THE DECREASE VALUE THEOREM WITH AN APPLICATION TO PERMUTATION STATISTICS

Dominique Foata and Guo-Niu Han

Dedicated to Dennis Stanton,
on the occasion of his sixtieth birthday.

ABSTRACT. The decrease value theorem is restated and given a specialization more adapted to Permutation Statistic Calculus. As an application, the computation of a factorial multivariable generating function for the wreath product of the cyclic group of finite order by the symmetric group is given in full detail.

1. Introduction

In one of our recent papers [FH07] we have derived the decrease value theorem, that makes the calculation of a fundamental multivariable statistical distribution on words possible. The multivariable statistic in question involves the basic notions of decrease and increase, whose definitions are now recalled, together with the classical descent and rise.

Let $[0, r]^*$ be the set of all words, whose letters belong to the finite alphabet $[0, r] = \{0, 1, \ldots, r\}$ and let $v = y_1y_2 \cdots y_n$ be such a word. An integer $i \in [1, n]$ is said to be a descent (or descent place) of $v$ if $y_i > y_{i+1}$; it is a decrease of $v$ if $y_i = y_{i+1} = \cdots = y_j > y_{j+1}$ for some $j$ such that $i \leq j \leq n - 1$. The letter $y_i$ is said to be a descent value and a decrease value of $v$, respectively. The set of all decreases (resp. descents) is denoted by $\text{DES}(v)$ (resp. $\text{DES}(v)$). Each descent is a decrease, so that $\text{DES}(v) \subset \text{DEC}(v)$.

In parallel with the notion of decrease, an integer $i \in [1, n]$ is said to be an increase (resp. a rise) of $v$ if $y_i = y_{i+1} = \cdots = y_j < y_{j+1}$ for some $j$ such that $i \leq j \leq n$ (resp. if $y_i < y_{i+1}$). By convention, $y_{n+1} = +\infty$. The letter $y_i$ is said to be an increase value (resp. a rise value) of $v$. Thus, the rightmost letter $y_n$ is always a rise and also an increase value. The set of all increases (resp. rises) of $v$ is denoted by $\text{INC}(v)$ (resp. $\text{RISE}(v)$). Each rise is an increase, so that $\text{RISE}(v) \subset \text{INC}(v)$.

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Furthermore, a position $i$ ($1 \leq i \leq n$) is said to be a record if $y_j \leq y_i$ for all $j$ such that $1 \leq j \leq i - 1$. The letter $y_i$ is said to be a record value. The set of all records of $v$ is denoted by $\text{REC}(v)$.

The multivariable statistic is now defined by means of six sequences of commuting variables $(X_i), (Y_i), (Z_i), (T_i), (Y'_i), (T'_i)$ ($i = 0, 1, 2, \ldots$): for each word $v = y_1y_2 \cdots y_n$ from $[0, r]^n$ the weight $\psi(v)$ is defined to be

$$
\psi(v) := \prod_{i \in \text{DES}} X_{y_i} \prod_{i \in \text{RISE} \setminus \text{REC}} Y_{y_i} \prod_{i \in \text{DEC} \setminus \text{DES}} Z_{y_i} \times \prod_{i \in (\text{INC} \setminus \text{RISE}) \setminus \text{REC}} T_{y_i} \prod_{i \in \text{RISE} \cap \text{REC}} Y'_{y_i} \prod_{i \in (\text{INC} \setminus \text{RISE}) \cap \text{REC}} T'_{y_i},
$$

where the argument "$(v)$" has not been written for typographic reasons. For example, $i \in \text{RISE} \setminus \text{REC}$ stands for $i \in \text{RISE}(v) \setminus \text{REC}(v)$. It is important to note that for each word $v = y_1y_2 \cdots y_n$ every integer $i \in [1, n]$ belongs to one and only one of the sets $\text{DES}(v)$, $(\text{RISE} \setminus \text{REC})(v)$, $(\text{DEC} \setminus \text{DES})(v)$, $((\text{INC} \setminus \text{RISE}) \setminus \text{REC})(v)$, $(\text{RISE} \cap \text{REC})(v)$, $((\text{INC} \setminus \text{RISE}) \cap \text{REC})(v)$.

Example. For the word $v = 324455531114135$ the sets DES, INC, RISE, REC of $v$ are indicated by bullets.

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We have $\psi(v) = X_3Y_2T'_4Y'_4Z_5X_5X_3T_1Y_1X_1Y_3Y'_5$.

Now, let $C$ be the $(r + 1) \times (r + 1)$ matrix

$$
C = \begin{pmatrix}
0 & X_1 & X_1 & \cdots & X_{r-1} & X_r \\
Y_0 & 1 - Z_1 & 1 - Z_2 & \cdots & 1 - Z_{r-1} & 1 - Z_r \\
1 - T_0 & 0 & 1 - Z_2 & \cdots & 1 - Z_{r-1} & 1 - Z_r \\
1 - T_0 & Y_1 & 0 & \cdots & 1 - Z_{r-1} & 1 - Z_r \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 - T_0 & Y_1 & Y_2 & \cdots & 0 & X_r \\
Y_0 & 1 - T_0 & 1 - T_1 & \cdots & X_r & 1 - Z_r \\
1 - T_0 & 1 - T_0 & 1 - T_1 & \cdots & Y_{r-1} & 0 \\
Y_0 & 1 - T_0 & 1 - T_1 & 1 - T_2 & \cdots & 1 - T_{r-1}
\end{pmatrix}.
$$
THE DECREASE VALUE THEOREM

**Theorem 1.1** (Decrease Value Theorem). The generating function for the set \([0, r]^*\) by the weight \(\psi\) is given by

\[
\sum_{w \in [0, r]^*} \psi(w) = \prod_{0 \leq j \leq r} \left(1 + \frac{Y'_j}{1 - T'_j}\right) \frac{1}{\det(I - C)},
\]

where \(I\) is the identity matrix of order \((r + 1)\).

The proof of the Decrease Value Theorem is fully given in our previous paper \([FH07]\), together with its two equivalent formulations:

\[
\sum_{w \in [0, r]^*} \psi(w) = \prod_{0 \leq j \leq r} \left(1 - \frac{1}{Z_j + X_j}\right) \frac{1}{\prod_{0 \leq j \leq r} \left(1 - \frac{1}{Z'_j + Y'_j}\right)}
\]

\[
\sum_{w \in [0, r]^*} \psi(w) = \prod_{1 \leq j \leq r} \left(1 + \frac{1}{Z_j + X_j}\right) \frac{1}{\prod_{1 \leq j \leq r} \left(1 + \frac{1}{T'_j + Y'_j}\right)}
\]

There is a specialization of (1.5) that deserves a special attention, which is the following. For convenience, introduce three sequences of commuting variables \((\xi_i), (\eta_i), (\zeta_i)\) \((i = 0, 1, 2, \ldots)\) and make the following substitutions:

\[
X_i \leftarrow \xi_i, \ Z_i \leftarrow \xi_i, \ Y_i \leftarrow \eta_i, \ T_i \leftarrow \eta_i, \ Y'_i \leftarrow \zeta_i, \ T'_i \leftarrow \zeta_i, \quad (i = 0, 1, 2, \ldots).
\]

