

The doubloon polynomial triangle

Dominique Foata and Guo-Niu Han

*Dedicated to George Andrews,
on the occasion of his seventieth birthday.*

ABSTRACT. The doubloon polynomials are generating functions for a class of combinatorial objects called normalized doubloons by the compressed major index. They provide a refinement of the q -tangent numbers and also involve two major specializations: the Poupard triangle and the Catalan triangle.

1. Introduction

The *doubloon* (*) *polynomials* $d_{n,j}(q)$ ($n \geq 1$, $2 \leq j \leq 2n$) introduced in this paper serve to globalize the *Poupard triangle* [Po89] and the classical *Catalan triangle* [Sl07]. They also provide a refinement of the *q-tangent numbers*, fully studied in our previous paper [FH08]. Finally, as generating polynomials for the doubloon model, they constitute a common combinatorial set-up for the above integer triangles. They may be defined by the following recurrence:

$$\begin{aligned}
 (D1) \quad & d_{0,j}(q) = \delta_{1,j} \text{ (Kronecker symbol);} \\
 (D2) \quad & d_{n,j}(q) = 0 \text{ for } n \geq 1 \text{ and } j \leq 1 \text{ or } j \geq 2n + 1; \\
 (D3) \quad & d_{n,2}(q) = \sum_j q^{j-1} d_{n-1,j}(q) \text{ for } n \geq 1; \\
 (D4) \quad & d_{n,j}(q) - 2d_{n,j-1}(q) + d_{n,j-2}(q) \\
 & = -(1-q) \sum_{i=1}^{j-3} q^{n+i+1-j} d_{n-1,i}(q) \\
 & \quad - (1+q^{n-1}) d_{n-1,j-2}(q) + (1-q) \sum_{i=j-1}^{2n-1} q^{i-j+1} d_{n-1,i}(q) \\
 & \text{for } n \geq 2 \text{ and } 3 \leq j \leq 2n.
 \end{aligned}$$

The polynomials $d_{n,j}(q)$ ($n \geq 1$, $2 \leq j \leq 2n$) are easily evaluated using (D1)–(D4) and form the *doubloon polynomial triangle*, as shown in Fig. 1.1 and 1.1'.

$$\begin{array}{cccccccc}
 & & & & & & & d_{1,2}(q) \\
 & & & & & & & d_{2,2}(q) & d_{2,3}(q) & d_{2,4}(q) \\
 & & & & & & & d_{3,2}(q) & d_{3,3}(q) & d_{3,4}(q) & d_{3,5}(q) & d_{3,6}(q) \\
 & & & & & & & d_{4,2}(q) & d_{4,3}(q) & d_{4,4}(q) & d_{4,5}(q) & d_{4,6}(q) & d_{4,7}(q) & d_{4,8}(q)
 \end{array}$$

Fig. 1.1. The doubloon polynomial triangle

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(*) Although the word “doubloon” originally refers to a Spanish gold coin, it is here used to designate a permutation written as a two-row matrix.

$$\begin{aligned}
d_{1,2}(q) &= 1; & d_{2,2}(q) &= q; & d_{2,3}(q) &= q + 1; & d_{2,4}(q) &= 1; \\
d_{3,2}(q) &= 2q^3 + 2q^2; & d_{3,3}(q) &= 2q^3 + 4q^2 + 2q; & d_{3,4}(q) &= q^3 + 4q^2 + 4q + 1; \\
d_{3,5}(q) &= 2q^2 + 4q + 2; & d_{3,6}(q) &= 2q + 2; \\
d_{4,2}(q) &= 5q^6 + 12q^5 + 12q^4 + 5q^3; & d_{4,3}(q) &= 5q^6 + 17q^5 + 24q^4 + 17q^3 + 5q^2; \\
d_{4,4}(q) &= 3q^6 + 15q^5 + 29q^4 + 29q^3 + 15q^2 + 3q; \\
d_{4,5}(q) &= q^6 + 9q^5 + 25q^4 + 34q^3 + 25q^2 + 9q + 1; \\
d_{4,6}(q) &= 3q^5 + 15q^4 + 29q^3 + 29q^2 + 15q + 3; \\
d_{4,7}(q) &= 5q^4 + 17q^3 + 24q^2 + 17q + 5; & d_{4,8}(q) &= 5q^3 + 12q^2 + 12q + 5.
\end{aligned}$$

Fig. 1.1'. The first doubleloon polynomials

Notice the different symmetries of the coefficients of the polynomials $d_{n,j}(q)$, which will be fully exploited in Section 4 (Corollaries 4.3, 4.7, 4.8). Various specializations are displayed in Fig. 1.2 below, where $C_n = \frac{1}{2n+1} \binom{2n}{n}$ stands for the celebrated Catalan number and t_n for the *reduced tangent number* occurring in the Taylor expansion

$$\begin{aligned}
\sqrt{2} \tan(u/\sqrt{2}) &= \sum_{n \geq 0} \frac{u^{2n+1}}{(2n+1)!} t_n \\
(1.1) \qquad \qquad &= \frac{u}{1!} 1 + \frac{u^3}{3!} 1 + \frac{u^5}{5!} 4 + \frac{u^7}{7!} 34 + \frac{u^9}{9!} 496 + \frac{u^{11}}{11!} 11056 + \dots
\end{aligned}$$

The symbol Σ attached to each vertical arrow has the meaning “make the summation over j ” and $d_n(q)$ is the polynomial further defined in (1.3).

$$\begin{array}{ccccc}
d_{n,j}(0) & \xleftarrow{q=0} & d_{n,j}(q) & \xrightarrow{q=1} & d_{n,j}(1) \\
\downarrow \Sigma & & \downarrow \Sigma & & \downarrow \Sigma \\
C_n & \xleftarrow{q=0} & d_n(q) & \xrightarrow{q=1} & t_n
\end{array}$$

Fig. 1.2. The specializations of $d_{n,j}(q)$

When $q = 1$, the (D1)–(D4) recurrence becomes

- (P1) $d_{0,j}(1) = \delta_{0,j}$ (Kronecker symbol);
- (P2) $d_{n,j}(1) = 0$ for $n \geq 1$ and $j \leq 1$ or $j \geq 2n + 1$;
- (P3) $d_{n,2}(1) = \sum_j d_{n-1,j}(1)$ for $n \geq 1$;
- (P4) $d_{n,j}(1) - 2d_{n,j-1}(1) + d_{n,j-2}(1) = -2d_{n-1,j-2}(1)$
for $n \geq 2$ and $3 \leq j \leq 2n$,

which is exactly the recurrence introduced by Christiane Poupard [Po89].

$$\begin{array}{cccccc}
& & & 1 & & \\
& & & 1 & 2 & 1 \\
& & 4 & 8 & 10 & 8 & 4 \\
34 & 68 & 94 & 104 & 94 & 68 & 34
\end{array}$$

Fig. 1.3. The Poupard triangle ($d_{n,j}(1)$)

The integers $d_{n,j}(1)$ are easily evaluated using (P1)–(P4) and form the *Poupart triangle*, as shown in Fig. 1.3.

