmatrix}\Bigr) = \begin{cases} P^C p^c X, & \text{if } z = x;\\ Q^C q^c Y, & \text{if } z = y.\end{cases}$$

Next, form the free monoid $\mathfrak{A}^*$ generated by $\mathfrak{A}$. It consists of all the three-row matrices, called *signed biwords*,

$$w = \begin{pmatrix} B \\ b \\ \varepsilon \end{pmatrix} = \begin{pmatrix} B(1)\, B(2) \dots B(n) \\ b(1)\ \ b(2)\ \dots\ b(n) \\ \varepsilon(1)\ \ \varepsilon(2)\ \dots\ \varepsilon(n) \end{pmatrix} \tag{22.13}$$

The *length* of w is n and denoted by $\ell(w)$. The signed biword of length 0 is the empty word, denoted by e. When $\varepsilon = x^n$ (resp. $\varepsilon = y^n$) and also when $B(1) < B(2) < \cdots < B(n)$ and $b(1) \le b(2) \le \cdots \le b(n)$, we say that the signed word w is *nondecreasing.* Let ND_x (resp. ND_y) be the set of all nondecreasing signed biwords, whose bottom row has only x's (resp. only y's). The set of the juxtaposition products $w'w''$, where $w' \in \mathrm{ND}_x$ and $w'' \in \mathrm{ND}_y$, is denoted by $\mathrm{ND}_x\,\mathrm{ND}_y$.

By defining $\Phi(w)$ as the product of the images under Φ of the triletters of w, we extend Φ, by linearity, to a homomorphism of $\mathfrak{A}^*$ into the ring of polynomials in the variables P, p, X, Q, q, Y.

Proposition 22.1. *Let*

$$G := \Big(1 - \sum_w (t-1)^{\ell(w)-1}\,(w)\Big)^{-1} \Big(\sum_{w'} (t-1)^{\ell(w')} w'\Big),$$

where w runs over $\mathrm{ND}_x\,\mathrm{ND}_y \setminus \{e\}$ and w' over ND_x. Then

$$F = \Phi(G).$$

Proof. It follows from (22.1) and (22.2) that

$$\mathbf{J}(X) := \mathbf{J}_k^K((1-t)X; P, p) = \sum_w (t-1)^{\ell(w)}\Phi(w) \quad (w \in \mathrm{ND}_x);$$

$$\mathbf{J}(Y) := \mathbf{J}_m^M((1-t)Y; Q, q) = \sum_w (t-1)^{\ell(w)}\Phi(w) \quad (w \in \mathrm{ND}_y);$$

$$\mathbf{J}(X)\mathbf{J}(Y) = \sum_w (t-1)^{\ell(w)}\Phi(w) \quad (w \in \mathrm{ND}_x\,\mathrm{ND}_y).$$

Hence, with w running over $\mathrm{ND}_x\,\mathrm{ND}_y$,

$$\frac{1-t}{-t + \mathbf{J}(X)\,\mathbf{J}(Y)} = \frac{1-t}{-t + \sum_w (t-1)^{\ell(w)}\,\Phi(w)} = \Big(1 - \sum_{w \ne e} (t-1)^{\ell(w)-1}\,\Phi(w)\Big)^{-1}.$$

We obtain the desired identity by multiplying by $\mathbf{J}(X)$ on the right. $\square$

Definition. Let w be a signed biword, as in (22.13), and i an integer. We say that i is a *rise* for w, if one of the following three conditions holds:

(i) $i=n$ and $\varepsilon(n)=x$;

(ii) $1\le i\le n-1$, $\varepsilon(i)=x$, $\varepsilon(i+1)=y$;

(iii) $1\le i\le n-1$, $\varepsilon(i)=\varepsilon(i+1)$, $B(i)<B(i+1)$ and $b(i)\le b(i+1)$.

The *number of rises* of the signed biword w is denoted by $\operatorname{rise} w$.

Proposition 22.2. *The following identity holds:*

$$G=\sum_w t^{\operatorname{rise} w}\, w \tag{22.14}$$

and consequently

$$F=\sum_w t^{\operatorname{rise} w}\, \Phi(w), \tag{22.15}$$

where w runs over all signed biwords in $\mathfrak{A}^*$.

To prove Proposition 22.2 we make use of a classical inversion formula that is stated and proved in various contexts. Let X^* be the free monoid generated by a set X and form the large algebra of X^* over a ring Ω of polynomials. The elements of that algebra are the formal sums $\sum c(w)\, w$, where w runs over X^* and $c(w)$ belongs to Ω. Let $a: X^2\to\Omega$ be a given mapping that we extend to a mapping $a: X^*\to\Omega$ by sending each word $w=x_1x_2\ldots x_n\in X^*$ to

$$a(w)=\begin{cases} a(x_1,x_2)\ldots a(x_{n-1},x_n), & \text{if } n\ge 2;\\ 1, & \text{if } n=0 \text{ or } 1.\end{cases}$$

Next define:

$$\overline{a}(w):=\begin{cases} (a(x_1,x_2)-1)\ldots (a(x_{n-1},x_n)-1), & \text{if } n\ge 2;\\ 1, & \text{if } n=0 \text{ or } 1.\end{cases}$$

Further, let U, V be two nonempty subsets of the alphabet X. The expressions U^+ and U^*V stand for the sets of all *nonempty* words $w=x_1x_2\ldots x_n$ *all* letters of which are in U (resp. whose *rightmost* letter x_n is in V and other letters are in U).

Lemma 22.3. *In the large algebra of X^* we have the identity:*

$$\sum_{w\in U^*V} a(w)\, w=\Bigl(1-\sum_{w\in U^+}\overline{a}(w)\, w\Bigr)^{-1}\times\sum_{w\in U^*V}\overline{a}(w)\, w. \tag{22.16}$$

Proof. Multiply each side by $(1-\sum_{w\in U^+}\overline{a}(w)\, w)$ on the left and verify that the coefficients of each word w on both sides are the same. $\square$

In formula (22.16) take the following ingredients: first, $X := \mathfrak{A} \cup \{\underline{\infty}\}$, where "$\underline{\infty}$" is the triletter (∞, ∞, ∞) (writing the triletter horizontally for typographical reasons). Let $u = (C, c, z)$ and $u' = (C', c', z')$ be two triletters belonging to X.

For "a" take the mapping:

$$a(u, u') = \begin{cases} t, & \text{if } z = x \text{ and } z' = \infty; \\ t, & \text{if } z = x \text{ and } z' = y; \\ t, & \text{if } z = z' \text{ and } C < C' \text{ and } c \le c'; \\ 1, & \text{otherwise.} \end{cases}$$

For V take the singleton $\{\underline{\infty}\}$ and for U the alphabet $\mathfrak{A}$. Formula (22.16) is rewritten, after simplification on the right by $\underline{\infty}$, as

$$\sum_{w \in \mathfrak{A}^*} a(w\,\underline{\infty})\, w = \Bigl(1 - \sum_{w \in \mathfrak{A}^* \setminus \{e\}} \overline{a}(w)\, w\Bigr)^{-1} \times \sum_{w \in \mathfrak{A}^*} \overline{a}(w\,\underline{\infty})\,(w). \tag{22.17}$$

Let $w = u^{(1)} u^{(2)} \ldots u^{(n)}$ be a signed biword of length n, written as a word with n triletters. Let $u^{(n+1)} := \underline{\infty}$. Then, the statistic $\operatorname{rise}(w)$ can also be expressed as:

$$\operatorname{rise}(w) = \sum_{1 \le i \le n} \chi\bigl(a(u^{(i)}\, u^{(i+1)}) = t\bigr). \tag{22.18}$$

For every signed biword w we then have:

$$a(w\,\underline{\infty}) = t^{\operatorname{rise} w},$$

$$\overline{a}(w) = \begin{cases} (t-1)^{\ell(w)-1}, & \text{if } w \in \mathrm{ND}_x \,\mathrm{ND}_y \text{ and } \ell(w) \ge 2; \\ 1, & \text{if } \ell(w) = 0 \text{ or } 1; \\ 0, & \text{otherwise;} \end{cases}$$

$$\overline{a}(w\,\underline{\infty}) = \begin{cases} (t-1)^{\ell(w)}, & \text{if } w \in \mathrm{ND}_x \text{ and } \ell(w) \ge 1; \\ 1, & \text{if } \ell(w) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

With those ingredients the left-hand side of identity (22.17) is equal to $\sum_{w \in \mathfrak{A}^*} t^{\operatorname{rise} w}\, w$ and the right-hand side is the expression found for G in Proposition 22.1. ∎

22.2. *Signed bipermutations.* Although Proposition 22.2 provides an expression for the fraction F in terms of *biwords* and involves one statistic, namely the *number of rises*, the combinatorial objects are still rudimentary. To obtain an expression in terms of combinatorial objects having a richer geometry, such as permutations or signed permutations, it is necessary to imagine a further correspondence. This is the purpose of this subsection and the next one.

The definition of a signed multipermutation $(\underline{\Sigma}, \underline{\sigma}, \varepsilon)$ was given in § 21.2. When $\underline{\Sigma}$ (resp. $\underline{\sigma}$) consists of a single permutation Σ (resp. σ), we speak of a *signed bipermutation* $(\Sigma, \sigma, \varepsilon)$, as in the present subsection. With such a signed bipermutation we associate the subset $I_{\varepsilon|x} := \{i_1, \ldots, i_{\ell(\varepsilon|x)}\}$ (resp. $I_{\varepsilon|y} := \{j_1, \ldots, j_{\ell(\varepsilon|y)}\}$) of all the integers i such that $\varepsilon(i) = x$ (resp. $\varepsilon(i) = y$).

Suppose that $(\Sigma, \sigma, \varepsilon)$ is *compatible* in the sense of Proposition 21.2 and consider the restriction of the bijection $i \mapsto \Sigma(i)$ to $I_{\varepsilon|y}$ (resp. to $I_{\varepsilon|x}$); it is a *bijection*, denoted by $\Sigma_{\varepsilon|y}$, of $I_{\varepsilon|y}$ onto the interval $[1, \ell(\varepsilon|y)]$ (resp. a bijection $\Sigma_{\varepsilon|x}$ of $I_{\varepsilon|x}$ onto the interval $[\ell(\varepsilon|y)+1, n]$). In the same manner, two restrictions of σ are considered: $\sigma_{\varepsilon|y}$ of $I_{\varepsilon|y}$ onto $[1, \ell(\varepsilon|y)]$ and $\sigma_{\varepsilon|x}$ of $I_{\varepsilon|x}$ onto $[\ell(\varepsilon|y)+1, n]$.

For instance,

$$\begin{matrix} \\ \Sigma = \\ \sigma = \\ \varepsilon = \end{matrix} \begin{pmatrix} \mathbf{1}\,\mathbf{2}\,3\,4\,5\,6\,\mathbf{7}\,\mathbf{8}\,\mathbf{9} \\ \mathbf{7}\,\mathbf{5}\,3\,4\,2\,1\,\mathbf{8}\,\mathbf{9}\,\mathbf{6} \\ \mathbf{6}\,\mathbf{7}\,2\,4\,3\,1\,\mathbf{5}\,\mathbf{8}\,\mathbf{9} \\ x\,x\,y\,y\,y\,y\,x\,x\,x \end{pmatrix}. \tag{22.19}$$

is a compatible signed bipermutation of order $n = 9$. Also $I_{\varepsilon|x} = \{\mathbf{1}, \mathbf{2}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$, $I_{\varepsilon|y} = \{3, 4, 5, 6\}$.

The restrictions $\Sigma_{\varepsilon|x}$, $\Sigma_{\varepsilon|y}$ $\sigma_{\varepsilon|x}$ $\sigma_{\varepsilon|y}$, written as two-row matrices, together with their inverses read:

$$\Sigma_{\varepsilon|x} = \begin{pmatrix} \mathbf{1\,2\,7\,8\,9} \\ \mathbf{7\,5\,8\,9\,6} \end{pmatrix}, \ \Sigma_{\varepsilon|y} = \begin{pmatrix} 3\,4\,5\,6 \\ 3\,4\,2\,1 \end{pmatrix}, \ \sigma_{\varepsilon|x} = \begin{pmatrix} \mathbf{1\,2\,7\,8\,9} \\ \mathbf{6\,7\,5\,8\,9} \end{pmatrix}, \ \sigma_{\varepsilon|y} = \begin{pmatrix} 3\,4\,5\,6 \\ 2\,4\,3\,1 \end{pmatrix},$$
$$\Sigma^{-1}_{\varepsilon|x} = \begin{pmatrix} \mathbf{5\,6\,7\,8\,9} \\ \mathbf{2\,9\,1\,7\,8} \end{pmatrix}, \ \Sigma^{-1}_{\varepsilon|y} = \begin{pmatrix} 1\,2\,3\,4 \\ 6\,5\,3\,4 \end{pmatrix}, \ \sigma^{-1}_{\varepsilon|x} = \begin{pmatrix} \mathbf{5\,6\,7\,8\,9} \\ \mathbf{7\,1\,2\,8\,9} \end{pmatrix}, \ \sigma^{-1}_{\varepsilon|y} = \begin{pmatrix} 1\,2\,3\,4 \\ 6\,3\,5\,4 \end{pmatrix}.$$

We do not repeat the definitions of "des" and "maj" for permutations, but introduce the definition of "rise" and "rmaj" (referring to the "rises" and no longer to the "descents"). If ρ is a permutation, written as a linear word $\rho = \rho(1)\rho(2)\ldots\rho(n)$. Then, define

$$\begin{aligned} \operatorname{rise} \rho &:= \sum_{1 \le i \le n-1} \chi(\rho(i) < \rho(i+1)); \\ \operatorname{rmaj} \rho &:= \sum_{1 \le i \le n-1} i\, \chi(\rho(i) < \rho(i+1)). \end{aligned}$$

The statistics "des" and "maj" will be determined for $\Sigma^{-1}_{\varepsilon|x}$, $\sigma^{-1}_{\varepsilon|y}$ and the statistics "rise" and "rmaj" for $\sigma^{-1}_{\varepsilon|x}$, $\sigma^{-1}_{\varepsilon|y}$, by means of word codes, as it was done in § 12.2 for the application of the MacMahon *Verfahren*. To avoid cumbersome notations suppose that those four bijections are identified with the bottom rows of their defining two-row matrices, respectively denoted by $S_x := S_x(1)\dots S_x(\ell(\varepsilon|x))$, $S_y := S_y(1)\dots S_x(\ell(\varepsilon|y))$, $s_x := s_x(1)\dots s_x(\ell(\varepsilon|x))$, $s_y := s_y(1)\dots s_y(\ell(\varepsilon|y))$.

For each $i = 1,\dots,\ell(\varepsilon|x)$ let $Z_x(i)$ be the number of j such that $i \le j \le \ell(\varepsilon|x) - 1$ and $S_x(j) > S_x(j+1)$, that is, to the *number of descents* to the right of $S_x(i)$ and form the word $Z_x := Z_x(1)\dots Z(\ell(\varepsilon|x))$. Define Z_y in the same way, replacing "x" by "y".

For s_x and s_y the word "descent" is to be replaced by "rise." For each $i = 1,\dots,\ell(\varepsilon|x)$ let $z_x(i)$ be the number of j such that $i \le j \le \ell(\varepsilon|x) - 1$ and $s_x(j) < s_x(j+1)$ and form the word $z_x := z_x(1)\dots z(\ell(\varepsilon|x))$. Define z_y in the same way, replacing "x" by "y".

With the above example we get

$$\begin{matrix} S_x = \\ Z_x = \end{matrix}\begin{pmatrix} \mathbf{2\,9\,1\,7\,8} \\ \mathbf{1\,1\,0\,0\,0} \end{pmatrix}, \quad \begin{matrix} S_y = \\ Z_y = \end{matrix}\begin{pmatrix} 6\,5\,3\,4 \\ 2\,1\,0\,0 \end{pmatrix}, \quad \begin{matrix} s_x = \\ z_x = \end{matrix}\begin{pmatrix} \mathbf{7\,1\,2\,8\,9} \\ \mathbf{3\,3\,2\,1\,0} \end{pmatrix}, \quad \begin{matrix} s_y = \\ z_y = \end{matrix}\begin{pmatrix} 6\,3\,5\,4 \\ 1\,1\,0\,0 \end{pmatrix}.$$

Proposition 22.4. *We have:*

$$\begin{aligned}
\operatorname{des}\Sigma^{-1}_{\varepsilon|x} &= Z_x(1); \quad \operatorname{maj}\Sigma^{-1}_{\varepsilon|x} = \operatorname{tot} Z_x = Z_x(1) + \cdots + Z_x(\ell(\varepsilon|x));\\
\operatorname{des}\Sigma^{-1}_{\varepsilon|y} &= Z_y(1); \quad \operatorname{maj}\Sigma^{-1}_{\varepsilon|y} = \operatorname{tot} Z_y = Z_y(1) + \cdots + Z_y(\ell(\varepsilon|y));\\
\operatorname{rise}\sigma^{-1}_{\varepsilon|x} &= z_x(1); \quad \operatorname{rmaj}\sigma^{-1}_{\varepsilon|x} = \operatorname{tot} z_x = z_x(1) + \cdots + z_x(\ell(\varepsilon|x));\\
\operatorname{rise}\sigma^{-1}_{\varepsilon|y} &= z_y(1); \quad \operatorname{rmaj}\sigma^{-1}_{\varepsilon|y} = \operatorname{tot} z_y = z_y(1) + \cdots + z_y(\ell(\varepsilon|y)).
\end{aligned}$$

Proof. With the words Z_x, Z_y, z_x and z_y we have another way of counting descents and rises. ∎

To further simplify the notations the following shorthand will be used:

$$\begin{aligned}
\operatorname{ides}_x\Sigma &:= \operatorname{des}\Sigma^{-1}_{\varepsilon|x}; \quad \operatorname{imaj}_x\Sigma := \operatorname{maj}\Sigma^{-1}_{\varepsilon|x};\\
\operatorname{ides}_y\Sigma &:= \operatorname{des}\Sigma^{-1}_{\varepsilon|y}; \quad \operatorname{imaj}_y\Sigma := \operatorname{maj}\Sigma^{-1}_{\varepsilon|y};\\
\operatorname{irise}_x\sigma &:= \operatorname{rise}\sigma^{-1}_{\varepsilon|x}; \quad \operatorname{irmaj}_x\sigma := \operatorname{rmaj}\sigma^{-1}_{\varepsilon|x};\\
\operatorname{irise}_y\sigma &:= \operatorname{rise}\sigma^{-1}_{\varepsilon|y}; \quad \operatorname{irmaj}_y\sigma := \operatorname{rmaj}\sigma^{-1}_{\varepsilon|y}.
\end{aligned}$$

With each compatible signed bipermutation $(\Sigma,\sigma,\varepsilon)$ is associated the eleven-variable monomial

$$\begin{aligned}
(22.20)\quad \Psi(\Sigma,\sigma,\varepsilon) &:= X^{\ell(\varepsilon|x)}Y^{\ell(\varepsilon|y)}t^{\operatorname{ddes}(\Sigma,\sigma,\varepsilon)}\\
&\times U^{\operatorname{ides}_x\Sigma}V^{\operatorname{ides}_y\Sigma}P^{\operatorname{imaj}_x\Sigma}Q^{\operatorname{imaj}_y\Sigma}u^{\operatorname{irise}_x\sigma}v^{\operatorname{irise}_y\sigma}p^{\operatorname{irmaj}_x\sigma}q^{\operatorname{irmaj}_y\sigma}.
\end{aligned}$$

Our purpose is to work out the generating function for the polynomials

$$\mathbf{B}_{\alpha,\beta} := \sum \Psi(\Sigma,\sigma,\varepsilon), \tag{22.21}$$

where the sum is over all compatible signed bipermutations $(\Sigma,\sigma,\varepsilon)$ such that $\ell(\varepsilon|x)=\alpha$ and $\ell(\varepsilon|x)=\beta$.

22.3. *Signed biwords and compatible bipermutations.* Using the notations of the previous subsection and Proposition 22.2, we prove the following theorem.

Theorem 22.5. *The factorial generating function for the polynomials* $\mathbf{B}_{\alpha,\beta}$ *defined in* (22.21) *is given by*

$$\sum_{\binom{K,M}{k,m}} U^K V^M u^k v^m \frac{(1-t)\,\mathbf{J}_k^K((1-t)X;P,p)}{-t+\mathbf{J}_k^K((1-t)X;P,p)\,\mathbf{J}_m^M((1-t)Y;Q,q)}$$
$$= \sum_{\alpha\ge 0,\,\beta\ge 0} e_P^{\alpha}(U)\,e_Q^{\beta}(V)\,e_p^{\alpha}(u)\,e_q^{\beta}(v)\mathbf{B}_{\alpha,\beta}.$$

By (22.12) and (22.15) the left-hand side of the previous identity is equal to

$$\sum_{\binom{K,M}{k,m}} R^K S^M r^k s^m\, F\binom{K,M}{k,m} = \sum_{\binom{K,M}{k,m}} U^K V^M u^k v^m \sum_{w\in\mathfrak{A}\binom{K,M}{k,m}^*} t^{\operatorname{rise} w}\Phi(w),$$

so that there remains to find an appropriate correspondence between signed biwords and compatible signed bipermutations.

Again use the notation (22.13) for the signed biword w and still let $I_{\ell(\varepsilon|x)}$ (resp. $I_{\ell(\varepsilon|y)}$) be the set of all i such that $\varepsilon(i)=x$ (resp. $\varepsilon(i)=y$). Then let $B|x := B_x(1)\dots B_x(\ell(\varepsilon|x))$ (resp. $B|y := B_y(1)\dots B_y(\ell(\varepsilon|y))$) be the *nonincreasing* rearrangement of all letters $B(i)$ such that $i\in I_{\ell(\varepsilon|x)}$ (resp. $i\in I_{\ell(\varepsilon|y)}$). Let $b|x := b_x(1)\dots b_x(\ell(\varepsilon|x))$ (resp. $b|y := b_y(1)\dots b_y(\ell(\varepsilon|y))$) have analogous definitions when B is replaced by b. Hence (see § 22.1) the image $\Phi(w)$ of the signed biword w reads:

$$\Phi(w) = P^{\operatorname{tot} B|x} Q^{\operatorname{tot} B|y} p^{\operatorname{tot} b|x} q^{\operatorname{tot} b|y} X^{\ell(\varepsilon|x)} Y^{\ell(\varepsilon|y)}. \tag{22.22}$$

For example, let $K=8$, $M=6$, $k=9$, $m=8$ and consider the signed biword $w\in\mathfrak{A}\binom{8,6}{9,8}$:

$$w = \begin{pmatrix} B\\ b\\ \varepsilon\end{pmatrix} = \begin{pmatrix} 4\,7\,1\,0\,4\,6\,3\,3\,5\\ 9\,5\,4\,0\,0\,6\,9\,1\,0\\ x\,x\,y\,y\,y\,y\,x\,x\,x\end{pmatrix}$$

Then $B|x = 7\,5\,4\,3\,3$, $B|y = 6\,4\,1\,0$, $b|x = 9\,9\,5\,1\,0$, $b|y = 6\,4\,0\,0$, $\ell(\varepsilon|x) = 5$, $\ell(\varepsilon|y) = 4$, so that $\Phi(w) = P^{22}Q^{11}p^{24}q^{10}X^5Y^4$. The rises of w are: 2 (because $\varepsilon(2) = y$, $\varepsilon(3) = x$), 4 and 5 (because $\varepsilon(4) = \varepsilon(5) = \varepsilon(6) = y$, $B(4) = 0 < B(5) = 4 < B(6) = 6$, $b(4) = 0 \le b(5) = 0 \le b(6) = 6$) and finally 9 (because $n = 9$ and $\varepsilon(9) = x$). Hence, $\operatorname{rise} w = 4$.

Definition of the mapping $w = (B, b, \varepsilon) \mapsto (\Sigma, \sigma, \varepsilon)$. First, the x, y-word ε remains the same. If $\varepsilon(i) = y$ let

$$\text{(22.23)}\quad \Sigma(i) := |\{j : 1 \le j \le n, B(j) > B(i), \varepsilon(j) = y\}| \\ + |\{j : 1 \le j \le i, B(j) = B(i), \varepsilon(j) = y\}|;$$

and if $\varepsilon(i) = x$ let

$$\text{(22.23')}\quad \Sigma(i) := \ell(\varepsilon|y) + |\{j : 1 \le j \le n, B(j) > B(i), \varepsilon(j) = x\}| \\ + |\{j : 1 \le i \le j, B(j) = B(i), \varepsilon(j) = x\}|.$$

Next for each $i = 1, 2, \ldots, n$ and $\varepsilon(i) = y$ let

$$\text{(22.24)}\quad \sigma(i) := |\{j : 1 \le j \le n, b(j) > b(i), \varepsilon(j) = y\}| \\ + |\{j : i \le j \le n, b(j) = b(i), \varepsilon(j) = y\}|.$$

When $\varepsilon(i) = x$, let

$$\text{(22.24')}\quad \sigma(i) := \ell(\varepsilon|y) + |\{j : 1 \le j \le n, b(j) > b(i), \varepsilon(j) = x\}| \\ + |\{j : i \le j \le n, b(j) = b(i), \varepsilon(j) = x\}|.$$

Notice that the only difference in the definitions of σ and Σ is the following: if $b(i) = b(j)$ (resp. $B(i) = B(j)$) and $i < j$, then $\sigma(i) > \sigma(j)$ (resp. $\Sigma(i) < \Sigma(j)$). Obviously, $(\Sigma, \sigma, \varepsilon)$ is a compatible *signed bipermutation.*

For instance, to the above signed biword $w = (B, b, \varepsilon)$ there corresponds the compatible signed permutation displayed in (22.19). Let us materialize the correspondence with the following matrix.

$$\text{(22.25)}\qquad \begin{array}{r} \\ B = \\ b = \\ \varepsilon = \\ \Sigma = \\ \sigma = \end{array} \begin{pmatrix} \mathbf{1}\,\mathbf{2}\,3\,4\,5\,6\,\mathbf{7}\,\mathbf{8}\,\mathbf{9} \\ \mathbf{4}\,\mathbf{7}\,1\,0\,4\,6\,\mathbf{3}\,\mathbf{3}\,\mathbf{5} \\ \mathbf{9}\,\mathbf{5}\,4\,0\,0\,6\,\mathbf{9}\,\mathbf{1}\,\mathbf{0} \\ x\,x\,y\,y\,y\,y\,x\,x\,x \\ \mathbf{7}\,\mathbf{5}\,3\,4\,2\,1\,\mathbf{8}\,\mathbf{9}\,\mathbf{6} \\ \mathbf{6}\,\mathbf{7}\,2\,4\,3\,1\,\mathbf{5}\,\mathbf{8}\,\mathbf{9} \end{pmatrix}.$$

Once the signed bipermutation $(\Sigma, \sigma, \varepsilon)$ has been obtained, we determine the words $S_x, Z_x, \ldots$ and the statistics ides_x, ides_y, $\ldots$, as they were introduced in the previous subsection; also, we let

$$\text{(22.26)}\qquad \begin{gathered} K' := K - \operatorname{ides}_x \Sigma; \quad M' := M - \operatorname{ides}_y \Sigma; \\ k' := k - \operatorname{irise}_x \sigma; \quad m' := m - \operatorname{ides}_y \sigma; \end{gathered}$$

and define the words:

$$\Lambda := B|x - Z_x, \quad \Theta := B|y - Z_y, \quad \lambda := b|x - z_x, \quad \theta := b|y - z_y,$$

where the difference is defined letter by letter. For instance,

$$\Lambda = \Lambda(1)\cdots\Lambda(\ell(\varepsilon|x)) := (B_x(1) - Z_x(1))\cdots(B_x(\ell(\varepsilon|x)) - Z_x(\ell(\varepsilon|x))).$$

Proposition 22.6. *The integers K', M', k', m' are nonnegative. Furthermore,*

(i) $K' \geq \Lambda(1) \geq \cdots \geq \Lambda(\ell(\varepsilon|x)) \geq 0; \quad M' \geq \Theta(1) \geq \cdots \geq \Theta(\ell(\varepsilon|y)) \geq 0;$

(ii) $k' \geq \lambda(1) \geq \cdots \geq \lambda(\ell(\varepsilon|x)) \geq 0; \quad m' \geq \theta(1) \geq \cdots \geq \theta(\ell(\varepsilon|y)) \geq 0;$

(iii) $\operatorname{tot} B|x = \operatorname{imaj}\Sigma_x + \operatorname{tot}\Lambda; \quad \operatorname{tot} B|y = \operatorname{imaj}\Sigma_y + \operatorname{tot}\Theta;$

(iv) $\operatorname{tot} b|x = \operatorname{imaj}\sigma_x + \operatorname{tot}\lambda; \quad \operatorname{tot} b|y = \operatorname{imaj}\sigma_y + \operatorname{tot}\theta.$

Also the rises of w and the descents of $(\Sigma, \sigma, \varepsilon)$ coincide. In particular,

(v) $$\operatorname{rise} w = \operatorname{ddes}(\Sigma, \sigma, \varepsilon).$$

Finally, the mapping

$$\left(\binom{K\,M}{k\,m}, w\right) \mapsto \left(\binom{K'M'}{k'\,m'}, \Lambda, \Theta, \lambda, \theta, (\Sigma, \sigma, \varepsilon)\right)$$

is bijective.

With our running example, the correspondence between w and $(\Sigma, \sigma, \varepsilon)$ was shown in (22.25). Also $K = 8$; $M = 6$, $k = 9$, $m = 8$. The other parameters are (see the first calculations just before Proposition 22.4): $K' = K - Z_x(1) = 8 - 1 = 7$, $M' = M - Z_y(1) = 6 - 2 = 4$, $k' = k - z_x(1) = 9 - 3 = 6$, $m' = m - z_y(1) = 8 - 1 = 7$. Finally, $\Lambda = B|x - Z_x = \mathbf{6\,4\,4\,3\,3}$, $\Theta = 4\,3\,1\,0$, $\lambda = \mathbf{6\,6\,3\,0\,0}$, $\theta = 5\,3\,0\,0$.

The proof of Proposition 22.6 follows the pattern derived earlier for the MacMahon *Verfahren*.

The final calculation uses all those items, as well as identity (22.3):

$$\begin{aligned}
&\sum_{\binom{K\,M}{k\,m}, w} U^K V^M u^k v^m t^{\operatorname{rise} w} \Phi(w)\\
&= \sum_{(\Sigma,\sigma,\varepsilon)} \Psi(\Sigma, \sigma, \varepsilon) \sum_{\binom{K'M'}{k'\,m'}} U^{K'} V^{M'} u^{k'} v^{m'} \sum_{\binom{\Lambda\Theta}{\lambda\,\theta}} P^{\operatorname{tot}\Lambda} Q^{\operatorname{tot}\Theta} p^{\operatorname{tot}\lambda} q^{\operatorname{tot}\theta}\\
&= \sum_{\alpha\geq 0, \beta\geq 0} \sum_{\substack{\ell(\varepsilon|x)=\alpha\\ \ell(\varepsilon|y)=\beta}} \sum_{(\Sigma,\sigma,\varepsilon)} \Psi(\Sigma, \sigma, \varepsilon) e_P^\alpha(U)\, e_Q^\beta(V)\, e_p^\alpha(u)\, e_q^\beta(v)\\
&= \sum_{\alpha\geq 0, \beta\geq 0} e_P^\alpha(U)\, e_Q^\beta(V)\, e_p^\alpha(u)\, e_q^\beta(v) \mathbf{B}_{\alpha,\beta}. \quad \square
\end{aligned}$$

22.4. *The last specializations.* With $X = 0$ in the identity of Theorem 22.5 we get:

$$
\begin{aligned}
(22.27)\quad &\sum_{\binom{M}{m}} V^M v^m \frac{1-t}{-t + \mathbf{J}_m^M((1-t)Y; Q, q)}\\
&= \sum_{\beta\ge 0} e_Q^\beta(V)\, e_q^\beta(v) Y^\beta \sum_{(\Sigma,\sigma)} t^{\operatorname{ddes}(\Sigma,\sigma)} V^{\operatorname{ides}\Sigma} Q^{\operatorname{imaj}\Sigma} v^{\operatorname{irise}\sigma} q^{\operatorname{irmaj}\sigma}.
\end{aligned}
$$

The sum on the right-hand side is over all pairs of permutations (Σ, σ). This can be regarded as the specialization sitting on vertex $(\cdot, q, J_k)$ in the diagram of Fig. 20.1.

When the parameters K, M, k, m are not finite, the identity of Theorem 22.5 is derived as follows: multiply both sides successively by $(1-U)$ and let $U = 1$, multiply by $(1-V)$ and let $V = 1$, multiply by $(1-u)$ and let $u = 1$, multiply by $(1-v)$ and let $v = 1$. We obtain the identity:

$$
\begin{aligned}
(22.28)\quad &\frac{(1-t)\,\mathbf{J}((1-t)X; P, p)}{-t + \mathbf{J}((1-t)X; P, p)\,\mathbf{J}((1-t)Y; Q, q)}\\
&= \sum_{\alpha\ge 0,\,\beta\ge 0} \frac{1}{(P;P)_\alpha}\frac{1}{(Q;Q)_\beta}\frac{1}{(p;p)_\alpha}\frac{1}{(q;q)_\beta} X^\alpha Y^\beta\\
&\quad\times \sum_{\substack{(\Sigma,\sigma,\varepsilon)\ (\text{comp.})\\ \ell(\varepsilon|x)=\alpha,\ \ell(\varepsilon|y)=\beta}} t^{\operatorname{ddes}(\Sigma,\sigma,\varepsilon)} P^{\operatorname{imaj}_x\Sigma} Q^{\operatorname{imaj}_y\Sigma} p^{\operatorname{irmaj}_x\sigma} q^{\operatorname{irmaj}_y\sigma}.
\end{aligned}
$$

The latter underlying generating function is over the *compatible* signed bipermutations only, while identity (21.6), that corresponds to vertex $(\operatorname{sgn}, q, J)$, is over *all* signed bipermutations and even multipermutations. However identity (22.8) involves *two* distinct Bessel functions $\mathbf{J}(\cdot; P, p)$, $\mathbf{J}(\cdot; Q, q)$, instead of only one. When we reduce the number of bases, say, when $(P, p) = (Q, q)$, we can still regain (21.6), as is now explained.