The new weight \(\psi'(v)\) attached to each word \(v = y_1y_2 \cdots y_n\) is then

\[
\psi'(v) = \prod_{i \in \text{DEC}(v)} \xi_i \prod_{i \in (\text{INC} \setminus \text{REC})(v)} \eta_i \prod_{i \in (\text{INCR})(v)} \zeta_i.
\]
and identity (1.5) becomes:

\[
\sum_{v \in [0,r]^*} \psi'(v) = \frac{\prod_{1 \leq j \leq r} (1 - \xi_j)}{\prod_{0 \leq j \leq r} (1 - \zeta_j)}.
\]

(1.7)

The further specializations of (1.6) and (1.7) require the following notations. For each word \( v = y_1 y_2 \cdots y_n \) let \( \lambda v \) designate its length \( (\lambda v = n) \) and \( \mathrm{tot} \; v = y_1 + y_2 + \cdots + y_n \) the sum of its letters. Next, given a positive integer \( l \), let \( \mathrm{dec}_l \; v \) be the number of letters of \( v \), which are \textit{decrease values and multiple of} \( l \); finally, for \( i = 0, 1, \ldots, l - 1 \) let \( |v|_i \mod l \) denote the number of letters of \( v \) \textit{congruent to} \( i \mod l \) and \( \mathrm{inrec}_i \; v \) the number of letters of \( v \), \textit{congruent to} \( i \mod l \), which are also increase and record values.

Now, let \( u, s, Y_i, Z_i \; (i = 0, 1, 2, \ldots) \) be a new set of variables and let \( \gamma \) denote the homomorphism defined by the following substitutions of variables:

\[
\begin{align*}
\xi_j & \leftarrow \begin{cases} uq^j s^l Z_0, & \text{if } j \equiv 0 \mod l; \\
uq^j s^l Z_i, & \text{if } j \equiv i \mod l \text{ and } 1 \leq i \leq l - 1; \\
\end{cases} \\
\eta_j & \leftarrow \begin{cases} uq^j s^l Y_i, & \text{if } j \equiv i \mod l \text{ and } 0 \leq i \leq l - 1; \\
\end{cases} \\
\zeta_j & \leftarrow \begin{cases} uq^j s^l Y_i Z_i, & \text{if } j \equiv i \mod l \text{ and } 0 \leq i \leq l - 1. \\
\end{cases}
\end{align*}
\]

(1.8)

It follows from (1.6) and (1.8) that

\[
\gamma \psi'(v) = u^{\lambda v} q^{\mathrm{tot} \; v} s^l \prod_{0 \leq i \leq l - 1} (s^i Z_i)^{|v|_i \mod l} Y_i^{\mathrm{inrec}_i \; v}.
\]

(1.9)

For each \( r \geq 0 \) consider the following multivariable generating function for the set \([0, r]^*\):

\[
F_r(u; q, s, (Y_i), (Z_i)) := \sum_{v \in [0,r]^*} u^{\lambda v} q^{\mathrm{tot} \; v} s^l \prod_{0 \leq i \leq l - 1} (s^i Z_i)^{|v|_i \mod l} Y_i^{\mathrm{inrec}_i \; v}.
\]

(1.10)

Using the traditional notations for the \textit{q-ascending factorials} \( (x; q)_n = 1 \) if \( n = 0 \) and \( (x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x) \) if \( n \geq 1 \), the first goal of the paper is to show that \( F_r(u; q, s, (Y_i), (Z_i)) \) can be expressed as an explicit hypergeometric series in the variable \( u \), as stated in the next theorem, where it is assumed that \( Z_l \equiv Z_0 \).
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**Theorem 1.2.** For each $r \geq 0$ the following evaluation holds:

(1.11) \( F_r(u; q, s, (Y_i), (Z_i)) = \prod_{1 \leq i \leq l} (uq^i s^i Z_i; q^i)_{(r-i)/l+1} \prod_{0 \leq i \leq l-1} (uq^i s^i Y_i Z_i; q^i)_{(r-i)/l+1} \) \( Z_0(1 - q^i s^i) \)

\( \times (Z_0 - Z_0 q^i s^i + \sum_{i=1}^l q^i s^i Z_i - \sum_{i=1}^l q^i s^i Z_i (uq^i s^i Z_0; q^i)_{(r-i)/l+1})^{-1}. \)

The proof of Theorem 1.2 is given in Section 2. It is based on the decrease value theorem and makes use of the traditional techniques of $q$-telescoping. The above identity on word generating series is next used to show that the infinite series $\sum_{r \geq 0} t^r F_r(w; q, s, (Y_i), (Z_i))$ can be expressed as a factorial series $\sum_{n \geq 0} A_n(s, t, q, (Y_i), (Z_i)) w^n / (t^i q^i)_{n+1}$, where each coefficient $A_n(s, t, q, (Y_i), (Z_i))$ is a generating polynomial for an algebraic structure by a well-defined multivariable statistic.

This algebraic structure is the following. Let $l$ be a positive integer and consider the wreath product $C_l \wr S_n$ of the cyclic group $C_l$ of order $l$ by the symmetric group $S_n$ of order $n$ (see, e.g., [RR06] for a complete description). The elements of $C_l \wr S_n$ may be viewed as ordered pairs $(w, \varepsilon)$, where $w = x_1 x_2 \cdots x_n$ is a permutation of $12 \cdots n$ and $\varepsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_n$ a word of length $n$, whose letters belong to $\{0, 1, \ldots, l-1\}$. Finally, $C_l \wr S_n$ is equipped with the total order “$<$” defined by:

$$(j, i) < (j', i') \text{ if and only if either } i > i', \text{ or } i = i' \text{ and } j < j'.$$

For each argument $A$ let $\chi(A) = 1$ or 0, depending on whether $A$ is true or false and let $|\varepsilon|_i$ denote the number of letters equal to $i$ in $\varepsilon$, so that $1 \cdot |\varepsilon|_1 + 2 \cdot |\varepsilon|_2 + \cdots + (l-1) \cdot |\varepsilon|_{l-1} = \text{tot} \varepsilon$. The statistics associated with $(w, \varepsilon)$ are the following:

- $\text{exc}(w, \varepsilon) := \# \{ j : 1 \leq j \leq n, x_j > j, \epsilon_j = 0 \}$;
- $\text{fexc}(w, \varepsilon) := l \cdot \text{exc}(w, \varepsilon) + \text{tot} \varepsilon$;
- $\text{des}(w, \varepsilon) := \# \{ j : 1 \leq j \leq n - 1, (x_j, \epsilon_j) > (x_{j+1}, \epsilon_{j+1}) \}$;
- $\text{fdes}(w, \varepsilon) := l \cdot \text{des}(w, \varepsilon) + \epsilon_1$;
- $\text{maj}(w, \varepsilon) := \sum_{1 \leq j \leq n-1} j \chi((x_j, \epsilon_j) > (x_{j+1}, \epsilon_{j+1}))$;
- $\text{fmaj}(w, \varepsilon) := l \cdot \text{maj}(w, \varepsilon) + \text{tot} \varepsilon$;
- $\text{fix}_i(w, \varepsilon) := \# \{ j : 1 \leq j \leq n, (x_j, \epsilon_j) = (j, i) \} \quad (0 \leq i \leq l - 1)$.