When $q = 0$, relation (D4) becomes:

$$d_{n,j}(0) - 2d_{n,j-1}(0) + d_{n,j-2}(0) = -d_{n-1,j-2}(0) + d_{n-1,j-1}(0),$$

which can be rewritten as

$$d_{n,j}(0) - d_{n,j-1}(0) - d_{n-1,j-1}(0) = d_{n,j-1}(0) - d_{n,j-2}(0) - d_{n-1,j-2}(0),$$

so that by induction

$$d_{n,j}(0) - d_{n,j-1}(0) - d_{n-1,j-1}(0) = d_{n,2}(0) = \sum_j q^{j-1} d_{n-1,j} \Big|_{q=0} = 0$$

using (D2) and (D3) when $n \geq 2$. Consequently, the integers $d_{n,j}(0)$ satisfy the recurrence relation

$$(C1) \quad d_{n,j}(0) = d_{n,j-1}(0) + d_{n-1,j-1}(0)$$

for $n \geq 2$ and $3 \leq j \leq 2n$ with the initial conditions

$$(C2) \quad \begin{aligned} d_{n,n+1}(0) &= 1 \quad (n \geq 1); \\ d_{n,j}(0) &= 0 \quad (n \geq 1 \text{ and } j \leq 1 \text{ or } j \geq 2n+1); \\ d_{n,2}(0) &= d_{n,3}(0) = \dots = d_{n,n}(0) = 0 \quad (n \geq 2). \end{aligned}$$

In view of (C1) and (C2) the integers $d_{n,j}(0)$ ($n \geq 1$, $n+1 \leq j \leq 2n$) obey the rules of the classical *Catalan triangle* that has been studied by many authors (see the sequence A00976 in Sloane [Sl07] and its abundant bibliography). They form the *Catalan triangle* displayed in Fig. 1.4.

$$\begin{array}{cccccccc} & & & & & & & 1 \\ & & & & & & & 0 & 1 & 1 \\ & & & & & & & 0 & 0 & 1 & 2 & 2 \\ & & & & & & & 0 & 0 & 0 & 1 & 3 & 5 & 5 \\ & & & & & & & 0 & 0 & 0 & 0 & 1 & 4 & 9 & 14 & 14 \\ & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 14 & 28 & 42 & 42 \end{array}$$

Fig. 1.4. The Catalan triangle ($d_{n,j}(0)$)

For each $n \geq 0$ let $A_n(t, q)$ be the Carlitz [Ca54, Ca75] q -analog of the Eulerian polynomial defined by the identity

$$(1.2) \quad \frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j+1]_q)^n,$$

where $(t; q)_{n+1} = (1-t)(1-tq) \dots (1-tq^n)$ and $[j+1]_q = 1+q+q^2+\dots+q^n$ are the traditional q -ascending factorials and q -analogs of the positive integers.

The polynomial $d_n(q)$ under study was introduced in [FH08]. It is defined by

$$(1.3) \quad d_n(q) = \frac{(-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n}, q)}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

It was shown to be a *polynomial* of degree $\binom{n-1}{2}$, with *positive integral* coefficients, having the following two properties:

$$(1.4) \quad d_n(1) = t_n;$$

$$(1.5) \quad d_n(0) = C_n \quad (n \geq 0);$$

(see the bottom row in the diagram of Fig. 1.2.)

The first values of the polynomials $d_n(q)$ are: $d_0(q) = d_1(q) = 1$; $d_2(q) = 2 + 2q$; $d_3(q) = 5 + 12q + 12q^2 + 5q^3$; $d_4(q) = 14 + 56q + 110q^2 + 136q^3 + 110q^4 + 56q^5 + 14q^6$.

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let $(d_{n,j}(q))$ be the set of polynomials defined by (D1)–(D4) and $d_n(q)$ be defined by (1.3). Then, the following identity holds*

$$(1.6) \quad \sum_j d_{n,j}(q) = d_n(q).$$

In other words, the diagram in Fig. 1.2 is commutative.

Even for $q = 1$ identity (1.6), which then reads $\sum_j d_{n,j}(1) = t_n$, is not at all straightforward. It was elegantly proved by Christiane Poupard [Po89] by means of the bivariable generating function

$$(1.7) \quad Z(u, v) = 1 + \sum_{n \geq 1} \sum_{1 \leq l \leq 2n-1} \frac{u^{2n-l}}{(2n-l)!} \frac{v^l}{l!} d_{n,l+1}(1).$$

She even obtained the following stronger result

$$(1.8) \quad Z(u, v) = \frac{\cos((u-v)/\sqrt{2})}{\cos((u+v)/\sqrt{2})},$$

so that

$$(1.9) \quad \left. \frac{\partial}{\partial u} Z(u, v) \right|_{\{v=0\}} = \sum_{n \geq 1} \frac{u^{2n-1}}{(2n-1)!} d_{n,2}(1) = \sqrt{2} \tan(u/\sqrt{2}),$$

which proves $\sum_j d_{n,j}(1) = t_n$ by appealing to (P3).

Finally, she obtains a combinatorial interpretation for the polynomial

$$(1.10) \quad d_n(s, 1) = \sum_j d_{n,j}(1) s^j$$

in terms of strictly ordered binary trees, or in an equivalent manner, of André permutations which are also alternating (see, e.g. [FS71], [FS71a], [FS73]).

For $q = 0$ identity (1.6) reads $\sum_j d_{n,j}(0) = C_n = \frac{1}{n+1} \binom{2n}{n}$. This is a consequence of the identities

$$(1.11) \quad d_{n,2n}(0) = \sum_j d_{n-1,j}(0);$$

$$(1.12) \quad d_{n,j}(0) = \binom{j-2}{n-1} - \binom{j-2}{n} = \frac{2n-j+1}{n} \binom{j-2}{n-1};$$

so that, in particular,

$$(1.13) \quad d_{n,2n}(0) = \frac{1}{n} \binom{2n-2}{n-1} = C_{n-1};$$

easily obtained from (C1) and (C2) by induction and iteration, as well as an expression for the generating function

$$(1.14) \quad \sum_{n \geq 1} u^n \sum_j d_{n,j}(0) v^j = \frac{1}{1-v-uv} \left(uv^2 - \frac{v}{2} \left(1 - \sqrt{1-4uv^2} \right) \right).$$

Again, the reader is referred to the excellent commented bibliography about the sequence A009766 in Sloane's On-Line Encyclopedia of Integer Sequences [Sl07], in particular the contributions made by David Callan [Cal05] and Emeric Deutsch [De04], where identities (1.11)–(1.14) are actually derived with other initial conditions.

The proof of Theorem 1.1 will be of *combinatorial nature*. In our previous paper [FH08] we proved that each polynomial $d_n(q)$ was a polynomial with positive integral coefficients by showing that it was the generating function for the class \mathcal{N}_{2n+1}^0 of permutations called *normalized doubletons*, by an integral-valued statistic “cmaj” called the *compressed major index*:

$$(1.15) \quad d_n(q) = \sum_{\delta \in \mathcal{N}_{2n+1}^0} q^{\text{cmaj } \delta}.$$

In particular, $d_n(1) = \#\mathcal{N}_{2n+1}^0 = t_n$ by (1.4). Normalized doubleton and “cmaj” will be fully described in Section 2. Here we just recall that each normalized doubleton $\delta \in \mathcal{N}_{2n+1}^0$ is a two-row matrix $\delta =$

$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$, where the word $\rho(\delta) = a_0 a_1 \cdots a_n b_n b_{n-1} \cdots b_0$ is a permutation of $012 \cdots (2n+1)$ having further properties that will be given shortly. The set of all normalized doubletons $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ such that $b_0 = j$ is denoted by $\mathcal{N}_{2n+1,j}^0$. It will be shown that $\mathcal{N}_{2n+1,j}^0$ is an empty set for $j = 0, 1$ and $2n+1$, so that the subsets $\mathcal{N}_{2n+1,j}^0$ ($j = 2, 3, \dots, 2n$) form a *partition* of \mathcal{N}_{2n+1}^0 . Consequently, the following identity holds:

$$(1.16) \quad d_n(q) = \sum_{j=2}^{2n} \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta}.$$

Accordingly, Theorem 1.1 is a simple corollary of the next theorem.

