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New hook length formulas for binary trees

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ABSTRACT. — We find two new hook length formulas for binary trees. The particularity of our formulas is that the hook length h_v appears as an exponent.

Consider the set $\mathcal{B}(n)$ of all binary trees with n vertices. It is well-known that the cardinality of $\mathcal{B}(n)$ is equal to the Catalan number (see, e.g., [11, p.220]):

$$\sum_{T\in\mathcal{B}(n)} 1 = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

For each vertex v of a binary tree $T \in \mathcal{B}(n)$ the *hook length* of v, denoted by $h_v(T)$ or h_v, is the number of descendants of v (including v). It is also well-known [5, p.67] that the number of ways to label the vertices of T with $\{1, 2, \ldots, n\}$, such that the label of each vertex is less than that of its descendants, is equal to $n!$ divided by the product of the h_v's ($v \in T$). On the other hand, each labeled binary tree with n vertices is in bijection with a permutation of order n [10, p.24], so that

$$\sum_{T\in\mathcal{B}(n)} n! \prod_{v\in T} \frac{1}{h_v} = n! \tag{2}$$

The following hook length formula for binary trees

$$\sum_{T\in\mathcal{B}(n)} \frac{n!}{2^n} \prod_{v\in T} \bigl(1+\frac{1}{h_v}\bigr) = (n+1)^{n-1} \tag{3}$$

is due to Postnikov [7]. Further combinatorial proofs and extensions have been proposed by several authors [1, 2, 3, 6, 9].

In the present Note we obtain the following two new hook length formulas for binary trees. The particularity of our formulas is that the hook length h_v appears as an exponent. Their proofs are based on the induction principle. It would be interesting to find simple bijective proofs.

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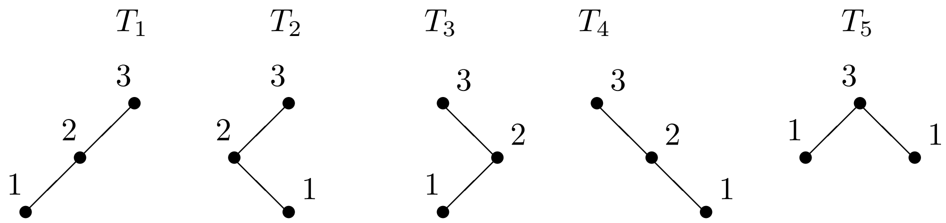
Theorem. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{h_v 2^{h_v-1}}=\frac{1}{n!} \tag{4}$$

and

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{(2h_v+1)2^{2h_v-1}}=\frac{1}{(2n+1)!}. \tag{5}$$

Example. There are five binary trees with $n=3$ vertices:



The hook lengths of T_1, T_2, T_3, T_4 are all the same $1,2,3$; but the hook lengths of T_5 are $1,1,3$. The left-hand side of (4) is then equal to

$$\frac{4}{1\cdot 2^0\cdot 2\cdot 2^1\cdot 3\cdot 2^2}+\frac{1}{1\cdot 2^0\cdot 1\cdot 2^0\cdot 3\cdot 2^2}=\frac{1}{3!}$$

and the left-hand side of (5) to

$$\frac{4}{3\cdot 2^1\cdot 5\cdot 2^3\cdot 7\cdot 2^5}+\frac{1}{3\cdot 2^1\cdot 3\cdot 2^1\cdot 7\cdot 2^5}=\frac{1}{7!}.$$

Proof. Let

$$P(n)=\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{h_v 2^{h_v-1}}.$$

With each binary tree $T\in\mathcal{B}(n)$ $(n\geq 1)$ we can associate a triplet (T', T'', u), where $T'\in\mathcal{B}(k)$ $(0\leq k\leq n-1)$, $T''\in\mathcal{B}(n-1-k)$ and u is a vertex of hook length $h_u=n$. In fact, u is the root of T; moreover, T' and T'' are the left and right subtrees of the root. Hence $P(0)=1$ and

$$P(n)=\sum_{k=0}^{n-1}P(k)P(n-1-k)\times\frac{1}{n\cdot 2^{n-1}}\quad(n\geq 1). \tag{6}$$

It is routine to verify that $P(n) = 1/n!$ for $n = 1$. Suppose that $P(k) = 1/k!$ for $k \le n-1$. Then

$$P(n) = \sum_{k=0}^{n-1} \frac{1}{k!(n-1-k)!n \cdot 2^{n-1}} = \frac{1}{2^{n-1}n!} \sum_{k=0}^{n-1} \binom{n-1}{k} = \frac{1}{n!}.$$

By induction, formula (4) is true for any positive integer n.