For each $n \geq 0$ consider the generating polynomial for $C_l \wr S_n$:

(1.12) \( W_n(s, t, q, (Y_i), (Z_i)) := \sum_{(w, \varepsilon) \in C_l \wr S_n} s^{\text{fexc}(w, \varepsilon)} t^{\text{fdes}(w, \varepsilon)} q^{\text{fmaj}(w, \varepsilon)} \prod_{0 \leq i \leq l-1} Z_i^{\text{fix}_i(w, \varepsilon)}. \)
Theorem 1.3. Let $F_r(u; q, s, (Y_i), (Z_i))$ be given by (1.11). Then, the following identity holds

\begin{equation}
\sum_{n \geq 0} (1 + t + \cdots + t^{l-1}) W_n(s, t, q, (Y_i), (Z_i)) \frac{u^n}{(t^l; q^l)^{n+1}} = \sum_{r \geq 0} t^r F_r(u; q, s, (Y_i), (Z_i)).
\end{equation}

The proof of Theorem 1.3 is made in two steps. First, the left-hand side of (1.13) is shown to be equal to the generating function for the so-called wreathed permutations, expressed as a series $\sum t^r G_r(u; q, s, (Y_i), (Z_i))$. This is the content of Section 3. Second, each coefficient of $t^r$ in the above series is shown to be equal to the generating series $F_r(u; q, s, (Y_i), (Z_i))$, given in (1.10). This is accomplished, in Section 4, by means of a bijection, whose construction is directly inspired by the classical standardization procedure. Finally, we derive several specializations of Theorem 1.3, in particular, the following theorem, which extends MacMahon’s classical result of the equidistribution of exceedances and descents on permutations (see [Lo83, Chap. 10]).

Theorem 1.4. The two statistics “fdes” and “fexc” are equidistributed over $C_l \wr S_n$. Let $W_{n,k}^{(l)}$ denote the number of elements $(w, \varepsilon)$ from $C_l \wr S_n$ such that $fdes(w, \varepsilon) = k$. Then, the following recurrence formula holds

\begin{equation}
W_{n,k}^{(l)} = (k + 1)W_{n-1,k}^{(l)} + \cdots + W_{n-1,k-(t-1)}^{(l)} + (ln - k)W_{n-1,k-l}^{(l)}
\end{equation}

for $n \geq 2$ and $0 \leq k \leq nl - 1$ and the initial conditions: $W_{0,0}^{(l)} = 1$, $W_{0,k}^{(l)} = 0$ for $k \neq 0$; $W_{1,k}^{(l)} = 1$ for $k = 0, 1, \ldots, l - 1$ and 0 for any other value of $k$.

2. Proof of Theorem 1.2

To obtain the right-hand side of (1.11) it suffices to calculate the image under $\gamma$ of the right-hand side of (1.7). First, remembering that $Z_l \equiv Z_0$, we have

$$
\gamma \prod_{1 \leq j \leq r} (1 - \xi_j) = \gamma \prod_{i=1}^{l} \prod_{j=0}^{[(r-i)/l]} (1 - \xi_{jl+i}) = \gamma \prod_{i=1}^{t-1} \prod_{j=0}^{[(r-i)/l]} (1 - \xi_{jl+i}) \times \prod_{j=0}^{[(r-t)/l]} (1 - \xi_{jl+t}) = \prod_{i=1}^{t-1} \prod_{j=0}^{[(r-i)/l]} (1 - uq^{jl+i}s^iZ_i) \times \prod_{j=0}^{[(r-t)/l]} (1 - uq^{jl+t} s^l Z_l)
$$

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In the same manner,

\[
\gamma \prod_{0 \leq j \leq r} (1 - \zeta_j) = \prod_{i=0}^{l-1} (uq^i s^i Z; q^l)_{[(r-i)/l]+1};
\]

\[
\gamma \prod_{1 \leq j \leq k-1} (1 - \xi_j) = \prod_{i=1}^{l} (uq^i s^i Z; q^l)_{[(k-1-i)/l]+1};
\]

\[
\gamma \prod_{0 \leq j \leq k-1} (1 - \eta_j) = \prod_{i=0}^{l-1} (uq^i s^i Z; q^l)_{[(k-1-i)/l]+1};
\]

so that the image of (1.7) under \( \gamma \) becomes

\[
(2.1) \quad \sum_{v \in [0,r]^*} \gamma \psi'_v(v) = \frac{\prod_{i=1}^{l} (uq^i s^i Z; q^l)_{[(r-i)/l]+1}}{\prod_{i=0}^{l-1} (uq^i s^i Y; q^l)_{[(r-i)/l]+1}} \times (1 - S)^{-1},
\]

where

\[
S = \sum_{1 \leq k \leq r} \frac{(uq^l s^l Z; q^l)_{[(k-1)/l]+1}}{(uZ; q^l)_{[(k-1)/l]+1}} \gamma(\xi_k),
\]

which can be rewritten:

\[
S = \sum_{i=1}^{l} \sum_{j=0}^{[(r-i)/l]} \frac{(uq^l s^l Z; q^l)_{j+1}}{(uZ; q^l)_{j+1}} uq^l q^i s^i Z_i.
\]

Introduce

\[
G(m) = \sum_{0 \leq j \leq m} \frac{(uq^l s^l Z; q^l)_{j}}{(uZ; q^l)_{j+1}} uq^l j,
\]

so that

\[
S = \sum_{i=1}^{l} q^i s^i Z_i G([r-l]/l]).
\]

As \( \frac{(uq^l s^l Z_0; q^l)_{j+1}}{(uZ_0; q^l)_{j+1}} - \frac{(uq^l s^l Z_0; q^l)_{j}}{(uZ_0; q^l)_{j+1}} = \frac{(uq^l s^l Z_0; q^l)_{j}}{(uZ_0; q^l)_{j+1}} uq^l j Z_0 (1 - q^l s^l) \), we have:

\[
G(m) = \frac{1}{Z_0 (1 - q^l s^l)} \left( \frac{(uq^l s^l Z_0; q^l)_{m+1}}{(uZ_0; q^l)_{m+1}} - 1 \right),
\]

and then

\[
S = \sum_{i=1}^{l} \frac{q^i s^i Z_i}{Z_0 (1 - q^l s^l)} \left( \frac{(uq^l s^l Z_0; q^l)_{[(r-i)/l]+1}}{(uZ_0; q^l)_{[(r-i)/l]+1}} - 1 \right).
\]

We obtain the righthand side of (1.11) by substituting the above \( S \) into (2.1). \( \square \)
3. Wreathed permutations

The generating polynomial $W_n(s, t, q; (Y_i), (Z_i))$ for the group $C_l \wr \mathfrak{S}_n$ has been defined in (1.12). Recall the notation for the $q$-binomial coefficient \[
\binom{n}{k}_q := (q; q)_n / ((q; q)_k (q; q)_{n-k}) \quad \text{for} \quad 0 \leq k \leq n \quad \text{and the classical identities}
\]
\[
\frac{1}{(t; q)_{n+1}} = \sum_{r \geq 0} \left[ \begin{array}{c} n + r \\ r \end{array} \right]_q t^r; \quad \left[ \begin{array}{c} n + r \\ r \end{array} \right]_q = \sum_{b \in \text{NIW}_n(r)} q^{\text{tot } b},
\]

where $\text{NIW}_n(r)$ (resp. $\text{NIW}_n$) denotes the set of all words $b = b_1 b_2 \cdots b_n$ of length $n$, whose letters are nonnegative integers satisfying $r \geq b_1 \geq b_2 \geq \cdots b_n \geq 0$ (resp. $b_1 \geq b_2 \geq \cdots b_n \geq 0$) (see [An76, Chap. 3]). Using those two identities we have:

\[
\frac{1 + t + \cdots + t^{l-1}}{(t; q)_n} = \sum_{r' \leq 0} (t^{r'} + t^{r'+1} + \cdots + t^{r'+l-1}) \left[ \begin{array}{c} n + r' \\ r' \end{array} \right]_q t^{r'} = \sum_{r \geq 0} \left[ \begin{array}{c} n + \lfloor r/l \rfloor \\ \lfloor r/l \rfloor \end{array} \right]_q t^r = \sum_{b \in \text{NIW}_n([r/l])} \sum_{l \leq r} q^{\text{tot } b}.
\]