Theorem 1.2. *Let $(d_{n,j}(q))$ be the set of polynomials in one variable q defined by (D1)–(D4). Then $d_{n,j}(q)$ is the generating polynomial for $\mathcal{N}_{2n+1,j}^0$ by the compressed major index. In other words,*

$$(1.17) \quad d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta}.$$

To prove that the polynomial $d_{n,j}(q)$, as expressed in (1.17), satisfies (D4), symmetry properties must be derived (see Corollaries 4.3, 4.7 and 4.8). This requires a careful geometric study of those doubletons and how the statistic “cmaj” evolves. All this is developed in Sections 3 and 4. The proof of the recurrence is completed in Section 5 and in the final Section concluding remarks are made.

2. Doubletons

A *doubleton* of order $2n+1$ is a $2 \times (n+1)$ -matrix $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ such that the word $\rho(\delta) = a_0 a_1 \cdots a_n b_n b_{n-1} \cdots b_0$, called the *reading* of δ , is a permutation of $012 \cdots (2n+1)$. Let \mathcal{D}_{2n+1} (resp. \mathcal{D}_{2n+1}^0) denote the set of all doubletons $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ of order $(2n+1)$ (resp. the subset of all doubletons such that $a_0 = 0$).

Let $1 \leq k \leq n$; each doubleton $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ is said to be *normalized at k* , if the following two conditions are satisfied:

(N1) *exactly one* of the two integers a_k, b_k lies between a_{k-1} and b_{k-1} (we also say that δ is *interlaced at k*);

(N2) either $a_{k-1} > a_k$ and $b_{k-1} > b_k$, or $a_{k-1} < b_k$ and $b_{k-1} < a_k$.

In an equivalent manner, δ is *normalized at k* , if one of the four following orderings holds:

$$(2.1) \quad \begin{aligned} & a_{k-1} < b_k < b_{k-1} < a_k; \\ & b_k < b_{k-1} < a_k < a_{k-1}; \\ & b_{k-1} < a_k < a_{k-1} < b_k; \\ & a_k < a_{k-1} < b_k < b_{k-1}. \end{aligned}$$

For each pair of distinct integers (i, j) the symbol $\mathcal{N}_{2n+1, j}^i$ will denote the set of all doubletons $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$, normalized at *every* $k = 1, 2, \dots, n$, such that $a_0 = i$ and $b_0 = j$. The doubletons belonging to $\mathcal{N}_{2n+1, j}^0$ for some j are simply called *normalized*. Also \mathcal{N}_{2n+1}^0 designates the union of the sets $\mathcal{N}_{2n+1, j}^0$'s. For further results on permutations studied as two-row matrices see [Ha92, Ha94, FH00].

Let $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ be a doubleton of order $(2n+1)$ and h an integer. For each $k = 0, 1, \dots, n$ let $a'_k = a_k + h$, $b'_k = b_k + h$ be expressed as *residues* mod $(2n+2)$. The two-row matrix $\begin{pmatrix} a'_0 & a'_1 & \cdots & a'_n \\ b'_0 & b'_1 & \cdots & b'_n \end{pmatrix}$, denoted by $\delta + h$, is still a doubleton. Let $T_h : \delta \mapsto \delta + h$.

Proposition 2.1. *The map T_h , restricted to $\mathcal{N}_{2n+1, j}^i$, is a bijection onto $\mathcal{N}_{2n+1, j+h}^{i+h}$ (superscript and subscript being taken mod $(2n+2)$).*

Proof. The four orderings in (2.1) are *cyclic* rearrangements of each other, so that if δ is normalized at each i , the doubleton $T_h\delta = \delta + h$ has the same property. \square

The *number of descents*, $\text{des } \delta$, (resp. the *major index*, $\text{maj } \delta$) of a doubleton $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ is defined to be the number of descents (resp. the major index) of the permutation $\rho(\delta) = a_0 a_1 \cdots a_n b_n b_{n-1} \cdots b_0$, so that if the descents ($a_i > a_{i+1}$, or $a_n > b_n$, or still $b_{i-1} > b_i$) occur at positions l_1, l_2, \dots, l_r in $\rho(\delta)$, then $\text{des } \delta = r$ and $\text{maj } \delta = l_1 + l_2 + \cdots + l_r$. The *compressed major index*, $\text{cmaj } \delta$, of δ is defined by

$$(2.2) \quad \text{cmaj } \delta = \text{maj } \delta - (n+1) \text{des } \delta + \binom{n}{2}.$$

For example, there is one normalized doubleton of order 3 ($n = 1$): $\delta = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$ and $\text{cmaj } \delta = \text{maj}(0312) - (1+1) \text{des}(0312) + \binom{1}{2} = 0$, so that $d_{1,2}(q) = 1$. There are four normalized doubletons of order 5 ($n = 2$) and the partition of \mathcal{N}_5^0 reads:

$$\mathcal{N}_{5,2}^0 = \left\{ \begin{pmatrix} 0 & 4 & 3 \\ 2 & 1 & 5 \end{pmatrix} \right\}, \quad \mathcal{N}_{5,3}^0 = \left\{ \begin{pmatrix} 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \right\}, \quad \mathcal{N}_{5,4}^0 = \left\{ \begin{pmatrix} 0 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix} \right\}.$$

We have $\text{cmaj}(043512) = \text{maj}(043512) - 3 \text{des}(043512) + \binom{2}{2} = (2+4) - 3 \times 2 + 1 = 1$, so that $d_{2,2}(q) = q$. Furthermore, $d_{2,3}(q) = 1 + q$ and $d_{2,4}(q) = 1$, as expected (see Fig. 1.2).

3. Operations on doubletons

In this section we study the actions of several operators on the statistic “cmaj.” First, we recall the action of the dihedral group on the traditional statistics “des” and “maj,” in particular characterize the images of

each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ under the *reversal* \mathbf{r} (resp. *complement* \mathbf{c}) that maps σ onto its mirror image $\mathbf{r}\sigma = \sigma(n)\cdots\sigma(2)\sigma(1)$ (resp. onto its complement $\mathbf{c}\sigma = (n+1-\sigma(1))(n+1-\sigma(2))\cdots(n+1-\sigma(n))$). The next properties are well-known (see, e.g., [FS78]): if σ belongs to \mathfrak{S}_n , then

$$(3.1) \quad \text{des } \mathbf{r}\sigma = n - 1 - \text{des } \sigma; \quad \text{maj } \mathbf{r}\sigma = \text{maj } \sigma - n \text{des } \sigma + \binom{n}{2};$$

$$(3.2) \quad \text{des } \mathbf{c}\sigma = n - 1 - \text{des } \sigma; \quad \text{maj } \mathbf{c}\sigma = \binom{n}{2} - \text{maj } \sigma;$$

$$(3.3) \quad \text{des } \mathbf{r}\mathbf{c}\sigma = \text{des } \sigma; \quad \text{maj } \mathbf{r}\mathbf{c}\sigma = n \text{des } \sigma - \text{maj } \sigma.$$

Now let i, j be the two integers defined by $\sigma(1) = n - j$, $\sigma(n) = n - i$ and σ' be the permutation mapping k onto

$$\sigma'(k) = \begin{cases} \sigma(k) + i, & \text{if } \sigma(k) + i \leq n; \\ \sigma(k) + i - n, & \text{if } \sigma(k) + i > n. \end{cases}$$

The operation $\sigma \mapsto \sigma'$ and Property (3.5) below already appear in [Ha92b] for the study of the Z -statistic.