For each signed bipermutation (not necessarily compatible) $(\Sigma, \sigma, \varepsilon)$ let

$$
\operatorname{inv}(\sigma_{\varepsilon|y}, \sigma_{\varepsilon|x}) := \#\{(i,j) : \varepsilon(i) = y,\ \varepsilon(j) = x,\ \sigma(i) > \sigma(j)\},
$$

a quantity equal to zero when the bipermutation is compatible. Also let

$$
\begin{aligned}
\operatorname{imaj}(\Sigma,\varepsilon) &:= \operatorname{imaj}_x(\Sigma,\varepsilon) + \operatorname{imaj}_y(\Sigma,\varepsilon) + \operatorname{inv}(\sigma_{\varepsilon|y}, \sigma_{\varepsilon|x})\\
\operatorname{irmaj}(\sigma,\varepsilon) &:= \operatorname{irmaj}_x(\sigma,\varepsilon) + \operatorname{irmaj}_y(\sigma,\varepsilon) + \operatorname{inv}(\sigma_{\varepsilon|y}, \sigma_{\varepsilon|x}).
\end{aligned}
$$

Now in (22.8) let $(P,p)=(Q,q)$; the right-hand side can then be rewritten as:

$$\sum_{n\ge 0}\frac{1}{(Q;Q)_n(q;q)_n}\sum_{\alpha+\beta=n}\begin{bmatrix}n\\ \alpha\end{bmatrix}_Q\begin{bmatrix}n\\ \alpha\end{bmatrix}_q X^\alpha Y^\beta \\ \times \sum_{\substack{(\Sigma,\sigma,\varepsilon)\ \text{(comp.)}\\ \ell(\varepsilon|x)=\alpha,\,\ell(\varepsilon|y)=\beta}} t^{\mathrm{ddes}(\Sigma,\sigma,\varepsilon)}Q^{\mathrm{imaj}(\Sigma,\varepsilon)}q^{\mathrm{irmaj}(\sigma,\varepsilon)}.$$

Because of the definition of "imaj" and "irmaj" for a signed permutation the polynomial $B'_n(X,Y,t,Q,q)$ defined by

$$(22.29)\quad B'_n(X,Y,t,Q,q):=\sum_{(\Sigma,\sigma,\varepsilon)}X^{\ell(\varepsilon|x)}Y^{\ell(\varepsilon|y)}t^{\mathrm{ddes}(\Sigma,\sigma,\varepsilon)}Q^{\mathrm{imaj}(\Sigma,\varepsilon)}q^{\mathrm{irmaj}(\sigma,\varepsilon)},$$

the sum being over all signed bipermutations of length n, is also equal to

$$\sum_{\alpha+\beta=n}\begin{bmatrix}n\\ \alpha\end{bmatrix}_Q\begin{bmatrix}n\\ \alpha\end{bmatrix}_q X^\alpha Y^\beta \sum_{\substack{(\Sigma,\sigma,\varepsilon)\ \text{(comp.)}\\ \ell(\varepsilon|x)=\alpha,\,\ell(\varepsilon|y)=\beta}} t^{\mathrm{ddes}(\Sigma,\sigma,\varepsilon)}Q^{\mathrm{imaj}\,\Sigma}q^{\mathrm{irmaj}\,\sigma}.$$

Hence, when $(P,p)=(Q,q)$ identity (22.28) takes the form

$$(22.30)\quad \frac{(1-t)\,\mathbf{J}((1-t)X;Q,q)}{-t+\mathbf{J}((1-t)X;Q,q)\,\mathbf{J}((1-t)Y;Q,q)} \\ =\sum_{n\ge 0}\frac{1}{(Q;Q)_n(q;q)_n}B'_n(X,Y,t,Q,q).$$

The comparison with (21.6) shows that the right-hand sides of (21.6) and (22.30) involve different statistics: $\mathrm{inv}(\Sigma,\varepsilon),\mathrm{coinv}(\sigma,\varepsilon)$ for (21.6) (when reduced to bipermutations) and $\mathrm{imaj}(\Sigma,\varepsilon),\mathrm{irmaj}(\sigma,\varepsilon)$ for (22.30). However, those two formulas imply the identity

$$(22.31)\qquad B_n(X,Y,t,Q,q)=B'_n(X,Y,t,Q,q).$$

In order to show that Theorem 22.5 that contains the most general extension sitting on (sgn,q,J_k) implies (21.6) sitting on (sgn,q,J) we prove identity (22.31) *combinatorially* by means of a direct bijection.

A bijection for $(\mathrm{inv},\mathrm{coinv})$ *and* $(\mathrm{imaj},\mathrm{irmaj})$. The main ingredient is the bijection Φ of Theorem 11.2 that is used in the following context. Let π be a bijection of a finite set $I=\{i_1<i_2<\cdots<i_k\}$, of integers onto a finite set of integers $\pi(I):=\{j_1<j_2<\cdots<j_k\}$. The

permutation π is presented as a two-row matrix $\begin{pmatrix} i_1 & \ldots & i_k \\ \pi(i_1) & \ldots & \pi(i_k) \end{pmatrix}$. Let $\mathbf{i}$ denote the correspondence that sends each bijection onto its inverse and consider the chain

$$\begin{pmatrix} i_1 & \ldots & i_k \\ \pi(i_1) & \ldots & \pi(i_k) \end{pmatrix} \xrightarrow{\mathbf{i}} \begin{pmatrix} j_1 & \ldots & j_k \\ \pi^{-1}(j_1) & \ldots & \pi^{-1}(j_k) \end{pmatrix}$$
$$\xrightarrow{\Phi} \begin{pmatrix} j_1 \ldots j_k \\ y_1 \ldots y_k \end{pmatrix} \xrightarrow{\mathbf{i}} \begin{pmatrix} i_1 & \ldots & i_k \\ \pi'(i_1) & \ldots & \pi'(i_k) \end{pmatrix},$$

that is rewritten:

$$\pi \xrightarrow{\mathbf{i}} \mathbf{i}\,\pi \xrightarrow{\Phi} \Phi\mathbf{i}\,\pi \xrightarrow{\mathbf{i}} \mathbf{i}\Phi\mathbf{i}\,\pi.$$

As proved in Theorem 11.2, the transformation Φ sends the *word* $\pi^{-1}(j_1)\ldots\pi^{-1}(j_k)$ onto a *rearrangement* $y_1\ldots y_k$ of that word, which is the bottom row of the third matrix above. Moreover,

$$\mathrm{maj}\big(\pi^{-1}(j_1)\ldots\pi^{-1}(j_k)\big) = \mathrm{inv}\big(y_1\ldots y_k\big),$$

which implies

$$\mathrm{maj}\,\mathbf{i}\,\pi = \mathrm{inv}\,\mathbf{i}\Phi\mathbf{i}\,\pi \quad \text{and} \quad \mathrm{rmaj}\,\mathbf{i}\,\pi = \mathrm{coinv}\,\mathbf{i}\Phi\mathbf{i}\,\pi.$$

Also $\mathrm{Iligne}\,\mathbf{i}\,\pi = \mathrm{Iligne}\,\Phi\mathbf{i}\,\pi$. Hence,

$$\text{(22.32).} \qquad \mathrm{Ligne}\,\pi = \mathrm{Ligne}\,\mathbf{i}\Phi\mathbf{i}\,\pi.$$

Now each signed bipermutation $(\Sigma,\sigma,\varepsilon)$ is characterized by the two subsets $I_{\varepsilon|x}$, $I_{\varepsilon|y}$ and its restrictions Σ_x, Σ_y, σ_x, σ_y to those two subsets, as they were defined in § 22.2. This gives a sense to the notation

$$[I_{\varepsilon|x}, I_{\varepsilon|y}, \Sigma_x, \Sigma_y, \sigma_x, \sigma_y]$$

to designate a signed bipermutation $(\Sigma,\sigma,\varepsilon)$.

Next form the composition product $\mathbf{i}\Phi\mathbf{i}$ described above and consider the mapping $\Delta : (\Sigma,\sigma,\varepsilon) \mapsto (\Sigma',\sigma',\varepsilon)$ defined by

$$(\Sigma,\sigma,\varepsilon) := [I_{\varepsilon|x}, I_{\varepsilon|y}, \Sigma_x, \Sigma_y, \sigma_x, \sigma_y];$$
$$(\Sigma',\sigma',\varepsilon) := [I_{\varepsilon|x}, I_{\varepsilon|y}, \mathbf{i}\Phi\mathbf{i}\,\Sigma_x, \mathbf{i}\Phi\mathbf{i}\,\Sigma_y, \mathbf{i}\Phi\mathbf{i}\,\sigma_x, \mathbf{i}\Phi\mathbf{i}\,\sigma_y].$$

Proposition 22.6. *The mapping Δ is a bijection of the set of signed bipermutations of length n onto itself, having the following properties:*

(i) $\mathrm{ddes}(\Sigma,\sigma,\varepsilon) = \mathrm{ddes}(\Sigma',\sigma',\varepsilon)$;

(ii) $\operatorname{imaj}(\Sigma,\varepsilon)=\operatorname{inv}(\Sigma',\varepsilon)$ *and* $\operatorname{irmaj}(\sigma,\varepsilon)=\operatorname{coinv}(\sigma',\varepsilon)$.

Proof. Property (i) follows from (22.32), since $\operatorname{Ligne}\sigma_x=\operatorname{Ligne}\mathbf{i}\Phi\mathbf{i}\,\sigma_x$ and analogous relations for the other restrictions. Finally,

$$\operatorname{inv}(\Sigma_{\varepsilon|y},\Sigma_{\varepsilon|x})=\operatorname{inv}(\Sigma'_{\varepsilon|y},\Sigma'_{\varepsilon|x})\ \text{and}\ \operatorname{inv}(\sigma_{\varepsilon|y},\sigma_{\varepsilon|x})=\operatorname{inv}(\sigma'_{\varepsilon|y},\sigma'_{\varepsilon|x}).$$

Then,

$$\begin{aligned}\operatorname{imaj}(\Sigma,\varepsilon)&=\operatorname{maj}\Sigma^{-1}_{\varepsilon|x}+\operatorname{maj}\Sigma^{-1}_{\varepsilon|y}+\operatorname{inv}(\Sigma_{\varepsilon|y},\Sigma_{\varepsilon|x})\\&=\operatorname{inv}\Sigma'_{\varepsilon|x}+\operatorname{inv}\Sigma'_{\varepsilon|y}+\operatorname{inv}(\Sigma'_{\varepsilon|y},\Sigma'_{\varepsilon|x})=\operatorname{inv}(\Sigma',\varepsilon)\,;\\ \operatorname{irmaj}(\sigma,\varepsilon)&=\operatorname{rmaj}\sigma^{-1}_{\varepsilon|x}+\operatorname{rmaj}\sigma^{-1}_{\varepsilon|y}+\operatorname{inv}(\sigma_{\varepsilon|y},\sigma_I)\\&=\operatorname{coinv}\sigma'_{\varepsilon|x}+\operatorname{coinv}\sigma'_{\varepsilon|y}+\operatorname{inv}(\sigma'_{\varepsilon|y},\sigma'_{\varepsilon|x})=\operatorname{coinv}(\sigma,\varepsilon).\end{aligned}$$ □

23. Eulerian Calculus; multi-indexed polynomials

As seen in section 10, the classical Eulerian polynomials $A_n(t)=\sum_{\sigma\in\mathfrak{S}_n}t^{\operatorname{des}\sigma}$ $(n\ge 0)$ have the following exponential generating function

$$1+\sum_{n\ge1}A_n(t)\frac{u^n}{n!}=\frac{1-t}{-t+\exp(u(t-1))},\tag{23.1}$$

so that

$$1+\sum_{n\ge1}t\,A_n(t)\frac{u^n}{n!}=\frac{1-t}{1-t\,\exp(u(1-t))}\tag{23.2}$$

provides the exponential generating function for the polynomials ${}^0\!A_n(t):=t\,A_n(t)=\sum_{\sigma\in\mathfrak{S}_n}t^{1+\operatorname{des}\sigma}$ $(n\ge1)$ and ${}^0\!A_0(t):=1$. In the previous three sections we have worked out various extensions of the rational function occurring in (23.1). In the present one we will rather use the second one, because the major results dealing with multi-indexed Eulerian polynomials have been introduced in that way and have become classical. The change, however, is minor.

23.1. *The bi-indexed Eulerian polynomials.* In the fraction occurring in (23.2) replace $\exp(u(1-t))$ by the product $e_p((1-t)u)\,e_q((1-t)v)$ where u, v, p, q are independent variables. Then expand the fraction as a series normalized by products of the form $(p;p)_n\,(q;q)_m$. We obtain:

$$\sum_{\substack{n\ge0,\\ m\ge0}}\frac{u^nv^m}{(p;p)_n\,(q;q)_m}A_{n,m}(t;p,q)=\frac{1-t}{1-te_p((1-t)u)\,e_q((1-t)v)}.\tag{23.3}$$

To see that $A_{n,m}(t;p,q)$ is a *polynomial* with *integral* coefficients we may rewrite (23.3) as $L = (1-R)^{-1}$. The identity $L(1-R) = 1$ yields the recurrence relation

$$A_{n,m}(t;p,q) = \sum_{i,j} \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} m \\ j \end{bmatrix}_q A_{i,j}(t;p,q)\, t(1-t)^{n-i+m-j-1}, \tag{23.4}$$

where $0 \le i \le n$, $0 \le j \le m$ and $(i,j) \ne (n,m)$. As $A_{0,0}(t;p,q) = 1$, relation (23.4) shows by induction that each $A_{n,m}(t;p,q)$ is a polynomial with integral coefficients. To show that the coefficients are indeed positive and of sum $(n+m)!$ requires more analysis and will be a consequence of the theorem below.

Of course, a product of more than two q-exponentials could have been taken and the results below would have been very similar. The advantage of keeping only two allows a much easier reading.

The second class of polynomials under study will be denoted by $A_{n,m}(t;p)$. There is no risk of confusion with the previous ones, as they are in two variables t and p instead of three. No confusion either with the q-Eulerian polynomials $A_n(t,q)$, as the present ones are double indexed. Those polynomials $A_{n,m}(t;p)$ are defined by

$$\sum_{\substack{n\ge 0,\\ m\ge 0}} \frac{u^n v^m}{(p;p)_n\, m!} A_{n,m}(t;p) = \frac{1-t}{1-te_p((1-t)u)\,\exp((1-t)v)}. \tag{23.5}$$

Let $q=1$ in (23.4). Then $A_{n,m}(t;p,q)$ is transformed into a polynomial in two variables t and p. Furthermore, the Gaussian polynomial $\left[{m \atop j}\right]_q$ becomes the ordinary binomial coefficient $\binom{m}{j}$. Hence (23.3) itself is transformed into identity (23.5). Thus

$$A_{n,m}(t;p) = A_{n,m}(t;p,1). \tag{23.6}$$

Notice that when $u=0$ (resp. $v=0$) identity (23.5) specializes into the exponential (resp. basic) generating function for the Eulerian polynomials ${}^0\!A_n(t)$ introduced above (resp. for the polynomials ${}^{\mathrm{inv}}\!A_n(t,q)$).

The ligne and inverse ligne of route "Ligne" and "Iligne," that have been defined in the previous sections for permutations, are being partitioned into two parts. Let (n,m) be an ordered pair of nonnegative integers and σ be a permutation of order $n+m$. Define:

$\mathrm{Iligne}_1\,\sigma := \{r : 1 \le r \le n-1,\ \sigma^{-1}(r) > \sigma^{-1}(r+1)\}$,
$\mathrm{Iligne}_2\,\sigma := \{r : n+1 \le r \le n+m-1,\ \sigma^{-1}(r) > \sigma^{-1}(r+1)\}$,

$$\begin{aligned}
\operatorname{imaj}_1 \sigma &:= \sum\{r : r \in \operatorname{Iligne}_1 \sigma\},\\
\operatorname{imaj}_2 \sigma &:= \sum\{r-n : r \in \operatorname{Iligne}_2 \sigma\},\\
\operatorname{inv}_1 \sigma &:= \#\{(r,r') : 1 \le r < r' \le n,\ \sigma^{-1}(r) > \sigma^{-1}(r')\},\\
\operatorname{inv}_2 \sigma &:= \#\{(r,r') : n+1 \le r < r' \le n+m,\ \sigma^{-1}(r) > \sigma^{-1}(r')\}.
\end{aligned}$$

When $m=0$ (resp. $n=0$), the statistics $\operatorname{imaj}_1 \sigma$ and $\operatorname{inv}_1 \sigma$ (resp. $\operatorname{imaj}_2 \sigma$ and $\operatorname{inv}_2 \sigma$) are the familiar inverse major index and inversion number of the permutation σ, respectively.

Theorem 23.1. *Let n, m be two nonnegative integers. The generating polynomial for the group $\mathfrak{S}_{n+m}$ of the permutations of order $n+m$ by the three-variable statistic $(1+\operatorname{des}, \operatorname{imaj}_1, \operatorname{imaj}_2)$ is equal to $A_{n,m}(t;p,q)$. In other words, if $A_{n,m}(t;p,q)$ is defined by identity (23.3), then*

$$A_{n,m}(t;p,q) = \sum_{\sigma\in\mathfrak{S}_{n+m}} t^{1+\operatorname{des}\sigma} p^{\operatorname{imaj}_1 \sigma} q^{\operatorname{imaj}_2 \sigma}. \tag{23.7}$$

We also have:

$$A_{n,m}(t;p,q) = \sum_{\sigma\in\mathfrak{S}_{n+m}} t^{1+\operatorname{des}\sigma} p^{\operatorname{inv}_1 \sigma} q^{\operatorname{inv}_2 \sigma}. \tag{23.8}$$

For the proof of (23.7) we use the Schur function algebra and the Robinson-Schensted correspondence, as was already done for the proof of Theorem 19.4. Take up again the previous notations with a pair (n,m) of integers and σ designating a permutation of order $n+m$. Let τ_1 (resp. τ_2) be the restriction of σ to the set $\sigma^{-1}([1,n])$ (resp. $\sigma^{-1}([n+1,n+m])$). Using the Robinson-Schensted correspondence (see § Corollary 19.2) each bijection τ_j $(j=1,2)$ is mapped onto a pair (P_j,Q_j) of Young tableaux of the same *shape* that we shall denote by λ_j.

It follows from the properties of the correspondence that the entries of P_1 (resp. P_2) are the elements of the interval $[1,n]$ (resp. $[n+1,n+m]$) and the entries of Q_1 (resp. Q_2) are those of the set $\sigma^{-1}([1,n])$ (resp. $\sigma^{-1}([n+1,n+m])$). Let $T_1 := P_1$ and T_2 be the Young tableau obtained from P_2 by replacing each entry r by $r-n$ and let U be the product $U = Q_1 \otimes Q_2$. This means that U is the skew tableau obtained by placing Q_2 *to the right of* Q_1 and just *under* it. Thus T_1 (resp. T_2) is a Young tableau of shape λ_1 (resp. λ_2) whose entries are $1,2,\ldots,n$ (resp. $1,2,\ldots,m$); the entries of U are $1,2,\ldots,n$ and the shape of U is the skew shape $\lambda_1\otimes\lambda_2$.

We summarize all this by writing :

$$\begin{aligned}
&\operatorname{shape} T_j = \lambda_j\ (j=1,2);\quad |\lambda_1| = n;\quad |\lambda_2| = m;\\
&\operatorname{shape} U = \lambda_1\otimes\lambda_2\,;\quad |\lambda_1|+|\lambda_2| = n+m.
\end{aligned} \tag{23.9}$$

For instance, let $(n,m)=(5,4)$ and

$$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\ 8&6&4&5&2&1&9&7&3\end{pmatrix};\quad \sigma^{-1}=\left(\begin{array}{ccccc|cccc}1&2&3&4&5&6&7&8&9\\ 6&5&9&3&4&2&8&1&7\end{array}\right).$$

Then

$$\tau_1=\begin{pmatrix}3&4&5&6&9\\ 4&5&2&1&3\end{pmatrix};\quad \tau_2=\begin{pmatrix}1&2&7&8\\ 8&6&9&7\end{pmatrix}.$$

Under the Robinson-Schensted correspondence

$$\tau_1\mapsto(P_1,Q_1)=\left(\begin{array}{lcl}4&&6\\ 2\,5&&5\,9\\ 1\,3&,&3\,4\end{array}\right);\quad \lambda_1=(2,2,1)\,;$$
$$\tau_2\mapsto(P_2,Q_2)=\left(\begin{array}{lcl}8\,9&&2\,8\\ 6\,7&,&1\,7\end{array}\right);\quad \lambda_2=(2,2)\,;$$

and

$$T_1=P_1=\begin{array}{l}4\\ 2\,5\\ 1\,3\end{array}\;;\quad T_2=P_2-n=\begin{array}{l}3\,4\\ 1\,2\end{array}\;;\quad U=Q_1\otimes Q_2=\begin{array}{ll}6&\\ 5\,9&\\ 3\,4&\\ &2\,8\\ &1\,7\end{array}\;.$$

The inverse ligne of route (see section 18) of U is $\operatorname{Iligne}U=\{1,2,4,5,7,8\}$, so that $\operatorname{imaj}U=27$ and $\operatorname{ides}U=6$.

It follows from Theorem 19.3 that the mapping

$$(\sigma,n,m)\mapsto(\lambda_1,\lambda_2;T_1,T_2;U)$$

is a bijection having properties (23.9) and also

$$\begin{gathered}\operatorname{Iligne}_1\sigma=\operatorname{Iligne}T_1\quad \operatorname{Iligne}_2\sigma-n=\operatorname{Iligne}T_2;\\ \operatorname{Ligne}\sigma=\operatorname{Iligne}U.\end{gathered}\tag{23.10}$$

With our working example $\operatorname{Iligne}_1\sigma=\{1,3\}=\operatorname{Iligne}T_1$; $\operatorname{Iligne}_2\sigma=\{2\}=\operatorname{Iligne}T_2$; $\operatorname{Ligne}\sigma=\{1,2,4,5,7,8\}=\operatorname{Iligne}U$.

It follows from (23.10) that

$$\operatorname{imaj}_j\sigma=\operatorname{imaj}T_j\quad(j=1,2);\qquad \operatorname{des}\sigma=\operatorname{ides}U.\tag{23.11}$$

Let $t\,{}^1\!A_n(t;p,q)$ denote the right-hand side of (23.7). Then by (23.10) and (23.11)

$$ {}^1\!A_n(t;p,q)=\sum_{(\lambda_1,\lambda_2)}\sum_{(T_1,T_2,U)} t^{\mathrm{ides}\,U}p^{\mathrm{imaj}\,T_1}q^{\mathrm{imaj}\,T_2}, \tag{23.12}$$

where the first sum is over the ordered pairs (λ_1,λ_2) of partitions satisfying $|\lambda_1|=n$, $|\lambda_2|=m$, and the second over all triples (T_1,T_2,U) satisfying (23.9).

We next use identity (18.7) as well as a specialization of (18.8) obtained by letting $q=1$, i.e.,

$$\sum_{r\ge 0} t^r s_{\nu/\theta}(1^{r+1})=\frac{1}{(1-t)^{n+1}}\sum_{T,\,T\ \text{of shape}\ \nu/\theta} t^{\mathrm{ides}\,T}, \tag{23.13}$$

where $s_{\nu/\theta}(1^{r+1})$ is the skew Schur function obtained by taking an alphabet of $(r+1)$ letters all equal to 1. We have :

$$\begin{aligned}
&\frac{{}^1\!A_n(t;p,q)}{(1-t)^{n+m+1}(p;p)_n(q;q)_m}\\
&\quad=\frac{1}{(1-t)^{n+m+1}(p;p)_n(q;q)_m}\sum_{(T_1,T_2,U)} t^{\mathrm{ides}\,U}p^{\mathrm{imaj}\,T_1}q^{\mathrm{imaj}\,T_2}\\
&\quad=\sum_{(\lambda_1,\lambda_2)}\sum_r t^r s_{\lambda_1\otimes\lambda_2}(1^{r+1})s_{\lambda_1}(1,p,p^2,\dots)s_{\lambda_2}(1,q,q^2,\dots)\\
&\quad=\sum_{(\lambda_1,\lambda_2)}\sum_r t^r s_{\lambda_1}(1^{r+1})s_{\lambda_1}(1,p,p^2,\dots)s_{\lambda_2}(1^{r+1})s_{\lambda_2}(1,q,q^2,\dots),
\end{aligned}$$

as U is of shape $\lambda_1\otimes\lambda_2$. In the last step we have used the fondamental multiplicative property of the Schur functions : $s_{\lambda\otimes\mu}(x)=s_\lambda(x)s_\mu(x)$. Now, the Cauchy identity (16.4) for Schur functions yields

$$\begin{aligned}
\sum_{\lambda_1} u^{|\lambda_1|}s_{\lambda_1}(1^{r+1})s_{\lambda_1}(1,p,p^2,\dots)&=\prod_{d\ge 1}\frac{1}{(1-up^{d-1})^{r+1}}=\frac{1}{(u;p)_\infty{}^{r+1}};\\
\sum_{\lambda_2} v^{|\lambda_2|}s_{\lambda_2}(1^{r+1})s_{\lambda_2}(1,q,q^2,\dots)&=\prod_{d\ge 1}\frac{1}{(1-vq^{d-1})^{r+1}}=\frac{1}{(v;q)_\infty{}^{r+1}}.
\end{aligned}$$

As $|\lambda_1|=m$, $|\lambda_2|=n$, we may write

$$\sum_{n,m}\frac{u^nv^m}{(1-t)^{n+m+1}(p;p)_n(q;q)_m}\,{}^1\!A_n(t;p,q)$$

$$
\begin{aligned}
&= \sum_{n,m} u^n v^m \sum_{(\lambda_1,\lambda_2)} \sum_r t^r s_{\lambda_1}(1^{r+1}) s_{\lambda_1}(1,p,\dots) s_{\lambda_2}(1^{r+1}) s_{\lambda_2}(1,q,\dots) \\
&= \sum_r t^r \sum_{(\lambda_1,\lambda_2)} u^{|\lambda_1|} v^{|\lambda_2|} s_{\lambda_1}(1^{r+1}) s_{\lambda_1}(1,p,\dots) s_{\lambda_2}(1^{r+1}) s_{\lambda_2}(1,q,\dots) \\
&= \sum_r t^r \Bigl(\sum_{\lambda_1} u^{|\lambda_1|} s_{\lambda_1}(1^{r+1}) s_{\lambda_1}(1,p,\dots)\Bigr) \\
&\qquad\qquad \times \Bigl(\sum_{\lambda_2} v^{|\lambda_2|} s_{\lambda_2}(1^{r+1}) s_{\lambda_2}(1,q,\dots)\Bigr) \\
&= \sum_r t^r \frac{1}{{(u;p)_\infty}^{r+1}} \frac{1}{{(v;q)_\infty}^{r+1}} = \frac{1}{-t + (u;p)_\infty (v;q)_\infty}.
\end{aligned}
$$

Next replace $u/(1-t)$ by u and $v/(1-t)$ by v. This yields

$$
\begin{aligned}
\sum_{n,m} \frac{u^n v^m}{(p;q)_n (q;q)_m}\, {}^1\!A_n(t;p,q) &= \frac{1-t}{-t + ((1-t)u;p)_\infty ((1-t)v;q)_\infty} \\
&= \frac{1-t}{-t + E_p((t-1)u) E_q((t-1)v)},
\end{aligned}
$$

using the notation of the second q-exponential. Since $E_p(u)e_p(-u) = 1$, we obtain:

$$
\begin{aligned}
1 + \sum_{\substack{(n,m)\\ \neq(0,0)}} \frac{u^n v^m}{(p;q)_n (q;q)_m} A_n(t;p,q) &= t \frac{1-t}{-t + E_p((t-1)u) E_q((t-1)v)} - t + 1 \\
&= \frac{1-t}{1 - t e_p((1-t)u) e_q((1-t)v)}.
\end{aligned}
$$

This proves identity (23.7).

There are two methods for proving identity (23.8): first, the *iterative method* that has already been used for the proofs of Theorems 10.1, 20.3 and 21.1. The recurrence relation (23.4) is to be considered and the summand to be combinatorially interpreted by introducing the notion of the longest increasing rightmost factor. As one of the previous proofs could be reproduced almost verbatim, the proof is omitted. The second proof consists of constructing a bijection $\sigma \mapsto \sigma'$ of $\mathfrak{S}_{n+m}$ onto itself having the property that

$$
\operatorname{Ligne} \sigma = \operatorname{Ligne} \sigma', \quad \operatorname{imaj}_j \sigma = \operatorname{inv}_j \sigma' \ (j = 1, 2).
$$

Again, the bijection Φ of Theorem 11.2 could also be used in the same way as was done in Proposition 22.6. As the similarity is so strong, the construction is omitted. $\square$

23.2. *The Désarménien Verfahren.* This method, based on the algebra of symmetric functions, is quite effective for deriving congruence properties of certain polynomials, especially the Eulerian polynomials and also the bi-indexed Eulerian polynomials introduced in the previous subsection.

Take up again the notations introduced in § 14.3, especially formulas (ii) and (ix) of Theorem 14.2, that merge into

$$H(u;x)=\prod_{j\ge 1}(1-ux_j)^{-1}=\exp\sum_{r\ge 1}u^r\,\frac{\mathbf{p}_r(x)}{r}, \tag{23.14}$$

where the "p" has been written in boldface as "$\mathbf{p}$" to avoid confusion with a further base denoted by "p." Also recall that a partition of an integer n is a non-increasing sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ or a word $\lambda=1^{m_1}2^{m_2}\dots$ (the multiplicative notation) with the meaning that λ has m_1 parts λ_i equal to 1, m_2 parts λ_i equal to 2, etc. As usual, to each partition λ is attached the constant

$$z_\lambda=1^{m_1}2^{m_2}\dots m_1!\,m_2!\,\dots$$

and the *power symmetric function*

$$\mathbf{p}_\lambda(x)=\mathbf{p}_{\lambda_1}(x)\mathbf{p}_{\lambda_2}(x)\cdots=\mathbf{p}_1(x)^{m_1}\mathbf{p}_2(x)^{m_2}\cdots$$

Finally, $|\lambda|=n$ means that λ is a partition of n and the notation $l(\lambda)$ stands for the number of parts of λ.

Now for each partition $\lambda=1^{m_1}2^{m_2}\dots$ of the integer n introduce the polynomial:

$$T_\lambda(q):=\frac{(q;q)_n}{\prod_j(1-q^j)^{m_j}}=(q;q)_n\,\mathbf{p}_\lambda(1,q,q^2,\dots). \tag{23.15}$$

By induction on the number of *distinct* parts of λ it is easily verified that $T_\lambda(q)$ is a polynomial of degree $n(n-1)/2$. In the following lemma needed for deriving congruences properties for the polynomials $T_\lambda(q)$ we denote the k-th cyclotomic polynomial by $\Phi_k(q)$ with the convention that $\Phi_1(q)=1-q$.

Lemma 23.2. *Let $n=ka+b$ $(0\le b\le k-1)$. Then*

$$\frac{(q;q)_n}{(1-q^k)^a}\equiv k^a a!\,(q;q)_b \bmod \Phi_k(q).$$

Proof. Write $(q;q)_n=\Bigl(\prod_{0\le j\le a-1}(q^{kj+1};q)_{k-1}\Bigr)\,(q^k;q^k)_a\,(q^{ka+1};q)_b$. As $q^k{\equiv}1 \bmod \Phi_k(q)$, we have $(q^{ka+1};q)_b{\equiv}(q;q)_b$ and $(q^{kj+1};q)_{k-1}{\equiv}(q;q)_{k-1}$.

When $q = \zeta$ is a k-th primitive root of unity, $(\zeta;\zeta)_{k-1} = \prod_{1\le l\le k-1}(1-\zeta^l)$, which is the value in $X = 1$ of the polynomial $(1-X^k)/(1-X) = 1 + X + \cdots + X^{k-1}$, that is, k. Therefore, $(q;q)_{k-1} \equiv k \bmod \Phi_k(q)$. Moreover, $(q^k;q^k)_a$ is divisible by $(1-q^k)^a$ and $\lim_{q\to\zeta} \dfrac{(q;q)_a}{(1-q^a)} = a!$ ∎

Proposition 23.3. *Let $n = ka + b$, $0 \le b \le k-1$ and $\lambda = 1^{m_1}2^{m_2}\ldots$ be a partition of n. Then the following congruences hold :*

(i) *if $m_k \ne a$, then $T_\lambda(q) \equiv 0 \bmod \Phi_k(q)$.*

(ii) *if $m_k = a$, let $\lambda^* = 1^{m_1}\ldots(k-1)^{m_{k-1}}(k+1)^{m_{k+1}}\ldots$ be the partition of b derived from λ by deleting the $m_k = a$ parts equal to k. Then*

$$T_\lambda(q) \equiv k^a a!\, T_{\lambda^*}(q) \bmod \Phi_k(q). \tag{23.16}$$

Proof. The proposition is an immediate consequence of the previous lemma and the definition of $T_\lambda(q)$ given in (23.15). ∎

Let $F(u)$ be a formal power series in one variable u. As the two series $e_q(u)e^v - 1$ and $e_q(u)e_p(v) - 1$ have no constant term, it makes sense to expand the following two series as

$$F\big(e_q(u)e^v - 1\big) = \sum_{n\ge0,\, m\ge0} \frac{u^n}{(q;q)_n}\frac{v^m}{m!}K_{n,m}(q); \tag{23.17}$$

$$F\big(e_q(u)e_p(v) - 1\big) = \sum_{n\ge0,\, m\ge0} \frac{u^n}{(q;q)_n}\frac{v^m}{(p;p)_m}K_{n,m}(p,q). \tag{23.18}$$

Theorem 23.4. *Let $n = ka + b$ with $0 \le b \le k-1$ and $m \ge 0$. Then*

$$K_{ka+b,m}(q) \equiv K_{b,m+a}(q) \bmod \Phi_k(q). \tag{23.19}$$

Furthermore, if $0 \le m \le k-1$, then

$$K_{ka+b,m}(q,p) \equiv K_{b,m+ka}(q,p) \bmod \Phi_k(q) \bmod \Phi_k(p), \tag{23.20}$$

where the double congruence means that the two congruences are to be taken in succession.

Notice that the subscripts in the polynomials K are different in (23.19) and (23.20), so that the first congruence is not a specialization of the second one.