Hence,

\[
(3.1) \quad \frac{1 + t + \cdots + t^{l-1}}{(t; q)_{n+1}} W_n(s, t, q; (Y_i), (Z_i)) = \sum_{r \geq 0} \sum_{b \in \text{NIW}_n, \text{tot } b \leq r} \sum_{(w, \epsilon) \in C_l \wr \mathfrak{S}_n} s^{\text{fexc}(w, \epsilon)} t^{\text{fdes}(w, \epsilon)} q^{\text{fmaj}(w, \epsilon)} \prod_{0 \leq i \leq l-1} Y_i^{\text{fix}_i(w, \epsilon)} Z_i^{\epsilon_i | i}.
\]

For each element $(w, \epsilon) \in C_l \wr \mathfrak{S}_n$ form the word $z = z_1 z_2 \cdots z_n$, where $z_j$ is defined to be the number of $k$ such that $j \leq k \leq n - 1$ and $(x_k, \epsilon_k) > (x_{k+1}, \epsilon_{k+1})$ with respect to the order imposed on $C_l \wr \mathfrak{S}_n$. In other words, $z_j$ is the number of descents in the right factor $(x_j, \epsilon_j)(x_{j+1}, \epsilon_{j+1}) \cdots (x_n, \epsilon_n)$. The next proposition is easy to verify.

**Proposition 3.1.** We have: $\text{des}(w, \epsilon) = z_1$, $\text{maj}(w, \epsilon) = \text{tot } z$.

Now, let $b = b_1 b_2 \cdots b_n \in \text{NIW}_n$ and $(w, \epsilon) = (x_1, \epsilon_1)(x_2, \epsilon_2) \cdots (x_n, \epsilon_n) \in C_l \wr \mathfrak{S}_n$ be given. We define the word $c = c_1 c_2 \cdots c_n$ by

\[
c_j := l(b_j + z_j) + \epsilon_j \quad (1 \leq j \leq n).
\]
As both words $b$ and $z$ are monotonic nonincreasing, we have $(b_j + z_j) \geq (b_{j+1} + z_{j+1})$ ($1 \leq j \leq n-1$). If the inequality is strict, then $c_j > c_{j+1}$ since $0 \leq \epsilon_j \leq l-1$. If $b_j + z_j = b_{j+1} + z_{j+1}$, then $z_j = z_{j+1}$ and, consequently, $(x_j, \epsilon_j) < (x_{j+1}, \epsilon_{j+1})$, so that $\epsilon_j \geq \epsilon_{j+1}$. Therefore, $c_j \geq c_{j+1}$. The word $c$ is then monotonic nonincreasing. We further have the following properties:

(i) $c_j = c_{j+1} \Rightarrow (x_j, \epsilon_j) < (x_{j+1}, \epsilon_{j+1})$;

(ii) $c_j \equiv \epsilon_j \pmod{l}$;

(iii) $c_1 = lb_1 + \text{fdes}(w, \epsilon)$;

(iv) $\text{tot} c = l \text{tot} b + \text{fmaj}(w, \epsilon)$.

Note that (iii) and (iv) are immediate consequences of Proposition 3.1 and the definition of $c$.

A triple $(c, w, \epsilon)$ such that $(w, \epsilon) = (x_1, \epsilon_1)(x_2, \epsilon_2) \cdots (x_n, \epsilon_n) \in C_l \wr \mathfrak{S}_n$ and $c = c_1c_2 \cdots c_n \in \text{NIW}_n$ and such that properties (i) and (ii) hold is called a wreathed permutation of order $n$. The set of all wreathed permutations $(c, w, \epsilon)$ of order $n$ is denoted by $\text{WP}_n$ and the subset of $\text{WP}_n$ of all $(c, w, \epsilon)$ such that $c_1 \leq r$ by $\text{WP}_n(r)$.

It follows from (3.2) that, for each $r \geq 0$ the mapping

$$(b, w, \epsilon) \mapsto (c, w, \epsilon)$$

provides a bijection of the set of all triples $(b, w, \epsilon)$ such that $b \in \text{NIW}_n$, $(w, \epsilon) \in C_l \wr \mathfrak{S}_n$ and $lb_1 + \text{fdes}(w, \epsilon) \leq r$ onto $\text{WP}_n(r)$, having properties (iii) and (iv).

By (3.1)

$$
\frac{1 + t + \cdots + t^{l-1}}{(t^l; q^l)_{n+1}} W_n(s, t, q, (Y_i), (Z_i))
= \sum_{r \geq 0} t^r \sum_{(c, w, \epsilon) \in \text{WP}_n(r)} s^{\text{fexc}(w, \epsilon)} q^{\text{tot} c} \prod_{0 \leq i \leq l-1} Y_i^{\text{fix}_i(w, \epsilon)} Z_i^{\epsilon_i},
$$

so that, if we let

(3.3) $G_r(w; s, q, (Y_i), (Z_i)) := \sum_{(c, w, \epsilon) \in \text{WP}(r)} u^w s^{\text{fexc}(w, \epsilon)} q^{\text{tot} c} \prod_{0 \leq i \leq l-1} Y_i^{\text{fix}_i(w, \epsilon)} Z_i^{\epsilon_i},$

we have the identity:

(3.4) $\sum_{n \geq 0} (1 + t + \cdots + t^{l-1}) W_n(s, t, q, (Y_i), (Z_i)) \frac{u^n}{(t^l; q^l)_{n+1}}
= \sum_{r \geq 0} t^r G_r(w; s, q, (Y_i), (Z_i)).$
Example. With $l = 3$ the monotonic nonincreasing word $c$ is calculated from the triple $(b, w, \epsilon)$:

\[
\begin{align*}
\text{Id} & = 1 
2 
3 
4 
5 
6 
7 
8 
9 
10 \\
\text{b} & = 5 
5 
4 
2 
2 
2 
0 
0 
0 
0 \\
\text{w} & = 1 
8 
7 
4 
10 
2 
6 
9 
5 
3 \\
\epsilon & = 0 
0 
1 
1 
1 
0 
2 
2 
1 
0 \\
\text{z} & = 3 
3 
2 
1 
1 
1 
0 
0 
0 
0 \\
\text{c} & = 24 
24 
19 
10 
10 
9 
2 
2 
1 
0 \\
\end{align*}
\]

Note that $\text{des}(w, \epsilon) = 3 = z_1$; $\text{maj}(w, \epsilon) = 2 + 3 + 6 = 11$; $\text{tot}_b = 20$; $\text{tot}_\epsilon = 8$; $\text{fdes}(w, \epsilon) = 3 \cdot \text{des}(w, \epsilon) + \epsilon_1 = 9$; $24 = 3 \cdot \text{tot}_b + \text{fdes}(w, \epsilon) = 3 \cdot 5 + 9$; $\text{fmaj}(w, \epsilon) = 3 \cdot \text{maj}(w, \epsilon) + \text{tot}_\epsilon = 3 \cdot 11 + 8 = 41$; $101 = \text{tot}_c = 3 \cdot \text{tot}_b + \text{fmaj}(w, \epsilon) = 3 \cdot 20 + 41$.