Lemma 3.1. *We have:*

$$(3.4) \quad \text{des } \sigma - \text{des } \sigma' = \begin{cases} 0, & \text{if } i < j; \\ 1, & \text{if } i > j; \end{cases}$$

$$(3.5) \quad \text{maj } \sigma - \text{maj } \sigma' = i.$$

Proof. As $\sigma(n) = n - i$, we can factorize the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ as a product $p_0q_1p_1\cdots q_r p_r$ ($r \geq 1$), where the letters of all p_k 's (resp. all q_k 's) are smaller than or equal to (resp. greater than) $n - i$. Also, let $p'_0q'_1p'_1\cdots q'_r p'_r$ be the corresponding factorization of σ' such that $\lambda p'_0 = \lambda p_0$, $\lambda q'_1 = \lambda q_1, \dots$ (λ being the word length). In particular, p_r (and p'_r) is never empty since $\sigma(n) = n - i$ (and $\sigma'(n) = n$), but p_0 (resp. and p'_0) is non empty if and only if $\sigma(1) + i = n - j + i \leq n$, that is, if $i < j$.

As the rightmost letter of each factor q_k is larger than the leftmost letter of p_k , we have

$$\text{des } \sigma = \text{des } p_0 + \text{des } q_1 + 1 + \text{des } p_1 + \cdots + \text{des } q_r + 1 + \text{des } p_r.$$

In σ' the rightmost letter of each p'_k is greater than the leftmost letter of q'_{k+1} , so that, if $i < j$,

$$\begin{aligned} \text{des } \sigma' &= \text{des } p'_0 + 1 + \text{des } q'_1 + \cdots + \text{des } p'_{r-1} + 1 + \text{des } q'_r + \text{des } p'_r \\ &= \text{des } p_0 + \text{des } q_1 + 1 + \cdots + \text{des } p_{r-1} + \text{des } q_r + 1 + \text{des } p_r \\ &= \text{des } \sigma, \end{aligned}$$

while, if $i > j$, the factor p'_0 is empty and

$$\begin{aligned} \text{des } \sigma' &= \text{des } q'_1 + \cdots + \text{des } p'_{r-1} + 1 + \text{des } q'_r + \text{des } p'_r \\ &= \text{des } \sigma - 1. \end{aligned}$$

However, in both cases we have:

$$\begin{aligned}
\text{maj } \sigma - \text{maj } \sigma' &= \lambda(p_0 q_1) + \lambda(p_0 q_1 p_1 q_2) + \cdots + \lambda(p_0 q_1 \cdots p_{r-1} q_r) \\
&\quad - (\lambda(p'_0) + \lambda(p'_0 q'_1 p'_1) + \cdots + \lambda(p'_0 q'_1 \cdots p'_{r-1})) \\
&= \lambda(q_1) + \lambda(q_2) + \cdots + \lambda(q_r) \\
&= \#\{k : \sigma(k) > n - i\} = i. \quad \square
\end{aligned}$$

Lemma 3.2. Let $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ be a doubleton such that $a_0 = i$ and $b_0 = j$. Also, let $\delta' = T_{-i}(\delta) = \delta - i = \begin{pmatrix} 0 & a_1 - i & \cdots & a_n - i \\ j - i & b_1 - i & \cdots & b_n - i \end{pmatrix}$. Then

$$(3.6) \quad \text{cmaj } \delta - \text{cmaj } \delta' = \begin{cases} -i, & \text{if } i < j; \\ n + 1 - i, & \text{if } i > j. \end{cases}$$

Proof. Add 1 to each letter of the reading $\rho(\delta) = a_0 a_1 \cdots a_n b_n b_{n-1} \cdots b_0$ of δ and do the same for the reading $\rho(\delta')$ of δ' . We obtain two permutations σ, σ' of $12 \cdots (2n + 2)$ of the form:

$$\begin{aligned}
\sigma &= (i + 1) \sigma(2) \cdots \sigma(2n + 1) (j + 1); \\
\sigma' &= 1 \sigma'(2) \cdots \sigma'(2n + 1) (j - i + 1).
\end{aligned}$$

When the transformation \mathbf{rc} is applied to each of them, we get

$$\begin{aligned}
\mathbf{rc} \sigma &= (2n + 2 - j) \cdots (2n + 2 - i); \\
\mathbf{rc} \sigma' &= (2n + 2 - j + i) \cdots (2n + 2).
\end{aligned}$$

It follows from Lemma 3.1 (with $(2n + 2)$ replacing n , $\mathbf{rc} \sigma$ instead of σ and $\mathbf{rc} \sigma'$ instead of σ') that

$$\text{des } \mathbf{rc} \sigma - \text{des } \mathbf{rc} \sigma' = \begin{cases} 0, & \text{if } i < j; \\ 1, & \text{if } i > j; \end{cases}$$

$$\text{maj } \mathbf{rc} \sigma - \text{maj } \mathbf{rc} \sigma' = i,$$

so that by (3.3)

$$\text{des } \sigma - \text{des } \sigma' = \begin{cases} 0, & \text{if } i < j; \\ 1, & \text{if } i > j; \end{cases}$$

$$\begin{aligned}
\text{maj } \sigma - \text{maj } \sigma' &= (2n + 2)(\text{des } \sigma - \text{des } \sigma') - (\text{maj } \mathbf{rc} \sigma - \text{maj } \mathbf{rc} \sigma') \\
&= (2n + 2)(\text{des } \sigma - \text{des } \sigma') - i.
\end{aligned}$$

As $\text{cmaj } \delta = \text{maj } \sigma - (n + 1) \text{des } \sigma + \binom{n}{2}$, we get

$$\begin{aligned}
\text{cmaj } \delta - \text{cmaj } \delta' &= (\text{maj } \sigma - \text{maj } \sigma') - (n + 1)(\text{des } \sigma - \text{des } \sigma') \\
&= (n + 1)(\text{des } \sigma - \text{des } \sigma') - i \\
&= \begin{cases} -i, & \text{if } i < j; \\ n + 1 - i, & \text{if } i > j. \end{cases} \quad \square
\end{aligned}$$

Lemma 3.3. Let $\delta = \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix} \mapsto \delta' = \begin{pmatrix} a'_0 & a'_1 & \cdots & a'_n \\ b'_0 & b'_1 & \cdots & b'_n \end{pmatrix}$ be the transformation mapping $\delta \in \mathcal{D}_{2n+1}^0$ onto the doubloon δ' obtained from δ by replacing each entry a_k (resp. b_k) by $a'_k = (2n+1) - a_k$ (resp. $b'_k = (2n+1) - b_k$). Then $\text{cmaj } \delta = n(n-1) - \text{cmaj } \delta'$.

Proof. Again, add 1 to each element of $\rho(\delta)$ and of $\rho(\delta')$ to obtain two permutations σ and σ' of $12 \cdots (2n+2)$. We have: $\sigma' = \mathbf{c} \sigma$. By (3.2) $\text{des } \sigma + \text{des } \sigma' = 2n+1$ and $\text{maj } \sigma + \text{maj } \sigma' = \binom{2n+2}{2} = (n+1)(2n+1)$. Hence, $\text{cmaj } \delta + \text{cmaj } \delta' = \text{maj } \sigma + \text{maj } \sigma' - (n+1)(\text{des } \sigma + \text{des } \sigma') + n(n-1) = (n+1)(2n+1) - (n+1)(2n+1) + n(n-1) = n(n-1)$. \square

Let $\Gamma : \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2n+2-a_1 & \cdots & 2n+2-a_n \\ 2n+2-b_0 & 2n+2-b_1 & \cdots & 2n+2-b_n \end{pmatrix} [\Gamma : \delta \mapsto \delta'' \text{ in short}]$ be the transformation mapping each doubloon $\delta = \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{D}_{2n+1}^0$ onto the doubloon $\delta'' = \begin{pmatrix} 0 & a''_1 & \cdots & a''_n \\ b''_0 & b''_1 & \cdots & b''_n \end{pmatrix}$ obtained from δ by replacing each entry a_k (resp. b_k) by the residue $a''_k = (2n+2) - a_k$ (resp. $b''_k = (2n+2) - b_k$).