Proof. Consider the following expansion in the variables u and v

$$\Bigl(\sum_{r\ge1} u^r\,\frac{\mathbf{p}_r(x)}{r}+v\Bigr)^N=\sum_{m=0}^N\binom{N}{m}v^m\Bigl(\sum_{r\ge1}u^r\,\frac{\mathbf{p}_r(x)}{r}\Bigr)^{N-m}$$

$$=\sum_{m=0}^n\binom{N}{m}v^m\sum_{n\ge0}u^n\sum_{m_1,m_2,\ldots}\frac{(N-m)!}{1!^{m_1}2!^{m_2}\ldots}\,\frac{\mathbf{p}_1(x)^{m_1}}{1^{m_1}}\,\frac{\mathbf{p}_2(x)^{m_2}}{2^{m_2}}\cdots$$

$$[n+m\ge N,\ m_1+m_2+\cdots=N-m\,;\ 1.m_1+2.m_2+\cdots=n.]$$

$$=N!\sum_{0\le m\le N\le n+m}\frac{v^m}{m!}u^n\sum_{\substack{|\lambda|=n\\ l(\lambda)=N-m}}\frac{\mathbf{p}_\lambda(x)}{z_\lambda}.$$

Replacing the alphabet x by the successive powers of q yields

$$\Bigl(\sum_{r\ge1}u^r\,\frac{\mathbf{p}_r(1,q,q^2,\ldots)}{r}+v\Bigr)^N=\sum_{0\le m\le N\le n+m}\frac{u^n}{(q;q)_n}\,\frac{v^m}{m!}K_{n,m}^{(N)}(q),\tag{23.21}$$

where

$$K_{n,m}^{(N)}(q)=N!\sum_{\substack{|\lambda|=n\\ l(\lambda)=N-m}}\frac{T_\lambda(q)}{z_\lambda}.\tag{23.22}$$

When $m_k=a$, then $l(\lambda)=l(\lambda^*)+a$ and $z_\lambda=z_{\lambda^*}\,k^a a!$, so that

$$K_{n,m}^{(N)}(q)\equiv N!\sum_{\substack{|\lambda^*|=b\\ l(\lambda^*)=N-m-a}}\frac{T_{\lambda^*}(q)}{z_{\lambda^*}}\bmod \Phi_k(q).$$

In particular by (23.22)

$$K_{b,m+a}^{(N)}(q)=N!\sum_{\substack{|\lambda^*|=b\\ l(\lambda^*)=N-m-a}}\frac{T_{\lambda^*}(q)}{z_{\lambda^*}}.$$

Therefore

$$K_{ka+b,m}^{(N)}(q)\equiv K_{b,m+a}^{(N)}(q)\bmod \Phi_k(q).\tag{23.23}$$

Now, by (23.14),

$$e_q(u)e^v=\prod_{j\ge1}(1-uq^{j-1})^{-1}e^v=\exp\Bigl(\sum_{r\ge1}u^r\,\frac{\mathbf{p}_r(1,q,q^2,\ldots)}{r}+v\Bigr)\Bigr),$$

so that, if $F(u)=\sum_{i\ge0}c_iu^i$, we get

$$\begin{aligned}
F(e_q(u)e^v-1) &= \sum_{i\ge 0} c_i(e_q(u)e^v-1)^i\\
&= \sum_{i\ge 0} c_i\Big(\exp\Big(\sum_{r\ge 1} u^r\,\frac{\mathbf{p}_r(1,q,q^2,\ldots)}{r}+v\Big)-1\Big)^i\\
&= \sum_{N\ge 0}\frac{C_N}{N!}\Big(\sum_{r\ge 1} u^r\,\frac{\mathbf{p}_r(1,q,q^2,\ldots)}{r}+v\Big)^N\\
&\qquad\qquad\text{[for some coefficients } C_N\ (N\ge 0);]\\
&= \sum_{N\ge 0}\frac{C_N}{N!}\sum_{0\le m\le N\le n+m}\frac{u^n}{(q;q)_n}\frac{v^m}{m!}K^{(N)}_{n,m}(q)\\
&= \sum_{n\ge 0,\, m\ge 0}\frac{u^n}{(q;q)_n}\frac{v^m}{m!}K_{n,m}(q),
\end{aligned}$$

where $K_{n,m}(q)=\displaystyle\sum_{N=0}^{n+m}\frac{C_N}{N!}K^{(N)}_{n,m}(q)$, since the formal series (23.21) is of order N in the variables u and v. Hence, the congruence property for $K_{n,m}(q)$ is a consequence of the congruence property (23.23) satisfied by the polynomials $K^{(N)}_{n,m}(q)$ themselves.

The proof of (23.20) is quite similar. This time we start with the expansion of the series $\displaystyle\sum_{r\ge 1}\Big(\frac{u^r\mathbf{p}_r(x)+v^r\mathbf{p}_r(y)}{r}\Big)^N$, where y stands for a second alphabet. We have

$$\begin{aligned}
\Big(\sum_{r\ge 1}\frac{u^r\mathbf{p}_r(x)+v^r\mathbf{p}_r(y)}{r}\Big)^N &= \sum_{A+B=N}\binom{N}{A}\Big(\sum_{r\ge 1}u^r\,\frac{\mathbf{p}_r(x)}{r}\Big)^A\Big(\sum_{r\ge 1}v^r\,\frac{\mathbf{p}_r(y)}{r}\Big)^B\\
&= \sum_{A+B=N}^{n}\binom{N}{A}\Big(\sum_{l(\lambda)=A}A!\,\frac{u^{|\lambda|}\mathbf{p}_\lambda(x)}{z_\lambda}\Big)\Big(\sum_{l(\mu)=B}B!\,\frac{v^{|\mu|}\mathbf{p}_\mu(y)}{z_\mu}\Big)\\
&= N!\sum_{\substack{n\ge 0,\, m\ge 0\\ n+m\ge N}}u^n v^m\sum_{\substack{|\lambda|=n,\,|\mu|=m\\ l(\lambda)+l(\mu)=N}}\frac{\mathbf{p}_\lambda(x)}{z_\lambda}\,\frac{\mathbf{p}_\mu(y)}{z_\mu}.
\end{aligned}$$

Again, if we replace the alphabet x (resp. y) by the successive powers of q (resp. of p), we obtain

$$\Big(\sum_{r\ge 1}\frac{u^r\mathbf{p}_r(1,q,\ldots)+v^r\mathbf{p}_r(1,p,\ldots)}{r}\Big)^N=\sum_{\substack{n\ge 0,\, m\ge 0\\ n+m\ge N}}\frac{u^n}{(q;q)_n}\frac{v^m}{(p;p)_m}K^{(N)}_{n,m}(q,p),$$

where

$$K_{n,m}^{(N)}(q,p)=N!\sum_{\substack{|\lambda|=n,\,|\mu|=m\\ l(\lambda)+l(\mu)=N}}\frac{T_\lambda(q)}{z_\lambda}\frac{T_\mu(p)}{z_\mu}. \tag{23.24}$$

In the last summation the sum of the lengths of the two partitions is to remain constant, equal to N. We then use the bijection $(\lambda,\mu)\mapsto(\lambda^*,\mu^+)$, where μ^+ is derived from μ by *adding* a parts equal to k. We have

$$\begin{aligned}K_{ka+b,m}^{(N)}(q,p)&\equiv N!\sum_{\substack{|\lambda^*|=b,\,|\mu|=m+ka\\ l(\lambda^*)+l(\mu^+)=N}}\frac{T_{\lambda^*}(q)}{z_{\lambda^*}}\frac{T_{\mu^+}(p)}{z_{\mu^+}}\ \operatorname{mod}\Phi_k(q)\ \operatorname{mod}\Phi_k(p)\\ &\equiv K_{b,m+ka}^{(N)}(q,p).\end{aligned}$$

For the coefficients $K_{n,m}(q,p)$ of the series $F(e_p(u)e_q(v)-1)$ we also find the expression

$$K_{n,m}(q,p)=\sum_{N=0}^{n+m}\frac{C_N}{N!}K_{n,m}^{(N)}(q,p)$$

for some coefficients C_N ($N\ge 0$). The congruence property for the $K_{n,m}(q,p)$ follows from (23.24). □

23.3. *Congruences for bi-indexed polynomials.* We apply the techniques of the previous subsection to the sequence of the bi-indexed Eulerian polynomials (in *two* variables) $A_{n,m}(t,q)$ defined by identity (23.5). Notice that it has two specializations

$$A_{n,0}(t,q)={}^{\mathrm{inv}}A_n(t,q),\quad A_{0,m}(t,q)={}^{0}A_m(t), \tag{23.25}$$

where ${}^{\mathrm{inv}}A_n(t,q)$ is the q-inv-Eulerian polynomial introduced in section 10 and ${}^{0}A_m(t)=1$ if $m=0$ and $tA_m(t)$ (with $A_m(t)$ being the traditional Eulerian polynomial) if $m\ge 1$. In particular, $A_{1,0}(t,q)=A_{0,1}(t,q)=t$. The recurrence relation (23.4) with $q=1$ provides the recurrence relation for the polynomials $A_{n,m}(t,q)$ and shows by induction that

$$A_{1,m}(t,q)={}^{\mathrm{inv}}A_{0,m+1}(t,q)={}^{0}A_{m+1}(t)\quad(m\ge 0). \tag{23.26}$$

Theorem 23.5. *Let n and k be two positive integers and let $n=ka+b$, $0\le b\le k-1$ be the Euclidean division of n by k. Then the following congruence holds:*

$$A_{ka+b,m}(t,q)\equiv(1-t)^{(k-1)a}A_{b,m+a}(t,q)\ \operatorname{mod}\Phi_k(q). \tag{23.27}$$

Furthermore, if $0\le m\le k-1$, then

$$A_{ka+b,m}(t;q,p) \equiv A_{b,m+ka}(t;q,p) \mod \Phi_k(q) \mod \Phi_k(p), \tag{23.28}$$

For $k=1$, $m=1$:

$$A_{a,1}(t,q) = A_{0,a+1}(t,q) = {}^0\!A_{a+1}(t); \tag{23.29}$$

for $k=2$, $b=0,1$, $m=0$:

$$\begin{aligned} {}^{\mathrm{inv}}\!A_{2a+b}(t,q) = A_{0,2a+b} &\equiv (1-t)^a A_{b,a}(t,q) \bmod (1+q)\\ &\equiv (1-t)^a\, {}^0\!A_{a+b}(t) \bmod (1+q). \end{aligned} \tag{23.30}$$

Proof. The generating function for the polynomials $A_{n,m}(t,q)$ may be rewritten as

$$\sum_{n\ge 0, m\ge 0} \frac{A_{n,m}(t,q)}{(1-t)^{n+m}} \frac{u^n}{(q;q)_n} \frac{v^m}{m!} = \sum_{i\ge 0} \Bigl(\frac{t}{1-t}\Bigr)^i \bigl(e_q(u)\exp v - 1\bigr)^i.$$

By Proposition 23.4 the coefficients $K_{n,m}(q) := A_{n,m}(t,q)/(1-t)^{n+m}$ satisfy the congruence (23.19):

$$\frac{A_{ka+b,m}(t,q)}{(1-t)^{ka+b+m}} \equiv \frac{A_{b,m+a}(t,q)}{(1-t)^{b+m+a}} \mod \Phi_k(q).$$

In an analogous manner

$$\sum_{n\ge 0, m\ge 0} \frac{A_{n,m}(t;q,p)}{(1-t)^{n+m}} \frac{u^n}{(q;q)_n} \frac{v^m}{(p;p)_m} = \sum_{i\ge 0} \Bigl(\frac{t}{1-t}\Bigr)^i \bigl(e_q(u)e_p(u) - 1\bigr)^i$$

and congruence (23.20) applies. □

23.4. *The signed Eulerian numbers.* Consider the q-inv-Eulerian polynomials ${}^{\mathrm{inv}}\!A_n(t,q) = \sum_{\sigma\in\mathfrak{S}_n} t^{1+\mathrm{des}\,\sigma} q^{\mathrm{inv}\,\sigma}$ introduced in section 10 and define the *signed Eulerian polynomial* to be

$$ {}^{\mathrm{sgn}}\!A_n(t) := {}^{\mathrm{inv}}\!A_n(t,-1) := \sum_{k=1}^{n} {}^{\mathrm{sgn}}\!A_{n,k}\, t^k. \tag{23.30}$$

The integers ${}^{\mathrm{sgn}}\!A_{n,k}$ $(1\le k\le n)$ are called the *signed Eulerian numbers.* Relation (23.30) shows that

$$ {}^{\mathrm{sgn}}\!A_{n,k} = \sum_{\sigma} \mathrm{sgn}\,\sigma, \tag{23.31}$$

where the sum ranges over all permutations $\sigma\in\mathfrak{S}_n$ such that $1+\mathrm{des}\,\sigma = k$. Their first values are displayed in Table 23.1. Recall that ${}^0\!A_n(t)$ is the generating polynomial for $\mathfrak{S}_n$ by the statistic "$1+\mathrm{des}$" (see section 10).

$k=$	1	2	3	4	5	6	7
$n=1$	1						
2	1	-1					
3	1	0	-1				
4	1	-1	-1	1			
5	1	2	-6	2	1		
6	1	1	-8	8	-1	-1	
7	1	8	-19	0	19	-8	-1

Table 23.1

Theorem 23.10. *Let ${}^{\mathrm{sgn}}A_n(t)$ be the signed Eulerian polynomial defined in* (23.30). *Then*

$$\text{(23.32a)}\qquad {}^{\mathrm{sgn}}A_{2n}(t) = (1-t)^n\, {}^{0}A_n(t)\,;$$

$$\text{(23.32b)}\qquad {}^{\mathrm{sgn}}A_{2n+1}(t) = (1-t)^n\, {}^{0}A_{n+1}(t).$$

Proof. This has already been proved in Theorem 23.5 with relation (23.30), since ${}^{\mathrm{sgn}}A_n(t) \equiv {}^{\mathrm{inv}}A_n(t,q) \bmod (1+q)$. Another proof consists in starting with the recurrence relation (10.11) for the polynomials ${}^{\mathrm{inv}}A_n(t,q)$ and let $q \mapsto -1$ in that relation. As

$$\lim_{q\to -1} \begin{bmatrix} 2m \\ 2i \end{bmatrix}_q = \lim_{q\to -1} \begin{bmatrix} 2m+1 \\ 2i \end{bmatrix}_q = \lim_{q\to -1} \begin{bmatrix} 2m+1 \\ 2i+1 \end{bmatrix}_q = \binom{m}{i},$$
$$\lim_{q\to -1} \begin{bmatrix} 2m \\ 2i+1 \end{bmatrix}_q = 0,$$

we are led to the recurrences

$$\text{(23.33)}\qquad {}^{\mathrm{sgn}}A_{2n}(t) = t(1-t)^{2n-1} + \sum_{1\le i\le n-1} \binom{n}{i} {}^{\mathrm{sgn}}A_{2i}(t)\, t\,(1-t)^{2n-1-2i}\,;$$

$$\text{(23.34)}\qquad {}^{\mathrm{sgn}}A_{2n+1}(t) = t(1-t)^{2n} + \sum_{1\le i\le n} \binom{n}{i} {}^{\mathrm{sgn}}A_{2i}(t)\, t\,(1-t)^{2n-2i} + \sum_{0\le i\le n-1} \binom{n}{i} {}^{\mathrm{sgn}}A_{2i+1}(t)\, t\,(1-t)^{2n-2i-1}.$$

When $q \to 1$ in (10.11) we get the recurrence relation for the polynomials ${}^{0}A_n(t)$, i.e.,

$$\text{(23.35)}\qquad {}^{0}A_n(t) = t(1-t)^{n-1} + \sum_{1\le i\le n-1} \binom{n}{i} {}^{0}A_i(t)\, t\,(1-t)^{n-1-i}.$$

By comparing (23.33) and (23.35) we see that the polynomials $(1-t)^{n\,0}A_n(t)$ satisfy the same recurrence relation as the polynomials ${}^{\mathrm{sgn}}A_{2n}(t)$. Hence (23.32a) holds. To establish (23.32b) an easy induction on n suffices. □

Having the classical relations for the (unsigned) Eulerian numbers $A_{n,k}$ at our disposal, as derived in (10.5)–(10.9), it is straightforward from Theorem 23.10 to prove the following relations:

$$\text{(23.36a)} \qquad {}^{\mathrm{sgn}}A_{2n,k} = {}^{\mathrm{sgn}}A_{2n-1,k} - {}^{\mathrm{sgn}}A_{2n-1,k-1}\,;$$

$$\text{(23.36b)} \qquad {}^{\mathrm{sgn}}A_{2n+1,k} = k\,{}^{\mathrm{sgn}}A_{2n,k} + (2n-k+2)\,{}^{\mathrm{sgn}}A_{2n,k-1}\,;$$

together with the analogs of the Worpitzky formulas

$$\sum_i \binom{2n+i}{i} {}^{\mathrm{sgn}}A_{2n,k-i} = k^n\ ;$$

$$\sum_i \binom{2n-1+i}{i} {}^{\mathrm{sgn}}A_{2n-1,k-i} = k^n;$$

and the polynomial relations

$${}^{\mathrm{sgn}}A_{2n}(t) = (1-t)\,{}^{\mathrm{sgn}}A_{2n-1}(t);$$

$${}^{\mathrm{sgn}}A_{2n+1}(t) = (2n+1)\,t\,{}^{\mathrm{sgn}}A_{2n}(t) + t(1-t)\,{}^{\mathrm{sgn}}A'_{2n}(t);$$

where ${}^{\mathrm{sgn}}A'_{2n}(t)$ denotes the derivative of ${}^{\mathrm{sgn}}A_{2n}(t)$.

24. The basic and bibasic trigonometric functions

By analogy with classical analysis start with the traditional Euler identity

$$\text{(24.1)} \qquad e_L = \cos_L + i \sin_L,$$

where e_L is an analog of the traditional complex exponential. When we choose $e_L = e_q(iu)$ (resp. $e_L = e_p(iu)e_q(iv)$), we obtain the definitions of the q-sine, $\sin_q(u)$, and q-cosine, $\cos_q(u)$, (resp. of the p,q-sine, $\sin_{p,q}(u,v)$, and p,q-cosine, $\cos_{p,q}(u,v)$.) Of course, the other q-exponential $E_q(u)$ could be used, but this does not lead to significantly different results.

The goal of this section is to define the other functions $\tan_L$ and $\sec_L$, to derive the recurrence formulas for the coefficients of their expansions and, finally, to obtain combinatorial interpretations for the coefficients.

24.1. *The basic and bibasic tangent and secant functions.* First, recall some easy results on the *p-* or *q-partial derivative.* If $A(u,v) =$

$A(u,v;p,q)=\sum\limits_{n\ge0,m\ge0} A_{n,m}\,u^nv^m/((p;p)_n(q;q)_m)$ is a (p,q)-bibasic series in the variables u and v, define:

$$\partial_p\, A(u,v) := \frac{A(u,v)-A(pu,v)}{u}; \tag{24.2}$$

$$\partial_q\, A(u,v) := \frac{A(u,v)-A(u,qv)}{u}. \tag{24.3}$$

The following relations are immediate

$$\partial_p\big(A(u,v)B(u,v)\big) = A(u,v)\,\partial_p\, B(u,v) + A(pu,v)\,\partial_p\, B(u,v); \tag{24.4}$$

$$\partial_p\,\frac{A(u,v)}{B(u,v)} = \frac{B(u,v)\,\partial_p\, A(u,v) - A(u,v)\,\partial_p\, B(u,v)}{B(u,v)B(pu,v)}; \tag{24.5}$$

$$= \frac{B(pu,v)\,\partial_p\, A(u,v) - A(pu,v)\,\partial_p\, B(u,v)}{B(u,v)B(pu,v)}; \tag{24.6}$$

and also the analogous relations for the q-partial derivative. If the series $A(u;p)$ and $B(u;p)$ are p-basic series of the variable u, we speak of *p-derivatives*. The above properties are similar and will not be restated.

Definition. The *p-sine* and *p-cosine* are defined by the identity

$$e_p(iu) =: \cos_p(u) + i\,\sin_p(u);$$

the *p,q-sine* and *p,q-cosine* by the identity

$$e_{p,q}(iu,iv) := e_p(iu)e_q(iu) =: \cos_{p,q}(u) + i\,\sin_{p,q}(u).$$

Proposition 24.1. *The following identities hold:*

$$e_{p,q}(-iu,-iv) = \cos_{p,q}(u,v) - i\,\sin_{p,q}(u,v); \tag{24.7}$$

$$\cos_{p,q}(u,v) = \frac{e_{p,q}(iu,iv)+e_{p,q}(-iu,-iv)}{2}; \tag{24.8}$$

$$\sin_{p,q}(u,v) = \frac{e_{p,q}(iu,iv)-e_{p,q}(-iu,-iv)}{2i}; \tag{24.9}$$

$$\partial_p\, e_{p,q}(\alpha\, u,v) = \alpha\, e_{p,q}(\alpha\, u,v); \tag{24.10}$$

$$\partial_p\, \sin_{p,q}(\alpha\, u,v) = \alpha\, \cos_{p,q}(\alpha\, u,v); \tag{24.11}$$

$$\partial_p\, \cos_{p,q}(\alpha\, u,v) = -\alpha\, \sin_{p,q}(\alpha\, u,v); \tag{24.12}$$

and analogous relations for the q-partial derivative. The identities for $e_p(-iu)$, $\sin_p(u)$, $\cos_p(u)$ are obtained by letting $v=0$.

Proof. As $e_p(iu) = \cos_p(u) + i\,\sin_p(u)$, relation (25.7) is proved by direct calculation. Next, (24.8) and (24.9) are banal. For deriving (24.10) remember that $e_p(\alpha\, u)=\prod_{n\ge0}(1-\alpha\, up^n)^{-1}$, so that $e_p(p\alpha\, u) =$

$(1-\alpha\, u)e_p(\alpha\, u)$. Then $\partial_p\, e_p(\alpha\, u) = \alpha\, e_p(\alpha\, u)$. The last two relations hold by linearity. $\square$

Definition. The *p-tangent*, *p*-secant, *p, q-tangent* and *p, q-secant* functions are defined by

$$\tan_p(u) := \frac{\sin_p(u)}{\cos_p(u)}; \qquad \tan_{p,q}(u,v) := \frac{\sin_{p,q}(u,v)}{\cos_{p,q}(u,v)}; \tag{24.13}$$

$$\sec_p(u) := \frac{1}{\cos_p(u)}; \qquad \sec_{p,q}(u,v) := \frac{1}{\cos_{p,q}(u,v)}. \tag{24.14}$$

Also define

$$\mathrm{Eul}_(\, u; p) := \frac{1+\sin_p(u)}{\cos_p(u)} = \sec_p(u) + \tan_p(u); \tag{24.15}$$

$$\mathrm{Eul}(u, v; p, q) := \frac{1+\sin_{p,q}(u,v)}{\cos_{p,q}(u,v)} = \sec_{p,q}(u,v) + \tan_{p,q}(u,v). \tag{24.16}$$

Theorem 24.2. *The following derivative relations hold:*

$$\partial_p\ \mathrm{Eul}(u; p) = 1 + \tan_p(u)\ \mathrm{Eul}(pu; p); \tag{24.17}$$

$$\partial_p\ \mathrm{Eul}(u, v; p, q) = 1 + \tan_{p,q}(u,v)\ \mathrm{Eul}(pu, v; p, q); \tag{24.18}$$

$$\partial_q\ \mathrm{Eul}(u, v; p, q) = 1 + \tan_{p,q}(u,v)\ \mathrm{Eul}(u, qv; p, q). \tag{24.19}$$

Proof. Relation (25.17) follows from (25.18) or can be proved in a straightforward manner. As $\mathrm{Eul}(u, v; p, q) = \mathrm{Eul}(v, u; q, p)$, it suffices to prove the first of those two relations. By (24.6)

$$\begin{aligned}
&\partial_p \frac{1+\sin_{p,q}(u,v)}{\cos_{p,q}(u,v)} \\
&\quad = \frac{\cos_{p,q}(pu,v)\, \partial_p(1+\sin_{p,q}(u,v)) - (1+\sin_{p,q}(u,v))\, \partial_p\, \cos_{p,q}(u,v)}{\cos_{p,q}(u,v)\ \cos_{p,q}(pu,q)} \\
&\quad = \frac{\cos_{p,q}(pu,v)\cos_{p,q}(u,v) + \sin_{p,q}(u,v) + \sin_{p,q}(u,v)\sin_{p,q}(pu,v)}{\cos_{p,q}(u,v)\ \cos_{p,q}(pu,q)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_p \frac{1+\sin_{p,q}(u,v)}{\cos_{p,q}(u,v)} &= 1 + \frac{\sin_{p,q}(u,v)}{\cos_{p,q}(u,v)} \frac{1+\sin_{p,q}(pu,v)}{\cos_{p,q}(pu,v)} \\
&= 1 + \tan_{p,q}(pu,v)\ \mathrm{Eul}(pu, v; p, q). \quad \square
\end{aligned}$$

Denote the expansions of $\operatorname{Eul}(u;p,q)$ and $\operatorname{Eul}(u,v;p,q)$ by

$$\operatorname{Eul}(u;p) := \sum_{n\ge 0} \operatorname{Eul}_n(p)\frac{u^n}{(p;p)_n}; \tag{24.20}$$

$$\operatorname{Eul}(u,v;p,q) := \sum_{n\ge 0,\, m\ge 0} \operatorname{Eul}_{n,m}(p,q)\frac{u^n}{(p;p)_n}\frac{v^m}{(q;q)_m}; \tag{24.21}$$

so that

$$\tan_p(u) = \operatorname{Eul}^{\text{odd}}(u;p) := \sum_{n\ge 0,\, n \text{ odd}} \operatorname{Eul}_n(p)\frac{u^n}{(p;p)_n}; \tag{24.22}$$

$$\tan_{p,q}(u) = \operatorname{Eul}^{\text{odd}}(u,v;p,q) := \sum_{\substack{n\ge 0,\, n\ge 0\\ n+m \text{ odd}}} \operatorname{Eul}_{n,m}(p,q)\frac{u^n}{(p;p)_n}\frac{v^m}{(q;q)_m}. \tag{24.23}$$

Proposition 24.3. *The coefficients* $\operatorname{Eul}_n(p)$ $(n\ge 0)$ *and accordingly the function* $\operatorname{Eul}(u;p)$ *are inductively defined by the following two relations:*

(i) $\operatorname{Eul}_0(p) = \operatorname{Eul}_1(p) = 1$*;*

(ii) *the recurrence formula valid for* $n\ge 1$

$$\operatorname{Eul}_{n+1}(p) = \sum_{\substack{0\le a\le n\\ a \text{ odd}}} \begin{bmatrix} n\\ a\end{bmatrix}_p p^{n-a}\operatorname{Eul}_a(p)\operatorname{Eul}_{n-a}(p). \tag{24.24}$$

The coefficients $\operatorname{Eul}_{n,m}(p,q)$ $(n\ge 0,\ m\ge 0)$ *and accordingly the function* $\operatorname{Eul}(u,v;p,q)$ *are inductively defined by the following three relations:*

(iii) $\operatorname{Eul}_{0,0}(p) = \operatorname{Eul}_{0,1}(p,q) = \operatorname{Eul}_{1,0}(p,q) = 1$*;*

(iv) *the recurrence formula valid for* $n\ge 1$

$$\operatorname{Eul}_{n+1,0}(p,q) = \sum_{\substack{0\le a\le n\\ a \text{ odd}}} \begin{bmatrix} n\\ a\end{bmatrix}_p p^{n-a}\operatorname{Eul}_{a,0}(p,q)\operatorname{Eul}_{n-a,0}(p,q). \tag{24.25}$$

(v) *the recurrence formula valid for* $n+m\ge 1$

(24.26)

$$\operatorname{Eul}_{n,m+1}(p,q) = \sum_{\substack{0\le a\le n, 0\le b\le m\\ a+b \text{ odd}}} \begin{bmatrix} n\\ a\end{bmatrix}_p \begin{bmatrix} m\\ b\end{bmatrix}_q q^{m-b}\operatorname{Eul}_{a,b}(p,q)\operatorname{Eul}_{n-a,m-b}(p,q).$$

Proof. Go back to identities (24.17) and (24.19), expand both sides and write that coefficients of the same monomial are equal on both sides. ∎

When (24.18) is taken into account, conditions (iv) and (v) can be replaced by

(iv′) the recurrence formula valid for $m \ge 1$

$$(24.25') \qquad \mathrm{Eul}_{0,m+1}(p,q) = \sum_{\substack{0\le b\le n\\ b \text{ odd}}} \begin{bmatrix} m\\ b\end{bmatrix}_q q^{m-b}\,\mathrm{Eul}_{0,b}(p,q)\,\mathrm{Eul}_{0,m-b}(p,q).$$

(v′) the recurrence formula valid for $n+m \ge 1$

(24.26′)
$$\mathrm{Eul}_{m+1,n}(p,q) = \sum_{\substack{0\le a\le n, 0\le b\le m\\ a+b \text{ odd}}} \begin{bmatrix} n\\ a\end{bmatrix}_p \begin{bmatrix} m\\ b\end{bmatrix}_q p^{n-a}\,\mathrm{Eul}_{a,b}(p,q)\,\mathrm{Eul}_{n-a,m-b}(p,q).$$

Finally, notice that both (24.25) and (24.25′) are the same recurrence formulas as (24.24).

As $\mathrm{Eul}_0(p) = \mathrm{Eul}_1(p) = 1$, all the coefficients $\mathrm{Eul}_n(p)$ can be determined by means of formula (24.24). The first values are the following:

$\mathrm{Eul}_0(p) = \mathrm{Eul}_1(p) = \mathrm{Eul}_2(p) = 1, \quad \mathrm{Eul}_3(p) = p(1+p),$
$\mathrm{Eul}_4(p) = p(1+p)^2 + p^4, \quad \mathrm{Eul}_5(p) = p^2(1+p)^2(1+p^2)^2,$
$\mathrm{Eul}_6(p) = p^2(1+p)^2(1+p^2+p^4)(1+p+p^2+2p^3) + p^{12},$
$\mathrm{Eul}_7(p) = p^3(1+p)^2(1+p^2)(1+p^3)(1+p+3p^2+2p^3+3p^4+2p^5+3p^6+p^7+p^8).$

The first polynomials $\mathrm{Eul}_{n,m}(p,q)$ can be derived by menas of Proposition 24.3. Because of the symmetry we only list the polynomials $\mathrm{Eul}_{n,m}(p,q)$ such that $n \le m$.

$\mathrm{Eul}_{0,0}(p,q) = \mathrm{Eul}_{0,1}(p,q) = \mathrm{Eul}_{0,2}(p,q) = \mathrm{Eul}_{1,1}(p,q) = 1;$
$\mathrm{Eul}_{0,3}(p,q) = q(1+q); \quad \mathrm{Eul}_{1,2}(p,q) = 1+q;$
$\mathrm{Eul}_{0,4}(p,q) = q(1+q)^2 + q^4; \quad \mathrm{Eul}_{1,3}(p,q) = 1+2q+2q^2;$
$\mathrm{Eul}_{2,2}(p,q) = 2+p+q+pq;$
$\mathrm{Eul}_{0,5}(p,q) = q^2(1+q)^2(1+q^2)^2; \quad \mathrm{Eul}_{1,4}(p,q) = 2q(1+q)^2(1+q^2);$
$\mathrm{Eul}_{2,3}(p,q) = (1+p)(1+q)^3;$
$\mathrm{Eul}_{0,6}(p,q) = q^2(1+q)^2(1+q^2+q^4)(1+q+q^2+2q^3) + q^{12};$
$\mathrm{Eul}_{1,5}(p,q) = (3q^7+6q^6+10q^5+13q^4+12q^3+9q^2+6q+2)q;$
$\mathrm{Eul}_{2,4}(p,q) = 3q^5+3pq^5+7q^4+6pq^4+7pq^3+8q^3+9q^2+7pq^2+4pq+5q+p+1;$
$\mathrm{Eul}_{3,3}(p,q) = 2p^3q+2p^3q^2+p^3q^3+p^3+4p^2+8p^2q^2+8p^2q+2p^2q^3+4p+ 8pq^2+8pq+2pq^3+4q^2+4q+2+q^3;$
$\mathrm{Eul}_{0,7}(p,q) = q^3(1+q)^2(1+q^2)(1+q^3)(1+q+3q^2+2q^3+3q^4+2q^5+3q^6+ q^7+q^8);$
$\mathrm{Eul}_{1,6}(p,q) = q^2(1+q)^2(1+q^2)(1+q^3)(3q^4+3q^3+5q^2+3q+3);$
$\mathrm{Eul}_{2,5}(p,q) = q(1+p)(1+q)^2(1+q^2)(3q^4+4q^3+3q^2+4q+3);$

$$\mathrm{Eul}_{3,4}(p,q) = (1+p)(1+q)^2(1+q^2)$$
$$\times\,(q^2p^2 + pq^2 + q^2 + 5qp + 3qp^2 + 3q + p^2 + p + 1).$$

24.2. *Alternating permutations.* We now use formula (24.24) to show that the polynomials $\mathrm{Eul}_n(p)$ are generating polynomials for sets of permutations, called *alternating*, by the statistic "inv" (and also by "imaj").

Definition. A *rising alternating* (resp. *falling alternating*) permutation is defined to be a permutation $\sigma = \sigma(1)\ldots\sigma(n)$ having the following properties:

$\sigma(1) < \sigma(2)$, $\sigma(2) > \sigma(3)$, $\sigma(3) < \sigma(4)$, etc. in an alternating way
(resp. $\sigma(1) > \sigma(2)$, $\sigma(2) < \sigma(3)$, $\sigma(3) > \sigma(4)$, etc. in an alternating way).

By $\mathcal{DR}_n$ (resp. $\mathcal{D}_n$) is denoted the set of rising alternating (resp. falling alternating) permutations of order n.

One of the consequences of the following theorem is the fact that the number of rising alternating (resp. falling alternating) permutations of order n is equal to $\mathrm{Eul}_n(1)$. Referring to the table of the polynomials $\mathrm{Eul}_n(q)$ given above, we see that: $\mathrm{Eul}_0(1) = \mathrm{Eul}_1(1) = \mathrm{Eul}_2(1) = 1$, $\mathrm{Eul}_3(1) = 2$, $\mathrm{Eul}_4(1) = 5$, $\mathrm{Eul}_5(1) = 16$, $\mathrm{Eul}_6(1) = 61$. The numbers $\mathrm{Eul}_{2n}(1)$ (resp. $\mathrm{Eul}_{2n+1}(1)$) are called *tangent numbers* (resp. *secant numbers*) and have been combinatorially studied for decades. The polynomials $\mathrm{Eul}_n(q)$ appear as q-analogs of those numbers.

The rising alternating permutations for $n = 1, 2, 3, 4$ are the following: 1 ; $1,2$; $1,3,2$; $2,3,1$; $1,3,2,4$; $1,4,2,3$; $2,3,1,4$; $2,4,1,3$; $3,4,1,2$.

Theorem 24.4. *For every $n \ge 0$ the polynomial $\mathrm{Eul}_n(p)$ is the generating function for the rising alternating permutations of length n by the number of inversions. In other words,*

$$\mathrm{Eul}_n(p) = \sum_{\sigma\in\mathcal{DR}_n} p^{\mathrm{inv}\,\sigma}.$$

When n is odd, we also have

$$\mathrm{Eul}_n(p) = \sum_{\sigma\in\mathcal{D}_n} p^{\mathrm{inv}\,\sigma}.$$

Proof. The result is banal for $n = 0$, 1 and 2. For $n \ge 2$ consider the set $S_{n+1,a+1}$ of rising alternating permutations σ of order $n+1$ such that $n+1$ is in the $(a+1)$-st position (i.e., $\sigma(a+1) = n+1$) with a odd. Such a permutation is characterized by the two rising alternating subpermutations $\sigma' = \sigma(1)\ldots\sigma(a)$ and $\sigma'' = \sigma(a+2)\ldots\sigma(n+1)$ that do not contain $n+1$. The inversions of σ fall into four groups:

(a) the inversions that correspond to the pairs of letters whose first letter is in σ' and the second one in σ''; if follows from Proposition 4.3 that their generating polynomial is equal to $\begin{bmatrix} n \\ a \end{bmatrix}_p$;

(b) the $n-a$ inversions between $n+1$ and each letter of σ'';

(c) the inversions inside σ'; by induction their generating polynomial is equal to $\mathrm{Eul}_a(p)$;

(d) the inversions inside σ''; their generating polynomial is equal to $\mathrm{Eul}_{n-a}(p)$.