4. Standardization

By comparison with (1.13) we see that Theorem 1.3 is proved if we show that $\text{F}_r(u; s, q, (Y_i), (Z_i))$, given by (1.10), is equal to $\text{G}_r(u; s, q, (Y_i), (Z_i))$, given by (3.3), for all $r \geq 0$, that is, if we prove

\[
\sum_{v \in [0, r]^n} u^{\lambda_v} q^{\text{tot}_v} s^{\text{dec}_v} \prod_{0 \leq i \leq l-1} (s^i Z_i)^{|v|_i \mod l} Y_i^{\text{inrec}_i v} = \sum_{(c, w, \epsilon) \in \text{WP}_n(r)} u^{\lambda_w} s^{\text{fexc}(w, \epsilon)} q^{\text{tot}_c} \prod_{0 \leq i \leq l-1} Y_i^{\text{fix}_i (w, \epsilon)} Z_i^{|\epsilon|_i}.
\]

To do this, it suffices to construct a bijection $v \mapsto (c, w, \epsilon)$ of $[0, r]^n$ (the set of all words of length $n$ with letters from the alphabet $[0, r]$) onto $\text{WP}_n(r)$ having the properties:

(i) $\text{tot}_v = \text{tot}_c$;
(ii) $\text{dec}_v = \text{exc}(w, \epsilon)$;
(iii) $|v|_i \mod l = |\epsilon|_i$ for $i = 0, 1, \ldots, l - 1$;
(iv) $\text{inrec}_v = \text{fix}_i (w, \epsilon)$ for $i = 0, 1, \ldots, l - 1$.

For such a bijection the word $c$ is to be the monotonic nonincreasing rearrangement of $v$, the permutation $w$ an adequate labelling from 1 to $n$ of the $n$ letters of $v$, and $\epsilon$ a word whose letters are the residues of the
letters of \( v \mod l \). Such a bijection appears to be a standardization of each word by a certain element from \( C_l \times S_n \). Earlier standardizations by the symmetric group \( S_n \) (resp. the hyperoctahedral group \( B_n \)) have been constructed by Gessel and Reutenauer [GR93] (resp. in [FH09]). The procedure we now develop proceeds from the same principle.

Recall that a nonempty word \( v = y_1y_2 \cdots y_n \) is a Lyndon word, if either \( n = 1 \), or \( n \geq 2 \) and, with respect to the lexicographic order, the inequality \( y_1y_2 \cdots y_n > y_1y_{i+1} \cdots y_n y_1 \cdots y_{i-1} \) holds for every \( i \) such that \( 2 \leq i \leq n \). Let \( v, v' \) be two nonempty primitive words (none of them can be written as \( v_0^a \) for \( a \geq 2 \) and some word \( v_0 \)). We write \( v \preceq v' \) if and only if \( v^a \preceq v'^a \) with respect to the lexicographic order for an integer \( a \) large enough. As shown for instance in [Lo83, Theorem 5.1.5] (also see [Ch58], [Sch65]) each nonempty word \( v \) can be written uniquely as a product \( l_1l_2 \cdots l_k \), called its Lyndon factorization, where each \( l_i \) is a Lyndon word and \( l_1 \preceq l_2 \preceq \cdots \preceq l_k \). In the example below the Lyndon factorization of \( v \) has been materialized by vertical bars.

Now, start with the Lyndon factorization \( l_1l_2 \cdots l_k \) of a word \( v \) from \([0, n]^{[n]} \). With such a \( v \) associate a permutation \( \sigma \) from \( S_n \) in the following manner: each letter \( y_i \) of \( v \) belongs to a Lyndon word factor \( l_h \), so that \( l_h = v'y_i v'' \). Then, form the infinite word \( A(y_i) := v'' v'y_i v'' v' \cdots \) If \( y_i \) and \( y_{i'} \) are two letters of \( v \), say that \( y_i \) precedes \( y_{i'} \) if \( A(y_i) > A(y_{i'}) \) for the lexicographic order, or if \( A(y_i) = A(y_{i'}) \) and \( y_i \) is to the right of \( y_{i'} \) in the word \( v \). This precedence determines a total order on the \( n \) letters of \( v \). The letter that precedes all the other ones is given label 1, the next one label 2, and so on. When each letter \( y_i \) of \( v \) is replaced by its label, say, \( \text{lab}(y_i) \), each Lyndon word factor \( l_j \) becomes a new word \( \tau_j \).

The essential property is that each \( \tau_j \) starts with its minimum element and those minimum elements read from left to right are in decreasing order. We can then interpret each \( \tau_j \) as the cycle of a permutation and the (juxtaposition) product \( \tau_1 \tau_2 \cdots \tau_k \) as the (functional) product of disjoint cycles. This product, said to be written in canonical form, defines a unique permutation \( \sigma \) from \( S_n \) ([Lo83], §10.2).

For example,

\[
\begin{align*}
v &= 2 | 3 \ 2 \ 1 \ 1 | 3 | 5 | 6 \ 4 \ 2 \ 1 \ 3 \ 2 \ 3 | 6 \ 6 \ 3 \ 1 \ 6 \ 6 \ 2 \ | 6 \\
\sigma &= 16 | 12 \ 18 \ 22 \ 21 | 10 | 7 | 4 \ 8 \ 17 \ 20 \ 11 \ 15 \ 9 | 2 \ 5 \ 13 \ 19 \ 3 \ 6 \ 14 | 1
\end{align*}
\]

The labels on the second row are obtained as follows: read the letters equal to 6 (the maximal letter) from left to right and form their associated infinite words: 64213236421 \( \cdots \), 66316626631 \( \cdots \), 6316621131 \( \cdots \), 662663166 \( \cdots \), 62663166 \( \cdots \), 6666 \( \cdots \). Those letters 6 read from left to right will be given the labels 4, 2, 5, 3, 6, 1. We continue the labellings by reading the letters equal to 5, then 4, \( \cdots \) in the above word \( v \).
No decrease \( y_i \) in \( v \) can be the rightmost letter of a Lyndon word factor \( l_i \). We have then \( l_i = \cdots y_i y_{i+1} \cdots y_j y_{j+1} \cdots \) with \( y_i \geq y_{i+1} \geq \cdots \geq y_j > y_{j+1} \). Consequently, \( A(y_i) > A(y_{i+1}) \) and \( \text{lab}(y_i) < \text{lab}(y_{i+1}) \). Conversely, if \( \text{lab}(y_i) < \text{lab}(y_{i+1}) \) and \( y_i, y_{i+1} \) belong to the same Lyndon factor, then \( y_i \) is a decrease in \( v \). To each decrease \( y_i \) in \( v \) there corresponds a unique cycle \( \tau_h \) of \( \sigma \) and a pair \( \text{lab}(y_i) \text{lab}(y_{i+1}) \) of successive letters of \( \tau_h \) such that \( \text{lab}(y_i) < \text{lab}(y_{i+1}) \) and \( \text{lab}(y_{i+1}) = \sigma(\text{lab}(y_i)) \).