Remark. If δ is interlaced at each k (condition (N1) holds at each k), the same property holds for $\Gamma(\delta)$. However, if δ is normalized at each k , the property is *not* preserved under the transformation Γ .

Lemma 3.4. For each $\delta \in \mathcal{D}_{2n+1}^0$ we have: $\text{cmaj } \delta + \text{cmaj } \Gamma(\delta) = n^2$.

Proof. First, transform δ into the doubloon δ' defined in Lemma 3.3, so that $\delta' = \begin{pmatrix} 2n+1 & a'_1 & \cdots & a'_n \\ 2n+1-j & b'_1 & \cdots & b'_n \end{pmatrix}$ and $\text{cmaj } \delta = n(n-1) - \text{cmaj } \delta'$. Next, apply Lemma 3.2 to δ' with $i = 2n+1$, so that the new doubloon is of the form $\delta'' = \begin{pmatrix} 0 & a''_1 & \cdots & a''_n \\ 2n+2-j & b''_1 & \cdots & b''_n \end{pmatrix}$. Hence, $\text{cmaj } \delta' - \text{cmaj } \delta'' = (n+1) - (2n+1) = -n$ and $\text{cmaj } \delta = n(n-1) - \text{cmaj } \delta'' + n = n^2 - \text{cmaj } \delta''$. \square

4. Further operations on doubloons

In our previous paper [FoHa08] we also introduced a class of transformations ϕ_i ($0 \leq i \leq n$) on \mathcal{D}_{2n+1} , called *micro flips*, which permute the entries in a given column. By definition,

$$\phi_i : \begin{pmatrix} a_0 & \cdots & a_{i-1} & a_i & a_{i+1} & \cdots & a_n \\ b_0 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_n \end{pmatrix} \mapsto \begin{pmatrix} a_0 & \cdots & a_{i-1} & b_i & a_{i+1} & \cdots & a_n \\ b_0 & \cdots & b_{i-1} & a_i & b_{i+1} & \cdots & b_n \end{pmatrix} \quad (0 \leq i \leq n).$$

Next, the *macro flips* Φ_i are defined by $\Phi_i = \phi_i \phi_{i+1} \cdots \phi_n$ ($1 \leq i \leq n$). Note that both ϕ_i and Φ_i are *involutions* of \mathcal{D}_{2n+1} . In particular,

$$(4.1) \quad \Phi_1 \Phi_2 \cdots \Phi_i = \prod_{\substack{j \text{ odd,} \\ j \leq i}} \phi_j.$$

By means of the transformation Γ (see Lemma 3.4 above) and the involutions Φ_i 's we now construct a bijection of $\mathcal{N}_{2n+1,j}^0$ onto $\mathcal{N}_{2n+1,2n+2-j}^0$.

Lemma 4.1. Let $\delta = \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ be a normalized doubleton and let $\Gamma(\delta) = \delta'' = \begin{pmatrix} 0 & a'_1 & \cdots & a'_n \\ b'_0 & b'_1 & \cdots & b'_n \end{pmatrix}$. Then, for each $i = 1, 2, \dots, n$ the doubleton $\Phi_1 \Phi_2 \cdots \Phi_i(\delta'')$ is normalized at each integer $1, 2, \dots, i$. In particular, $\Phi_1 \Phi_2 \cdots \Phi_n(\delta'')$ is normalized.

Proof. Using (4.1) we have:

$$\delta^* = \Phi_1 \Phi_2 \cdots \Phi_i(\delta'') = \begin{cases} \begin{pmatrix} 0 & b'_1 & a'_2 & \cdots & a'_{i-1} & b'_i \\ b'_0 & a'_1 & b'_2 & \cdots & b'_{i-1} & a'_i \end{pmatrix}, & \text{if } i \text{ odd;} \\ \begin{pmatrix} 0 & b'_1 & a'_2 & \cdots & b'_{i-1} & a'_i \\ b'_0 & a'_1 & b'_2 & \cdots & a'_{i-1} & b'_i \end{pmatrix}, & \text{if } i \text{ even.} \end{cases}$$

The doubleton δ'' is not normalized (see the Remark before Lemma 3.4), but is interlaced at each i , so that the relations $a''_{j-1} < a''_j$, $b''_{j-1} < b''_j$ or $a''_{j-1} > a''_j$, $b''_{j-1} > b''_j$ hold for all j . This shows that δ^* is normalized whenever j is odd or even. \square

For the next Proposition we need the following property proved in our previous paper [FoHa08, Theorem 3.5]:

(4.2) Let $\delta = \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ be an interlaced doubleton, normalized at i . Then $\text{cmaj } \Phi_i(\delta) - \text{cmaj } \delta = n - i + 1$ ($1 \leq i \leq n$).

Proposition 4.2. The transformation $\Phi_1 \Phi_2 \cdots \Phi_n \Gamma : \delta \mapsto \delta^*$ is a bijection of $\mathcal{N}_{2n+1, j}^0$ onto $\mathcal{N}_{2n+1, 2n+2-j}^0$ having the property

$$(4.3) \quad \text{cmaj } \delta + \text{cmaj } \delta^* = \binom{n}{2}.$$

Proof. Let $\delta'' = \Gamma(\delta)$. As $\Phi_1 \Phi_2 \cdots \Phi_i(\delta'')$ is normalized at i , we have

$$\text{cmaj } \Phi_i \Phi_1 \Phi_2 \cdots \Phi_i(\delta'') - \text{cmaj } \Phi_1 \Phi_2 \cdots \Phi_i(\delta'') = n - i + 1$$

by (4.2). Summing over all $i = 1, 2, \dots, n$ we get

$$\sum_{i=1}^n (\text{cmaj } \Phi_1 \Phi_2 \cdots \Phi_{i-1}(\delta'') - \text{cmaj } \Phi_1 \Phi_2 \cdots \Phi_i(\delta'')) = \sum_{i=1}^n (n - i + 1)$$

$$\text{cmaj } \delta'' - \text{cmaj } \Phi_1 \Phi_2 \cdots \Phi_n(\delta'') = n^2 - \binom{n}{2}.$$

Using Lemma 3.4 we conclude that $(n^2 - \text{cmaj } \delta) - \text{cmaj } \delta^* = n^2 - \binom{n}{2}$ and then $\text{cmaj } \delta + \text{cmaj } \delta^* = \binom{n}{2}$. \square

Example. For $n = 2$ we have $\Phi_1 \Phi_2 = \phi_1$. The four normalized doubletons of order 5 ($n = 2$) displayed at the end of Section 2 are mapped under $\Phi_1 \Phi_2 \Gamma = \phi_1 \Gamma$ as shown in Fig. 4.1. Next to each doubleton appears the value of its “cmaj”.

$$\begin{array}{cccc}
\begin{pmatrix} 0 & 4 & 3 \\ 2 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix}, 0 \\
\downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma \\
\begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 1 \end{pmatrix}, 3 & \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}, 4 & \begin{pmatrix} 0 & 2 & 4 \\ 3 & 5 & 1 \end{pmatrix}, 3 & \begin{pmatrix} 0 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix}, 4 \\
\downarrow \phi_1 & \downarrow \phi_1 & \downarrow \phi_1 & \downarrow \phi_1 \\
\begin{pmatrix} 0 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 4 & 3 \\ 2 & 1 & 5 \end{pmatrix}, 1
\end{array}$$

Fig. 4.1. The transformation $\Phi_1 \Phi_2 \cdots \Phi_n \Gamma$

Let $d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta}$. The final purpose is to show that $d_{n,j}(q)$ satisfies relations (D1) – (D4). This will be done in Section 5. In the rest of this section we state some symmetry properties of the $d_{n,j}(q)$'s.