The generating polynomial for $S_{n+1,a+1}$ is then equal to

$$\sum_{\sigma \in S_{n+1,a+1}} q^{\mathrm{inv}\,\sigma} = \begin{bmatrix} n \\ a \end{bmatrix} q^{n-a} \,\mathrm{Eul}_a(p)\,\mathrm{Eul}_{n-a}(p).$$

Hence,

$$\mathrm{Eul}_{n+1}(p) = \sum_{0 \le a \le n,\, a \text{ odd}} \begin{bmatrix} n \\ a \end{bmatrix}_p p^{n-a} \,\mathrm{Eul}_a(p)\,\mathrm{Eul}_{n-a}(p),$$

that is exactly the induction formula (24.24).

When n is odd, the transformation $\mathbf{rc}$, already introduced (see the end of section 12), that sends the permutation σ onto the permutation $\mathbf{rc}\,\sigma$ defined by

$$\mathbf{rc}\,\sigma(i) := n+1-\sigma(n+1-i) \quad (1 \le i \le n),$$

maps $\mathcal{DR}_n$ onto $\mathcal{D}_n$ in a one-to-one manner. Moreover, under the transformation the number of inversions "inv" remains alike. This proves the second part of the theorem. $\square$

Saying that a permutation σ is rising alternating (resp. falling alternating) is equivalent to saying that $\mathrm{Ligne}\,\sigma = \{2,4,6,\dots\}$ (resp. $\mathrm{Ligne}\,\sigma = \{1,3,5,\dots\}$). Hence, Corollary 11.5 implies the following proposition.

Proposition 24.5. *For every $n \ge 1$ we have:*

$$\sum_{\sigma \in \mathcal{DR}_n} p^{\mathrm{inv}\,\sigma} = \sum_{\sigma \in \mathcal{DR}_n} p^{\mathrm{imaj}\,\sigma} \quad \text{and} \quad \sum_{\sigma \in \mathcal{D}_n} p^{\mathrm{inv}\,\sigma} = \sum_{\sigma \in \mathcal{D}_n} p^{\mathrm{imaj}\,\sigma}.$$

24.3. *Combinatorics of the bibasic secant and tangent.* In section 23 we have studied the class of the bi-indexed Eulerian polynomials defined by

$$A_{n,m}(t,p,q) := \sum_{\sigma \in \mathfrak{S}_{n+m}} t^{1+\mathrm{des}\,\sigma} p^{\mathrm{inv}_1\,\sigma} q^{\mathrm{inv}_2\,\sigma}.$$

As the statistic "inv" (also the statistic "imaj"), when defined on the subset of alternating permutations, leads to a combinatorial interpretation of the q-tangent and q-secant numbers, the natural question arises: what can be said about the polynomials

$$(24.27)\quad D_{n,m}(p,q) := \sum_{\sigma\in\mathcal{DR}_{n+m}} p^{\mathrm{inv}_1\sigma} q^{\mathrm{inv}_2\sigma}, \quad E_{n,m}(p,q) := \sum_{\sigma\in\mathcal{D}_{n+m}} p^{\mathrm{inv}_1\sigma} q^{\mathrm{inv}_2\sigma}$$

when the pair $(\mathrm{inv}_1, \mathrm{inv}_2)$ is restricted to the subset $\mathcal{DR}_{n+m}$ of the *rising* (resp. the subset $\mathcal{D}_{n+m}$ of the *descending*) alternating permutations. No use keeping the variable t as all the permutations in $\mathcal{DR}_{n+m}$ (resp. $\mathcal{D}_{n+m}$) have the same ligne of route, i.e., $\{2,4,6,\ldots\}$ (resp. $\{1,3,5,\ldots\}$).

Again, let (n,m) be an ordered pair of nonnegative integers. By convention, $D_{0,0}(p,q) := 1$. Let $n+m\ge 1$ and let σ be a permutation of order $n+m$. The statistics "inv_1" and "inv_2" (see Theorem 23.1) can be redefined as follows:

$$\begin{aligned}
\mathrm{inv}_1\sigma &:= \#\{(i,j) : 1\le i<j\le n+m,\ n\ge\sigma(i)>\sigma(j)\ge 1\},\\
\mathrm{inv}_2\sigma &:= \#\{(i,j) : 1\le i<j\le n+m,\ n+m\ge\sigma(i)>\sigma(j)\ge n+1\}.
\end{aligned}$$

To emphasize the fact that inv_1 and inv_2 are defined by means of the ordered pair (n,m), we also use the notations:

$$\mathrm{inv}_1^{(n,m)} := \mathrm{inv}_1; \qquad \mathrm{inv}_2^{(n,m)} := \mathrm{inv}_2\,.$$

As has been defined in § 24.1,

$$\begin{aligned}
\mathrm{Eul}(u,v;p,q) &= \sec_{p,q}(u,v) + \tan_{p,q}(u,v)\\
&= \sum_{n\ge 0,\, m\ge 0} \mathrm{Eul}_{n,m}(p,q)\frac{u^n}{(p;p)_n}\frac{v^m}{(q;q)_m}.
\end{aligned}$$

The purpose of this subsection is to prove the following theorem.

Theorem 24.6. *For each pair of nonnegative integers (n,m) we have:*

$$(24.28)\qquad D_{n,m}(p,q) = \mathrm{Eul}_{n,m}(p,q);$$

when $n+m$ is odd, we further have:

$$(24.29)\qquad E_{n,m}(p,q) = \mathrm{Eul}_{n,m}(p,q);$$

Before proving the theorem and making some comment about the symmetry of the polynomials $\mathrm{Eul}_{n,m}(p,q)$ let us establish the following lemma.

Lemma 24.7. *Let σ be a permutation of order $n+m \geq 1$ and $\sigma' := \mathbf{r}\,\mathbf{c}\,\sigma$. Then*

$$(\mathrm{inv}_1^{(n,m)}, \mathrm{inv}_2^{(n,m)})\,\sigma = (\mathrm{inv}_2^{(m,n)}, \mathrm{inv}_2^{(m,n)})\,\sigma'.$$

Proof. By definition of $\mathbf{r}$ and $\mathbf{c}$ we have

$$\sigma'(n+m+1-i) = n+m+1-\sigma(i) \quad (1 \leq i \leq m+n).$$

Hence the following statements are equivalent:

(i) $1 \leq i < j \leq n+m \quad \& \quad n \geq \sigma(i) > \sigma(j) \geq 1$;

(ii) $1 \leq n+m+1-j < n+m+1-i \leq n+m$ $\&\ n+m \geq n+m+1-\sigma(j) > n+m+1-\sigma(i) \geq m+1$;

(iii) $1 \leq j' < i' \leq n+m \quad \& \quad n+m \geq \sigma'(j') > \sigma'(i') \geq m+1$ with $j' := n+m+1-j$, $i' := n+m+1-i$;

Thus, $\mathrm{inv}_1^{(n,m)}\,\sigma = \mathrm{inv}_2^{(m,n)}\,\sigma'$. Also, $\mathrm{inv}_2^{(n,m)}\,\sigma = \mathrm{inv}_1^{(m,n)}\,\sigma'$ by permuting the roles of n and m. ∎

If σ is a rising alternating permutation of order $n+m$, then $\sigma' = \mathbf{r}\,\mathbf{c}\,\sigma$ is rising alternating or descending alternating, depending on whether $n+m$ is even or odd. By the previous lemma we then have

$$D_{n,m}(p,q) = \begin{cases} D_{m,n}(q,p), & \text{if } n+m \text{ is even;} \\ E_{m,n}(q,p), & \text{if } n+m \text{ is odd.} \end{cases} \tag{24.30}$$

Thus, when $n+m$ is even, the symmetry of the polynomials $D_{n,m}(p,q)$ is obvious combinatorially. When $n+m$ is *odd*, it is only a consequence of Theorem 24.6.

In order to prove identity (24.28) of Theorem 24.6 we show that the sequence $(D_{n,m}(p,q))$ satisfies the relations (iii), (iv), (v) of Proposition 24.3 that uniquely *define* the sequence $(\mathrm{Eul}_{n,m}(p,q))$. This is the content of the next proposition.

Proposition 24.8. *Let $D_{n,m}(p,q)$ be the generating polynomial for the set $\mathcal{DR}_{n+m}$ of the rising alternating permutations of order $n+m$ by the two-variable statistic* $(\mathrm{inv}_1, \mathrm{inv}_2)$. *Then the sequence* $(D_{n,m}(p,q))$ $(n \geq 0, m \geq 0)$ *is inductively defined by the following three relations:*

(iii) *the initial conditions $D_{0,0}(p,q) = D_{1,0}(p,q) = D_{0,1}(p,q) = 1$;*

(iv) *the recurrence formula valid for $n \geq 1$:*

$$D_{n+1,0}(p,q) = \sum_{\substack{0 \leq a \leq n, \\ a \text{ odd}}} \begin{bmatrix} n \\ a \end{bmatrix}_p p^{n-a} D_{a,0}(p,q)\, D_{n-a,0}(p,q);$$

(v) *and the following formula valid for $n+m \geq 1$:*

$$(24.31)\ D_{n,m+1}(p,q)=\sum_{\substack{0\le a\le n, 0\le b\le m\\ a+b\ \text{odd}}}\begin{bmatrix}n\\ a\end{bmatrix}_p\begin{bmatrix}m\\ b\end{bmatrix}_q q^{m-b}D_{a,b}(p,q)D_{n-a,m-b}(p,q).$$

Proof. Relation (iii) is banal, while relation (iv) is the recurrence formula for the p-tangent or p-secant number derived in the previous subsection.

Let $\sigma=\sigma(1)\sigma(2)\ldots\sigma(n+m+1)$ be a rising alternating permutation of order $(n+m+1)$. As the permutation starts with a rise $\sigma(1)<\sigma(2)$, the letter $(n+m+1)$ in the word σ occurs in an even position, say, $2k+2$ $(k\ge 0)$. Let

$$\begin{aligned}
A&:=\{\sigma(l):1\le l\le 2k+1, 1\le\sigma(l)\le n\};\\
B&:=\{\sigma(l):1\le l\le 2k+1, n+1\le\sigma(l)\le n+m\};\\
C&:=\{\sigma(l):2k+3\le l\le n+m+1, 1\le\sigma(l)\le n\}=[n]\setminus A;\\
D&:=\{\sigma(l):2k+3\le l\le n+m+1, n+1\le\sigma(l)\le n+m\}\\
&\quad=[n+1,n+m]\setminus B.
\end{aligned}$$

Also, let $\#A:=a$, $\#B:=b$, $\#C:=c=n-a$, $\#D:=d=m-b$, so that $a+b=2k+1$ and $c+d=n+m-(2k+2)$. The two words $\sigma':=\sigma(1)\ldots\sigma(2k+1)$ and $\sigma'':=\sigma(2k+3)\ldots\sigma(n+m+1)$ are alternating permutations of the sets $A+B$ and $C+D$, respectively. The *reduction* of σ' is defined to be the permutation $\tau'=\tau'(1)\ldots\tau'(2k+1)$, where the letter $\tau'(l)$ is equal to m if and only if $\sigma(l)$ is the smallest m-th letter of $A+B$. The reduction $\tau''=\tau''(1)\ldots\tau''(n+m+1-2k-2)$ of σ'' is defined in the same way. Of course, both τ' and τ'' are rising alternating permutations.

Now σ is completely characterized by the four-sequence (A,B,τ',τ''). The inversions of σ of the form $n\ge\sigma(i)>\sigma(j)\ge 1$ $(1\le i<j\le n+m)$ are of three kinds:

(i) $1\le i<j\le 2k+1$; they are counted by $\mathrm{inv}_1^{(a,b)}\,\tau'$;

(ii) $2k+3\le i<j\le n+m$; they are counted by $\mathrm{inv}_1^{(c,d)}\,\tau''$;

(iii) $1\le i<2k+2<j\le n+m+1$; using the notations of § 4.4 they are counted by $\mathrm{inv}(\gamma(A)\gamma(C))$.

Thus,

$$\mathrm{inv}_1^{(n,m)}=\mathrm{inv}_1^{(a,b)}\,\tau'+\mathrm{inv}_1^{(c,d)}\,\tau''+\mathrm{inv}(\gamma(A)\gamma(C));$$

and in the same way

$$\mathrm{inv}_2^{(n,m)}=\mathrm{inv}_2^{(a,b)}\,\tau'+\mathrm{inv}_2^{(c,d)}\,\tau''+\mathrm{inv}(\gamma(B)\gamma(D))+d.$$

The "d" that has been added takes the inversions $\sigma(2k+2)=n+m>\sigma(l)\ge n+1$ $(2k+3\le l\le n+m+1)$ into account.

When the pair (A, B) is fixed, so are the ordered partitions (A, C) and (B, D) of $[n]$ and $[n+1, n+m]$, respectively. Hence

$$\sum_{\tau', \tau''} p^{\mathrm{inv}_1^{(a,b)} \tau' + \mathrm{inv}_1^{(c,d)} \tau''} q^{\mathrm{inv}_2^{(a,b)} \tau' + \mathrm{inv}_2^{(c,d)} \tau''} = D_{a,b}(p,q) D_{c,d}(p,q).$$

Now, when A (resp. B) runs over all subsets of $[n]$ (resp. of $[n+1, n+m]$) of cardinality a (resp. b), the pairs (A, C) and (B, D) range over all ordered partitions of $[n]$ and of $[n+1, n+m]$, respectively, into two blocks such that $\#A = a$, $\#C = c$, and $\#B = b$, $\#D = d$. By Proposition 4.3

$$\sum_{(A,B)} p^{\mathrm{inv}(\gamma(A)\gamma(C))} q^{\mathrm{inv}(\gamma(B)\gamma(D))+d} = \begin{bmatrix} n \\ a \end{bmatrix}_p \begin{bmatrix} m \\ b \end{bmatrix}_q q^d.$$

The final summation with respect to a, b yields (24.31). $\square$

For proving (24.29) consider (24.31) when $n+m$ is odd. Using (24.30) we can rewrite (24.31) as

$$E_{n+1,m}(p,q) = \sum_{\substack{0 \le b \le m, 0 \le a \le n \\ b+a \text{ odd}}} \begin{bmatrix} m \\ b \end{bmatrix}_q \begin{bmatrix} n \\ a \end{bmatrix}_p q^{m-b} E_{b,a}(q,p) E_{m-b,m-a}(q,p).$$

This is exactly the recurrence formula (24.16′) written for $\mathrm{Eul}_{n+1,m}(q,p)$. As the analogous relations (iii) and (iv′) hold for $E_{m,n}(0,p)$, identity (24.29) is proved. $\square$

Exercises and examples

1. *Rogers-Szegö Polynomials.* The product $\exp(xu)\exp(u)$ is the exponential generating function for polynomials $H_n(x)$, whose expression is easy to derive.

In the same manner, the product of the two q-exponentials $e_q(xu)\, e_q(u)$ is the factorial generating function $\sum H_n(x,q)u^n/(q;q)_n$ for polynomials $H_n(x,q)$, called the Rogers-Szegö polynomials. They can be expressed by means of the q-binomial coefficients.

We have $H_{2n+1}(-1,q) = 0$, $H_{2n}(-1;q) = (q;q^2)_n$, then $H_n(q^{1/2},q) = (-q^{1/2};q^{1/2})_n$ and finally the induction formula

$$H_{n+1}(x,q) = (1+x)H_n(x,q) - (1-q^n)x\, H_{n-1}(x,q).$$

2. *The Ramanujan sum.* The sum in question is:

$$\sum_{n=-\infty}^{+\infty} \frac{(a;q)_n}{(b;q)_n} u^n = \frac{(au;q)_\infty\,(qa^{-1}u^{-1};q)_\infty\,(q;q)_\infty\,(ba^{-1};q)_\infty}{(u;q)_\infty\,(ba^{-1}u^{-1};q)_\infty\,(b;q)_\infty\,(qa^{-1};q)_\infty}.$$

On the left-hand side each term is a series in the variable q and is defined for each integer, positive and negative. In the proof of the identity there will be no ambiguity in the definition of the sum.

Start with the sum

$$h(b) = \sum_{n=-\infty}^{+\infty} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} u^n = \frac{(b;q)_\infty}{(a;q)_\infty} \sum_{n=-\infty}^{+\infty} \frac{(a;q)_n}{(b;q)_n} u^n$$

and use the trivial relation $(bq^n;q)_\infty = (1-bq^n)(bq^{n+1};q)_\infty$. We obtain the q-recurrence

$$h(b) = (1-ba^{-1})\, h(bq) + ba^{-1}u^{-1}\, h(b),$$

and by itération

$$h(b) = h(bq^n)\frac{(ba^{-1};q)_n}{(ba^{-1}u^{-1};q)_n}$$

for every $n \geq 0$. As soon as $n \geq j+1$ the latter identity implies that the coefficients of $u^i q^j$

$$\text{in} \quad h(bq^n)\frac{(ba^{-1};q)_n}{(ba^{-1}u^{-1};q)_n} \quad \text{and in} \quad h(0)\frac{(ba^{-1};q)_\infty}{(ba^{-1}u^{-1};q)_\infty}$$

are identical. Hence, $h(b) = h(0)(ba^{-1};q)_\infty/(ba^{-1}u^{-1};q)_\infty$ and also $h(q) = h(0)(qa^{-1};q)_\infty/(qa^{-1}u^{-1};q)_\infty$. Going back to the original definition of $h(b)$ and using the q-binomial theorem we obtain

$$h(q) = \frac{(q;q)_\infty\,(au;q)_\infty}{(a;q)_\infty\,(u;q)_\infty}.$$

By combining the last three formulas in an evident manner we obtain the Ramanujan sum.

3. *A maj-inv bijection for permutations.* Let $x = (x_1, x_2, \ldots, x_n)$ be the inv-coding of a permutation $\sigma \in \mathfrak{S}_n$. With $x_{n+1} := \sigma(n+1) = 0$ define a sequence $y = (y_1, y_2, \ldots, y_n)$ by $y_i := x_i - x_{i+1} + i\,\chi(\sigma(i) > \sigma(i+1))$ for each $i = 1, 2, \ldots, n$. Then the sequence y is subexcedent, the mapping $x \mapsto y$ is bijective and $\operatorname{tot} y = y_1 + y_2 + \cdots + y_n$ is equal to $\operatorname{maj}\sigma$. Let σ' be the permutation whose inv-coding is y. Then $\sigma \mapsto \sigma'$ is a bijection of $\mathfrak{S}_n$ onto itself having the property: $\operatorname{maj}\sigma = \operatorname{inv}\sigma'$.

4. *A numerical example.* Calculate the inv-coding and the maj-coding of $\sigma = 259478361$ and the number of inversions and the Major Index of σ.

Find the permutations having for inv-coding and maj-coding the subexcedent sequence $x = 001304645$, respectively.

5. *Another maj-coding.* Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation. With $\sigma(n+1) := +\infty$ define a sequence $z = z_1 z_2 \cdots z_n$ by

$$z_i := \#\{1 \leq j < i \mid \sigma(j) \in \rrbracket\sigma(i), \sigma(i+1)\rrbracket\},$$

using the notation $\rrbracket \ldots \rrbracket$ for a cyclic interval (see section 2).

(a) The sequence z is subexcedent.

(b) Determine the sequence z that corresponds to $\sigma = 259478361$.

(c) The transformation $\sigma \mapsto z$ is invertible. Find the inverse of $z = 001304645$.

(d) Let $x = x_1 x_2 \cdots x_n$ be the inv-coding of σ and $x_{n+1} = 0$. The following identity holds:

$$z_i = x_i - x_{i+1} + i\chi(\sigma_i > \sigma_{i+1}).$$

(e) We have: $z_1 + z_2 + \cdots + z_n = \operatorname{maj}\sigma$. Thus, the transformation $\sigma \mapsto y$ defines another maj-coding.

6. *A counter-example.* For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ define a sequence $y = y_1y_2\cdots y_n$ by $y_i := (i-1)\chi(\sigma(i-1) > \sigma(i))$. The transformation $\sigma \mapsto y$ is not a maj-coding although $\operatorname{tot} y = \operatorname{maj}\sigma$.

7. *Number of vector subspaces.* Let q be the power of a prime number and let $\mathbb{F}_q$ denote the field having q elements. The number of vector subspaces of dimension n of the vector space ${\mathbb{F}_q}^{N+n}$ is equal to $\left[{N+n \atop n}\right]$.

First, the number of sets $\{v_1, v_2, \ldots, v_n\}$ of n vectors $v_1, v_2, \ldots, v_n$ of ${\mathbb{F}_q}^{N+n}$, that are linearly independent, is equal to

$$q^{(N+n)(N+n-1)/2}(q^{N+n}-1)(q^{N+n-1}-1)\cdots(q^{N+1}-1);$$

then, the number of such sets that generate a given vector subspace of dimension n in ${\mathbb{F}_q}^{N+n}$ is equal to $q^{n(n-1)/2}(q^n-1)(q^{n-1}-1)\cdots(q-1)$.

8. *The use of the q-Pascal Triangle.* Derive the expansions of $1/(u;q)_N$ and of $(-u;q)_n$ (formulas (3.9) and (3.10)) by means of the q-Pascal Triangle formulas (3.5) and (3.6).

9. *Nondecreasing sequences of integers.* Calculate the generating function for nondecreasing sequences of N integers at most equal to n by "tot" (formula (4.5)) by using the q-Pascal Triangle formulas (3.5) and (3.6)

10. *Binary words.* Calculate the generating function for binary words of length $(N+n)$ having N letters equal to 1 and n letters equal to 0 by their number of inversions (formula (4.17)) by using the q-Pascal Triangle formulas (3.5) and (3.6)

11. *Partitions of integers and q-binomial coefficients.* The interpretation of the q-binomial coefficients in terms of partitions of integers (see § 4.1) makes it easy to prove the following identities.

(a) By classifying the partitions in at most n parts, all being at most equal to N, according the size of the smallest n-th part (possibly 0) we get:

$$\sum_{j=0}^{N} \begin{bmatrix} N-j+n-1 \\ N-j \end{bmatrix} q^{jn} = \begin{bmatrix} N+n \\ N \end{bmatrix}.$$

(b) Also

$$\begin{aligned}\sum_{j=0}^{N} \begin{bmatrix} n+j \\ j \end{bmatrix} \begin{bmatrix} m-1+N-j \\ N-j \end{bmatrix} q^{jm} &= \begin{bmatrix} m+n+N \\ N \end{bmatrix} \\ &= (-1)^N q^{N(n+m)+N(N+1)/2} \begin{bmatrix} -m-n-1 \\ N \end{bmatrix}.\end{aligned}$$

12. *The Eulerian numbers.* They are denoted by $A_{n,k}$ and are defined by the recurrence relation (see (10.8))

$$\text{(E12.1)}\quad A_{n,k}=(k+1)A_{n-1,k}+(n-k)A_{n-1,k-1}\quad (1\le k\le n-1)\,;$$
$$A_{n,0}=1\quad (n\ge 0)\,;\qquad A_{n,k}=0\quad (k\ge n).$$

Their first values are shown in Table E12.1.

k=	0	1	2	3	4	5	6
n=1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

Table E12.1

(a) For each $n\ge 0$ we have $\sum\limits_{k\ge 0} A_{n,k}=n!$ and $A_{n,k}=A_{n,n-1-k}$.

(b) The number of descents, $\operatorname{des}\sigma$, of a permutation $\sigma=\sigma(1)\ldots\sigma(n)$ is defined to be the number of integers j such that $1\le j\le n-1$ and $\sigma(j)>\sigma(j+1)$. For every $k\ge 0$ and every $n\ge 0$ the number $A_{n,k}$ is equal to the number of permutations $\sigma\in\mathfrak{S}_n$ having k descents.

(c) The *Eulerian polynomial* $A_n(t)$ is defined by $A_n(t):=\sum\limits_{0\le k\le n-1} A_{n,k}t^k$. As said in Definition 10.2, relations (10.7) and (10.8) are equivalent.

13. *The Eulerian Polynomials.* Consider the sequence $(A_n(t))$ $(n\ge 0)$ of formal series in the variable t (they will appear to be *polynomials* and more precisely the Eulerian Polynomials defined in Exercice 12) defined by

$$\text{(E13.1)}\qquad \frac{A_n(t)}{(1-t)^{n+1}}:=\sum_{j\ge 0} t^j\,(j+1)^n.$$

(a) If D designates the derivative-operator for the formal series, we have: $A_n(t)=(1+(n-1)t)A_{n-1}(t)+t(1-t).DA_{n-1}(t)\quad (n\ge 1)$.

(b) For $n\ge 0$ let $A_n(t):=\sum\limits_{k\ge 0} A_{n,k}t^k$. Then the relations listed in (E12.1) hold.

(c) It follows from the definition $A_n(t) = \sum_{k\ge 0} A_{n,k}t^k$ and (E13.1) that

$$A_{n,k} = \sum_{0\le i\le k} (-1)^i (k-i+1)^n \binom{n+1}{i}.$$

(d) The "inverse" of the previous formula is the *Worpitzky* formula:

$$x^n = \sum_{0\le k\le n-1} \binom{x+k}{n} A_{n,k}.$$

[Start with the expansion of $A_n(t)(1-t)^{-(n+1)}$, then use Exercise 12 (a) and (d) to recover (E12.1).]

(e) The exponential generating function

$$\sum_{n\ge 0} A_n(t)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t-1))}$$

is a consequence of (E13.1). Thus, the five definitions (10.5)—(10.9) are shown to be equivalent.

14. *A less unwieldy definition for the Denert statistic.* The Denert statistic "den" has been defined in section 2 by means of the cyclic intervals. There is an alternative definition that is the following. Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ be a permutation of order n. If $1 \le i \le n-1$ and $\sigma(i) > i$, say that i is an *excedence-place* for σ, and $\sigma(i)$ is an *excedence-letter* for σ. Let $i_1 < i_2 < \cdots < i_k$ be the increasing sequence of the excedence places and $j_1 < j_2 < \cdots < j_{n-k}$ the increasing sequence of *non-excedence places.* The subwords $\operatorname{Exc}\sigma = \sigma(i_1)\ldots\sigma(i_k)$ and $\operatorname{Nexc}\sigma = \sigma(j_1)\ldots\sigma(j_{n-k})$ are referred to as the *excedence-letter* and *non-excedence-letter* subwords.

Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ be a permutation. Let $i_1, \ldots, i_k$ be its excedence-place sequence. Then the Denert statistic of σ is also given by

$$\operatorname{den}\sigma = i_1 + i_2 + \cdots + i_k + \operatorname{inv}\operatorname{Exc}\sigma + \operatorname{inv}\operatorname{Nexc}\sigma.$$

For instance, for the permutation $\sigma = \begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\ 7\,1\,5\,4\,9\,2\,6\,3\,8\end{pmatrix}$ shown in § 2.3 we have $i_1 = 1$, $i_2 = 3$, $i_3 = 5$, $\operatorname{Exc}\sigma = 7,5,9$, $\operatorname{Nexc}\sigma = 1,4,2,6,3,8$, $\operatorname{inv}\operatorname{Exc}\sigma = 1$, $\operatorname{inv}\operatorname{Nexc}\sigma = 3$, and then $\operatorname{den}\sigma = 1+3+5+1+3 = 13$.

15. *Euler-Mahonian Statistics.* In Definition 10.1 the q-maj-Eulerian ${}^{\mathrm{maj}}A(t,q) = \sum_{k\ge 0} t^k\, {}^{\mathrm{maj}}A_{n,k}(q)$ have been introduced in four different ways.

To simplify the notations the superscript "maj" will be deleted in this Exercise. For an easy reference rewrite the recurrence for the coefficents $A_{n,k}(q) = {}^{\mathrm{maj}}A_{n,k}(q)$:

$$(E15.1)\qquad A_{n,k}(q) = [k+1]_q\, A_{n-1,k}(q) + q^k\, [n-k]_q\, A_{n-1,k-1}(q),$$

for $1 \le k \le n-1$ with the initial conditions $A_{n,0}(q) = 1$ for $n \ge 0$ and $A_{n,k}(q) = 0$ for $k \ge n$.

Let $E = (E_n)$ $(n \ge 0)$ be a family of finite sets such that $\operatorname{card} E_n = n!$ for all $n \ge 0$. A family $(f, g) = (f_n, g_n)$ $(n \ge 0)$ is said to be *Euler-Mahonian* on E, if $f_0 = g_0 = 0$, $f_1 = g_1 = 0$ and if for every $n \ge 2$ both f_n and g_n are integral-valued mappings defined on E_n and if there exists a bijection $\psi_n : (w', j) \mapsto w$ of $E_{n-1} \times [0, n-1]$ onto E_n having the following properties:

$$(E15.2)\qquad \begin{aligned} &g_n(w) = g_{n-1}(w') + j;\\ &f_n(w) = \begin{cases} f_{n-1}(w'), & \text{if } 0 \le j \le f_{n-1}(w');\\ f_{n-1}(w') + 1, & \text{if } f_{n-1}(w') + 1 \le j \le n-1. \end{cases} \end{aligned}$$

Every such a pair (f_n, g_n) is called a *Euler-Mahonian* statistic on E_n.

(a) Let (f, g) be a Euler-Mahonian family on E and for each triple (n, k, l) let $A_{n,k,l}$ be the number of elements $w \in E_n$ such that $f_n(w) = k$ and $g_n(w) = l$ and let $A_{n,k}(q) := \sum_l A_{n,k,l}\, q^l$. Then $(A_{n,k}(q))$ satisfies recurrence (E15.1).

(b) On the set SE_n of the subexcedent sequences of length n (see Proposition 2.1) we know that "tot" is Mahonian. Further, define the *Eulerian value* "$\mathrm{eul}\, x$" of a sequence $x = (x_1, \ldots, x_n) \in \mathrm{SE}_n$ by $\mathrm{eul}\, x = 0$ if x is of length 1 and for $n \ge 2$

$$\mathrm{eul}\, x := \begin{cases} \mathrm{eul}(x_1, \ldots, x_{n-1}), & \text{if } x_n \le \mathrm{eul}(x_1, \ldots, x_{n-1})\ ;\\ \mathrm{eul}(x_1, \ldots, x_{n-1}) + 1, & \text{if } x_n \ge \mathrm{eul}(x_1, \ldots, x_{n-1}) + 1. \end{cases}$$

Then the pair $(\mathrm{eul}, \mathrm{tot})$ is a Euler-Mahonian statistic on the family (SE_n) $(n \ge 0)$.

(c) The pair $(\mathrm{des}, \mathrm{maj})$ is a Euler-Mahonian statistic on the family $(\mathfrak{S}_n)$ $(n \ge 0)$.

(d) Let $n \ge 2$ and $\sigma' = x_1 \ldots x_{n-1}$ be a permutation having k excedences, that is, there are k integers i such that $1 \le i \le n-1$ and $i < x_i$. In short, $\operatorname{exc} \sigma' = k$. Let $(x_{i_1} > \cdots > x_{i_k})$ be the *decreasing* sequence of the excedence values $x_i > i$ and let $(x_{i_{k+1}} < \cdots < x_{i_{n-1}})$ be the *increasing* sequence of the non-excedence values $x_i \le i$. By convention, $x_{i_0} := n$.

Define $\psi_n(\sigma',0) := x_1x_2\dots x_{n-1}n$. If $1 \le j \le n-1$, let $y_j := \min\{x_{i_m} : x_{i_m} \ge x_{i_j}\}$. Replace each letter x_{i_m} $(1 \le m \le j)$ in σ' such that $x_{i_m} \ge x_{i_j}$ by $x_{i_{m-1}}$, leave the other letters alike and insert y_j into the x_{i_j}-th position in σ'. Let $\sigma = \psi_n(\sigma',j)$ denote the permutation derived by that procedure.

For example, $\sigma' = 3\,2\,5\,4\,1$ has $k=2$ excedences $x_3 = 5 > 3$, $x_1 = 3 > 1$ (in decreasing order) and three non-excedences $x_5 = 1 \le 5$, $x_2 = 2 \le 2$, $x_4 = 4 \le 4$ (in increasing order), so that $(i_1,i_2,i_3,i_4,i_5) = (3,1,5,2,4)$. With $j=1$ we have $i_j = 3$ and $x_3 = 5$. For getting $\psi_6(\sigma',1)$ replace $x_{i_1} = 5$ by $x_{i_0} = 6$, leave the other letters alike and insert $x_{i_1} = 5$ into the x_{i_1}-th $=5$-th position. Thus, $\psi_6(\sigma',1) = 3\,2\,6\,4\,5\,1$.

For $j = 3$ we have $i_j = 5$ and $x_5 = 1$. As $j = 3 > k = 2$, replace $x_{i_1} = x_3$ by $x_{i_0} = 6$, then $x_{i_2} = x_1 = 3$ by $x_{i_1} = 5$, leave the other letters alike and insert $x_{i_k} = x_{i_2} = x_1 = 3$ into the x_{i_3}-th $=$ 1-st position to obtain $\psi_6(\sigma',3) = 3\,5\,2\,6\,4\,1$.

With $(f,g) = (\mathrm{exc},\mathrm{den})$ (see section 2) properties (E15.2) hold for ψ_n, so that $(\mathrm{exc},\mathrm{den})$ is a Euler-Mahonian statistic on the family $(\mathfrak{S}_n)$ $(n \ge 0)$.

Let us illustrate the latter property with the running example. The statistic "den" is calculated by using the definition of Exercice 14. We have: $\sigma' = \begin{pmatrix}\mathbf{1}\,2\,\mathbf{3}\,4\,5\\ 3\,2\,5\,4\,1\end{pmatrix}$, so that $\mathrm{exc}\,\sigma' = 2$ and $\mathrm{den}\,\sigma' = (\mathbf{1}+\mathbf{3})+0+2 = 6$. Next $\psi_6(\sigma',1) = \begin{pmatrix}\mathbf{1}\,2\,\mathbf{3}\,4\,5\,6\\ 3\,2\,6\,4\,5\,1\end{pmatrix}$, so that $\mathrm{exc}\,\psi_6(\sigma',1) = 2$ and $\mathrm{den}\,\psi_6(\sigma',1) = (\mathbf{1}+\mathbf{3})+0+3 = 7 = \mathrm{den}\,\sigma'+1$. Finally, $\psi_6(\sigma',3) = \begin{pmatrix}\mathbf{1}\,\mathbf{2}\,3\,\mathbf{4}\,5\,6\\ 3\,5\,2\,6\,4\,1\end{pmatrix}$, so that $\mathrm{exc}\,\psi_6(\sigma',3) = 3$ and $\mathrm{den}\,\psi_6(\sigma',3) = (\mathbf{1}+\mathbf{2}+\mathbf{4})+0+2 = 9 = \mathrm{den}\,\sigma'+3$.