In the same manner, let

$$Q(n) = \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{1}{(2h_v+1)2^{2h_v-1}}.$$

Using the previous decomposition we have $Q(0) = 1$ and

$$Q(n) = \sum_{k=0}^{n-1} Q(k)Q(n-1-k) \times \frac{1}{(2n+1)\cdot 2^{2n-1}} \quad (n \ge 1). \tag{7}$$

It is routine to verify that $Q(n) = 1/(2n+1)!$ for $n = 1$. Suppose that $Q(k) = 1/(2k+1)!$ for $k \le n-1$. Then

$$\begin{aligned} Q(n) &= \sum_{k=0}^{n-1} \frac{1}{(2k+1)!(2n-2k-1)!(2n+1)\cdot 2^{2n-1}} \\ &= \frac{2}{2^{2n}(2n+1)!} \sum_{k=0}^{n-1} \binom{2n}{2k+1}. \end{aligned}$$

Since

$$\sum_{k=0}^{n-1} \binom{2n}{2k+1} = \sum_{k=0}^{n} \binom{2n}{2k} = \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n-1},$$

we get

$$Q(n) = \frac{2}{2^{2n}(2n+1)!} 2^{2n-1} = \frac{1}{(2n+1)!}.$$

By induction, formula (5) is true for any positive integer n. $\square$

Concluding and Remarks. The two hook length formulas presented here were originally discovered by using the *expansion technique*, developed in [4]. They are also re-proved by using a more compact notation, see [4, Theorems 6.3 and 7.2]. In [12] Yang has extended formulas (4) and (5) to binomial families of trees. A probabilistic proof of (4) is given by Sagan [8]. Unlike formula (2), we still do not have any simple combinatorial proofs of the two formulas.

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References

[1] Chen, William Y.C.; Yang, Laura L.M., On Postnikov's hook length formula for binary trees, *European Journal of Combinatorics*, **29** (2008), pp. 1563–1565.

[2] Du, Rosena R. X.; Liu, Fu, (k, m)-Catalan Numbers and Hook Length Polynomials for Plane Trees, *European J. Combin*, **28** (2007), pp. 1312–1321.

[3] Gessel, Ira M.; Seo, Seunghyun, A refinement of Cayley's formula for trees, *Electron. J. Combin.*, **11**(2004/06), no. 2, Research Paper 27, 23 pp.

[4] Han, G.-N., Discovering hook length formulas by an expansion technique, *Electron. J. Combin.*, Vol. **15(1)**, Research Paper #R133, 41 pp, 2008.

[5] Knuth, Donald E., *The Art of Computer Programming,* **vol. 3**, *Sorting and Searching, 2nd ed.*, Addison Wesley Longman, 1998.

[6] Moon, J. W.; Yang, Laura L. M., Postnikov identities and Seo's formulas, *Bull. Inst. Combin. Appl.*, **49** (2007), pp. 21–31.

[7] Postnikov, Alexander, Permutohedra, associahedra, and beyond, *arXiv:math. CO/0507163*, 2004.

[8] Sagan, Bruce E., Probabilistic proofs of hook length formulas involving trees, *Sém. Lothar. Combin.*, vol. **61**, Article B61Ab, 2009, 10 pages.

[9] Seo, Seunghyun, A combinatorial proof of Postnikov's identity and a generalized enumeration of labeled trees, *Electron. J. Combin.*, **11**(2004/06), no. 2, Note 3, 9 pp.

[10] Stanley, Richard P., *Enumerative Combinatorics, vol. 1*, Wadsworth & Brooks /Cole, 1986.

[11] Stanley, Richard P., *Enumerative Combinatorics, vol. 2*, Cambridge university press, 1999.

[12] Yang, Laura L.M., Generalizations of Han's Hook Length Identities, *arXiv: 0805.0109 [math.CO]*, 2008.

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