Corollary 4.3. *For $2 \leq j \leq 2n$ we have:*

$$(4.4) \quad d_{n,j}(q) = q^{\binom{n}{2}} d_{n,2n+2-j}(q^{-1}).$$

Proof. This follows from the previous proposition:

$$d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta} = \sum_{\delta^* \in \mathcal{N}_{2n+1,2n+1-j}^0} q^{\binom{n}{2} - \text{cmaj } \delta^*} = q^{\binom{n}{2}} d_{n,2n+2-j}(q^{-1}). \quad \square$$

In the next lemma we study the action of the sole transposition ϕ_0 that permutes the leftmost entries of each doubleton.

Lemma 4.4. *For each normalized doubleton δ*

$$(4.5) \quad \text{cmaj } \delta - \text{cmaj } \phi_0(\delta) = -n.$$

Proof. Let $\rho(\delta) = 0a_1 \cdots a_n b_n \cdots b_1 b_0$ be the reading of δ , so that $\rho(\phi_0(\delta)) = b_0 a_1 \cdots a_n b_n \cdots b_1 0$. As δ is normalized (and, in particular, interlaced), we have: $0 < b_1 < b_0 < a_1$. Thus, $\rho(\delta)$ starts and ends with a rise ($0 < a_1$ and $b_1 < a_0$), while $\rho(\phi_0(\delta))$ starts with a rise $b_0 < a_1$ and ends with a descent $b_1 > 0$. Hence, $\text{des } \rho(\delta) = \text{des } \rho(\phi_0(\delta)) - 1$ and $\text{maj } \rho(\delta) = \text{maj } \rho(\phi_0(\delta)) - (2n + 1)$ and $\text{cmaj } \delta - \text{cmaj } \rho(\phi_0(\delta)) = -(2n + 1) - (n + 1) - 1 = -n \quad \square$

The involution $\phi_0 \Phi_1 = \Phi_1 \phi_0 = \phi_0 \phi_1 \cdots \phi_n$ transposes the two rows of each doubleton $\delta \in \mathcal{D}_{2n+1}$.

Lemma 4.5. *For each normalized doubleton δ*

$$(4.5) \quad \text{cmaj } \delta = \text{cmaj } \phi_0 \Phi_1(\delta).$$

Proof. First, $\text{cmaj } \delta = \text{cmaj } \phi_0(\delta) - n$ by the previous lemma. Furthermore, as $\phi_0\Phi_1(\delta)$ remains normalized at 1, relation (4.2) implies that $\text{cmaj } \Phi_1(\phi_0\Phi_1(\delta)) - \text{cmaj } \phi_0\Phi_1(\delta) = n - 1 + 1$. Thus,

$$\begin{aligned} \text{cmaj } \phi_0\Phi_1(\delta) &= \text{cmaj } \phi_0(\delta) - n \\ &= \text{cmaj } \delta + n - n = \text{cmaj } \delta. \quad \square \end{aligned}$$

We next study the joint action of the operators T_{-j} (introduced in Lemma. 3.2) ϕ_0 and Φ_1 .

Proposition 4.6. *The transformation $T_{-j}\phi_0\Phi_1 : \delta \mapsto \delta'$ is a bijection of $\mathcal{N}_{2n+1,j}^0$ onto $\mathcal{N}_{2n+1,2n+2-j}^0$ having the property that*

$$(4.6) \quad \text{cmaj } \delta - \text{cmaj } \delta' = n + 1 - j.$$

Proof. Let $\delta \in \mathcal{N}_{2n+1,j}^0$. Then, $\phi_0\Phi_1(\delta)$ is normalized at each $i = 1, 2, \dots, n$ and also $T_{-j}\phi_0\Phi_1(\delta)$ by the remark made before Lemma 3.2. Thus, $\delta' \in \mathcal{N}_{2n+1,2n+2-j}^0$ and the map $\delta \mapsto \delta'$ is bijective. Furthermore, $\text{cmaj } \delta = \text{cmaj } \phi_0\Phi_1(\delta)$ by Lemma 4.5 and $\text{cmaj } \phi_0\Phi_1(\delta) = \text{cmaj } \delta' + n + 1 - j$ by Lemma 3.2. \square

Example. Again consider the four normalized doubletons of order 5. We get the display of Fig. 4.2. with the value of “cmaj” next to each doubleton.

$$\begin{array}{cccc} \begin{pmatrix} 0 & 4 & 3 \\ 2 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix}, 0 \\ \downarrow \phi_0\Phi_1 & \downarrow \phi_0\Phi_1 & \downarrow \phi_0\Phi_1 & \downarrow \phi_0\Phi_1 \\ \begin{pmatrix} 2 & 1 & 5 \\ 0 & 4 & 3 \end{pmatrix}, 1 & \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 4 \end{pmatrix}, 0 & \begin{pmatrix} 3 & 1 & 5 \\ 0 & 4 & 2 \end{pmatrix}, 1 & \begin{pmatrix} 4 & 2 & 1 \\ 0 & 5 & 3 \end{pmatrix}, 0 \\ \downarrow T_{-2} & \downarrow T_{-3} & \downarrow T_{-3} & \downarrow T_{-4} \\ \begin{pmatrix} 0 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, 0 & \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}, 1 & \begin{pmatrix} 0 & 4 & 3 \\ 2 & 1 & 5 \end{pmatrix}, 1 \end{array}$$

Fig. 4.2. The transformation $T_{-j}\phi_0\Phi_1$

Corollary 4.7. *For $2 \leq j \leq 2n$ we have:*

$$(4.7) \quad d_{n,j}(q) = q^{n+1-j} d_{n,2n+2-j}(q).$$

The corollary immediately follows from Proposition 4.6. In its turn the next corollary is a consequence of both Corollaries 4.3 and 4.7.

Corollary 4.8. *For $2 \leq j \leq 2n$ we have:*

$$(4.8) \quad d_{n,j}(q) = q^{\binom{n+1}{2}+1-j} d_{n,j}(q^{-1}).$$

As could be seen in Fig. 1.2 for the first values, the polynomial $d_{n,j}(q)$ is a multiple of $d_{n,2n+2-j}(q)$ (Corollary 4.7) and Corollary 4.8 indicates a symmetry between its coefficients.

5. The recurrence itself

In this section we prove Theorem 1.2, that is, we show that relations (D1) – (D4) hold for $d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta}$. The first two relations (D1) – (D2) are evidently true. For proving that relations (D3) – (D4) hold for such a $d_{n,j}(q)$ we start with a doubleton $\delta \in \mathcal{N}_{2n+1}^0$, drop its leftmost column and compare the compressed major indices of δ and of the doubleton obtained after deletion. In so doing we get the following result.