(e) Let (f,g) be a Euler-Mahonian family on $E = (E_n)$. For each $w \in E_n$ $(n \ge 2)$ let $\psi_n^{-1}(w) := (w', j_n)$, $\psi_{n-1}^{-1}(w') := (w'', j_{n-1})$, … , $\psi_2^{-1}(w^{(n-2)}) := (w^{(n-1)}, j_2)$ and $j_1 := 0$; the sequence $\Psi(w) := (j_1, j_2, \dots, j_{n-1}, j_n)$ is subexcedent and Ψ is a bijection if E_n onto SE_n such that $f(w) = \mathrm{tot}\,\Psi(w)$ and $g(w) = \mathrm{eul}\,\Psi(w)$. The bijection Ψ is said to be an *(f,g)-coding E_n*.

Let $\Psi_{(\mathrm{des},\mathrm{maj})}$ (resp. $\Psi_{(\mathrm{exc},\mathrm{den})}$) be the $(\mathrm{des},\mathrm{maj})$-coding (see question (c))) (resp. the $(\mathrm{exc},\mathrm{den})$-coding (see question (d)) of $\mathfrak{S}_n$. Then $\Theta := \Psi^{-1}_{(\mathrm{des},\mathrm{maj})} \circ \Psi_{(\mathrm{exc},\mathrm{den})}$ is a bijection of $\mathfrak{S}_n$ onto itself having the property: $(\mathrm{exc},\mathrm{den})\,w = (\mathrm{des},\mathrm{maj})\,\Theta(w)$.

16. *Binary words.* Let $\mathrm{BW}(N,n)$ be the set of binary words of length $(N+n)$ containing N times 1 and n times 0. If $x = x_1x_2\dots x_{N+n}$ is such a word, define

$$\mathrm{rise}\,x := \sum_{1\le i\le N+n-1} \chi(x_i < x_{i+1}) \quad \text{and} \quad \mathrm{rmaj}\,x := \sum_{1\le i\le N+n-1} i\,\chi(x_i < x_{i+1}).$$

Next, let $\mathrm{DES}\, x := \mathrm{des}\, x + \mathrm{rise}\, x$ and $\mathrm{MAJ}\, x := \mathrm{maj}\, x + \mathrm{rmaj}\, x$.

(a) For each of the words in $\mathrm{BW}(2,3)$ write the values of the six statistics "des", "maj", "rise", "rmaj", "DES", "MAJ".

(b) $\displaystyle\sum_{x\in \mathrm{BW}(N,n)} q^{\mathrm{rmaj}\, x} = \begin{bmatrix} N+n \\ n \end{bmatrix}.$

(c) $\displaystyle\sum_{x\in \mathrm{BW}(N,n)} t^{\mathrm{rise}\, x} q^{\mathrm{rmaj}\, x} = \sum_{x\in \mathrm{BW}(N,n)} t^{\mathrm{des}\, x} q^{\mathrm{maj}\, x}.$

(d) For $x \in \mathrm{BW}(N,n)$ let $x' := x_1 x_2 \cdots x_{N+n-1}$. Then

$$\begin{cases} \mathrm{maj}\, x = \mathrm{rise}\, x + N = \mathrm{rmaj}\, x' + N, & \text{if } x_{N+n} = 1; \\ \mathrm{rmaj}\, x = \mathrm{maj}\, x + n = \mathrm{rmaj}\, x' + n, & \text{if } x_{N+n} = 0. \end{cases}$$

(e) $\displaystyle\sum_{x\in \mathrm{BW}(N,n)} q^{\mathrm{MAJ}\, x} = \begin{bmatrix} N+n \\ n \end{bmatrix}_{q^2} \frac{q^N + q^n}{1 + q^{N+n}}.$

17. *The Z-statistic is a Mahonian statistic.* Two transformations are described, the *global cycling* "gcyc" for manipulating the Major Index and the *local cycling* "lcyc" for dealing with the Z-statistic itself.

(a) Let $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ be a *multiplicity*, that is, a sequence of positive integers and $R(\mathbf{m})$ be the class of all the rearrangements of the word $1^{m_1} 2^{m_2} \ldots r^{m_r}$. Let $\mathbf{n}$ be a rearrangement of $\mathbf{m}$. Construct a bijection $\theta_{\mathbf{m},\mathbf{n}}$, defined on $R(\mathbf{m})$, with values in $R(\mathbf{n})$, preserving "maj." [It suffices to give the construction when $\mathbf{m}$ and $\mathbf{n}$ differ by two *consecutive* letters x and $y = x+1$; in other words, construct a bijection θ of $R(m_1, \ldots, m_x, m_y, \ldots, m_r)$ onto $R(m_1, \ldots, m_y, m_x, \ldots, m_r)$.]

(b) The Z-statistic is defined, for each word $w = x_1 x_2 \ldots x_m$, by

$$Z(w) := \sum_{i<j} \mathrm{maj}\, w_{ij},$$

where w_{ij} stands for the subword of w made of all the letters i and j. For example, for the word $w = 2412131242$, the subwords $w_{12} = 2121122$, $w_{13} = 1131$, … , are to be considered, the Major Indices are to be calculated and their sum to be added up. Calculate $Z(2412131242)$.

(c) For each word $w = x_1 x_2 \cdots x_m$ in $R(\mathbf{m})$ and each letter $x = 1, 2 \ldots, r$ the *global cycling* $\mathrm{gcyc}_x(w) = y_1 y_2 \cdots y_m$ and the *local cycling* $\mathrm{lcyc}_x(w) = z_1 z_2 \cdots z_m$ are defined by

$$y_i := \begin{cases} x_i - x, & \text{si } x_i > x; \\ x_i - x + r, & \text{sinon.} \end{cases} \qquad z_i := \begin{cases} x_i, & \text{si } x_i < x; \\ x_i - 1, & \text{si } x_i > x; \\ r, & \text{si } x_i = x. \end{cases}$$

Let $\mathbf{m}^x$ denote the multiplicity of $\mathrm{gcyc}_x(w)$ and $\mathbf{m}_x$ the multiplicity of $\mathrm{lcyc}_x(w)$. Characterize those two multiplicities, that is to say, express $\mathbf{m}^x$ and $\mathbf{m}_x$ as rearrangements of the sequence $\mathbf{m} = (m_1, m_2, \ldots, m_r)$.

(d) For the word $w = 2412131242$ calculate the differences $\mathrm{maj}\, w - \mathrm{maj}\, \mathrm{gcyc}_2(w)$ et $Z(w) - Z(\mathrm{lcyc}_2(w))$.

(e) If $x = x_m$ is the last letter of w, show that

$$\mathrm{maj}\, w - \mathrm{maj}(\mathrm{gcyc}_x(w)) = m_{x+1} + m_{x+2} + \cdots + m_r.$$

(f) If $x = x_m$ is the last letter of w, show that

$$Z(w) - Z(\mathrm{lcyc}_x(w)) = m_{x+1} + m_{x+2} + \cdots + m_r.$$

(g) Construct a bijection of $R(\mathbf{m})$ onto itself having the property

$$\mathrm{maj}\, w = Z(\Phi(w)),$$

thus proving that the Z-statistic is *Mahonian* on each class of rearrangements.

18. *The $t = 1$ Lemma.* Let (b_r) $(r \geq 0)$ be a sequence of formal series belonging to an algebra $\mathfrak{A}$ of formal series in one or several variables. Let t be a new variable; we can form the series $b(t) := \sum_{r \geq 0} b_r t^r$, that belongs to the algebra $\mathfrak{A}[[t]]$. 'Let $t = 1$" in $b(t)$ does not always make sense. However, if the series $\sum_r b_r$ converges for the topology of the formal series in $\mathfrak{A}$, that is, if there exists $a \in \mathfrak{A}$ such that the order $o((b_0 + b_1 + \cdots + b_r) - a)$ tends to $+\infty$ with r, we can *define* "let $t = 1$ in $b(t)$" and then $b(1)$ by: $b(1) := a = \sum_{r \geq 0} b_r$.

(a) Let (a_r) $(r \geq 0)$ be a sequence of formal series in $\mathfrak{A}$ such that $\lim_r a_r = a$, that is, such that $o(a - a_r)$ tends to infinity with r. We define: $b(t) := (1 - t) \cdot \sum_{r \geq 0} a_r t^r$. Then $b(1) = a$.

(b) Deduce (6.13) from (7.8).

(c) Deduce (12.3) from (13.7).

19. *The dihedral group.* On the group $\mathfrak{S}_n$ of the permutations on order n three transformations $\mathbf{i}$, $\mathbf{r}$ and $\mathbf{c}$ can be defined in the following way. First, $\mathbf{i}$ is the bijection that maps each permutation σ onto its *inverse* σ^{-1}. Write σ as a linear word $\sigma = \sigma(1) \ldots \sigma(n)$. Then, define

$$\mathbf{c}\, \sigma := (n + 1 - \sigma(1))(n + 1 - \sigma(2)) \ldots (n + 1 - \sigma(n);$$
$$\mathbf{r}\, \sigma := \sigma(n)\, \ldots\, \sigma(2)\sigma(1).$$

We say that $\mathbf{c}$ is the *complement to* $(n + 1)$ and $\mathbf{r}$ the *reverse image.*

(a) $\mathbf{r} = \mathbf{i}\,\mathbf{c}\,\mathbf{i}$.

(b) The group acting on $\mathfrak{S}_n$ generated by $\{\mathbf{i}, \mathbf{c}\}$ is isomorphic to the dihedral group D_4 of order 8 of the rotations of the square.

(c) $\mathbf{r}\,\mathbf{c} = \mathbf{c}\,\mathbf{r}$, $\mathbf{i}\,\mathbf{r} = \mathbf{c}\,\mathbf{i}$ et $\mathbf{i}\,\mathbf{r}\,\mathbf{c} = \mathbf{r}\,\mathbf{c}\,\mathbf{i}$.

(d) For each $\sigma \in \mathfrak{S}_n$ the following relations hold: $\operatorname{Ligne} \mathbf{c}\,\sigma = [n-1] \setminus \operatorname{Ligne} \sigma$ and $\operatorname{Ligne} \mathbf{r}\,\mathbf{c}\,\sigma = n - \operatorname{Ligne}\sigma = \{n - i : i \in \operatorname{Ligne}\sigma\}$.

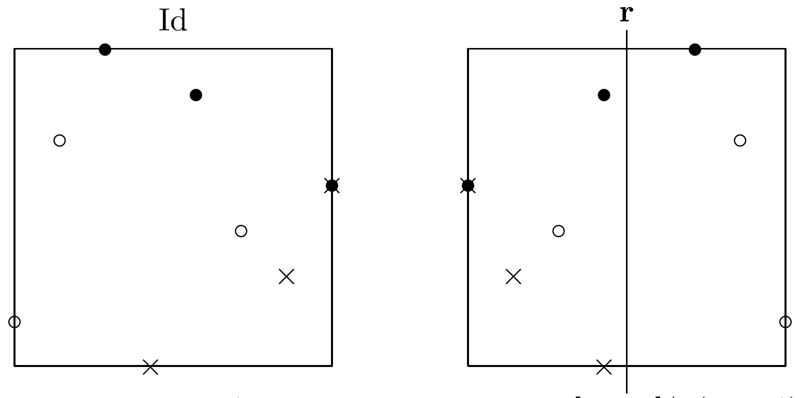
20. *Action of the dihedral group.* By means of an example the action of the dihedral group on several set-valued statistics, such as "Ligne," "Iligne," the various extremum letter or place subsets (see § 11.4) is examined.

In Fig. E20 the dihedral group of order 8 is acting on the permutation $\sigma = 2, 6, 8, 1, 7, 4, 3, 5$, whose graph appears on the first square entitled "Id". The ligne of route, $\operatorname{Ligne}\sigma$, of σ (the set of the "places" where a descent occurs) is denoted by A. Here $A = \{3, 5, 6\}$. The inverse ligne of route, $\operatorname{Iligne}\sigma$, of σ (the set of the "letters" i occurring to the right of the letters $(i+1)$) is denoted by B; here $B = \{1, 3, 5, 7\}$.

The **R**ight to left **M**aximum **l**etter set, $\operatorname{Rmals}\sigma$, of σ (the set of the letters greater than all the letters to their right) is denoted by C. The elements of $\operatorname{Rmals}\sigma$ are the ordinates of the bullets "$\bullet$". Here $C = \{5, 7, 8\}$.

The **R**ight to left **m**inimum **l**etter set, $\operatorname{Rmil}\sigma$, of σ (the set of the letters less than all the letters to their right) is denoted by D. The elements of $\operatorname{Rmals}\sigma$ are the ordinates of the crosses "$\times$". Here $D = \{1, 3, 5\}$.

Each graph corresponds to the action of an element Id, $\mathbf{r}$, $\mathbf{c}$, $\mathbf{r}\,\mathbf{c}$, $\mathbf{i}$, $\mathbf{i}\,\mathbf{r}$, $\mathbf{i}\,\mathbf{c}$, $\mathbf{i}\,\mathbf{r}\,\mathbf{c}$ of the dihedral group on the permutation σ. Notice that $n - A$ is to be understood as the set $\{n - x : x \in A\}$. Under each graph the ligne of route and the inverse ligne of route have been determined. Under graph $\mathbf{r}\,\mathbf{c}$, for instance, $\text{Ligne} = n - A$ and $\text{Iligne} = n - B$ are to be read $\operatorname{Ligne} \mathbf{r}\,\mathbf{c}\,\sigma = n - A$ and and $\operatorname{Iligne} \mathbf{r}\,\mathbf{c}\,\sigma = n - B$.



$\text{Ligne} = A$ | $\text{Ligne} = [n-1] \setminus (n - A)$

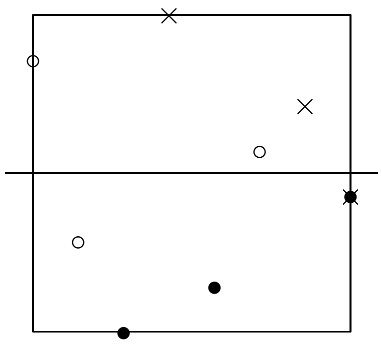
$\text{Iligne} = B$ | $\text{Iligne} = [n-1] \setminus B$

$\text{Rmals} = C$ | $\text{Lmals} = C$

$\text{Rmil} = D$ | $\text{Lmil} = D$

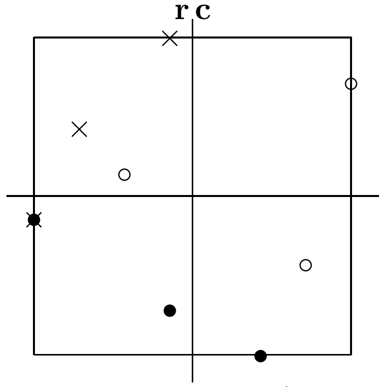
Fig. E20

c



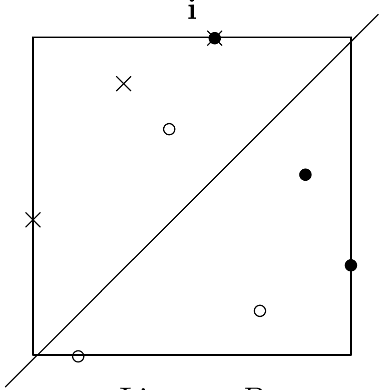
Ligne $= [n-1] \setminus A$
Iligne $= [n-1] \setminus (n-B)$
Rmil $= n+1-C$
Rmals $= n+1-D$

r c



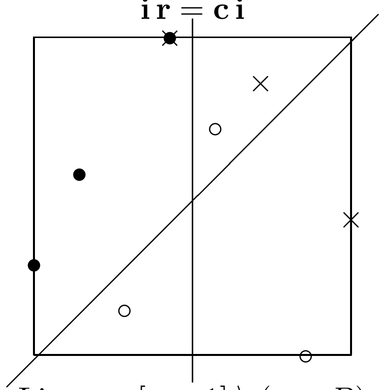
Ligne $= n-A$
Iligne $= n-B$
Lmil $= n+1-C$
Lmals $= n+1-D$

i



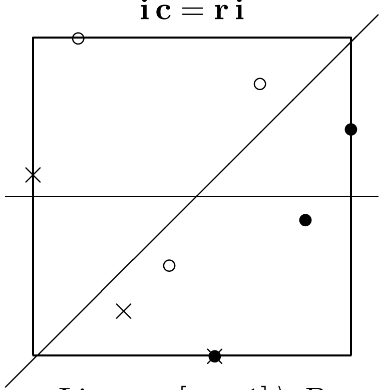
Ligne $= B$
Iligne $= A$
Rmap $= C$
Lmap $= D$

i r = c i



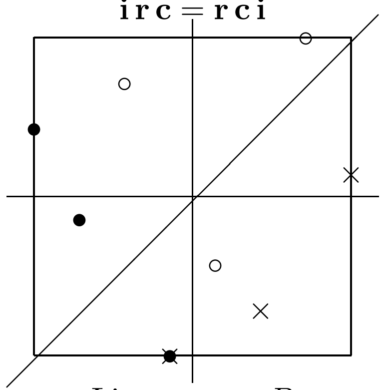
Ligne $= [n-1] \setminus (n-B)$
Iligne $= [n-1] \setminus A$
Lmap $= n+1-C$
Rmap $= n+1-D$

i c = r i



Ligne $= [n-1] \setminus B$
Iligne $= [n-1] \setminus (n-A)$
Rmip $= C$
Lmip $= D$

i r c = r c i



Ligne $= n-B$
Iligne $= n-A$
Lmip $= n+1-C$
Rmip $= n+1-D$

Notice that whenever the transformation “**i**” is applied the “l” for “letter” is replaced by the “p” for “place”. Under graph **i**, for instance, $\mathrm{Rmap} = C$ means that the **R**ight-to-left **M**aximum **p**lace of $\mathbf{i}\,\sigma$ is equal to C, i.e., $\mathrm{Rmap}\,\mathbf{i}\,\sigma = C$.

21. *Variations on the cycle number polynomial.* Let $\mathrm{cyc}\,\sigma$ denote the number of cycles of a permutation σ and let $C_n(x) := \sum_\sigma x^{\mathrm{cyc}\,\sigma}$ $(\sigma \in \mathfrak{S}_n)$.

(a) Then, $C_n(x) = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$.

(b) As in Exercice 20 let $\mathrm{Rmals}\,\sigma$ (resp. $\mathrm{Rmil}\,\sigma$) be the *set* of the *letters* greater than (resp. smaller than) all the letters located on their right in $\sigma = \sigma(1)\cdots\sigma(n)$. Let $\mathrm{Rmap}\,\sigma$ (resp. $\mathrm{Rmip}\,\sigma$) be the *set* of the *places* of those letters, respectively. Recall that $\#\,\mathrm{Rmals}\,\sigma$ (resp. $\#\,\mathrm{Rmil}\,\sigma$) designates the *number* of those letters. For instance, with $\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 3\,1\,5\,6\,2\,4 \end{pmatrix}$ we have $\mathrm{Rmals}\,\sigma = \{6,4\}$, $\mathrm{Rmap}\,\sigma = \{4,6\}$. $\mathrm{Rmil}\,\sigma = \{1,2,4\}$, $\mathrm{Rmip}\,\sigma = \{2,5,6\}$, so that $\#\,\mathrm{Rmals}\,\sigma = \#\,\mathrm{Rmap}\,\sigma = 2$, $\#\,\mathrm{Rmil}\,\sigma = \#\,\mathrm{Rmip}\,\sigma = 3$. For each $n \ge 1$ we have: $C_n(x) = \sum_\sigma x^{\#\,\mathrm{Rmals}\,\sigma} = \sum_\sigma x^{\#\,\mathrm{Rmap}\,\sigma} = \sum_\sigma x^{\#\,\mathrm{Rmil}\,\sigma} = \sum_\sigma x^{\#\,\mathrm{Rmip}\,\sigma}$ $(\sigma \in \mathfrak{S}_n)$.

(c) Let $C_n(x,y,q) := \sum_\sigma x^{\#\,\mathrm{Rmap}\,\sigma} y^{\#\,\mathrm{Rmip}\,\sigma} q^{\mathrm{inv}\,\sigma}$ $(\sigma \in \mathfrak{S}_n)$. Then, $C_n(x,y,q) = xy(y+qx)(y+q+q^2x)\cdots(y+q+q^2+\cdots+q^{n-2}+q^{n-1}x)$.

(d) Let $w = x_1x_2\ldots x_n$ be the maj-coding of $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. Then, $\mathrm{Rmil}\,\sigma = \{i : 1 \le i \le n, x_i = 0\}$.

(e) Let $D_n(y,q) := \sum_\sigma y^{\#\,\mathrm{Rmil}\,\sigma} q^{\mathrm{maj}\,\sigma}$ $(\sigma \in \mathfrak{S}_n)$. Then, $D_n(y,q) = y(y+q)(y+q+q^2)\cdots(y+q+q^2+\cdots+q^{n-1})$.

(f) Starting with the expression obtained in (c) for $C_n(x,y,q)$ derive

$$\sum_{n\ge 0} C_n(q,x,y)\frac{u^n}{(q;q)_n} = 1 - \frac{xy}{x+y-1} + \frac{xy}{x+y-1}\,\frac{\Bigl(\dfrac{u}{1-q} - ux; q\Bigr)_\infty}{\Bigl(uy + \dfrac{uq}{1-q}; q\Bigr)_\infty}.$$

22. *A lower-record extension of the q-maj-Eulerian polynomial.* For each $n \ge 0$ let $A_n(t,q,y) := \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}\,w} q^{\mathrm{maj}\,w} y^{\#\,\mathrm{Rmil}\,\sigma} = \sum_k t^k A_{n,k}$.

(a) The following induction formula holds

$$A_{n,k} = (y+q+q^2+\cdots+q^k)A_{n-1,k} + (q^k+\cdots+q^{n-1})A_{n-1,k-1},$$

that implies

$$\begin{aligned}(1-q)&A_n(t,q,1,y)\\ &= \bigl(y(1-q)+q-tq^n\bigr)A_{n-1}(t,q,1,y) - q(1-t)A_{n-1}(tq,q,1,y),\end{aligned}$$

for $n \geq 1$ and $A_0(t,q,1,y)=0$.

(b) Next, derive

$$\sum_{n\geq 0} A_n(t,q,y)\frac{u^n}{(t;q)_{n+1}} = \sum_{s\geq 0} t^s \frac{\Bigl(\dfrac{uq}{1-q+uq^{s+1}};q\Bigr)_{s+1}}{\Bigl(u\dfrac{y(1-q)+q}{1-q+uq^{s+1}};q\Bigr)_{s+1}}$$

that specializes into the generating function (10.2a) for the q-maj-Eulerian polynomials ${}^{\mathrm{maj}}A_n(t,q)$ when $y=1$.

23. *The tableau emptying-filling involution.* Consider a standard Young tableau P of order n whose entries are the integers $1, 2, \ldots, n$, for example the following one with $n=5$ and consider the tableaux successively derived from P:

$$P = \begin{array}{|lll} 3 & 4 & \\ 1 & 2 & 5 \\ \hline \end{array}$$

$$\mapsto \begin{array}{|lll} 3 & 4 & \\ \bullet & 2 & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} 3 & 4 & \\ 2 & \bullet & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} 3 & \bullet & \\ 2 & 4 & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} 3 & \mathbf{5} & \\ 2 & 4 & 5 \\ \hline \end{array} \qquad \text{(new entry: } \mathbf{5}\text{)}$$

$$\mapsto \begin{array}{|lll} 3 & \mathbf{5} & \\ \bullet & 4 & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} \bullet & \mathbf{5} & \\ 3 & 4 & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 3 & 4 & 5 \\ \hline \end{array} \qquad \text{(new entry: } \mathbf{4}\text{)}$$

$$\mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ \bullet & 4 & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 4 & \bullet & 5 \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 4 & 5 & \bullet \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 4 & 5 & \mathbf{3} \\ \hline \end{array} \qquad \text{(new entry: } \mathbf{3}\text{)}$$

$$\mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ \bullet & 5 & \mathbf{3} \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 5 & \bullet & \mathbf{3} \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ 5 & \mathbf{2} & \mathbf{3} \\ \hline \end{array} \qquad \text{(new entry: } \mathbf{2}\text{)}$$

$$\mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ \bullet & \mathbf{2} & \mathbf{3} \\ \hline \end{array} \mapsto \begin{array}{|lll} \mathbf{4} & \mathbf{5} & \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \hline \end{array} =: P^J \qquad \text{(new entry: } \mathbf{1}\text{)}$$

The entries are taken from the following set $\{1,2,3,4,5,\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\bullet\}$. When a tableau contains a bullet "$\bullet$," adopt the convention that the *neighbors* of the bullet are the *roman typed* entries just above or on the right of it. There are zero, one or two such neighbors. The passage "$\mapsto$" to the next tableau is defined as follows:

(i) if the tableau has no bullet, either the tableau has no *roman typed* entries and the final tableau P^J is reached, or there is such an entry, in particular on the leftmost bottom corner; call it x, then replace x by the bullet "$\bullet$";

(ii) if the tableau contains a bullet, consider the neighbors of the bullet; if there is no neighbor and the number of boldface typed entries is i, replace the bullet by the boldface typed $(\mathbf{n}-\mathbf{i})$; if there is one neighbor, say, x, permute the two entries "$\bullet$" and "x"; if there are two neighbors, let x be the smallest entry of the two and again permute the two entries "$\bullet$" and "x."

It is obvious that P^J is a *standard Young tableau* having the *same shape* as P. Let $P \mapsto P^T$ be the *transposition* operation acting on tableaux. Further properties are the following:

(i) *the map $P \mapsto P^J$ is an involution (called the tableau emptying-filling involution) having the property:* $\operatorname{Iligne} P^J = n - \operatorname{Iligne} P$;

(ii) *J and T commute;*

(iii) *if $\sigma \mapsto PQ$ under the Robinson-Schensted correspondence, then $\mathbf{r}\,\sigma \mapsto P^T Q^{JT}$.*

(The operations $\mathbf{r}$, as well as $\mathbf{c}$ and $\mathbf{i}$, are defined in Exercice 19.)

Properties (i) and (iii) are not straightforward, the J-algorithm being not easy to handle (see [Sch63, 77] for a proof or use the *jeu de taquin* approach as developed, for instance, in [Lo02, chap. 5]).

(a) Using the above notations, if $\sigma \mapsto PQ$ (and $\mathbf{r}\,\sigma \mapsto P^T Q^{JT}$ by (iii)), then $\mathbf{c}\,\sigma \mapsto P^{JT} Q^T$.

(b) Define the involutions $\mathbf{t}$ and $\mathbf{j}$ of $\mathfrak{S}_n$ by $\sigma \mapsto PQ \mapsto P^T Q^T \mapsto \mathbf{t}\,\sigma$, and $\sigma \mapsto PQ \mapsto PQ^J \mapsto \mathbf{j}\,\sigma$, where, in each case, the last arrow "$\mapsto$" refers to the inverse of the Robinson-Schensted correspondence. Then $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{t}^2 = (\mathbf{i}\,\mathbf{j})^4 = 1$, $\mathbf{i}\,\mathbf{t} = \mathbf{t}\,\mathbf{i}$, $\mathbf{j}\,\mathbf{t} = \mathbf{t}\,\mathbf{j}$. Also $\mathbf{r} = \mathbf{t}\,\mathbf{j} = \mathbf{j}\,\mathbf{t}$, $\mathbf{c} = \mathbf{i}\,\mathbf{j}\,\mathbf{t}\,\mathbf{i}$, so that the group generated by $\{\mathbf{i}, \mathbf{t}, \mathbf{j}, \mathbf{r}, \mathbf{c}\}$ is of order 16, in fact, the direct product of the dihedral group D_4 generated by $\{\mathbf{i}, \mathbf{j}\}$ by the group of two elements $\{1, \mathbf{t}\}$.

(c) Let $PQ = \begin{array}{|lll}3&4&\\1&2&5\\\hline\end{array}\ \begin{array}{|lll}2&4&\\1&3&5\\\hline\end{array}$ be the pair of standard Young tableaux associated with the permutation $\sigma = 3\,1\,4\,2\,5$ under the Robinson-Schensted correspondence. Determine P^J, Q^J, P^T, Q^T, P^{JT}, Q^{JT} and the sixteen pairs of standard Young tableaux associated with the sixteen permutations in the orbit of σ with respect to the group defined in (b).

(d) As already described in section 19 the following two properties hold: $\operatorname{Ligne} \mathbf{j}\,\sigma = n - \operatorname{Ligne} \sigma$ and $\operatorname{Iligne} \mathbf{j}\,\sigma = \operatorname{Iligne} \sigma$.

24. *A classical tool for symmetry proving.* Let σ be a permutation of $1, \ldots, r$ and $\mathbf{m} = (m_1, \ldots, m_r)$ be a sequence of nonnegative integers. Let $R(\mathbf{m})$ (resp. $R(\sigma\mathbf{m})$) denote the class of the rearrangements of $1^{m_1} \ldots r^{m_r}$ (resp. the rearrangements of $1^{m_{\sigma(1)}} \ldots r^{m_{\sigma(r)}}$). As noticed in Remark 8.2, $A_{\mathbf{m}}(t, q) = A_{\sigma\mathbf{m}}(t, q)$. A combinatorial proof of that property can be made as follows.

It suffices to prove the property when σ is a transposition $(i, i+1)$ of two adjacents integers ($1 \le i \le r - 1$). Consider a word w in $R(\mathbf{m})$ and write all its factors of the form $(i+1)i$ in bold-face; then replace all the maximal factors of the form $i^p(i+1)^q$ that do not involve any boldface letters by $i^q(i+1)^q$. Finally, rewrite all the boldface letters in roman type. Clearly, the transformation is a bijection that maps each word w in $R(\mathbf{m})$ onto a word w' in $R((i,i+1)\mathbf{m})$ with the property that $(\mathrm{des},\mathrm{maj})\,w = (\mathrm{des},\mathrm{maj})\,w'$.

If we take the statement of Theorem 17.1 as the definition of the Schur function $s_\lambda(x)$, the symmetric nature of $s_\lambda(x)$ does not appear. To still prove that the Schur function is symmetric, we can use a similar argument.

25. *Line of route-indexed Eulerian polynomials.* Let $L = \{\ell_1 < \cdots < \ell_k\}$ be a subset of $\{1,2,\ldots,n-1\}$ with the conventions $\ell_0 := 0$ and $\ell_{k+1} := n$. The L-indexed Eulerian polynomial $A_L(t,q)$ is defined to be

$$A_L(t,q) := \sum_\sigma t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma} \quad (\sigma \in \mathfrak{S}_n,\ \mathrm{Ligne}\,\sigma = L).$$

Form the $(k+1)\times(k+1)$ matrix $N_r(L,n)$:

$$N_r(L,n) = \begin{pmatrix} \begin{bmatrix}\ell_1-\ell_0+r\\ r\end{bmatrix} & \begin{bmatrix}\ell_2-\ell_0+r\\ r\end{bmatrix} & \ldots & \begin{bmatrix}\ell_k-\ell_0+r\\ r\end{bmatrix} & \begin{bmatrix}\ell_{k+1}-\ell_0+r\\ r\end{bmatrix} \\ 1 & \begin{bmatrix}\ell_2-\ell_1+r\\ r\end{bmatrix} & \ldots & \begin{bmatrix}\ell_k-\ell_1+r\\ r\end{bmatrix} & \begin{bmatrix}\ell_{k+1}-\ell_1+r\\ r\end{bmatrix} \\ 0 & 1 & \ldots & \begin{bmatrix}\ell_k-\ell_2+r\\ r\end{bmatrix} & \begin{bmatrix}\ell_{k+1}-\ell_2+r\\ r\end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \begin{bmatrix}\ell_{k+1}-\ell_k+r\\ r\end{bmatrix} \end{pmatrix}.$$

In particular, $N_r(\emptyset,n) = \begin{bmatrix}n+r\\ r\end{bmatrix}$. The purpose of this exercise is to prove the formula:

$$(E22.1) \qquad \frac{A_L(t,q)}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \det N_r(L,n).$$

(a) Let $W_r(L,n)$ denote the set of the words $w = w_1w_2\ldots w_n$, of length n, whose letters are nonnegative integers satisfying the identities

$$(*)_{(L,n)} \quad r \ge w_1 \ge \cdots \ge w_{\ell_1} \ge 0; \quad r \ge w_{\ell_1+1} \ge \cdots \ge w_{\ell_2} \ge 0; \quad \ldots \quad r \ge w_{\ell_k+1} \ge \cdots \ge w_n \ge 0;$$
$$w_{\ell_1} < w_{\ell_1+1}, \quad w_{\ell_2} < w_{\ell_2+1}, \quad \ldots, \quad w_{\ell_k} < w_{\ell_k+1}.$$

Then $\displaystyle\sum_{w\in W_r(L,n)} q^{\mathrm{tot}\,w} = \det N_r(L,n)$. [By induction on k.]

(b) Let $S_r(L,n)$ be the set of all pairs (σ,s), where σ is a permutation of $1,2,\ldots,n$, of ligne of route L, and where $s=s_1s_2\ldots s_n$ is a nonincreasing word having the properties:

$$\text{(E22.2)}\qquad \begin{gathered} r\ge s_1\ge s_2\ge\cdots\ge s_n\ge 0;\\ \operatorname{Ligne}\sigma=L;\quad i\in\operatorname{Iligne}\sigma\Rightarrow s_i>s_{i+1}. \end{gathered}$$

Each word $w=w_1w_2\ldots w_n\in W_r(L,n)$ is mapped onto a pair $(\sigma,s)\in S_r(L,n)$ in the following way: suppose that the nondecreasing rearrangement of w is of the form $i_1^{a_1}\ldots i_m^{a_m}$, with $i_1<\cdots<i_m$ and $a_1\ge 1,\ldots,a_m\ge 1$. Read w *from left to right* and give the labels $1,2,\ldots,a_m$ to the a_m letters equal to i_m; continue, again from left to right, giving the labels $a_m+1,\ldots,a_m+a_{m-1}$ to the a_{m-1} letters equal to i_{m-1} and so on. Reading those labels from left to right yields a permutation $\sigma=\sigma(1)\sigma(2)\ldots\sigma(n)$. The word s is the nonincreasing rearrangement of w.