Consider the monotonic nonincreasing rearrangement \( c = c_1 c_2 \cdots c_n \) of \( v \) and form the three-row matrix

\[
1 \quad 2 \quad \cdots \quad n \\
c_1 \quad c_2 \quad \cdots \quad c_n \\
\sigma(1) \quad \sigma(2) \quad \cdots \quad \sigma(n)
\]

Then, if \( y_i \) is a decrease in \( v \), the \( \text{lab}(y_i) \)-th column of the previous matrix is of the form

\[
\begin{array}{c}
\text{lab}(y_i) \\
y_i \\
\text{lab}(y_{i+1})
\end{array}
\quad \text{with } \quad \text{lab}(y_i) < \text{lab}(y_{i+1}).
\]

Consequently, \( y_i \) is a decrease in \( v \) if and only if \( \sigma(\text{lab}(y_i)) > \text{lab}(y_i) \). As \( \sigma(i) > \sigma(i + 1) \Rightarrow c_i > c_{i+1} \), the above three-row matrix can be expressed as

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Id} & 1 & m_1 & m_1 + 1 & \cdots & m_1 + m_2 & \cdots & m_1 + \cdots + m_k - 1 + 1 & n \\
c & a_1 & a_1 & a_1 & \cdots & a_1 & a_2 & \cdots & a_k \\
\sigma & \sigma(1) & \sigma(m_1) & \sigma(m_1 + 1) & \cdots & \sigma(m_1 + m_2) & \cdots & \sigma(m_1 + \cdots + m_k - 1 + 1) & \sigma(n)
\end{array}
\]

where \( a_1 > a_2 > \cdots > a_k \geq 0 \) and \( m_1 \geq 1, m_2 \geq 1, \ldots, m_k \geq 1 \) and \( \sigma(1) < \cdots < \sigma(m_1), \sigma(m_1 + 1) < \cdots < \sigma(m_1 + m_2), \ldots, \sigma(m_1 + \cdots + m_k - 1 + 1) < \cdots < \sigma(n) \). For each \( i = 1, \ldots, k \) let \( \overline{a}_i \) be the residue of \( a_i \) mod \( l \) and let

\[
\left( \begin{array}{cc}
w & \epsilon \\
\overline{a}_1 & \overline{a}_1 \\
\overline{a}_2 & \overline{a}_2 \\
\vdots & \vdots \\
\overline{a}_k & \overline{a}_k
\end{array} \right)
\]

It then follows that \((c, w, \epsilon)\) is a wreathed permutation and properties (i), (ii) and (iii) hold. For the proof of (iv) we note that a letter \( y_i \) of \( v \) is an increase and record value if and only if \( y_i \) is a one-letter factor in the Lyndon factorization of \( v \), that is, if and only if lab \( y_i \) is a cycle of length 1 of \( w \), or equivalently, a fixed point of \( w \). All the steps previously described are perfectly reversible. This achieves the proof of Theorem 1.3.

With the running example take \( l = 3 \) we can form the table:

\[
\begin{array}{cccccccccccccccccccccc}
v & = & 2 & | & 3 & 2 & 1 & 1 & | & 3 & | & 5 & | & 6 & 4 & 2 & 1 & 3 & 2 & 3 & | & 6 & 6 & 3 & 1 & 6 & 6 & 2 & | & 6 \\
c & = & 6 & 6 & 6 & 6 & 6 & | & 5 & | & 4 & 3 & 3 & 3 & 3 & | & 2 & 2 & 2 & 2 & 2 & | & 1 & 1 & 1 & 1 \\
\epsilon & = & 0 & 0 & 0 & 0 & 0 & 0 & | & 2 & | & 1 & | & 0 & 0 & 0 & 0 & | & 2 & 2 & 2 & 2 & 2 & | & 1 & 1 & 1 & 1 \\
\end{array}
\]
The word $v$ is given, with its corresponding Lyndon factorization; the word $\sigma$ is obtained from $v$ by replacing each letter $y_i$ by its label $\text{lab}(y_i)$ and to be regarded as the product of the cycles duly materialized; $c$ is just the monotonic nonincreasing rearrangement of $v$; $w$ is the sequence $\sigma(1)\sigma(2)\cdots\sigma(n)$; $\epsilon$ is derived from $c$ by replacing each $c_i$ by its residue mod 3.

The eight decreases of $v$ which are multiple of 3, and the eight exceedances of $(w,\epsilon)$ have been reproduced in boldface. The four increase and record values of $v$ are reproduced in italic, together with the four one-letter factors of $\sigma$, and the four fixed points of $w$, that is, 16, 10, 7, 1.

5. Specializations

Let $W_n(s, t, q)$ (resp. $F_r(u; q, s)$) be the polynomial $W_n(s, t, q, (Y_i), (Z_i))$ (resp. the series $F(u; q, s, (Y_i), (Z_i))$), when $Y_i = Z_i = 1$ for all $i$, so that

\begin{equation}
W_n(s, t, q) = \sum_{(w, \epsilon) \in \mathcal{C}_t \times \mathcal{S}_n} s^{\text{foxc}(w, \epsilon)} t^{\text{des}(w, \epsilon)} q^{\text{fmaj}(w, \epsilon)};
\end{equation}

\begin{equation}
F_r(u; q, s) = \frac{(uq^l s^l; q^l)[r/l]}{(u; q^l)[r/l]+1} (1 - q^l s^l)
\end{equation}

\[\times \left(1 - q^l s^l + \sum_{i=1}^{l} q^i s^i - \sum_{i=1}^{l} q^i s^i \frac{(uq^l s^l; q^l)[(r-i)/l]+1}{(u; q^l)[(r-i)/l]+1} \right)^{-1}.
\]

Theorem 1.3 yields the following result.

**Theorem 5.1.** The generating function for the polynomials $W_n(s, t, q)$ reads:

\begin{equation}
\sum_{n \geq 0} (1 + t + \cdots + t^{l-1}) W_n(s, t, q) \frac{u^n}{(t^l; q^l)_{n+1}} = \sum_{r \geq 0} t^r F_r(u; q, s).
\end{equation}

When $n$ tends to infinity, remember that $1/(u; q)_n$ tends to the $q$-exponential series $e_q(u) = \sum_{n \geq 0} u^n/(q; q)_n$ (see, e.g., [An76], chap. 2). Accordingly, when $r$ tends to infinity, $F_r(u; q, s)$ tends to

\[F_{\infty}(u; q, s) = \frac{e_q(u)}{e_q(uq^l s^l)} (1 - q^l s^l) \left(1 - q^l s^l + \sum_{i=1}^{l} q^i s^i - \sum_{i=1}^{l} q^i s^i \frac{e_q(u)}{e_q(uq^l s^l)} \right)^{-1}.
\]

\[= \frac{e_q(u)}{e_q(uq^l s^l)} (1 - q^l s^l) \left((1 - q^l s^l) + \frac{qs(1 - q^l s^l)}{1 - qs} \left(1 - \frac{e_q(u)}{e_q(uq^l s^l)} \right) \right)^{-1}.
\]
\[\begin{align*}
&= \frac{e_q(u)}{e_q(uq^l s^l)} (1 - qs) \left( (1 - qs) + qs \left( 1 - \frac{e_q(u)}{e_q(uq^l s^l)} \right) \right)^{-1} \\
&= \frac{e_q(u)}{e_q(uq^l s^l)} (1 - qs) \left( 1 - \frac{e_q(u)}{e_q(uq^l s^l)} \right)^{-1} \\
&= \frac{(1 - qs)e_q(u)}{e_q(uq^l s^l) - qse_q(u)}.
\end{align*}\]