Lemma 5.1. *To each doubleton $\delta \in \mathcal{N}_{2n+1,j}^0$ there corresponds a unique triplet (k, l, δ') such that $k \geq j - 1$, $l \leq j - 2$ and $\delta' \in \mathcal{N}_{2n-1,l}^k$ having the property that*

$$(5.1) \quad \text{cmaj } \delta = \text{cmaj } \delta' + (n - 1).$$

Proof. Let $\delta = \begin{pmatrix} 0 & a_1 & \dots & a_n \\ & b_1 & \dots & b_n \end{pmatrix} \in \mathcal{N}_{2n+1,j}^0$ and define

$$a'_i = \begin{cases} a_i - 1, & \text{if } a_i < j; \\ a_i - 2, & \text{if } j < a_i \leq 2n + 1; \end{cases}$$

with an analogous definition for the b'_i , the b 's replacing the a 's. Then, $\delta' = \begin{pmatrix} a'_1 & \dots & a'_n \\ b'_1 & \dots & b'_n \end{pmatrix}$ is a normalized doubleton of order $(2n + 1)$, but its left-top corner a'_1 is not necessarily 0. Call it the *reduction* of δ . As $a_1 > j > b_1$, we see that $b'_1 = b_1 - 1 < j - 1$ and $a'_1 = a_1 - 2 > j - 2$, that is, $b'_1 \leq j - 2 < j - 1 \leq a'_1$. Thus $\delta' \in \mathcal{N}_{2n-1,l}^k$ with $k \geq j - 1$ and $l \leq j - 2$. Conversely, given the triplet (δ', k, l) with the above properties, we can uniquely reconstruct the normalized doubleton δ .

As δ is normalized at 1, the inequalities $0 < b_1 < j < a_1$ hold, so that $\rho(\delta)$ starts and ends with a rise: $0 < a_1, b_1 < j$. In particular, $\text{des } \rho(\delta) = \text{des } \rho(\delta')$. However, $\text{maj } \rho(\delta) = \text{maj } \rho(\delta') + \text{des } \delta$, as the first letter 0 is dropped when going from $\rho(\delta)$ to $\rho(\delta')$. Hence, $\text{cmaj } \delta - \text{cmaj } \delta' = \text{des } \delta - ((n + 1) - n) \text{des } \delta + \binom{n}{2} - \binom{n-1}{2} = n - 1$. \square

The previous Lemma yields the first passage from the generating polynomial $d_{n,j}(q)$ for the normalized doubletons $\delta \in \mathcal{N}_{2n+1,j}^0$ by “cmaj” to the generating polynomial for doubletons of order $(2n - 1)$, as expressed in the next Corollary.

Corollary 5.2. *For $2 \leq j \leq 2n$ we have:*

$$(5.2) \quad d_{n,j}(q) = q^{n-1} \sum_{k \geq j-1, l \leq j-2} q^{n-k} d_{n-1,2n+l-k}(q).$$

Proof. By the previous Lemma we can write:

$$d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta} = q^{n-1} \sum_{k \geq j-1, l \leq j-2} \sum_{\delta' \in \mathcal{N}_{2n-1,l}^k} q^{\text{cmaj } \delta'}.$$

With each $\delta' \in \mathcal{N}_{2n-1,l}^k$ associate $\delta'' = T_{-k}\delta' = \delta' - k$. By Lemma 3.2 we get $\delta'' \in \mathcal{N}_{2n-1,2n+l-k}^0$ (note that the residue mod(2n + 2) must be considered in the subscript for $l < k$) and $\text{cmaj } \delta' - \text{cmaj } \delta'' = n - k$. Hence,

$$\sum_{\delta' \in \mathcal{N}_{2n-1,l}^k} q^{\text{cmaj } \delta'} = q^{n-k} \sum_{\delta'' \in \mathcal{N}_{2n-1,2n+l-k}^0} q^{\text{cmaj } \delta''} = q^{n-k} d_{n-1,2n+l-k}(q). \quad \square$$

When $j = 2$ in (5.2), we get:

$$\begin{aligned} d_{n,2}(q) &= \sum_{k \geq 2} q^{2n-k-1} d_{n-1,2n-k}(q) \\ &= \sum_{k \geq 2} q^{i-1} d_{n-1,i}(q) \quad [\text{by the change of variables } i = 2n - k] \end{aligned}$$

and also

$$d_{n,2}(q) = q^{n-1} \sum_{i \geq 2} d_{n-1,i}(q) \quad [\text{by using (4.7).}]$$

Consequently, relation (D3) holds for the polynomial $d_{n,j}(q) = \sum_{\delta \in \mathcal{N}_{2n+1,j}^0} q^{\text{cmaj } \delta}$. Using (5.2) it is also easy to derive the identity:

$$q d_{n,3}(q) = (q + 1) d_{n,2}(q).$$

Proposition 5.2. For $2 \leq j \leq 2n$ we have:

$$(5.3) \quad d_{n,j}(q) = \sum_i \frac{q^{\max(0, i+1-j)} - q^{\min(i, 2n+1-j)}}{1 - q} d_{n-1,i}(q).$$

Proof. Let $i = 2n + l - k$ with $0 \leq l \leq j - 2$ and $j - 1 \leq k \leq 2n - 1$. This implies $0 \leq i + k - 2n \leq j - 2$, or still $2n - i \leq k \leq 2n + j - i - 2$. Taking the two relations keeping k within bounds into account we get the double inequality:

$$(5.4) \quad \max(2n - i, j - 1) \leq k \leq \min(2n + j - i - 2, 2n - 1).$$

Identity (5.2) may then be rewritten as

$$(5.5) \quad d_{n,j}(q) = q^{2n-1} \sum_i d_{2n-1,i}(q) \sum_k q^{-k},$$

with k ranging over the interval defined in (5.4). The geometric sum over k is equal to

$$\frac{q^{-\max(2n-i, j-1)} - q^{-\min(2n+j-i-2, 2n-1)-1}}{1 - q^{-1}}.$$

Now, $2n - \max(2n - i, j - 1) = 2n + \min(-2n + i, -j + 1) = \min(i, 2n - j + 1)$ and $2n - 1 - \min(2n + j - i - 2, 2n - 1) = \max(0, -j + i + 1)$. Accordingly,

$$q^{2n-1} \sum_{k \text{ subject to (5.4)}} q^{-k} = \frac{q^{\max(0, i+1-j)} - q^{\min(i, 2n+1-j)}}{1 - q}.$$

This proves (5.3) when reporting the latter expression into (5.5). \square

For getting rid of “max” and “min” from identity (5.3) we decompose the sum into four subsums, assuming $j \geq 3$ (when $j = 2$ identity (5.3) gives back (D3)). We obtain:

$$(5.6) \quad (1 - q)d_{n,j}(q) = \sum_{i=0}^{j-1} d_{n-1,i}(q) + \sum_{i=j}^{2n+1} q^{i+1-j} d_{n-1,i}(q) \\ - \sum_{i=0}^{2n-j} q^i d_{n-1,i}(q) - \sum_{i=2n-j+1}^{2n+1} q^{2n+1-j} d_{n-1,i}(q).$$

By means of (5.6) we calculate $d_{n,j}(q) - d_{n,j-1}(q)$ and then $-d_{n,j-1}(q) + d_{n,j-2}(q)$, whose sum is the left-hand side of (D4). In the computation of $(1 - q)(d_{n,j}(q) - d_{n,j-1}(q))$ the contribution of the first subsum is simply $d_{n-1,j-1}(q)$. Next, $q^{2n-j+1}d_{n-1,2n-j+1}(q)$ is the contribution of the second subsum. For the third subsum we get

$$-q d_{n-1,j-1}(q) + \sum_{i=j}^{2n+1} q^{i-j+1}(1 - q)d_{n-1,i}(q)$$

and for the fourth one

$$-q^{2n+1-j}d_{n-1,2n+1-j}(q) + \sum_{i=2n-j+2}^{2n+1} q^{2n+1-j}(q - 1)d_{n-1,i}(q).$$