Example. Let $n=9$, $r=8$, $L=\{2,5,7\}$. The word $w\in N_8(L,9)$ and the associated pair $(\sigma,s)\in S_8(L,9)$ are shown in the following table:

$$\begin{pmatrix} L=\\ w=\\ \sigma=\\ \operatorname{Iligne}=\\ s=\end{pmatrix}\begin{pmatrix} \ \ 2\ \ \ 5\ \ 7\ \ \\ 5\,5\,7\,4\,1\,3\,0\,4\,4\\ 2\,3\,\mathbf{1}\,4\,8\,\mathbf{7}\,9\,5\,\mathbf{6}\\ \mathbf{1}\qquad\ \ \mathbf{6}\,\mathbf{7}\qquad\\ 7\,5\,5\,4\,4\,4\,3\,1\,0\end{pmatrix}.$$

The map $w\mapsto(\sigma,s)$ is a bijection of $W_r(L,n)$ onto $S_r(L,n)$.

(c) The following identity holds

$$\frac{\displaystyle\sum_{\sigma,\,\operatorname{Ligne}\sigma=L} t^{\operatorname{ides}\sigma}\,q^{\operatorname{imaj}\sigma}}{(t;q)_{n+1}}=\sum_{r\ge 0}t^r\sum_{w\in W_r(L,n)}q^{\operatorname{tot}w},$$

so identity (E22.1) also holds by (a) and (c).

(d) Let $A_L(q):=\sum_\sigma q^{\operatorname{imaj}\sigma}$ $(\operatorname{Ligne}\sigma=L)$. Then $A_L(q)/(q;q)_n=\det N$, where $N=(N_{i,j})$ $(1\le i,j\le k+1)$ is the $(k+1)\times(k+1)$-matrix

$$N_{i,j}:=\begin{cases}1/(q;q)_{\ell_j-\ell_{i-1}}, & \text{if } i\le j;\\ 1, & \text{if } i=j+1;\\ 0, & \text{else.}\end{cases}$$

(e) We also have $A_L(q):=\sum_\sigma q^{\operatorname{inv}\sigma}$ $(\operatorname{Ligne}\sigma=L)$.

26. *Partitions, permutations and descents.* The statistic "ligne of route" ("Ligne") is defined in § 11.4. The notations used in (b) refer to § 14.1.

(a) Let $(a_1, a_2, \dots, a_k)$ be a *composition* of the integer n, that is, an ordered sequence of integers such that $a_1 \ge 1$, $a_2 \ge 1$, ... , $a_k \ge 1$ and $a_1 + a_2 + \cdots = a_k = n$. The number of permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, of order n, such that $\operatorname{Ligne}\sigma \subset \{a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_{k-1}\}$ is equal to $\binom{n}{a_1, a_2, \dots, a_k} = n!/(a_1!\, a_2!\, \dots\, a_k!)$.

(b) For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each integer $i = 1, 2, \dots, n$ let $d_i(\sigma)$ denote the number of *descents* $(\sigma(j) > \sigma(j+1))$ occurring in the *right factor* $\sigma(i)\sigma(i+1)\dots\sigma(n)$ and let

$$\mathbf{q}^{\mathbf{d}(\sigma)} := q_1^{d_1(\sigma)} q_2^{d_2(\sigma)} \cdots q_n^{d_n(\sigma)},$$

where $q_1, q_2, \dots, q_n$ is a finite sequence of variables. Notice that $d_1(\sigma)$ is the usual number of descents of σ. Furthermore, if $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ is a partition, whose number $l(\lambda)$ of *positive* parts is at most equal to n, let $m_i(\lambda)$ designate its number of parts equal to i, for $i = 0, 1, \dots, n$. Then, define

$$\binom{n}{\mathbf{m}(\lambda)} := \binom{n}{m_0(\lambda), m_1(\lambda), \dots, m_n(\lambda)}, \quad \mathbf{q}^{\lambda} := q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_n^{\lambda_n}.$$

The following identity holds:

$$\sum_{l(\lambda) \le n} \binom{n}{\mathbf{m}(\lambda)} \mathbf{q}^{\lambda} = \frac{\sum\limits_{\sigma \in \mathfrak{S}_n} \mathbf{q}^{\mathbf{d}(\sigma)}}{(1 - q_1)(1 - q_1 q_2) \cdots (1 - q_1 q_2 \cdots q_n)}.$$

[Expand the right-side of the previous identity and use (a).]

27. *A further extension of the MacMahon Verfahren.* Let $U := (S_<, S_\le, L_<, L_\le)$ be a partition of the alphabet $X = \{1, \dots, r\}$ such that $S_< \cup S_\le = \{1, \dots, h\}$ (the *small* letters) and $L_< \cup L_\le = \{h+1, \dots, r\}$ (the *large* letters) for a certain h $(0 \le h \le r)$. Let $w = x_1 x_2 \dots x_m$ be a word in the alphabet and let $x_{m+1} := h + \frac{1}{2}$. An integer i such that $1 \le i \le m$ is said to be a *U-descent* in w, if either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and $x_i \in S_\le \cup L_\le$. Because of the convention $x_{m+1} := h + \frac{1}{2}$ there is a U-descent in position m iff $x_m \in L_< \cup L_\le$, so that the four blocks of the partition determine different kinds of descents.

Let $\operatorname{des}_U w$ (resp. $\operatorname{maj}_U w$) denote the *number* (resp. the *sum*) of the U-descents in w. For each sequence $\mathbf{m} = (m_1, \dots, m_r)$ consider the generating polynomial for the class $R(\mathbf{m})$ by the pair $(\operatorname{des}_U, \operatorname{maj}_U)$, i.e.,

$A^U_{\mathbf m}(t,q) = \sum_w t^{\mathrm{des}_U w}\, q^{\mathrm{maj}_U w}$ $(w \in R(\mathbf m))$ (see section 6). Then the extensions of identities (7.7), (7.8) and (6.13) read:

$$(E27.1)\quad \frac{1}{(t;q)_{1+\|\mathbf m\|}} A^U_{\mathbf m}(t,q) = \sum_{s\ge 0} t^s \prod_{i\in S_<} \begin{bmatrix} m_i+s \\ m_i \end{bmatrix} \prod_{i\in S_\le} q^{\binom{m_i}{2}} \begin{bmatrix} s+1 \\ m_i \end{bmatrix}$$
$$\times \prod_{i\in L_<} q^{m_i} \begin{bmatrix} m_i+s-1 \\ m_i \end{bmatrix} \prod_{i\in L_\le} q^{\binom{m_i+1}{2}} \begin{bmatrix} s \\ m_i \end{bmatrix};$$

$$(E27.2)\quad \sum_{\mathbf m} A^U_{\mathbf m}(t,q) \frac{\mathbf u^{\mathbf m}}{(t;q)_{1+\|\mathbf m\|}} = \sum_{s\ge 0} t^s \frac{\prod_{i\in S_\le} (-u_i;q)_{s+1} \prod_{i\in L_\le} (-q\,u_i;q)_s}{\prod_{i\in S_<} (u_i;q)_{s+1} \prod_{i\in L_<} (qu_i;q)_s};$$

$$(E27.3)\quad \sum_{\mathbf m} A^U_{\mathbf m}(1,q) \frac{\mathbf u^{\mathbf m}}{(q;q)_{\|\mathbf m\|}} = \frac{\prod_{i\in S_\le} (-u_i;q)_\infty \prod_{i\in L_\le} (-q\,u_i;q)_\infty}{\prod_{i\in S_<} (u_i;q)_\infty \prod_{i\in L_<} (qu_i;q)_\infty};$$

where $\|\mathbf m\| = m_1 + \cdots + m_r$.

28. *A maj-inv transformation for signed words.* The purpose is to extend the transformation, introduced in section 11, to signed words. A first extension has already been given in § 20.5 for signed permutations. Keep the notations of Exercise 27, in particular the definitions of des_U and of maj_U, and for each word $w = x_1x_2\ldots x_m$ with letters in $S_< \cup S_\le \cup L_< \cup L_\le$ define

$$\mathrm{inv}_U\, w := \sum_{i<j} \Bigl(\chi(x_i > x_j) + \chi(x_i = x_j \in S_\le \cup L_\le)\Bigr) + \#\{i : x_i \in L_< \cup L_\le\}.$$

There are then four kinds of U-inversions. Let ${}^{\mathrm{inv}}A^U_{\mathbf m}(q) := \sum_{w\in R(\mathbf m)} q^{\mathrm{inv}_U w}$.

(a) The following identity holds

$${}^{\mathrm{inv}}A^U_{\mathbf m}(q) = \begin{bmatrix} \|\mathbf m\| \\ m_1,\ldots,m_r \end{bmatrix} \prod_{i\in S_\le} q^{\binom{m_i}{2}} \prod_{i\in L_<} q^{m_i} \prod_{i\in L_\le} q^{m_i+\binom{m_i}{2}}.$$

It then follows that the generating function $\sum_{\mathbf m} {}^{\mathrm{inv}}A^U_{\mathbf m}(q)\, \mathbf u^{\mathbf m}/(q;q)_{\|\mathbf m\|}$ is equal to the right-hand side of (E27.3). Thus, ${}^{\mathrm{inv}}A^U_{\mathbf m}(q) = A^U_{\mathbf m}(1,q)$, the generating polynomial for $R(\mathbf m)$ by maj_U.

(b) By using the transformation Φ of Theorem 11.3 build a bijection Ψ of $R(\mathbf m)$ onto itself having the property: $\mathrm{maj}_U\, w = \mathrm{inv}_U\, \Psi(w)$.

(c) Let $S_< = \{1,3\}$, $S_\le = \{2,4\}$, $L_< = \{5,7\}$, $L_\le = \{6,8\}$ and consider the word $w = 6,2,2,3,1,7,7,4,8,6$ whose U-descents occur at positions 1, 2, 4, 7, 9 and 10, so that $\operatorname{maj}_U w = 1+2+4+7+9+10 = 33$. Notice that the factor $2,2$ involves a descent because $2 \in S_\le$, but not the factor $7,7$ because $7 \in L_<$. There is a descent at the right end because $6 \in L_\le$. Determine $\Psi(w)$ and verify that $\operatorname{inv}_U \Psi(w) = 33$.

29. *The Brenti homomorphism.* The homomorphism materializes the similarity between the classical identities for the symmetric functions and the identities found for the Eulerian polynomials and their extensions. Recall that the generating functions for the elementary symmetric functions, on the one hand, and for homogeneous symmetric functions, on the other hand, read (see (14.1) and (14.2))

$$E(u) = \sum_{r\ge 0} e_r u^r = \prod_{i\ge 1}(1 + x_i u); \tag{$E24.1$}$$

$$H(u) = \sum_{r\ge 0} h_r u^r = \prod_{i\ge 1}(1 - x_i u)^{-1}; \tag{$E24.2$}$$

so that

$$E(-u)H(u) = 1. \tag{$E24.3$}$$

On the other hand, the exponential generating for the Eulerian polynomials $A_n(t)$ (see Exercise 13) can be put into the form:

$$\Bigl(1 + \sum_{k\ge 1} \frac{1 - t^{k-1}}{k!}(-u)^k\Bigr)\Bigl(\sum_{k\ge 0} \frac{A_k(t)}{k!} u^k\Bigr) = 1. \tag{$E24.4$}$$

The comparison of $(E24.3)$ and $(E24.4)$ shows that the following two statements are equivalent:

$$\xi(e_k) = \frac{1 - t^{k-1}}{k!} \ (k \ge 1), \quad \xi(e_0) = 1;$$
$$\xi(h_k) = \frac{A_k(t)}{k!} \quad (k \ge 0).$$

The mapping ξ is called the *Brenti homomorphism.*

(a) For each sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of positive integers such that $\lambda_1 + \cdots + \lambda_l = n$ (in particular, for each partition of n) and each permutation $\sigma = \sigma(1)\dots\sigma(n)$ define $\operatorname{des}_\lambda \sigma$ to be the number of descents $\sigma(i) > \sigma(i+1)$ with i different from λ_1, $\lambda_1 + \lambda_2$, $\dots$, $\lambda_1 + \cdots + \lambda_l$. Then form $A_\lambda(t) := \sum_\sigma t^{\operatorname{des}_\lambda \sigma}$ $(\sigma \in \mathfrak{S}_n)$. Then $\xi(h_\lambda) = A_\lambda(t)/n!$

(b) The factorial generating function for the polynomials ${}^{\mathrm{inv}}A_n(t,q)$ (see (10.10)) may be rewritten as:

$$\Bigl(1+\sum_{k\ge 1}\frac{(1-t^{k-1})q^{\binom{k}{2}}}{(q;q)_k}(-u)^k\Bigr)\Bigl(\sum_{k\ge 0}\frac{{}^{\mathrm{inv}}A_k(t,q)}{(q;q)_k}u^k\Bigr)=1.$$

so that the following two statements are equivalent:

$$\xi_q(e_k)=\frac{(1-t^{k-1})q^{\binom{k}{2}}}{(q;q)_k}\ (k\ge 1),\quad \xi_q(e_0)=1;$$
$$\xi_q(h_k)=\frac{A_k(t,q)}{(q;q)_k}\quad (k\ge 0).$$

Determine $\xi_q(h_\lambda)$.

(c) Let $B_n(t,Q,q)=\sum\limits_{\Sigma,\sigma} t^{\mathrm{ddes}(\Sigma,\sigma)}Q^{\mathrm{inv}\,\Sigma}q^{\mathrm{coinv}\,\sigma}\quad ((\Sigma,\sigma)\in\mathfrak{S}_n\times\mathfrak{S}_n)$ be the generating polynomial for the pairs of permutations by the three-variable statistic (ddes, inv, coinv). By specializing Theorem 21.1 we have

$$\frac{1-t}{-t+\mathbf{J}((1-t)u;Q,q)}=\sum_{n\ge 0}\frac{B_n(t,Q,q)}{(Q;q)_n(q;q)_n}u^n.$$

The following two statements are also equivalent:

$$\xi_{Q,q}(e_k)=\frac{(1-t^{k-1})Q^{\binom{k}{2}}}{(Q;q)_k(q;q)_k}\ (k\ge 1),\quad \xi_{Q,q}(e_0)=1;$$
$$\xi_{Q,q}(h_k)=\frac{B_k(t,Q,q)}{(Q;q)_,(q;q)_k}\quad (k\ge 0).$$

(d) Let $P(u)=\sum_{r\ge 1}p_r u^{r-1}$ be the generating series for the power symmetric functions. Using identity $E'(u)/E(u)=P(-u)$ derived in Theorem 14.2 (vi), determine $\xi(p_{k+1})$ $(k\ge 0)$.

(e) Using the relation $P(u)=E'(u)H(u)$ determine $\xi_q(p_n)$ and $\xi_{Q,q}(p_n)$ $(n\ge 1)$.

(f) By Theorem 20.1 the exponential generating function for the generating polynomials $B_n(X,Y,t)$ for *signed permutations* by number of descents is

$$\frac{(1-t)\,\exp((t-1)X)}{-t+\exp((t-1)(X+Y))}=\sum_{n\ge 0}\frac{1}{n!}\,B_n(X,Y,t),$$

a relation that may be written as

$$\Bigl(1+\sum_{n\ge 1}\frac{(1-t)^{n-1}}{n!}(-tX^n+(-Y)^n)\Bigr)\cdot\sum_{n\ge 0}\frac{1}{n!}\,B_n(X,Y,t)=1.$$

Make the changes $X \leftarrow XY$, $Y \leftarrow Y$. Then $-tX^n + (-Y)^n$ is changed into $(-Y)^n((-1)^{n-1}tX^n+1)$, and $B_n(X,Y,t)$ into $Y^nB_n(X,1,t)$, so that the following two statements are equivalent:

$$\xi_B(e_n) = \frac{(1-t)^{n-1}}{n!}((-1)^{n-1}tX^n+1)\ (n\ge 1),\quad \xi_B(e_0)=1;$$
$$\xi_B(h_n) = \frac{B_n(X,1,t)}{n!}\ (n\ge 0).$$

With the changes $X \leftarrow -X$, $Y \leftarrow XY$ the two statements are equivalent:

$$\xi_b(e_n) = \frac{(1-t)^{n-1}}{n!}(Y^n - t)\ (n\ge 1),\quad \xi_b(e_0)=1;$$
$$\xi_b(h_n) = \frac{B_n(-1,Y,t)}{n!}\ (n\ge 0).$$

30. *The des-length distribution for signed permutations.* The *length* l_{Cox} of a signed permutation (σ,ε) is defined by

$$l_{\mathrm{Cox}}(\sigma,\varepsilon) := \#\{i<j : \varepsilon(i)=\varepsilon(j), \sigma(i)>\sigma(j)\} + \#\{i<j : \varepsilon(i)=x, \varepsilon(j)=y\} + \sum_{\varepsilon(i)=x} .\sigma(i)$$

Let B''_n be the generating function for the group of the signed permutations of order n by the pair $(\mathrm{des}, l_{\mathrm{Cox}})$ expressed in the form

$$B''_n = \sum_{(\sigma,\varepsilon)} X^{\ell(\varepsilon|x)} t^{\mathrm{des}(\sigma,\varepsilon)} q^{l_{\mathrm{Cox}}(\sigma,\varepsilon)}.$$

Then

$$(*)\qquad \frac{1-t}{1-t\,e_q(u(1-t))}\sum_{n\ge 0}\frac{(u(1-t))^n}{(-Xq;q)_n\,(q;q)_n} = \sum_{n\ge 0}\frac{u^n}{(-Xq;q)_n\,(q;q)_n}B''_n.$$

The first values of B''_n are the following:

$B''_1 = 1+tX$; $\quad B''_2 = (t+q)+tq(1+q)^2X + tq^3(t+q)X^2$;

$B''_3 = (t^2q^3+2tq^2+2tq+1) + t(1+q+q^2)(1+q+2q^2+tq+tq^3)X$
$\quad + t^2(1+q+q^2)(1+q^2+2tq+tq^2+tq^3)X^2$
$\quad + tq^6(t^2q^3+2tq^2+2tq+1)X^3$.

Introduce the polynomials

$$A_n^k(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}_k\sigma}q^{\mathrm{inv}\,\sigma}\qquad (0\le k\le n),$$

where

$$\mathrm{des}_k\,\sigma := \begin{cases}\mathrm{des}\,\sigma, & \text{if } \sigma(n)\le n-k;\\ 1+\mathrm{des}\,\sigma, & \text{if } \sigma(n)\ge n-k+1.\end{cases}$$

With the notations of section 23

$$A_n^n(t,q) = {}^{\mathrm{inv}}A_n(t,q).$$

An easy derivation yields

$$B_n'' = \sum_{0\le k\le n} X^k\, q^{\binom{k+1}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q A_n^k(t,q).$$

By Theorem 10.1 the left-hand side of $(*)$ can be rewritten as

$$\sum_{l\ge 0} \frac{u^l}{(q;q)_l} A_l^l(t,q) \sum_{m\ge 0} \frac{(u(1-t))^m}{(-Xq;q)_m\,(q;q)_m},$$

that is denoted by

$$\sum_{n\ge 0} \frac{u^n}{(-Xq;q)_n\,(q;q)_n} B_n^{(3)}.$$

Proving $(*)$ amounts to proving the identity $B_n'' = B_n^{(3)}$.

31. *The Wachs Involution.* Let ${}^0A_{2n}(t) := \sum_\sigma t^{1+\mathrm{des}\,\sigma}$ and ${}^{\mathrm{sgn}}A_{2n}(t) := \sum_\sigma \mathrm{sgn}\,\sigma\, t^{1+\mathrm{des}\,\sigma}$ $(\sigma \in \mathfrak{S}_n)$ be the two polynomials introduced in Theorem 23.7, where the two identities ${}^{\mathrm{sgn}}A_{2n}(t) = (1-t)^n\, {}^0A_n(t)$ and ${}^{\mathrm{sgn}}A_{2n+1}(t) = (1-t)^n\, {}^0A_{n+1}(t)$ were proved analytically. A combinatorial proof can be derived by imagining an involution on $\mathfrak{S}_{2n}$ preserving the ligne of route of each permutation and changing the signature of some of them in such a way that the generating polynomial for the remaining permutations is precisely $(1-t)^n\, {}^0A_n(t)$. This is accomplished by the *Wachs involution* ϕ defined as follows: consider a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n)$ and let j be the *smallest* integer such that $1\le j\le n$ and the letters $2j-1$ and $2j$ are *not* adjacent in the word σ. Then define

$$\phi(\sigma) := \begin{cases} (2j-1, 2j)\,\sigma, & \text{if such an integer } j \text{ exists;}\\ \sigma, & \text{otherwise.}\end{cases}$$

The involution preserves the ligne of route and changes the signature, so that if $\mathfrak{F}_{2n}$ designates the set of its fixed points, we have: ${}^{\mathrm{sgn}}A_{2n}(t) = \sum_\sigma \mathrm{sgn}\,\sigma\, t^{1+\mathrm{des}\,\sigma}$ $(\sigma \in \mathfrak{F}_{2n})$. Now, if σ is in $\mathfrak{F}_{2n}$, there is a unique permutation $\tau = \tau(1)\tau(2)\ldots\tau(n)$ of order n such that for each $i = 1,2,\ldots,n$ the *unordered* pairs $\{\sigma(2i-1),\sigma(2i)\}$ and $\{2\tau(i), 2\tau(i)-1\}$ are identical. Let $E(\sigma)$ be the set of all integers i such that the two-letter factor $2\tau(i),(2\tau(i)-1)$ occurs in σ (and *not* the factor $(2\tau(i)-1), 2\tau(i)$). Then $\sigma \mapsto (\tau, E(\sigma))$ is a bijection of $\mathfrak{F}_{2n}$ onto the Cartesian product $\mathfrak{S}_n \times \mathfrak{P}(\{1,2,\ldots,n\})$, where $\mathfrak{P}(\{1,2,\ldots,n\})$ denotes the power set of $\{1,2,\ldots,n\}$, such that

$$\mathrm{inv}\,\sigma = 4\ \mathrm{inv}\,\tau + \#E(\sigma),$$
$$\mathrm{Ligne}\,\sigma = \{2i : i \in \mathrm{Ligne}\,\tau\} \cup \{2i-1 : i\in E(\sigma)\}.$$

Accordingly,

$$\begin{aligned}
{}^{\mathrm{sgn}}A_{2n}(t) &= \sum_{\sigma\in\mathfrak{F}_{2n}} \operatorname{sgn}\sigma\, t^{1+\operatorname{des}\sigma} = \sum_{\tau\in\mathfrak{S}_n, E\subset\{1,\dots,n\}} (-1)^{\#E} t^{\operatorname{des}\tau+\#E}\\
&= \sum_{E\subset\{1,\dots,n\}} (-t)^{\#E} \sum_{\tau\in\mathfrak{S}_n} t^{1+\operatorname{des}\tau} = (1-t)^n\, {}^0\!A_n(t).
\end{aligned}$$

Let ${}^{\mathrm{maj}}A_n(t,q)$ be the q-maj-Eulerian polynomial (see section 10) and ${}^{\mathrm{sgn}}A_n(t,q) := \sum_\sigma \operatorname{sgn}\sigma\, t^{1+\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}$ $(\sigma\in\mathfrak{S}_n)$ be the *signed q-Eulerian polynomial.* Using the above procedure it is shown that

$${}^{\mathrm{sgn}}A_{2n}(t,q) = (tq;q^2)_n\, t\, {}^{\mathrm{maj}}A_n(t,q^2).$$

32. *The divisibility of the q-tangent coefficients.* As introduced in section 24 the coefficients of the q-tangent functions are denoted by $D_{2n+1}(q)$ $(n\ge 0)$. Each integer $n\ge 1$ can be written as $n=m2^l$, where m is odd and $l\ge 0$. Define $Ev_n(q) := \prod\limits_{0\le j\le l}(1+q^{m2^j})$ and also

$$F_n(q) := \begin{cases} \prod\limits_{1\le i\le n} Ev_i(q), & \text{if } n \text{ is odd;}\\ (1+q^2)\prod\limits_{1\le i\le n} Ev_i(q), & \text{if } n \text{ is even.}\end{cases}$$

In an equivalent manner, let $Ev_0(q) = F_0(q) := 1$, $F_1(q) := 1+q$, $F_2(q) := (1+q)^2(1+q^2)^2$ and $F_n(q) := F_{n-2}(q)\,Ev_{n-1}(q)\,Ev_n(q)$ for $n\ge 3$. Define the exponents $a(n,i)$ by $F_n(q) := \prod\limits_{1\le i\le n}(1+q^i)^{a(n,i)}$.

(a) Give the table of the coefficients $a(n,i)$ $(1\le i\le n)$ for $1\le n\le 8$.

(b) For $i\ge 1$ let $\Phi_i(q)$ be the i-th cyclotomic polynomial with $\phi_1(q)=1-q$, so that $1-q^i=\prod\limits_{d\,|\,i}\phi_d(q)$. For $n=m2^l$ (m odd, $l\ge 0$) and $j=0,1,\dots,l$ let

$$A_j := \{d : d\,|\,m2^{j+1},\ d\nmid m2^j\},\quad B := \{d : d\,|\,m2^{l+1},\ d \text{ even}\}.$$

Then $1+q^{m2^j} = \prod\limits_{d\in A_j}\phi_d(q)\quad (0\le j\le l)$.

(c) Also $Ev_n(q) = \prod\limits_d \phi_d(q)\quad (d\,|\,2n,\ d \text{ even})$.

(d) For each $n\ge 1$ let $Od_n(q) := \prod\limits_d \phi_d(q)\quad (d\,|\,2n,\ d \text{ odd})$, so that $1-q^{2n} = Od_n(q)Ev_n(q)$. Using that factorization we see that the product $\begin{bmatrix} 2n\\ 2k+1\end{bmatrix}\dfrac{Ev_0(q)\,Ev_1(q)\,\dots\,Ev_k(q)}{Ev_{n-k}(q)\,Ev_{n-k+1}(q)\,\dots\,Ev_n(q)}$ is a polynomial in q.

(e) Finally, $F_n(q)$ divides $D_{2n+1}(q)$.

33. *Congruences for the q-secant coefficients.* The purpose is to prove the congruence
$$D_{2n}(q) \equiv q^{2n(n-1)} \bmod (q+1)^2$$
by using a combinatorial argument.

(a) Let $1 \le i \le 2n-1$. A rising alternating permutation $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$ is said to be *i-balanced* if the word σ contains the factor $i\,(i+1)$, or if the letter $(i+1)$ lies on the left of i in σ and not adjacent to i. For example, the permutation $\sigma = 2\,6\,3\,4\,1\,5$ is i-balanced only for $i = 1, 3, 5$. If σ is i-balanced for every $i = 1, 2, \dots, (2n-1)$, it is said to be *balanced.*

The permutation $(2n-1)\,2n\,(2n-3)\,(2n-2)\,\dots\,3\,4\,1\,2$ is the unique balanced rising alternating permutation of order $2n$.

(b) Let σ be a nonbalanced rising alternating permutation. The greatest integer i such that σ is not i-balanced is denoted by i'. Then σ is necessarily of the form $\sigma = w\,i'\,w'\,(i'+1)w''$, where w' is a nonempty factor. The permutation $\Phi(\sigma) := w\,(i'+1)\,w'\,i'\,w''$ is still rising alternating and $\operatorname{inv}\Phi(\sigma) = \operatorname{inv}\sigma + 1$.

(c) The polynomial $D_{2n}(q)$ is monic of degree $2n(n-1)$.

(d) For $n \ge 2$ let A_{2n} (resp. B_{2n}) be the set of all rising alternating permutations of length $2n$ beginning with the factor $(2n-1)\,2n$ (resp. ending with $1\,2$). By induction on n,
$$\sum_{\sigma \in A_{2n} \cup B_{2n}} q^{\operatorname{inv}\sigma} \equiv q^{2n(n-1)} \bmod (q+1)^2.$$

(e) Let E_{2n} be the complement of $A_{2n} \cup B_{2n}$. Then
$$\sum_{\sigma \in E_{2n}} q^{\operatorname{inv}\sigma} \equiv 0 \bmod (q+1)^2.$$

Answers to the exercises

1. Clearly, $\exp(xu)\exp(u) = \exp((1+x)u)$ is the exponential generating function for the polynomials $H_n(x) = (1+x)^n = \sum_{0\le n\le n} \binom{n}{j} x^j$. In the case of q-series the calculation is to be made explicitly: $e_q(xu)\, e_q(u) = \sum_{n\ge 0}\sum_{0\le j\le n} \frac{(xu)^j}{(q;q)_j}\frac{u^{n-j}}{(q;q)_{n-j}} = \sum_{n\ge 0}\frac{u^n}{(q;q)_n}\sum_{0\le j\le n}\begin{bmatrix} n\\ j\end{bmatrix} x^j$ $= \sum_{n\ge 0}\frac{u^n}{(q;q)_n} H_n(x,q)$. With $x=-1$ we get $\sum_{n\ge 0}\frac{u^n}{(q;q)_n} H_n(-1,q) = \frac{1}{(-u;q)_\infty\,(u;q)_\infty} = \frac{1}{(-u^2;q^2)_\infty} = \sum_{n\ge 0}(-1)^n\frac{u^{2n}}{(q^2;q^2)_n}$, hence $H_{2n+1}(-1,q)=0$ and $H_{2n}(-1,q) = (q;q)_{2n}/(q^2;q^2)_n = (q;q^2)_n$. With $x=q^{1/2}$ it turns into $\sum_{n\ge 0}\frac{u^n}{(q;q)_n}H_n(q^{1/2},q) = \frac{1}{(uq^{1/2};q)_\infty\,(u;q)_\infty} = \frac{1}{(u;q^{1/2})_\infty} = \sum_{n\ge 0}\frac{u^n}{(q^{1/2};q^{1/2})_n}$, hence $H_n(q^{1/2};q) = \frac{(q;q)_n}{(q^{1/2};q^{1/2})_n} = \frac{(q;q)_n\,(-q^{1/2};q^{1/2})_n}{(q^{1/2};q^{1/2})_n\,(-q^{1/2};q^{1/2})_n} =$ $(q^{-1/2};q^{1/2})_n$. Start with $f(x,u) = \sum_{n\ge 0}\frac{u^n}{(q;q)_n}H_n(x,q)$. Then $f(x,u) - f(x,qu) = x\sum_{n\ge 0}\frac{u^n}{(q;q)_n}H_{n+1}(x,q)$. On the other hand, $f(x,u) - f(x,qu) = \frac{1}{(u;q)_\infty}\frac{1}{(xu;q)_\infty}(1-(1-u)(1-xu)) =$ $f(x,u)u(1+x-xu)$. Therefore $\sum_{n\ge 0}\frac{u^n}{(q;q)_n}H_{n+1}(x,q) = (1+x)$ $\times\sum_{n\ge 0}\frac{u^n}{(q;q)_n}H_n(x,q) - \sum_{n\ge 0}\frac{u^{n+1}}{(q;q)_n}xH_n(x,q)$. Hence the recurrence for $n\ge 0$ with $H_{-1}(x,q)=0$.

2. No further comment.

3. By definition of y we have $y_i = i - \sum_{1\le j\le i}\chi(\sigma(i)>\sigma(j)>\sigma(i+1))$ or $\sum_{1\le j\le i-1}\chi(\sigma(i+1)>\sigma(j)>\sigma(i))$, depending on whether $\sigma(i)>\sigma(i+1)$ or $\sigma(i)<\sigma(i+1)$. The sequence y is then subexcedent. Now, given a subexcedent sequence y we can reconstruct the permutation σ in a unique way, starting with $\sigma(n)$ from $y_n = n - \sum_{1\le j\le n}\chi(\sigma(n)>\sigma(j))$.

The mapping $y \mapsto \sigma \mapsto x$ is then bijective. Finally, $\operatorname{tot} y = \sum_{1\le i\le n} (x_i - x_{i+1}) + i\chi(\sigma(i) > \sigma(i+1)) = \sum_{1\le i\le n} i\chi(\sigma(i) > \sigma(i+1)) = \operatorname{maj}\sigma$, since $x_0 = x_{n+1} = 0$.

4. We have inv-coding$(\sigma) = (0,0,0,2,1,1,5,3,8)$ and $\operatorname{inv}\sigma = 20$; furthermore, maj-coding$(\sigma) = (0,1,1,3,4,1,2,2,3)$, $\operatorname{maj}\sigma = 17$. Finally, inv-coding$(687293154) = x$ and maj-coding$(938164275) = x$.

5. (a) $z_i \le \#\{1 \le j < i\} = i$; (b) $z = 001102238$; (c) $\sigma = 756312984$.

(d) Distinguish two cases. If $\sigma(i) > \sigma(i+1)$, the components of this new maj-coding satisfy the relations:

$$\begin{aligned} &x_i - x_{i+1} + i\chi(\sigma(i) > \sigma(i+1)) \\ &\qquad = \{j < i \mid \sigma(j) > \sigma(i)\} - \{j < i+1 \mid \sigma(j) > \sigma(i+1)\} + i \\ &\qquad = \{j < i \mid \sigma(j) > \sigma(i)\} + \{j < i+1 \mid \sigma(j) \le \sigma(i+1)\} \\ &\qquad = \{j < i \mid \sigma(j) > \sigma(i)\} + \{j < i \mid \sigma(j) \le \sigma(i+1)\} \\ &\qquad = \#\{1 \le j < i \mid \sigma(j) \in \rrbracket\sigma(i), \sigma(i+1)\rrbracket\} = z_i. \end{aligned}$$

If $\sigma(i) < \sigma(i+1)$, then

$$\begin{aligned} &x_i - x_{i+1} + i\chi(\sigma(i) > \sigma(i+1)) \\ &\qquad = \{j < i \mid \sigma(j) > \sigma(i)\} - \{j < i+1 \mid \sigma(j) > \sigma(i+1)\} \\ &\qquad = \{j < i \mid \sigma(j) > \sigma(i)\} - \{j < i \mid \sigma(j) > \sigma(i+1)\} \\ &\qquad = \#\{1 \le j < i \mid \sigma(j) \in \rrbracket\sigma(i), \sigma(i+1)\rrbracket\} = z_i. \end{aligned}$$

6. The subsequence y is subexcedent and $y_1 + y_2 + \dots y_n = \operatorname{maj}\sigma$. However the transformation $\sigma \mapsto y$ is not bijective; for instance, the two permutations 213 and 312 have the same image 010.

7. The vectors of a set $\{v_1, v_2, \dots, v_n\}$ are linearly independent, first if v_1 is not zero (there are $(q^{N+n} - 1)$ possible choices), then if v_2 is not proportional to v_1 (there are $(q^{N+n} - q)$ possible choices), more generally, if for $i = 2, \dots, n$ the vector v_i does not belong to the subspace of dimension $(i-1)$ generated by v_1, v_2, … , v_{i-1} (there are $(q^{N+n} - q^{i-1})$ possible choices). The number of such sets $\{v_1, v_2, \dots, v_n\}$ is then equal to $(q^{N+n} - 1)(q^{N+n} - q)\cdots(q^{N+n} - q^{n-1})$. The second enumeration is derived by the same reasoning. Finally, the number of sets $\{v_1, v_2, \dots, v_n\}$ having only linearly independent vectors is equal to the number of subspaces of dimension n, multiplied by the number of such sets that span a given space of dimension n.