**Theorem 5.2.** Let

\[W_n(s, 1, q) = \sum_{(w, \epsilon) \in C_l \wr S_n} s^{\text{fexc}(w, \epsilon)} q^{\text{fmaj}(w, \epsilon)}.\]

Then, the following identity holds:

\[(5.4) \sum_{n \geq 0} W_n(s, 1, q) \frac{u^n}{(q^l; q^l)_n} = \frac{(1 - qs)e_q(u)}{e_q(uq^l s^l) - qse_q(u)}.\]

**Proof.** There suffices to multiply both sides of identity (5.3) by \((1 - t)\) and let \(t\) tend to 1. The right-hand side tends to \(F_\infty(u; q, s)\), which has just been calculated.

Let \(u = (1 - q^l)u\) and then \(q \to 1\) in (5.4). This leads to the identity:

\[(5.5) \sum_{n \geq 0} W_n(s, 1, 1) \frac{u^n}{n!} = \frac{1 - s}{-s + \exp(u(s^l - 1))},\]

where

\[(5.6) W_n(s, 1, 1) = \sum_{(w, \epsilon) \in C_l \wr S_n} s^{\text{fexc}(w, \epsilon)}.
\]

The next step is to calculate \(F_r(u; q, 1)\). First, note that

\[
\frac{(uq^l; q^l)_m}{(u; q^l)_m} = \frac{1}{1 - u} \quad \text{and} \quad \frac{(uq^l; q^l)_m}{(u; q^l)_m} = \frac{1 - uq^{ml}}{1 - u}.
\]

Now, let \(s = 1\) in (5.2). We get:

\[
F_r(u; q, 1) = \frac{(uq^l; q^l)_{[r/l]}}{(u; q^l)_{[r/l] + 1}} (1 - q^l)
\]

\[
\times \left( 1 - q^l + \sum_{i=1}^l q^i - \sum_{i=1}^l q^i \frac{(uq^l; q^l)_{(r - i)/l + 1}}{(u; q^l)_{(r - i)/l + 1}} \right)^{-1}
\]

\[
= \frac{1 - q^l}{1 - u} \times \left( 1 - q^l + \sum_{i=1}^l q^i - \sum_{i=1}^l q^i \frac{1 - uq^l_{(l - i)/l + 1}}{1 - u} \right)^{-1}
\]

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\[
\frac{1 - q^l}{1 - u} \times \left(1 - q^l - u \sum_{i=1}^{l} q^i \frac{1 - q^{l \left((r-i)/l\right)+1}}{1 - q^l}\right)^{-1} = \left(1 - u - u \sum_{i=1}^{l} q^i \frac{1 - q^{l \left((r-i)/l\right)+1}}{1 - q^l}\right)^{-1}.
\]

Let \( r = kl + s \) with \( 0 \leq s \leq l - 1 \). Then

\[
l((r-i)/l) + 1 = l((kl + s - i)/l) + 1.
\]

Hence,

\[
\sum_{i=1}^{l} q^i \frac{1 - q^{l \left((r-i)/l\right)+1}}{1 - q^l} = \sum_{i=1}^{s} q^i \frac{1 - q^{l(k+1)}}{1 - q^l} + \sum_{i=s+1}^{l} q^i \frac{1 - q^{lk}}{1 - q^l}
\]

\[
= \frac{1 - q^{l(k+1)}}{1 - q^l} \frac{q - q^{s+1}}{1 - q} + \frac{1 - q^{lk} q^{s+1} - q^{l+1}}{1 - q^l} \frac{1}{1 - q}
\]

\[
= \frac{1}{1 - q^l} \frac{1}{1 - q} \left(q(1 - q^l) - q^{lk+s+1}(1 - q^l)\right)
\]

\[
= \frac{q}{1 - q} \left(1 - q^{lk+s}\right) = \frac{q}{1 - q} q^{r}.
\]

Consequently, \( F_r(u; q, 1) = \left(1 - u - uq \frac{1 - q^{r}}{1 - q}\right)^{-1} = \left(1 - u \frac{1 - q^{r+1}}{1 - q}\right)^{-1} \), so that, using the traditional notation for the \( q \)-analogs of integers,

\[(5.7) \quad F_r(u; q, 1) = \left(1 - u[r+1]_q\right)^{-1}.
\]

The following theorem has then been proved.

**Theorem 5.3.** The factorial generating function for the polynomials

\[(5.8) \quad W_n(1, t, q) = \sum_{(w, \epsilon) \in C, t \in S_n} t^{\text{des}(w, \epsilon)} q^{\text{maj}(w, \epsilon)} \quad (n \geq 0)
\]

reads

\[(5.9) \quad \sum_{n \geq 0} (1 + t + \cdots + t^{l-1}) W_n(1, t, q) u^n \frac{1}{(t; q^l)_{n+1}} = \sum_{r \geq 0} t^r (1 - u[r+1]_q)^{-1}.
\]

Several authors ([AR01], [ABR01], [ABR05], [ABR06], [CHGe07], [HLR05], [FH09]) have derived identity (5.9) in the particular case \( l = 2 \)
(hyperoctahedral group). Theorem 5.3 has several consequences. Write $W^{(l)}(1, t, q) := W_n(1, t, q)$ to indicate that the polynomial also depends on $l$. In particular,

\[(5.10) \quad W^{(l)}_n(1, t, q) = A_n(t, q) = \sum_{\sigma \in S_n} t^{\text{des}\sigma} q^{\text{maj}\sigma}, \]

which is precisely the $q$-Eulerian polynomial introduced by Carlitz [Ca54], and also combinatorially interpreted by him [Ca75]. As (5.9) holds for every $l$, we also have

\[(5.11) \quad \sum_{n \geq 0} A_n(t, q) \frac{u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r (1 - u[r+1]_q)^{-1}, \]

so that

\[(5.12) \quad W^{(l)}_n(1, t, q) = \frac{(t^l q^l ; q^l)_n}{(t; q)_n} A_n(t, q) \quad (n \geq 0). \]

In view of (5.12) there is no use working out the other formulas for $W^{(l)}(1, t, q)$ from scratch. We just have to report to Carlitz’s original paper [Ca54], dealing with the polynomials $A_n(t, q)$, and use (5.12). First,

\[(5.13) \quad (1 - q)W^{(l)}_n(1, t, q) = (1 - t^l q^l)W^{(l)}_{n-1}(1, t, q)\]

\[- (-q + tq(1 - q) + \cdots + t^{l-1} q^{l-1} (1 - q) + t^l q^l) W^{(l)}_{n-1}(1, tq, q). \]

Next, let $W^{(l)}(1, t, q) = \sum_k W^{(l)}_{n,k}(q) t^k$ and look for the coefficients of $t^k$ on both sides. We get:

\[(5.14) \quad W^{(l)}_{n,k}(q) = [k + 1]_q W^{(l)}_{n-1,k}(q) + q^{k+1} W^{(l)}_{n-1,k-1}(q)\]

\[+ q^{k+2} W^{(l)}_{n-1,k-2}(q) + \cdots + q^{k+l-1} W^{(l)}_{n-1,k-(l-1)}(q) + q^{k}[nl-k]_q W^{(l)}_{n-1,k-l}(q). \]

With $q = 1$ in (5.14) we obtain the recurrence formula (1.14) of Theorem 1.4.