Altogether

$$(5.7) \quad d_{n,j}(q) - d_{n,j-1}(q) \\ = d_{n-1,j-1}(q) + \sum_{i=j}^{2n+1} q^{i-j+1}d_{n-1,i}(q) - \sum_{i=2n-j+2}^{2n+1} q^{2n+1-j}d_{n-1,i}(q);$$

$$(5.8) \quad -d_{n,j-1}(q) + d_{n,j-2}(q) \\ = -d_{n-1,j-2}(q) - \sum_{i=j-1}^{2n+1} q^{i-j+2}d_{n-1,i}(q) + \sum_{i=2n-j+3}^{2n+1} q^{2n+2-j}d_{n-1,i}(q).$$

Summing (5.7) and (5.8) we get:

$$\begin{aligned}
& d_{n,j}(q) - 2d_{n,j-1}(q) + d_{n-1,j-2}(q) \\
&= d_{n-1,j-1}(q) - d_{n-1,j-2}(q) \\
&\quad - q d_{n-1,j-1}(q) + (1-q) \sum_{i=j}^{2n+1} q^{i-j+1} d_{n-1,i}(q) \\
&\quad - q^{2n+1-j} d_{n-1,2n-j+2}(q) - (1-q) \sum_{i=2n-j+3}^{2n+1} q^{2n+1-j} d_{n-1,i}(q).
\end{aligned}$$

Now, $-q^{2n+1-j} d_{n-1,2n-j+2}(q) = -q^{n-1} d_{n-1,j-2}(q)$ by Corollary 4.7. Also

$$\begin{aligned}
\sum_{i=2n-j+3}^{2n+1} q^{2n+1-j} d_{n-1,i}(q) &= \sum_{i=2n-j+3}^{2n-1} q^{2n+1-j} d_{n-1,i}(q) \\
&= \sum_{k=1}^{j-3} q^{2n+1-j} d_{n-1,2n-k}(q) \quad [k = 2n - i] \\
&= \sum_{i=1}^{j-3} q^{n+i+1-j} d_{n-1,i}(q). \quad [\text{by Corollary 4.7}]
\end{aligned}$$

$$\begin{aligned}
& \text{Hence, } d_{n,j}(q) - 2d_{n,j-1}(q) + d_{n-1,j-2}(q) \\
&= -(1-q) \sum_{i=1}^{j-3} q^{n+i+1-j} d_{n-1,i}(q) \\
&\quad - (1+q^{n-1}) d_{n-1,j-2}(q) + (1-q) \sum_{i=j-1}^{2n-1} q^{i-j+1} d_{n-1,i}(q)
\end{aligned}$$

for $n \geq 2$ and $3 \leq j \leq 2n$, which is precisely relation (D4).

6. Concluding remarks

As mentioned in the Introduction, there has been a great number of papers dealing with the Catalan Triangle (the numbers $d_{n,j}(0)$). The recurrence for the numbers $d_{n,j}(1)$, namely the set of conditions (P1)–(P4), is definitely due to Christiane Poupard [Po89] and was recently rediscovered by Graham and Zang [GZ08]. More exactly, the latter authors introduced the notion of *split-pair arrangement*. To show that the number of such arrangements of order n was equal to the reduced tangent number t_n they set up an algebra for the coefficients $d_{n,j}(1)$ and again produced the recurrence (P1)–(P4), but their proof of the identity $\sum_j d_{n,j}(1) = t_n$ was more elaborate than the original one made by Christiane Poupard [Po89], shortly sketched in the Introduction.

Referring to Section 4 we say that two doubleons $\delta, \delta' \in \mathcal{D}_{2n+1}$ are *equivalent* if there is a sequence $\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k}$ of micro flips such that

$\delta' = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k}(\delta)$. A doubleton $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ is said to be *minimal*, if $a_k < b_k$ holds for every k . Clearly, every equivalence class of doubletons contains one and only one minimal doubleton. Also, as proved in [FH08], each doubleton $\delta \in \mathcal{D}_{2n+1}^0$ is equivalent to one and only one normalized doubleton.

The link between *split-pair arrangements* and *equivalence classes of interlaced doubletons* is the following: start with a *interlaced* and *minimal* doubleton $\delta = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$ and define the word $w = x_1 x_2 \cdots x_{2n}$ by $x_{l+1} = l$ if and only if a_k or b_k is equal to l . For instance,

$$\delta = \begin{pmatrix} 0 & 6 & 3 & 1 & 9 \\ 4 & 2 & 8 & 7 & 5 \end{pmatrix} \mapsto w = 1423152435.$$

As δ is interlaced (condition (N1)), exactly one of the integers a_k, b_k lies between a_{k-1} and b_{k-1} . If $a_k = l < l' = b_k$, then $x_{l+1} = x_{l'+1} = k$. If $a_{k-1} = m < a_k = l < b_{k-1} = m' < b_k = l'$, then $x_{m+1} = x_{m'+1} = k-1$ and there is exactly one letter equal to k , namely x_{l+1} between the two occurrences of $(k-1)$, namely x_{m+1} and $x_{m'+1}$. Same conclusion if $a_{k-1} < a_k < b_{k-1} < b_k$. Those words w of length $2n$ having the property that exactly one letter equal to k lies between two occurrences of $(k-1)$ for each $k = 2, 3, \dots, n$ was called *split-pair arrangements* by Graham and Zang [GZ08]. The mapping $\delta \mapsto w$ provides a bijection between equivalence classes of interlaced doubletons and those arrangements.

A permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(2n+1)$ of $12 \cdots (2n+1)$ is said to be *alternating* if $\sigma(2i) < \sigma(2i-1), \sigma(2i+1)$ for each $i = 1, 2, \dots, n$. For each even integer $2i$ ($1 \leq i \leq n$) let w'_i (resp. w''_i) be the longest right factor of $\sigma(1) \cdots \sigma(2i-1)$ (resp. longest left factor of $\sigma(2i+1) \cdots \sigma(2n+1)$), all letters of which are greater than $\sigma(2i)$. Let $\min w'_i$ (resp. $\min w''_i$) denote the minimum letter of w'_i (resp. of w''_i). Then σ is called an *alternating André* permutation if it is alternating and satisfies $\min w'_i > \min w''_i$ for every $i = 1, 2, \dots, n$. The number of alternating André permutations of order $(2n+1)$ is equal to t_n (see, e.g., [FS71], Property 2.6 and (5.4), or [Po89], p. 370, also [FS74]). The alternating André permutations of order 5 are the following: 53412, 51423, 41523, 31524. Let $A_{2n+1,j}$ denote the set of alternating André permutations of order $(2n+1)$ ending with j .

Thanks to the identity

$$(6.1) \quad d_{n,j}(1) = \sum_{i \geq 0} \binom{2n+1-j}{2i+1} t_i \sum_{k=0}^{j-1} d_{n-i-1,k}(1),$$

valid for $n \geq 1, 2 \leq j \leq 2n-1$ ($d_{0,j}(1) = \delta_{0,j}$) derived from the bivariable generating function (1.8), Christiane Poupard proved that

$$(6.2) \quad \# A_{2n+1,j} = d_{n,j}(1).$$

Three questions arise:

- (1) Construct a natural (?) bijection of $\mathcal{N}_{2n+1,j}^0$ onto $A_{2n+1,j}$.
- (2) Find an adequate statistic “stat” on the alternating André permutation such that holds the identity

$$d_{n,j}(q) = \sum_{\sigma \in A_{2n+1,j}} q^{\text{stat } \sigma}.$$

- (3) As mentioned in the Introduction, we have an expression for the generating function for $d_{n,j}(1)$ (formula (1.8)) and for $d_{n,j}(0)$ (formula (1.14)). Find the exponential (or factorial) generating function for $d_{n,j}(q)$.

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Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
foata@math.u-strasbg.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université Louis Pasteur et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu@math.u-strasbg.fr