8. The two formulas are banal for $N = 1$. For proving (3.9) use (3.6) with the substitutions $n \leftarrow N+n$, $i \leftarrow n$, so that $(1 - uq^N) \sum\limits_{n\ge0} \begin{bmatrix} N+n \\ n \end{bmatrix} u^n = \sum\limits_{n\ge0} (\begin{bmatrix} N+n \\ n \end{bmatrix} - q^N \begin{bmatrix} N+n-1 \\ n-1 \end{bmatrix}) u^n = \sum\limits_{n\ge0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix} u^n = (u;q)_N^{-1}$ (by induction on N). Hence $\sum\limits_{n\ge0} \begin{bmatrix} N+n \\ n \end{bmatrix} u^n = (u;q)_N^{-1}(1 - uq^N)^{-1} = (u;q)_{N+1}^{-1}$. For (3.10) make use of (3.5) with the substitutions $n \leftarrow N+1$, $i \leftarrow n$, so that
$(-u;q)_{N+1} = (\sum\limits_{0\le n\le N} q^{n(n-1)/2} \begin{bmatrix} N \\ n \end{bmatrix} u^n).(1 + uq^N) =$
$1 + \sum\limits_{1\le n\le N} (\begin{bmatrix} N \\ n \end{bmatrix} q^{n(n-1)/2} + \begin{bmatrix} N \\ n-1 \end{bmatrix} q^{(n-1)(n-2)/2+N}) u^n + q^{(N+1)N/2} u^{N+1} =$
$1 + \sum\limits_{1\le n\le N} (\begin{bmatrix} N \\ n \end{bmatrix} + q^{(N+1)-n} \begin{bmatrix} N \\ n-1 \end{bmatrix}) q^{n(n-1)/2} u^n + q^{(N+1)N/2} u^{N+1} =$
$\sum\limits_{0\le n\le N+1} \begin{bmatrix} N+1 \\ n \end{bmatrix} q^{n(n-1)/2} u^n.$

9. By induction on $N+n$. Let $\mathrm{ND}(N,n;q) := \sum\limits_{b\in \mathrm{ND}(N,n)} q^{\mathrm{tot}\, b}$. Then $\mathrm{ND}(N,n;q) = \sum\limits_{b_1=0} q^{\mathrm{tot}\, b} + \sum\limits_{b_1\ge1} q^{\mathrm{tot}\, b}$. Let $a_i := b_i - 1$ $(i = 1, \ldots, N)$ in the second sum. With the convention that $\mathrm{ND}(N,n;q) = 1$ when $N = 0$ we get:

$$\begin{aligned} \mathrm{ND}(N,n;q) &= \sum_{0\le b_2\le\cdots\le b_N\le n} q^{\|b\|} + \sum_{0\le a_1\le\cdots\le a_N\le n-1} q^{N+\|a\|} \\ &= \mathrm{ND}(N-1,n;q) + q^N\, \mathrm{ND}(N,n-1;q) \\ &= \begin{bmatrix} N+n-1 \\ n \end{bmatrix} + q^N \begin{bmatrix} N+n-1 \\ n-1 \end{bmatrix} = \begin{bmatrix} N+n \\ n \end{bmatrix}. \end{aligned}$$

10. Proceed by induction on $N+n$. Let $\mathrm{BW}(N,n;q) := \sum\limits_{x\in \mathrm{BW}(N,n)} q^{\mathrm{inv}\, x}$. Then $\mathrm{BW}(N,n;q) = \sum\limits_{x_1=0} q^{\mathrm{inv}\, x} + \sum\limits_{x_1=1} q^{\mathrm{inv}\, x}$. Let $y_i := x_{i+1}$ $(i = 1, \ldots, N+n-1)$ in each of the sums and let y be the word $y_1 y_2 \ldots y_{N+n-1}$. In the first (resp. the second) sum we have $\mathrm{inv}\, x = \mathrm{inv}\, y$ and $y \in \mathrm{BW}(N, n-1)$ (resp. $\mathrm{inv}\, x = n + \mathrm{inv}\, y$ and $y \in \mathrm{BW}(N-1, n)$). With the convention that $\mathrm{BW}(N,n;q) = 1$ when $N = 0$,

$$\begin{aligned} \mathrm{BW}(N,n;q) &= \sum_{y\in \mathrm{BW}(N,n-1)} q^{\mathrm{inv}\, y} + \sum_{y\in \mathrm{BW}(N-1,n)} q^{n+\mathrm{inv}\, y} \\ &= \mathrm{BW}(N,n-1;q) + q^n\, \mathrm{BW}(N-1,n;q) \\ &= \begin{bmatrix} N+n-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} N+n-1 \\ n \end{bmatrix} = \begin{bmatrix} N+n \\ n \end{bmatrix}. \end{aligned}$$

11. (a) As shown in Fig. A1, for each partition π in at most n parts, all at most equal to N denote the size of the n-th part of π (the smallest one being possibly 0) by j. Withdraw j from all the parts of π: there remains a partition in at most $(n-1)$ parts, all at most equal to $(N-j)$. The latter partitions have a generating function equal to $\begin{bmatrix} N-j+n-1 \\ N-j \end{bmatrix}$.

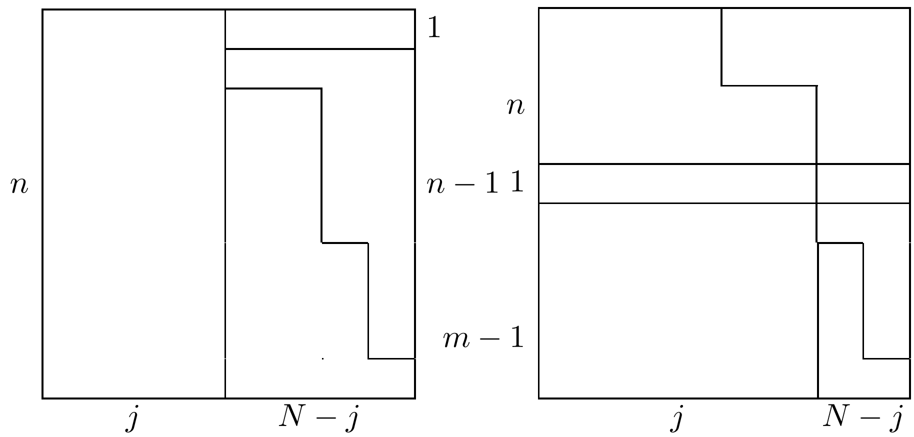


Fig. A1 Fig. A2

(b) Report to Fig. A2. For each partition π in at most $(n+m)$ parts, all at most equal to N, denote the size of the smallest $(n+1)$-st part by j. To such a partition π associate the partition π' in at most n parts, all at most equal to j and a partition π'' in at most $(m-1)$ parts, all at most equal to $(N-j)$.
The two q-binomial coefficients occurring on the left-hand side of the identity are precisely the generating functions for the partitions π' and π'', respectively. The identity is then a consequence of the fact that the weight of π is equal to the sum of the weights of π' and of π'', plus the size of the rectangle of dimension $j \times m$.

12. (a) By induction.
(b) If n is inserted onto the left of a permutation $\sigma' = \sigma'(1)\ldots\sigma'(n-1)$ or between two letters $\sigma(i)$, $\sigma(i+1)$ such that $\sigma(i) < \sigma(i+1)$, the number of descents increases by one. In the other case the number of descents remains alike. To obtain the set $S_{n,k}$ of all permutations of order n having k descents it suffices

either to take each permutation belonging to $S_{n-1,k-1}$ and insert n into the $(n-1)-(k-1)$ positions that create a new descent; we then obtain $(n-k)A_{n_1,k-1}$ permutations of order n;

or to take each permutation belonging to $S_{n-1,k}$ and insert n into the $(k+1)$ positions that create no descent; $(k+1)A_{n-1,k}$ permutations of order n having k descents are derived in that way.

13. (a) The right-hand side of the identity to be proved is equal to: $(1-t)^n \sum_{n\geq 0} t^j(j+1)^{n-1}(1+(n-1)t-nt+(1-t)j) = (1-t)^n \sum_{n\geq 0} t^j(j+1)^{n-1}(1-t)(j+1) = (1-t)^{n+1} \sum_{n\geq 0} t^j(j+1)^n = A_n(t)$.

(b) First $A_0(t) = (1-t)/(1-t) = 1$ and the constant coefficient of each series $A_n(t)$ is $A_{n,0} = 1$. Consider the coefficient of t^k $(k \geq 1)$ on each of sides of (E13.1). Then $A_{n,k} = A_{n-1,k} + (n-1)A_{n-1,k-1} + kA_{n-1,k} - (k-1)A_{n-1,k-1}$, which is the desired recurrence relation. Finally, the recurrence implies that $A_{n,k} = 0$ for $k \geq n \geq 1$.

(c) It suffices to calculate the coefficient of t^k in $(1-t)^{n+1} \sum_{n\geq 0} t^n(j+1)^n$.

(d) $A_n(t)(1-t)^{-(n+1)} = \sum_{k\geq 0} \binom{n+k}{n} t^k \sum_{0\leq l\leq n-1} A_{n,l}t^l = \sum_{j\geq 0} t^j \sum_{0\leq l\leq \min(j,n-1)} A_{n,l}\binom{n+j-l}{n} = \sum_{j\geq 0} t^j \sum_{0\leq l\leq n-1} A_{n,n-1-l}\binom{(j+1)+(n-l-1)}{n} = \sum_{j\geq 0} t^j(j+1)^n$, using $\binom{n+j-l}{n} = 0$ when $j \leq n-2$ and $l = j+1, \ldots, n-1$ and the Worpitzky for the last step.

(e) From (E13.1) $\sum_{n\geq 0} A_n(t)(1-t)^{-(n+1)}u^n/n! = \sum_{n\geq 0} u^n/n! \sum_{j\geq 0} t^j(j+1)^n = \sum_{j\geq 0} t^j \sum_{n\geq 0} (u(j+1))^n/n! = e^u \sum_{j\geq 0} (te^u)^j = e^u/(1-te^u)$. With $u := v(1-t)$ we get $\sum_{n\geq 0} A_n(t)v^n/n! = (1-t)\exp(v(1-t))/(1-t\exp(v(1-t)))$ and the desired generating function.

14. Let $\sigma = \sigma(1)\ldots\sigma(n)$ be a permutation. If $\sigma(i) < i$, say that i is an *accedence-letter* of σ. Let $A(\sigma)$ (resp. $E(\sigma)$) denote the set of all accedence-letters (resp. excedence-letters) of σ. First, verify that for each $j = 1, \ldots, n$

$$\#\{i \in A(\sigma) : j \in]\sigma(i), i]\} = \#\{i \in E(\sigma) : j \in]i, \sigma(i)]\}.$$

Next, verify that

$$\#\{(l,k) : l, k \in E(\sigma), k \in [l, \sigma(l)[\} = \#\{(l,k) : l, k \in E(\sigma), \sigma(k) \in]l, \sigma(l)]\};$$

so that

$$\#\{(l,k) : l \in A(\sigma), k \in E(\sigma), \sigma(k) \in]\sigma(l), l]\} = \#\{(l,k) : l, k \in E(\sigma), k \in]l, \sigma(l)]\};$$

This is sufficient to show that

$$\begin{aligned}&\#\{(l,k): l<k, \sigma(k)<\sigma(l)\le k\}+\#\{(l,k): l<k, \sigma(l)\le k<\sigma(k)\}\\&\qquad=\#\{(l,k): l<k, \sigma(k)<\sigma(l)\le l\}+\#\{(l,k): l\le k, k<\sigma(k)\}.\end{aligned}$$

Merge the third term into the first and the fourth into the second:

$$\begin{aligned}&\#\{(l,k): l<\sigma(l), \sigma(k)<k, \sigma(k)<\sigma(l)\le k\}\\&\qquad\qquad=\#\{(l,k): l<\sigma(k), k<\sigma(l), l\le k<\sigma(l)\}.\end{aligned}$$

15. (a) and (b) are easy; (c) is a rewriting of the maj-coding discussed in § 2.2. The difficult part is (d) and requires a deeper study. A numerical example is discussed in details inviting the reader to reconstruct the proof himself. For a complete formal proof see [Ha90b]. Finally, (e) is merely making the appropriate composition products.

16. (a) The table of the six statistics for $\mathrm{BW}(2,3)$:

	des	maj	rise	rmaj	DES	MAJ
11000	1	2	0	0	1	2
10100	2	4	1	2	3	6
10010	2	5	1	3	3	8
01100	1	3	1	1	2	4
01010	2	6	2	4	4	10
00110	1	4	1	2	2	6
10001	1	1	1	4	2	5
01001	1	2	2	5	3	7
00101	1	3	2	6	3	9
00011	0	0	1	3	1	3

(b) The bijection $x\mapsto y$ of $\mathrm{BW}(N,n)$ onto $\mathrm{BW}(n,N)$ is defined as follows. For $1\le i\le N+n$ let $y_i=1-x_i$. Then $\operatorname{rmaj} x=\operatorname{maj} y$ and
$$\sum_{x\in\mathrm{BW}(N,n)} q^{\operatorname{rmaj} x}=\sum_{y\in\mathrm{BW}(n,N)} q^{\operatorname{maj} y}=\begin{bmatrix}N+n\\ n\end{bmatrix}.$$

(c) A bijection $x\mapsto z$ of $\mathrm{BW}(N,n)$ onto itself is constructed as follows. First, for each $x\in\mathrm{BW}(N,n)$ apply the bijection of (b) to get $y\in\mathrm{BW}(n,N)$. Then factorize y in a unique manner under the form $y=v_1 10 v_2 10\cdots v_{k-1}10v_k$, where v_i is a nondecreasing binary word of the form 0^a1^b; moreover, the length $l(v_i)$ of v_i satisfies $l(v_i)\ge 1$ for $2\le i\le k-1$ and $l(v_1)\ge 0$, $l(v_k)\ge 0$. To each nondecreasing word $v_i = 0^a 1^b$ associate the nondecreasing word $u_i = 0^b 1^a$. Then, let $z := u_1 10 u_2 10 \cdots u_{k-1} 10 u_k$.

For example with $x = 10110001011 \in \mathrm{BW}(6,5)$ we get $y = 01001110100 \in \mathrm{BW}(5,6)$. The factorization of $y = 0(10)011(10)(10)0$ yields $z = 1(10)001(10)(10)1 = 11000110101 \in \mathrm{BW}(6,5)$. We verify that $\operatorname{rmaj} x = \operatorname{maj} y = \operatorname{maj} z$ and $\operatorname{rise} x = \operatorname{des} y = \operatorname{des} z$.

(d) Look for the difference between rmaj and maj.

(e)
$$\begin{aligned}
\sum_{x\in \mathrm{BW}(n,N)} q^{\mathrm{MAJ}\,x} &= \sum_{x\in \mathrm{BW}(n,N),x_{n+N}=0} q^{\mathrm{MAJ}\,x} + \sum_{x\in \mathrm{BW}(n,N),x_{n+N}=1} q^{\mathrm{MAJ}\,x}\\
&= \sum_{x\in \mathrm{BW}(n-1,N)} q^{2\operatorname{maj}' x+N} + \sum_{x\in \mathrm{BW}(n,N-1)} q^{2\operatorname{maj} x+n}\\
&= q^N \begin{bmatrix} n+N-1 \\ n-1, N \end{bmatrix}_{q^2} + q^n \begin{bmatrix} n+N-1 \\ n, N-1 \end{bmatrix}_{q^2}\\
&= q^N \begin{bmatrix} n+N \\ n, N \end{bmatrix}_{q^2} \frac{1-q^{2n}}{1-q^{2N+2n}} + q^n \begin{bmatrix} n+N \\ n, N \end{bmatrix}_{q^2} \frac{1-q^{2N}}{1-q^{2N+2n}}\\
&= \begin{bmatrix} n+N \\ n, N \end{bmatrix}_{q^2} \frac{q^N(1-q^{2n})+q^n(1-q^{2N})}{1-q^{2N+2N}}\\
&= \begin{bmatrix} n+N \\ n, N \end{bmatrix}_{q^2} \frac{q^n+q^N}{1+q^{n+N}}.
\end{aligned}$$

17. (a) Let w be a word of $R(\mathbf{m})$ and let $1 \le x < y = x+1 \le r$. Replace all the factors yx of w by a special letter "$\sim$". In the new word the *maximal* factors containing the two letters x and y are of the form $x^a y^b$ ($a \ge 0$, $b \ge 0$). Change all those factors into $x^b y^a$ and replace every "$\sim$" by yx to obtain a word w' of the second set. For instance, with $w = 122322233243213 \in R(2,7,5,1)$, $x = 2$ and $y = 3$ we successively get:

$$\begin{aligned}
w = {}& 122322233243213 \mapsto 122{\sim}223{\sim}4{\sim}13\\
&\mapsto 133{\sim}233{\sim}4{\sim}12 \mapsto 133322333243212 = w' \in R(2,5,7,1).
\end{aligned}$$

That transformation is bijective and $\operatorname{maj} w = \operatorname{maj} w'$ holds.

(b) $Z(w) = \operatorname{maj}(2121122) + \operatorname{maj}(1131) + \operatorname{maj}(41114) + \operatorname{maj}(22322) + \operatorname{maj}(242242) + \operatorname{maj}(434) = 4+3+1+3+7+1 = 19$.

(c) The multiplicities of $\operatorname{gcyc}_x(w)$ and $\operatorname{lcyc}_x(w)$ are respectively:
$\mathbf{m}^x = (m_{x+1}, m_{x+2}, \cdots, m_r, m_1, m_2, \cdots, m_{x-1}, m_x)$ and
$\mathbf{m}_x = (m_1, m_2, \cdots, m_{x-1}, m_{x+1}, m_{x+2}, \cdots, m_r, m_x)$.

(d) The global and local cycling are equal to $\operatorname{gcyc}_2(w) = 4234313424$ and $\operatorname{lcyc}_2(w) = 4214121434$. But $\operatorname{maj}\operatorname{gcyc}_2(w) = 18$ and $Z(\operatorname{lcyc}_2(w)) = 16$. As $\operatorname{maj} w = 21$ and $Z(w) = 19$, the two differences are equal to 3.

(e) Consider the unique factorization $w = p_0q_1p_1q_2p_2 \ldots q_sp_s$ of w, where the p_i's are words all letters of which are at most equal to x and where the q_i's are words whose letters are all greater than x with $|p_i| \geq 1$, $|q_i| \geq 1$ for all $i \geq 1$ and $|p_0| \geq 0$. The last letter of each factor p_i (resp. q_i) is less (resp. greater) than the first letter of the following factor q_{i+1} (resp. p_i). Say that there is a *rise* at the end of the factor p_i and a *descent* at the end of the factor q_i. In the word $\mathrm{gcyc}_x(w)$ those rises become descents and the descents become rises, the other rises or descents remaining invariant. Therefore $\mathrm{maj}\, w - \mathrm{maj}(\mathrm{gcyc}_x(w)) = |q_1| + |q_2| + \cdots + |q_s| = m_{x+1} + m_{x+2} + \cdots + m_r$.

(f) Let $a = 1, 2 \ldots, r$ be a letter occurring in w and denote the image of a under the local cycling $u := \mathrm{lcyc}_x(w)$ by a'. If $i < j$ and $i \neq x$, the subwords w_{ij} and $u_{i'j'}$ are identical up to the reduction. On the other hand, for $x < j$, item (e) implies $\mathrm{maj}\, w_{xj} - \mathrm{maj}(\mathrm{gcyc}_x(w))_{x'j'} = m_j$. As the two cyclings are identical up to the reduction for the words with two letters, it follows that $Z(w) - Z(u) = \sum\limits_{x<j} \big(\mathrm{maj}(w_{xj}) - \mathrm{maj}(u_{x'j'})\big) = m_{x+1} + m_{x+2} + \cdots + m_r$.

(g) The desired bijection Φ is defined, for each word $w \in R(\mathbf{m})$ and each letter x, by the composition product:

$$\Phi(wx) := \big(\mathrm{lcyc}_x^{-1} \circ \Phi \circ \theta_{\mathbf{m}^x, \mathbf{m}_x} \circ \mathrm{gcyc}_x(w)\big) x \ .$$

It is readily verified that $\Phi(wx)$ is a rearrangement of wx; furthermore, $\mathrm{maj}\, w = Z(\Phi(w))$ holds.

18. (a) Let $b := \sum\limits_{r \geq 0} b_r\, t^r$. Then $b_0 = a_0$ and $b_r = a_r - a_{r-1}$ for $r \geq 1$. But $a - (b_0 + b_1 + \cdots + b_r) = a - (a_0 + a_1 - a_0 + \cdots + a_r - a_{r-1}) = a - a_r$ and by assumption the order of that difference tends to infinity with r.

(b) The right-hand side of (7.8) is of the form $\sum_s b_s\, t^s$, where $b_s = 1/(\mathbf{u}; q)_{s+1}$. But $\lim_s b_s = 1/(\mathbf{u}; q)_\infty$. Then let $b(t) := (1-t) \cdot \sum_s b_s\, t^s$. It follows that $b(1) = 1/(\mathbf{u}; q)_\infty$. Multiply the left-hand side of (7.9) by $(1-t)$ and put $t = 1$ afterwards. It is found that $\sum_{\mathbf{m}} A_{\mathbf{m}}(t = 1, q)\mathbf{u}^{\mathbf{m}}/(q; q)_m$. This yields (6.13).

(c) This time (a) is to be applied twice in succession. The coefficient of t_1^r, that is, $\sum_s t_2^s/(u; q_1, q_2)_{r+1, s+1}$, tends to $\sum_s t_2^s/(u; q_1, q_2)_{\infty, s+1}$ (obvious definition) when r tends to infinity. Multiply the left-hand side of (13.7) by $(1 - t_1)$ and put $t_1 = 1$; we get the expression $\sum_s t_2^s/(u; q_1, q_2)_{\infty, s+1}$. As $1/(u; q_1, q_2)_{\infty, s+1}$ tends to $1/(u; q_1, q_2)_{\infty, \infty}$, when s tends to infinity when the previous sum is multiplied by $(1 - t_2)$ and when t_2 is replaced by 1, we get: $1/(u; q_1, q_2)_{\infty, \infty}$. By multiplying the right-hand side by $(1-t_1)$ and $(1-t_2)$ successively and by making the substitutions $t_1 = 1$, then $t_2 = 1$, we obtain: $\sum_n A_n(q_1, q_2)\, u^n/((q_1;q_1)_n\,(q_2;q_2)_n)$.

19. (a) If $\sigma(i) = j$, then $\mathbf{i}\,\sigma(j) = i$ and $\mathbf{c}\,\mathbf{i}\,\sigma(j) = (n+1-i)$. Hence $\mathbf{i}\,\mathbf{c}\,\mathbf{i}\,\sigma(n+1-i) = j$ and $\mathbf{r}\,\sigma(n+1-i) = \sigma(i) = j$.

(b) Write the n points $(1,\sigma(1))$, $(2,\sigma(2))$, ... , $(n,\sigma(n))$ onto the square $\{1,\ldots,n\} \times \{1,\ldots,n\}$. The reflection of the square about the horizontal (resp. vertical, resp. major diagonal) axis transforms the graph of σ onto the graph of the permutation $\mathbf{c}\,\sigma$ (resp. $\mathbf{r}\,\sigma$, resp. $\mathbf{i}\,\sigma$). But those reflections generate the dihedral group D_4 and there is a one-to-one correspondence between those reflections and the operations $\mathbf{i}$, $\mathbf{c}$ and $\mathbf{r}$.

(c) The relations can be verified either by computation, or geometrically.

(d) Lets $\mathbf{c}\,\sigma := \sigma'$ and $\mathbf{r}\,\mathbf{c}\,\sigma := \sigma''$. Then $j \in \mathrm{Ligne}\,\sigma$ if and only if $j \in [n-1]$ and $\sigma(j) > \sigma(j+1)$; hence $\sigma'(j) < \sigma'(j+1)$ and also $\sigma''(j'') < \sigma''(j'')$ with $j'' = n+1-j-1$. Consequently, $j \in \mathrm{Ligne}\,\sigma$ if and only if $j \in [n-1]$ and, in an equivalent manner, $j \notin \mathrm{Ligne}\,\mathbf{r}\,\sigma$ or $n-j \in \mathrm{Ligne}\,\mathbf{r}\,\mathbf{c}\,\sigma$.

20. The verification of all those properties is easy.

21. (a) First $C_1(x) = x$; by induction $xC_{n-1}(x)$ is the generating polynomial for the permutations having n as a unit cycle; moreover, $(n-1)C_{n-1}(x)$ is the generating polynomial for the other permutations, as the element n can be inserted in $(n-1)$ manners into each permutation of order $(n-1)$ to create a permutation of order n, where n is not a unit cycle.

(b) Several proofs can be made. If we use $\#\,\mathrm{Rmals}$, then $xC_{n-1}(x)$ is the generating polynomial for the permutations ending by 1. Moreover, $(n-1)C_{n-1}(x)$ is the generating polynomial for the other permutations, because the insertion of 1 into each one of the $(n-1)$ possible slots (excluding the rightmost one) of a permutation of $2, 3, \ldots, n$, does not modify the right-to-left maximum letter set.

(c) Consider the following modification of the Lehmer-coding: if $\sigma = \sigma(1)\ldots\sigma(n)$, is a permutation, let $w := x_1x_2\ldots x_n$ be the (modified) Lehmer coding defined by $x_i := \#\{j : i+1 \le j \le n, \sigma(i) > \sigma(i+1)\}$. Then, $\mathrm{inv}\,\sigma = \mathrm{tot}\,w$, $\mathrm{Rmip}\,\sigma = \{i : 1 \le i \le n\, x_i = 0\}$ and $\mathrm{Rmap}\,\sigma = \{i : 1 \le i \le n\, x_i = n-i\}$. First, $C_1(x,y,q) = xy$;

then, by induction,

$$\begin{aligned}
C_n(x,y,q) &= \sum_{w\in \mathrm{SE}_n} x^{\sum_{1\le i\le n}\chi(x_i=n-i)} y^{\sum_{1\le i\le n}\chi(x_i=0)} q^{\sum_{1\le i\le n} x_i}\\
&= \sum_{x_1=0}^{n-1} x^{\chi(x_1=n-1)} y^{\chi(x_1=0)} q^{x_1}\\
&\quad \times \sum_{w'\in \mathrm{SE}_{n-1}} x^{\sum_{2\le i\le n}\chi(x_i=n-i)} y^{\sum_{2\le i\le n}\chi(x_i=0)} q^{\sum_{2\le i\le n} x_i}\\
&= (y+q+q^2+\cdots+q^{n-2}+q^{n-1}x)C_{n-1}(x,y,q).
\end{aligned}$$

(d) In the maj-coding $x_1x_2\ldots x_n$ of a permutation σ we have $x_i=0$ if and only if i is on the right of the subpermutation of σ reduced to the letters $1,2,\ldots,(i-1)$, i.e., if and only if $i\in \mathrm{Rmals}\,\sigma$.

(e) First, $D_1(y,q)=y$ and for $n\ge 2$

$$\begin{aligned}
D_n(y,q) &= \sum_{w\in \mathrm{SE}_n} y^{\sum_{1\le i\le n}\chi(x_i=0)} q^{\mathrm{tot}\,w}\\
&= \sum_{w'\in \mathrm{SE}_{n-1}} y^{\sum_{1\le i\le n-1}\chi(x_i=0)} q^{\mathrm{tot}\,w'} \sum_{x_n=0}^{n-1} y^{\chi(x_n=0)} q^{x_n}\\
&= D_{n-1}(y,q)(y+q+q^2+\cdots+q^{n-2}+q^{n-1}).
\end{aligned}$$

(f) First, notice that, for $k\ge 1$,

$$y+q+q^2+\cdots+q^k x = \frac{y(1-q)+q}{1-q}\Bigl(1-q^k\frac{1-x(1-q)}{y(1-q)+q}\Bigr)$$

and $\dfrac{y(1-q)+q}{1-q}\Bigl(1-\dfrac{1-x(1-q)}{y(1-q)+q}\Bigr)=x+y-1$. Therefore,
$C_n(q,x,y)=\Bigl(\dfrac{1-x(1-q)}{y(1-q)+q};q\Bigr)_n\Bigl(\dfrac{y(1-q)+q}{1-q}\Bigr)^n\dfrac{xy}{x+y-1}$, so that

$$\begin{aligned}
&\sum_{n\ge 0} C_n(q,x,y)\frac{u^n}{(q;q)_n}\\
&\qquad = 1+\frac{xy}{x+y-1}\sum_{n\ge 1}\Bigl(\frac{1-x(1-q)}{y(1-q)+q};q\Bigr)_n\Bigl(u\frac{y(1-q)+q}{1-q}\Bigr)^n\\
&\qquad = 1+\frac{xy}{x+y-1}\left(\frac{\Bigl(\dfrac{u}{1-q}-ux;q\Bigr)_\infty}{\Bigl(uy+\dfrac{uq}{1-q};q\Bigr)_\infty}-1\right),
\end{aligned}$$

by using the q-binomial theorem (Theorem 1.1).

22. (a) The polynomial $A_{n,k}$ is the generating function for the permutations of order n having k descents by $(\mathrm{maj}, \#\,\mathrm{Rmil})$. When the letter n is inserted into a permutation of order $(n-1)$, the number of descents remains alike, if n is inserted into a descent and is increased by one otherwise; moreover, the statistic "$\#\,\mathrm{Rmil}$" is increased by one, if n is placed at the end of the permutation.

(b) Form the two factorial generating functions:

$$A(t,q,y;u) := \sum_{n\ge 0} A_n(t,q,y)\frac{u^n}{(t;q)_n};$$
$$B(t,q,y;u) := \sum_{n\ge 0} A_n(t,q,y)\frac{u^n}{(t;q)_{n+1}}.$$

From (a) we deduce:

$$A(t,q,y;u)=1+yuB(t,q,y;u)+\frac{uq}{1-q}A(t,q,y;u)-\frac{uq}{1-q}A(tq,q,y;u).$$

As $A(t,q,y;u) = B(t,q,y;u) - tB(t,q,y;qu)$, we get

$$\Bigl(1-\frac{uq}{1-q}-uy\Bigr)B(t,q,y;u)+\Bigl(-t+\frac{utq}{1-q}\Bigr)B(t,q,y;qu)$$
$$+\frac{uq}{1-q}B(tq,q,y;u)-\frac{utq^2}{1-q}B(tq,q,y;qu)=1.$$

Let $B(t,q,y;u) = \sum_{s\ge 0} t^s\, G_s(q,y;u)$ and work out a recurrence for the coefficients $G_s(q,y;u)$:

$$(1-q-uq-yu+yqu)\sum_{s\ge 0} t^s\, G_s(q,y;u)$$
$$+(-1+q+qu)\sum_{s\ge 1} t^s\, G_{s-1}(q,y;qu)$$
$$+uq\sum_{s\ge 0} t^s q^s\, G_s(q,y;u) - u\sum_{s\ge 1} t^s q^{s+1}\, G_{s-1}(q,y;qu) = 1-q.$$

By taking the coefficients of t^s on both sides we obtain:

$$G_0(q,y;u) = \frac{1}{1-yu};$$
$$G_s(q,y;u) = \frac{1-u(q+q^2+\cdots+q^s)}{1-u(y+q+q^2+\cdots+q^s)}G_{s-1}(q,y;qu) \quad (s\ge 1),$$

so that $G_s(q,y;u)=\displaystyle\prod_{0\le k\le s}\frac{1-u(q^{k+1}+\cdots+q^s)}{1-u(yq^k+q^{k+1}+\cdots+q^s)}$. As

$$1-u(yq^k+q^{k+1}+\cdots+q^s)=\frac{1-q+uq^{s+1}}{1-q}\Bigl(1-q^k u\frac{y(1-q)+q}{1-q+uq^{s+1}}\Bigr),$$

for $k=0,1,\ldots,s$, we get

$$G_s(q,y;u)=\prod_{0\le k\le s}\frac{1-q^k u\dfrac{q}{1-q+uq^{s+1}}}{1-q^k u\dfrac{y(1-q)+q}{1-q+uq^{s+1}}}=\frac{\Bigl(\dfrac{uq}{1-q+uq^{s+1}};q\Bigr)_{s+1}}{\Bigl(u\dfrac{y(1-q)+q}{1-q+uq^{s+1}};q\Bigr)_{s+1}}.$$

23. (a) As $\mathbf{c}=\mathbf{i}\,\mathbf{r}\,\mathbf{i}$, properties (i)—(iii) and relation (19.4) imply that if $\sigma\mapsto PQ$, then $\mathbf{i}\,\sigma\mapsto QP$, $\mathbf{r}\,\mathbf{i}\,\sigma\mapsto Q^TP^{JT}$ and $\mathbf{c}\,\sigma=\mathbf{i}\,\mathbf{r}\,\mathbf{i}\,\sigma\mapsto P^{JT}Q^T$.

(b) All the relations are proved by direct calculation.

(c) $P^J=\begin{array}{|l}4\ 5\\ 1\ 2\ 3\\ \hline\end{array}$, $Q^J=\begin{array}{|l}3\ 5\\ 1\ 2\ 4\\ \hline\end{array}$, $P^T=\begin{array}{|l}5\\ 2\ 4\\ 1\ 3\\ \hline\end{array}$, $Q^T=\begin{array}{|l}5\\ 3\ 4\\ 1\ 2\\ \hline\end{array}$,

$P^{JT}=\begin{array}{|l}3\\ 2\ 5\\ 1\ 4\\ \hline\end{array}$, $Q^{JT}=\begin{array}{|l}4\\ 2\ 5\\ 1\ 3\\ \hline\end{array}$. Notice that $\mathbf{i}$, $\mathbf{c}$, $\mathbf{r}$ correspond to *geometric* transformations on permutations, but neither $\mathbf{t}$, nor $\mathbf{j}$.

Tableaux	Permutations	Tableaux	Permutations
PQ	$\sigma=3\,1\,4\,2\,5$	P^TQ^T	$\mathbf{t}\,\sigma=2\,5\,1\,4\,3$
PQ^J	$\mathbf{j}\,\sigma=3\,4\,1\,5\,2$	P^TQ^{JT}	$\mathbf{r}\,\sigma=5\,2\,4\,1\,3$
P^JQ	$4\,1\,5\,2\,3$	$P^{JT}Q^T$	$\mathbf{c}\,\sigma=3\,5\,2\,4\,1$
P^JQ^J	$1\,4\,2\,5\,3$	$P^{JT}Q^{JT}$	$3\,2\,5\,1\,4$
QP	$\mathbf{i}\,\sigma=2\,4\,1\,3\,5$	Q^TP^T	$3\,1\,5\,4\,2$
Q^JP	$3\,5\,1\,2\,4$	$Q^{JT}P^T$	$4\,2\,5\,3\,1$
QP^J	$2\,4\,5\,1\,3$	Q^TP^{JT}	$5\,3\,1\,4\,2$
Q^JP^J	$1\,3\,5\,2\,4$	$Q^{JT}P^{JT}$	$4\,2\,1\,5\,3$

24. For each Ferrers diagram λ with m boxes and each vector $\mathbf{m}=(m_1,m_2,\ldots,m_r)$ of positive integers such that $m_1+m_2+\cdots+m_r=m$ let $\mathcal{K}(\lambda,\mathbf{m})$ denote the set of Young tableaux containing m_1 1's, m_2 2's, … , m_r r's. Let σ be a permutation of $1,2,\ldots,r$.