Note that for $l = 1$ identity (1.14) becomes the classical recurrence formula for the Eulerian numbers $A_{n,k} := W^{(1)}_{n,k}$

\[A_{n,k} = (k + 1) A_{n-1,k} + (n - k) A_{n-1,k-1} \]

(see, e.g., [FS70]), which are the coefficients of the classical Eulerian polynomial $A_n(t, 1) = \sum_k A_{n,k} t^k$. For $q = 1$ identity (5.12) becomes:

\[(5.15) \quad W^{(l)}_n(1, t, 1) = \frac{(1 - t^l)^n}{(1 - t)^n} A_n(t, 1). \]
THE DECREASE VALUE THEOREM

As the exponential generating function for the Eulerian polynomials (see, e.g., [FS70]) reads

\[ \sum_{n \geq 0} A_n(t, 1) \frac{u^n}{n!} = \frac{1 - t}{-t + \exp(u(t - 1))}, \]

identity (5.16) implies that

\[ \sum_{n \geq 0} W_n^{(l)}(1, t, 1) \frac{u^n}{n!} = \frac{1 - t}{-t + \exp(u(t^l - 1))}. \]

From identities (5.5) and (5.17) we conclude that

\[ W_n^{(l)}(t, 1, 1) = W_n^{(l)}(1, t, 1) = \sum_{k=0}^{n-1} W_{n,k}^{(l)} l^k, \]

the coefficients \( W_{n,k}^{(l)} \) satisfying recurrence (5.15). This proves Theorem 1.4.

A good exercise of Combinatorics consists of proving directly that both coefficients \( W_n^{(l, \text{ides})}_{n,k} := \# \{ (w, \epsilon) \in C_l \wr S_n, \text{ides}(w, \epsilon) = k \} \) and \( W_n^{(l, \text{exc})}_{n,k} := \# \{ (w, \epsilon) \in C_l \wr S_n, \text{exc}(w, \epsilon) = k \} \) satisfy recurrence formula (1.14). For \( W_n^{(l, \text{ides})} \) it suffices to analyze the impact of the insertion of the biletter \( (n) \) into the \( n \) slots of an element from \( C_l \wr S_{n-1} \). For \( W_n^{(l, \text{exc})} \) the variation of “exc” is to be analyzed when the following operation is performed: replace the \( j \)-th biletter \( (x_j \epsilon_j) \) of an element \( (x_1 \ldots x_j \ldots x_{n-1}) \) from \( C_l \wr S_{n-1} \) by \( (n) \), and insert the biletter \( (n') \) to the right. We do not reproduce the solution of this exercise!

6. Concluding remarks

The Decrease Value Theorem, stated and proved in [FH07], was regarded as our Ur-result in our studies on \( q \)-calculus of permutation statistics. The motivation of this paper has been to extend its application to another group structure, namely the wreath product \( C_l \wr S_n \). This theorem makes it possible to have a full control of all decreases in each word. By means of a standardization procedure, the decreases can then be carried over to exceedances of the underlying permutations.

As was done in our two papers [FH08] and [FH09] dealing with the symmetric, and hyperoctahedral group, respectively, we could as well have made use of the so-called V-word decomposition theorem for words, a theorem directly inspired from a result by Kim-Zeng [KZ01] valid for permutations, for deriving Theorem 1.2. The Decrease Value Theorem has the advantage of providing the adequate identity immediately without any further combinatorial construction.
There is also an extension of Theorem 1.3, where the set \{0, 1, \ldots , l-1\} is split into two subsets \(I, J\) such that \(0 \in I\) and the total order imposed on \(C_l \wr S_n\) has the properties:

\[
\begin{align*}
(a) \quad (j, i) < (j', i') \text{ when } i > i' \text{ for all } j, j'; \\
(b) \quad (1, i) < (2, i) < \cdots < (n, i) \text{ when } i \in I; \\
(c) \quad (n, i) < \cdots < (2, i) < (1, i) \text{ when } i \in J.
\end{align*}
\]

When \(J = \emptyset\), we recover the total order used in the previous sections. For \(l = 2\), \(I = \{0\}\), \(J = \{1\}\), we get the natural order defined on the hyperoctahedral group \(B_n\), namely, \(-n < \cdots < -1 < 1 < \cdots < n\) with the convention: \(-j \equiv (j, 1), j \equiv (j, 0)\) for all \(j = 1, 2, \ldots , n\).

Referring to formula (1.11) let

\[
H_r(u; q, s, (Z_i)) := Z_0(1 - q^l s^l)
\]

\[
\times \left(Z_0 - Z_0 q^i s^i + \sum_{i=1}^{l} q^i s^i Z_i - \sum_{i=1}^{l} q^i s^i Z_i \frac{(u q^i s^i Z_0; q^l)_{(r-i)/l+1}}{(u Z_0; q^l)_{(r-i)/l+1}}\right)^{-1},
\]

a series that does not depend on \((Y_i)\). The extension of (1.13) for the total order defined in (6.1) reads:

\[
(6.2) \quad \sum_{n \geq 0} (1 + t + \cdots + t^{l-1}) W_n(s, t, q, (Y_i), (Z_i)) \frac{u^n}{(t^l; q^l)_{n+1}}
\]

\[
= \sum_{r \geq 0} t^r \prod_{i \in J} (-u s^i q^i Y_i Z_i; q^l)_{(r-i)/l+1} \prod_{i \in (I \setminus \{0\}) \cup \{l\}} (u q^i s^i Z_i; q^l)_{(r-i)/l+1}
\]

\[
\times \prod_{i \in I} (u q^i s^i Y_i Z_i; q^l)_{(r-i)/l+1}
\]

\[
\times H_r(u; q, s, (Z_i)).
\]

We do not reproduce the proof of this extension. Note that it fully implies the result we had derived in [FH09] for the hyperoctahedral group. Also note that the first statistical study of the latter group has been made by Reiner [Re93a, Re93b, Re93c, Re95a].

Several papers have recently been published dealing with statistics on wreath products. The first analysis made by Bagno [B04] was followed by Haglund et al. [HLR05], who studied another aspect of permutation statistics on wreath products in connection with the theory of perfect matchings and rook placement \(q\)-counting. Bernstein [DB06] has worked out a theory for the so-called \(C_a \wr S_n\) \(q\)-maj Euler-Mahonian \(bivariable\) polynomials and obtained a solid formulary. The definitions he took for his generalized descent and major index do not coincide with the ones presented here. Bagno and Garber [BG06] generalize results by
Ksavrelof and Zeng [KZ03] on the multidistribution of the excedance number associated with the numbers of fixed points and cycles. Some more or less explicit three-variable statistical distributions are derived. Regev and Roichman [RR06] have worked out recurrence formulas of binomial-Stirling type for wreath product statistics related to left minima. Finally, Mendes and Remmel [MR07] have developed a brick-tabloid symmetric function approach for calculating generating functions for statistics for tuples of permutations.

References

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