We construct a one-to-one correspondence between $\mathcal{K}(\lambda, \mathbf{m})$ and $\mathcal{K}(\lambda, \sigma\,\mathbf{m})$ as follows. Let $\mathbf{m} = (m_1, \dots, m_i, m_{i+1}, \dots, m_r)$ and $\mathbf{m}' = (m_1, \dots, m_{i+1}, m_i, \dots, m_r)$ differ only by a transposition of two adjacent terms and consider a tableau T in $\mathcal{K}(\lambda, \mathbf{m})$ in its planar representation. Write all the pairs $i, i+1$ in boldface whenever those two integers occur in the same column with $(i+1)$ just above i. The remaining i's and $(i+1)$'s in T occur as horizontal blocks $i^a\, j^b$ ($a \geq 0$, $b \geq 0$). We define a bijection $T \mapsto T'$ of $\mathcal{K}(\lambda, \mathbf{m})$ onto $\mathcal{K}(\lambda, \mathbf{m}')$ by replacing each block $i^a\, j^b$ in T by $i^b\, j^a$ and rewriting the vertical pairs $i, i+1$ in roman type.

25. (a) The identity is true for $k = 1$. For $k \geq 2$ expand the determinant of $N_r(L, n)$ by the cofactors of the last row. By induction

$$\det N_r(L,n) = \det N_r(\{\ell_1, \dots, \ell_{k-1}\}, \ell_k) \begin{bmatrix} \ell_{k+1} - \ell_k + r \\ r \end{bmatrix} \\ - \det N_r(\{\ell_1, \dots, \ell_{k-1}\}, n).$$

The first (resp. second) term on the right-hand side is the generating polynomial for the words $w = w_1 w_2 \dots w_n$ satisfying the conditions $(*)_{(L,n)}$, with the possible (resp. sole) exception of the subcondition $w_{\ell_k} < w_{\ell_k+1}$.

(b) The construction of the bijections may be regarded as another version of the *MacMahon Verfahren.*

(c) Similar to the proof of Theorem 18.2. Start with a pair (σ, s) having the properties $(E22.2)$ and let $d = d_1 d_2 \dots d_n$ be the word whose letters d_i are defined by

$$d_i := \begin{cases} s_i - s_{i+1} - \chi(i \in \operatorname{Iligne} \sigma), & \text{if } 1 \leq i \leq n-1; \\ s_n, & \text{if } i = n. \end{cases}$$

Then $d_1 + \dots + d_n + \operatorname{ides} \sigma = s_1$, so that it makes sense to put $d_0 := r - s_1 \geq 0$ and $d_0 + d_1 + \dots + d_n + \operatorname{ides} \sigma = r$. On the other hand, $1 \cdot d_1 + 2 \cdot d_2 + \dots + n \cdot d_n + \operatorname{imaj} \sigma = \operatorname{tot} s$. The mapping $(\sigma, s) \mapsto (\sigma, (d_0, d_1, \dots, d_n))$ is a bijection of the pairs (σ, s) satisfying $(E22.2)$ onto the pairs $(\sigma, (d_0, d_1, \dots, d_n))$ such that $\operatorname{Ligne} \sigma = L$ and where the integers d_0, d_1, ... , d_n are nonnegative and of sum $d_0 + d_1 + \dots + d_n = r - \operatorname{ides} \sigma$. Hence,

$$\sum_{r \geq 0} t^r \det N_r(L,n) \\ = \sum_{r \geq 0} t^r \sum_{w \in W_r(L,n)} q^{\operatorname{tot} w} = \sum_{r \geq 0} t^r \sum_{(\sigma, s)} q^{\operatorname{tot} s}$$

$$= \sum_{r\ge 0}\sum_{\sigma}\sum_{d_0,\dots,d_n} t^{d_0+\dots+d_n+\operatorname{ides}\sigma} q^{1.d_1+\dots+n.d_n+\operatorname{imaj}\sigma}$$
$$= \sum_{\sigma} t^{\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma} \sum_{r\ge \operatorname{ides}\sigma} \sum_{\substack{d_0+\dots+d_n\\ =r-\operatorname{ides}\sigma}} t^{d_0}(tq)^{d_1}\cdots(tq^n)^{d_n}$$
$$= \sum_{\sigma} t^{\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma} \frac{1}{(t;q)_{n+1}}.$$

(c) This is the specialization for $t = 1$.

(d) A direct proof can be found (see [St86, p. 70]); it is also a consequence of Corollary 11.5.

26. (a) Distribute the n integers $1, 2, \dots, n$ into k classes of sizes $a_1, a_2, \dots$, a_k. Write the elements in each class in increasing order and justapose those class increasing sequences in the following order: first class 1, then class 2, … , class k. This yields a permutation σ having the following property: $\operatorname{Ligne}\sigma \subset \{a_1, a_1+a_2, \dots, a_1+a_2+\dots+a_{k-1}\}$. Furthermore, there are $n!/(a_1!\,a_2!\,\cdots\,a_k!)$ ways of making up such permutations.

(b) The expansion of the right-hand side reads $\sum\limits_{\sigma,\mathbf{b}} q_1^{b_1(\sigma)} q_2^{b_2(\sigma)} \cdots q_n^{b_n(\sigma)}$, where the summation is over all permutations σ and the sequences of integers $\mathbf{b} = (b_1(\sigma) \ge b_2(\sigma) \ge \dots \ge b_n(\sigma) \ge 0)$ such that the following property holds: $k \in \operatorname{Ligne}\sigma \Rightarrow b_k(\sigma) > b_{k+1}(\sigma)$. But the sequence $\mathbf{b}$ may be written as $i_1^{a_1} i_2^{a_2} \dots i_k^{a_k}$, where $i_1 > i_2 > \dots > i_k \ge 0$ and $a_1 \ge 1$, $a_2 \ge 1$, … , $a_k \ge 1$, so that the monomial $q_1^{b_1(\sigma)} q_2^{b_2(\sigma)} \cdots q_n^{b_n(\sigma)}$ is still equal to $(*)$ $q_1^{i_1} \cdots q_{a_1}^{i_1} q_{a_1+1}^{i_2} \cdots q_{a_1+a_2}^{i_2} \cdots q_{a_1+\dots+a_{k-1}+1}^{i_k} \cdots q_{a_1+\dots+a_k}^{i_k}$. But the sequence $i_1^{a_1} i_2^{a_2} \dots i_k^{a_k}$ is nothing but a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ having at most n positive parts, and a_1 parts equal to i_1, a_2 parts equal to i_2, … , a_k parts equal to i_k, so that the monomial $(*)$ is also equal to $\mathbf{q}^\lambda$. Hence,

$$\sum_{\sigma,\mathbf{b}} q_1^{b_1(\sigma)} q_2^{b_2(\sigma)} \cdots q_n^{b_n(\sigma)}$$
$$= \sum_{\lambda} \mathbf{q}^\lambda \, \#\{\sigma : \operatorname{Ligne}\sigma \subset \{a_1, a_1+a_2, \dots, a_1+\dots+a_{k-1}\}\}$$
$$= \sum_{\lambda, l(\lambda)\le n} \mathbf{q}^\lambda \frac{n!}{a_1!\,a_2!\,\cdots\,a_k!} = \sum_{\lambda, l(\lambda)\le n} \mathbf{q}^\lambda \binom{n}{\mathbf{m}(\lambda)}.$$

27. The left-hand side of $(E27.1)$ is equal to the sum of the series $\sum t^{s'+\operatorname{des}_U w} q^{\|\mathbf{a}\|+\operatorname{maj}_U w}$ over all triples $(s', \mathbf{a}, w)$. By using Proposition 3.1 the right-hand side is equal to the sum of the series

$\sum t^s\, q^{\|\mathbf{a}^{(1)}\|+\cdots+\|\mathbf{a}^{(r)}\|}$, where each $\mathbf{a}^{(i)} = (a_{i,1}, \ldots, a_{i,c_i})$ is a sequence of integers satisfying $s \ge a_{i,1} \ge \cdots \ge a_{i,c_i} \ge 0$, if $i \in S_<$; $s \ge a_{i,1} > \cdots > a_{i,c_i} \ge 0$, if $i \in S_\le$; $s \ge a_{i,1} \ge \cdots \ge a_{i,c_i} \ge 1$, if $i \in L_<$; $s \ge a_{i,1} > \cdots > a_{i,c_i} \ge 1$, if $i \in L_\le$. The bijection $(s', \mathbf{a}, w) \mapsto (s, \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(r)})$ such that $s = s' + \operatorname{des}_U w$ and $\|\mathbf{a}\| + \operatorname{maj}_U w = \|\mathbf{a}^{(1)}\| + \cdots + \|\mathbf{a}^{(r)}\|$ can be constructed by rewriting the *MacMahon Verfahren* developed in sections 6 and 7 almost verbatim. To obtain $(E27.2)$ use the manipulation on q-series as at the end of section 7 for going from identity (7.7) to (7.8).To derive $(E27.3)$ multiply $(E27.2)$ by $(1-t)$ and make $t=1$.

28. (a) The identity follows from Theorem 5.1. The letters belonging to $S_<$ bring no further U-inversions; those belonging to $S_\le \cup L_\le$ bring $\prod_{i \in S_\le \cup L_\le} q^{\binom{m_i}{2}}$ extra U-inversions when they are compared between themselves; finally, the U-inversion number must be increased by the number of letters belonging to $L_< \cup L_\le$, that is, $\sum_{i \in L_< \cup L_\le} m_i$. To show that ${}^{\mathrm{inv}}A^U_{\mathbf{m}}(q) = A^U_{\mathbf{m}}(1,q)$ is equal to the right-hand side of (E27.3) it suffices to expand the q-multinomial coefficient and to make use of (3.7) and (3.8).

(b) Let $w = x_1 x_2 \ldots x_m$ be a word in $R(\mathbf{m})$. For each $i \in S_\le \cup L_\le$ replace the m_i occurrences of the letter i in w by the letters (i, m_i), $(i, m_i - 1)$, … , $(i,1)$, when reading the word *from left to right.* For each $i \in S_< \cup L_<$ replace all the m_i occurrences of i by the same letter $(i,1)$. Finally, let $* := (h + \frac{1}{2}, 1)$. By convention, $(i,k) < (j,l)$ if either $i < j$, or $i = j$ and $k < l$, so that the new word, say, $\overline{w}$, has its letters in a totally ordered alphabet.
Apply the transformation Φ to $\overline{w}\,*$ to obtain a word of the form $\Phi(\overline{w}\,*) = \overline{w}'\,*$. In $\overline{w}'$ replace each letter (i,k) by the letter i, to obtain a word w' in the class $R(\mathbf{m})$. Define $\Psi(w) = w'$.

(c) Apply the algorithm described in (b). In the first operation write i_k instead of (i,k) and obtain $\overline{w} = 6_1, 2_2, 2_1, 3_1, 1_1, 7_1, 7_1, 4_1, 8_1, 5_1$. Here $* = 4 + \frac{1}{2}$; apply Φ to $\overline{w}\,*$, then derive $\Psi(w)$:

$$
\begin{aligned}
\overline{w}\,* = {} & 6_1, 2_2, 2_1, 3_1, 1_1, 7_1, 7_1, 4_1, 8_1, 5_1, * \\
& \mid 6_1 \\
& 6_1 \mid 2_2 \\
& 6_1, 2_2 \mid 2_1 \\
& 6_1, 2_2 \mid 2_1 \mid 3_1 \\
& 2_2, 6_1, 2_1, 3_1 \mid 1_1 \\
& 2_2, 6_1, 2_1, 3_1, 1_1 \mid 7_1 \\
& 2_2, 6_1, 2_1, 3_1, 1_1, 7_1 \mid 7_1
\end{aligned}
$$

$$\begin{aligned}
&2_2, 6_1 \mid 2_1, 3_1, 1_1, 7_1 \mid 7_1 \mid 4_1\\
&6_1, 2_2, 7_1, 2_1, 3_1, 1_1, 7_1, 4_1 \mid 8_1\\
&6_1 \mid 2_2, 7_1 \mid 2_1, 3_1, 1_1, 7_1 \mid 4_1, 8_1 \mid 5_1\\
&6_1 \mid 7_1 \mid 2_2, 7_1 \mid 2_1, 3_1, 1_1, 8_1 \mid 4_1, 5_1 \mid *\\
\Phi(\overline{w}\,*) &= 6_1, 7_1, 7_1, 2_2, 8_1, 2_1, 3_1, 1_1, 5_1, 4_1, *\\
\Psi(w) &= 6, 7, 7, 2, 8, 2, 3, 1, 5, 4
\end{aligned}$$

The number of U-inversions is $0+0+0+3+0+5+4+7+4+5 = 28$, plus the number of elements in $L_< \cup L_\le$, which is 5. Thus, $\operatorname{inv}_U \Psi(w) = 33$.

29. (a) By induction on the number l of parts of λ.
(b) Again by induction on l we get $\xi_q(h_\lambda) = {}^{\operatorname{inv}}A_\lambda(t,q)/(q;q)_n$, where ${}^{\operatorname{inv}}A_\lambda := \sum_\sigma t^{\operatorname{des}_\lambda \sigma} q^{\operatorname{inv}\sigma}$ $(\sigma \in \mathfrak{S}_n)$.
(c) Same derivation as in (a).
(d) Let $\mathcal{E}(u) := (-t + \exp(u(t-1)))/(1-t) = 1 + \sum\limits_{k\ge1} (1-t)^{k-1} u^k/k!$
Then $\mathcal{E}(u)\bigl(1 + \sum\limits_{k\ge1} t\, A_k(t)(-u)^k/k!\bigr) = \mathcal{E}'(u)$ (the derivative of $\mathcal{E}(u)$).
(e) $\xi_q(p_n) = \dfrac{1}{(q;q)_n} \sum\limits_{1\le k\le n} q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix} k(t-1)^{k-1}\, {}^{\operatorname{inv}}A_{n-k}(t,q)$;
$\xi_{Q,q}(p_n) = \dfrac{1}{(Q;Q)_n (q;q)_n} \sum\limits_{1\le k\le n} Q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_Q \begin{bmatrix} n\\ k\end{bmatrix}_q k(t-1)^{k-1} B_{n-k}(t,Q,q)$.
(f) No more hints.

30. The generating function for the $B_n^{(3)}$ is equivalent to the recurrence formula

$$B_n^{(3)} = \sum_{0\le k\le n} X^k\, q^{\binom{k+1}{2}} \sum_{0\le m\le n-k} \begin{bmatrix} n\\ m\end{bmatrix} A_{n-m}^{n-m}(t,q) \begin{bmatrix} n-m\\ k\end{bmatrix} q^{mk}(1-t)^m.$$

Hence, to show that $B_n''' = B_n^{(3)}$, it suffices for $0 \le k \le n$ to prove the recurrence formula

$$A_n^k(t,q) = \sum_{0\le m\le n-k} \begin{bmatrix} n-k\\ m\end{bmatrix}_q A_{n-m}^{n-m}(t,q)\, q^{mk}\,(1-t)^m.$$

Use the iterative method already illustrated in § 21.3. If a permutation of order n ends with a term at most equal to $(n-k)$, we can

consider its longest increasing rightmost factor (*l.i.r.f.*). If the permutation ends with a term greater than or equal to $(n-k+1)$, make the convention that its *l.i.r.f.* is of length zero.
For $m = 0, 1, \ldots, n-k$ designate the generating function for those permutations whose *l.i.r.f.* is of length m by F_m and let $G_m := F_m + \cdots + F_{n-k}$. Then $u_m = \begin{bmatrix} n-k \\ m \end{bmatrix}_q A_{n-m}^{n-m}(t,q)\, q^{mk}$ is the generating function for the permutations whose *l.i.r.f.* is of length at least equal to m, adding a supplementary descent when the *l.i.r.f.* is of length greater than or equal to $(m+1)$. We then have

$$u_m = F_m + t\, G_{m+1};$$

and

$$G_m = u_m + (1-t)G_{m+1}.$$

As $A_{n,k}(t,q) = G_0$ and as $G_{n-k} = A_{k,k}(t,q)\, q^{(n-k)k}$, we obtain the recurrence formula by iteration starting with G_0.

31. No comment for the first part. The relation involving ${}^{\mathrm{sgn}}\!A_{2n+1}(t)$ is treated in the same way. For the last part notice that $\operatorname{maj}\sigma = 2 \operatorname{maj}\tau + \sum_{i\in E(\sigma)}(2i-1)$. Hence

$$\begin{aligned}
{}^{\mathrm{sgn}}\!A_n(t,q) &= \sum_{\sigma} \operatorname{sgn}\sigma\, t^{1+\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} \\
&= \sum_{(\tau,E(\sigma))} (-1)^{\#E(\sigma)}\, t^{1+\operatorname{des}\tau+\#E(\sigma)} q^{2\operatorname{maj}\tau+\Sigma_{i\in E(\sigma)}(2i-1)} \\
&= \sum_{E\subset\{1,\ldots,n\}} (-t)^{\#E} q^{\Sigma_{i\in E}(2i-1)} \sum_{\tau\in\mathfrak{S}_n} t^{1+\operatorname{des}\tau} q^{2\operatorname{maj}\tau} \\
&= (tq;q^2)_n\, t^{\,\mathrm{maj}}\!A_n(t,q^2).
\end{aligned}$$

32. (a) The table of the first $a(n,i)$ is the following:

$i=$	1	2	3	4	5	6	7	8
$n=1$	1							
2	2	2						
3	2	1	1					
4	3	3	1	1				
5	3	2	1	1	1			
6	3	3	2	1	1	1		
7	3	2	2	1	1	1	1	
8	4	4	2	2	1	1	1	1

(b) By definition $1-q^i=\prod\{\Phi_d(q) : d\,|\,i\}$ for each $i\ge 1$. As $(1-q^{2i})=(1-q^i)(1+q^i)$, we have $1+q^i=\prod\{\Phi_d(q) : d\,|\,2i,\ d{\not|}\,i\}$. In particular, if $0\le j\le l$, then $1+q^{m2^j}=\prod\{\Phi_d(q) : d\,|\,m2^{j+1}, d{\not|}\,m2^j\}=\prod\{\Phi_d(q) : d\in A_j\}$.

(c) As the sets A_j are two by two disjoint, it suffices to show that B is the union of the A_j's. But if $d\,|\,m2^{j+1}$, $d{\not|}\,m2^{j+1}$ for some j with $0\le j\le l$, then $d\,|\,2n$ (equal to $m2^{l+1}$) and d is even. Thus d belongs to B. Conversely, suppose $d\,|\,2n$ and d even. Then $d=m'2^{j+1}$ with m' odd, $m'\,|\,m$ and $0\le j\le l$. Consequently, d is an element of A_j.

(d) The product is zero if $k\ge n$. When $0\le k\le n-1$, the Gaussian polynomial is the product of the two factors $P=\dfrac{Od_n(q)(1-q^{2n-1})Od_{n-1}(q)\cdots(1-q^{2n-2k+1})Od_{n-k}(q)}{(1-q^{2k+1})Od_k(q)(1-q^{2k-1})\cdots Od_1(q)(1-q)}$ and $Q=\dfrac{Ev_n(q)\,Ev_{n-1}(q)\,\ldots\,Ev_{n-k}(q)}{Ev_k(q)\,Ev_{k-1}(q)\,\ldots\,Ev_1(q)}$. When numerators and denominators are expressed in terms of cyclotomic polynomials, then P and Q involve cyclotomic polynomials Φ_d with d odd for P and d even for Q. As $\left[{2n \atop 2k+1}\right]$ is a polynomial and the cyclotomic polynomials are irreducible, each of those two factors is also a polynomial. But P is precisely equal to the expression under consideration.

(e) Rewrite the recurrence relation (24.5)

$$D_{2n+1}(q)=\sum_{0\le k\le n-1}\begin{bmatrix}2n\\2k+1\end{bmatrix}q^{2k+1}D_{2k+1}(q)\,D_{2n-2k-1}(q).$$

First, $D_1(q)=1$. Proceed by induction on $n\ge 1$. For $0\le k\le n-1$ the product $\begin{bmatrix}2n\\2k+1\end{bmatrix}\dfrac{D_{2k+1}(q)D_{2n-2k-2}(q)}{Ev_1(q)\ldots Ev_n(q)}$ is a polynomial because it may be factorized as $\begin{bmatrix}2n\\2k+1\end{bmatrix}\dfrac{Ev_0(q)\cdots Ev_k(q)}{Ev_{n-k}(q)\cdots Ev_n(q)}\times\dfrac{D_{2k+1}(q)}{Ev_0(q)\cdots Ev_k(q)}\times\dfrac{D_{2n-2k-1}(q)}{Ev_1(q)\cdots Ev_{n-k-1}(q)}$. The first factor is a polynomial by (d), so are the other two by the induction hypothesis. Hence, when n is odd, each term in the sum in the quadratic recurrence above is divisible by $F_n(q)=Ev_1(q)Ev_2(q)\cdots Ev_n(q)$. Thus $F_n(q)\,|\,D_{2n+1}(q)$. When n is even, rewrite the recurrence formula by grouping the terms two by two to give

$$D_{2n+1}(q)=\sum_k\begin{bmatrix}2n\\2k+1\end{bmatrix}q^{2k+1}(1+q^{2(n-2k-1)})D_{2k+1}(q)\,D_{2n-2k-1}(q),$$

where this time k runs over the interval $[0, n/2-1]$. As n is even, the binomial $1+q^{2(n-2k-1)}$ is divisible by $1+q^2$; and the product $\begin{bmatrix}2n\\2k+1\end{bmatrix} D_{2k+1}(q)D_{2n-2k-2}(q)$ is divisible by $Ev_1(q)\cdots Ev_n(q)$. Hence, each term in the sum is divisible by $F_n(q)=(1+q^2)Ev_1(q)\cdots Ev_n(q)$.

33. (a) Let $\sigma = x_1x_2\ldots x_{2n}$ be a balanced rising alternating permutation (*b.r.a.p.*) of order $2n$. As $2n$ is necessarily a peak of σ, it must be the leftmost one, i.e., $x_2 = 2n$. If $x_1 = i$ with $i \le 2n-2$, then $(i+1)$ would occur to the right of i and σ would not be balanced. Thus σ is of the form $\sigma = (2n-1)(2n)x_3\cdots x_{2n-1}x_{2n}$ and is balanced if and only if the right factor $x_3\cdots x_{2n-1}x_{2n}$ is i-balanced for every $i = 1, 2, \ldots, 2n-3$. By induction, $(2n-3)(2n-2)\cdots 3412$ is the only *b.r.a.p.* of order $2n-2$, so that the unique *b.r.a.p.* of order $2n$ is $(2n-1)(2n)(2n-3)(2n-2)\cdots 3412$.

(b) If σ contains the factor $(i'+1)i'$, then all the letters greater than $(i'+1)$ are to the left of $(i'+1)$. But as σ is alternating of even length, the letter i' is between two letters greater than i', which is a contradiction.

(c) The relation $\operatorname{inv}\Phi(\sigma) = \operatorname{inv}\sigma + 1$ implies that a rising alternating permutation with a maximal number of inversions is necessarily balanced. As there is only one such a permutation whose number of inversions is equal to $2n(n-1)$, the property is proved.

(d) As $\sum\{q^{\operatorname{inv}\sigma} : \sigma \in A_{2n}\} = \sum\{q^{\operatorname{inv}\sigma} : \sigma \in B_{2n}\} = q^{4(n-1)}E_{2n-2}(q)$ and $\sum\{q^{\operatorname{inv}\sigma} : \sigma \in A_{2n}\cap B_{2n}\} = q^{4(n-2)+4(n-1)}E_{2n-4}(q)$, the induction on n implies $\sum\{q^{\operatorname{inv}\sigma} : \sigma \in A_{2n}\cup B_{2n}\} \equiv 2q^{4(n-1)}q^{2(n-1)(n-2)} - q^{4(n-2)+4(n-1)}q^{2(n-2)(n-3)} \equiv q^{2n(n-1)} \bmod (q+1)^2$.

(e) Let C_{2n} be the complement of $A_{2n}\cup B_{2n}$, that is, the set of rising alternating permutations of length $2n$ having $(2n)$ and $(2n-1)$ among their peaks and 1 and 2 among their troughs. There remains to show that $\sum\{q^{\operatorname{inv}\sigma} : \sigma \in C_{2n}\} \equiv 0 \bmod (q+1)^2$. Let τ, τ' be the transpositions $(2n-1, 2n)$ and $(1,2)$, respectively, and G be the group of order 4 generated by $\{\tau, \tau'\}$. The group G acts on C_{2n} and the generating polynomial for the four elements in each orbit by the number of inversions is divisible by $(q+1)^2$. Therefore, the generating polynomials for all elements of C_{2n} is also divisible by $(q+1)^2$.

Notes

The use of the algebra of q-series in Combinatorics goes back to MacMahon [Mac13, 15, 78], who realized that certain closed formulas in Enumeration could only be expressed in that context. He had a great talent for deriving many of his results in a very intuitive manner. In our to-day's more systematic approach a memoir on q-series in Combinatorics has to begin with a chapter on basic hypergeometric series, at least on the fundamental result of that theory, which is the q-binomial theorem. More material can be found in Gasper and Rahman [GaRa90], also in the old book by Slater [Sl66], or in the more recent one by Fine [Fi88]. The second chapter in Andrews' book [An76] also covers all that is needed on this subject. The combinatorial aspects of the hypergeometric series identities are developed in [JoSt87], but not discussed in those Notes. Published in the seventies Gessel's, Ph.D. thesis [Ge77] must be viewed as an excellent memoir on q-series and combinatorics.

Coding permutations by sequences of integers for computer purposes goes back to Lehmer [Le60]. The inv-coding, as such, belongs to him. The maj-coding is implicit in the early papers by Carlitz [Ca54, 59, 75] on Eulerian numbers and made explicit in Rawlings [Ra79, 80]. In studying the genus zeta function of local minimal hereditary orders, Denert [Den90] introduced a new permutation statistic, which was later christened "den" and was shown to be equidistributed with the major index or inversion number. The den-coding in § 2.3 is taken from [FoZe90].

The algebra of the q-binomial coefficients is classical; see, e.g. [An71, § 3.3]. It was convenient to devote a full chapter, essentially chap. 4, to presenting the main combinatorial structures counted by those coefficients. Chapters 5 and 6 may be regarded as an extension of section 3.4 of Andrews's monograph [An71]. However, the proof of the fact that the Major Index is a q-multinomial statistic, a result that is due to MacMahon [Mac13], is given in greater detail; it involves the so-called *MacMahon Verfahren*, that can be viewed as a transformation on two-row matrices. That transformation is based upon a *commutation rule* that preserves the statistics under study. Notice that the *MacMahon Verfahren* has been extended by Stanley [St72] in his theory of (P, ω)-partitions.

As is often the case, an explicit combinatorial tool like the *MacMahon Verfahren* opens the way to significant extensions, such as the study of bivariable polynomials $A_{\mathbf{m}}(t,q)$, indexed by sequences $\mathbf{m}=(m_1,\ldots,m_r)$ of r nonnegative integers. Formula (7.7) already appears in MacMahon [Mac15, vol. 2, p. 211]. Chapters 7, 8 and 9 should be regarded as a systematic approach to combinatorial q-calculus. The material has been taken from [ClFo95a and b, Fo95, FoKr95].

Several sources have been used for chapter 10: [Ri58, p. 38–39 and 213–216, FoSc70] for the traditional Eulerian polynomials, [ChMo71, St76] for the q-inv Eulerian polynomials, [Fo76, Ga79] for the joint study of the two q-extensions. The *iteration method* developed for Lemma 10.2 appears in several different forms, for instance in [Ge82, Ze80a, FoZe91].

The joint combinatorial study of the statistics "maj" and "inv" on a class of rearrangements of an arbitrary sequence, as written in chapter 11, is borrowed from [Fo68, FoSc78]. However the construction of the fundamental transformation is made in a very different manner, as there is a canonical way of extending each bijection valid for binary words to a bijection over an arbitrary class of rearrangements. Notice that Björner and Wachs have proposed an interesting extension to Poset Theory [BjW88] and derived further properties of the fundamental transformation. Properties (d) and (e) in Theorem 11.3 are basically theirs.

The expansion of the *infinite* product $1/\prod_{i\ge 0,j\ge 0}(1-uq_1^iq_2^j)$, made in chapter 12, can be found in [Ca56] and also in [Ro74], where a first combinatorial interpretation was given. See also [St76].

The expansion of the *finite* product $1/\prod_{0\le i\le r,0\le j\le s}(1-uq_1^iq_2^j)$ naturally leads to the study of a four-variable generating polynomial for the permutation group. Identity (13.7) first appeared in [GG78]. Other proofs can be found in [Ra80, DeFo85]. Specializations had been anticipated in [Ca76].

There is no originality in discussing some basic facts on symmetric functions, as done in chapters 14, 15, 16. The best source of study remains Macdonald's book [Ma95]. To-day we rejoice at the coming out of Lascoux's treatise [La03] with its creative approach to the subject, as started in [LaSch81, 84].

The combinatorial definition of the Schur function given in chapter 17 is taken from Proctor [Pr89], but has been rewritten in a more systematic way, under the superb guidance of our friend Jean-Pierre Jouanolou.

Corollary 18.3 can be found in [St76], also in [DeFo85]. The idea of keeping a several-variable statistic y instead of the one-variable "imaj" is due to Adin et al. [ABR01]. This yields Theorem 18.2, but the resulting identities involve a *non-ring* linear homomorphism.

The Robinson-Schensted correspondence [Ro38, Sc61], that has been so popular in the seventies and eighties, is described in the more general set-up developed by Knuth [Kn70]. The geometric properties of that correspondence are due to Schützenberger [Sch73, 77]. An excellent exposition of its various properties is given in [Kn70, p. 48–72]. Theorem 19.4 may be regarded as an Adin-Brenti-Roichman extension of identity (13.7).

The generating function for the signed permutations by number of descents (Theorem 20.1) has been calculated by various authors: [St92, StE93, Br94, Re93a, Re93b, Re93c, Re95a, ClFo94]. The proof of Theorem 20.1 is taken from the last reference. Theorem 20.3 is due to Carlitz et al. [Ca76], Theorem 21.1 to [FoHa97]; however its specialization to the symmetric group appears already in [St76]. The idea of taking *finite analogs* of the Bessel functions for enumerating pairs of permutations is due to Fedou and Rawlings [Fe95, FeRa94, FeRa95]. The extension to the group of signed permutations and accordingly Theorem 22.5 can be found in [FoHa96]. The proof of the theorem relies upon a very convenient inversion formula, which has appeared in various contexts [GoJa83, p. 131, St86, p. 266, Vi86, HuWi75, Ze81]. The inversion formula stated here (Lemma 22.3) is borrowed from [FeRa94, FeRa95].

The proof of Theorem 23.1 is taken from [FoZe81]. The *Désarménien Verfahren* was developed in the two papers [De82 and 83]. Lemma 23.2, Proposition 23.3, as well congruences (23.19) and (23.27) are due to Désarménien. Congruences (23.20) and (23.28) are apparently new. Section 23.4 on signed Eulerian numbers are taken from [DeFo92]. Those numbers were introduced by Loday [Lod89] in a study of the cyclic homology of commutative algebras.

The theory of basic trigonometric functions is due to Jackson [Ja04] (see the detailed bibliography in Gasper and Rahman's book [GaRa90].) The study of *bibasic* trigonometric functions proposed in this memoir, as well as its combinatorial counterpart, are apparently new.

For Ex. 1 see [De82a] or [An76, chap. 3, Examples 1–6]. The proof of the Ramanujan sum (Ex. 2) reproduces Ismail's derivation [Is77]. The maj-inv bijection for permutations described in Ex. 3 belongs to common knowledge. For Ex. 12 and 13 refer to [Ri58, p. 38–39 and 213–216, FoSc70], as mentioned above. In Ex. 14 the solution made by Clarke [Cl95] has been borrowed. Ex. 14 reproduces the techniques developed in [Ha90a, 90b]. Notice that the study of the Denert statistics for arbitrary words, as developed in [Ha94, 95] has not been touched in those Notes. See, e.g. [Lo02, chap. 10].

The Z-statistic has been introduced in [ZeBr85] for the proof of the Andrews q-Dyson conjecture. The combinatorial approach in Ex. 17 is due to Han [Ha92]. The *tableau emptying-filling involution* was invented by Schützenberger [Sch73] and used in [FoSc78]. Ex. 20 is taken from [FoSc78]. Ex. 25 can be regarded as the t-extension of § 11.4 in the book by Lothaire [Lo02]. The identity of Ex. 26 is taken from [ABR01]. The calculation in Ex. 27 was made in [FoKr95], as well as the content of Ex. 28.

The Brenti homomorphism (see Ex. 29) has been used in [BecRe95], other applications of the Brenti homomorphism are proposed in the Exercise. The formula used for the length in the group of the signed permutations is due to Brenti [Br94], the generating function for the pair (length, number of descents) derived in Ex. 30 is due to Reiner [Re95a]. The involution in Ex. 31 belongs to Wachs [Wa92]. Finally, Ex. 32 and 33 are taken from [Fe81] and [AnFo82].

The algebra of symmetric functions, especially the Cauchy identity for Schur functions, has been a powerful tool for deriving various generating counting series. We have not mentioned other tools, such as the Hook Young diagrams developed by Berele, Regev and Remmel [BeRe85, 87] and the derived (k, l)-Schur functions [Rem83, 84, 87]. In most cases it seems that the Schur function model suffices for the derivations [DeFo91].

We have not discussed operator techniques, as developed in [An71] or [Ze80b]. The subject of permutation statistics is in full expansion, so that the present Notes can only be regarded as a partial aspect of to-day's state of the art. There have been interesting studies by the Californian school (see [Bec95, 96]) and by the vigorous Israeli school ([AR01, ABR01, ReRo03a, ReRo03b]. The goal is to find the most appropriate maj-analogs for the further Weyl groups, or to work out with set-valued statistics [FoHa02]. The first results found in [AR01] are very promising. Finally, let us mention the paper by Babson and Steingrímsson [BaSt00] on the classification of the patterns that lead to a Mahonian distribution.

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