A SYMMETRICAL q-EULERIAN IDENTITY

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Abstract. We find a q-analog of the following symmetrical identity involving binomial coefficients $\binom{n}{m}$ and Eulerian numbers $A_{n,m}$:

$$\sum_{k\geq 0}\binom{a+b}{k}A_{k,a-1}=\sum_{k\geq 0}\binom{a+b}{k}A_{k,b-1},$$

which was published by Chung, Graham and Knuth (J. of comb., Vol. 1, Number1, 29-38, 2010). We shall give two proofs using generating function and bijections, respectively.

1. Introduction

The *Eulerian polynomials* $A_n(t)$ are defined by the exponential generating function

$$\sum_{n\geq 0}A_n(t)\frac{z^n}{n!}=\frac{(1-t)e^z}{e^{zt}-te^z}. \tag{1.1}$$

The classical *Eulerian numbers* $A_{n,k}$ are the coefficients of the polynomial $A_n(t)$, i.e., $A_n(t)=\sum_{k=0}^n A_{n,k}t^k$. Recently, Chung, Graham and Knuth [2] noticed that if we modify the value of $A_0(t)$, which is 1 by (1.1), by taking the convention that $A_0(t)=A_{0,0}=0$, then the following symmetrical identity holds:

$$\sum_{k\geq 0}\binom{a+b}{k}A_{k,a-1}=\sum_{k\geq 0}\binom{a+b}{k}A_{k,b-1}\qquad (a,b>0). \tag{1.2}$$

Equivalently, instead of (1.1), we define the Eulerian polynomials by the generating function

$$\sum_{n\geq 0}A_n(t)\frac{z^n}{n!}=\frac{(1-t)e^z}{e^{zt}-te^z}-1=\frac{e^z-e^{tz}}{e^{zt}-te^z}. \tag{1.3}$$

At the end of [2], the authors asked for, among other unsolved problems, a q-analog of (1.2). The aim of this paper is to give such an extension and provide two proofs, of which one is analytical and another one is combinatorial.

We first introduce some q-notations. The q-shifted factorial $(z;q)_n$ is defined by $(z;q)_n := \prod_{i=0}^{n-1}(1-zq^i)$ for any positive integer n and $(z;q)_0=1$. The q-exponential function $e(z;q)$ is defined by

$$e(z;q):=\sum_{n\geq 0}\frac{z^n}{(q;q)_n}.$$

Several q-analogs of (1.1) have been proposed in the literature (see [6]). Inspired by the recent work of Shareshian and Wachs [6] we consider the following q-analog of (1.3):

$$\sum_{n\geq 0} A_n(t,q)\frac{z^n}{(q;q)_n} = \frac{e(z;q)-e(tz;q)}{e(tz;q)-t\,e(z;q)}. \tag{1.4}$$

The *q-Eulerian polynomials* $A_n(t,q)$ have many remarkable properties analogous to Eulerian polynomials, see Shareshian and Wachs [6] and Foata and Han [4]. The *q-Eulerian numbers* $A_{n,k}(q)$ are then defined by

$$A_n(t,q) = \sum_{k=0}^{n} A_{n,k}(q)t^k \qquad (n\geq 0).$$

The first few terms of $A_{n,k}(q)$ are as follows:

$$A_{0,0}(q)=0,\ A_{1,0}(q)=1,\ A_{2,0}(q)=1,\ A_{2,1}(q)=1,\ A_{3,0}(q)=1,\ A_{3,1}(q)=2+q+q^2$$

and $A_{3,2}(q)=1$. Also, replacing t by t^{-1} and z by tz in (1.4) yields that $t^nA_n(t^{-1},q) = tA_n(t,q)$. Thus we have the symmetrical property

$$A_{n,k}(q) = A_{n,n-k-1}(q). \tag{1.5}$$

Recall that the q-binomial numbers $\begin{bmatrix} n\\ k\end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} \qquad \text{for} \quad 0\leq k\leq n,$$

and $\begin{bmatrix} n\\ k\end{bmatrix}_q = 0$ if $k<0$ or $k>n$.

The following symmetrical identity involving both the q-binomial coefficients $\begin{bmatrix} n\\ k\end{bmatrix}_q$ and q-Eulerian numbers $A_{n,k}(q)$ is a true q-analog of (1.2).

Theorem 1. *We have the q-symmetrical identity*

$$\sum_{k\geq 0}\begin{bmatrix} a+b\\ k\end{bmatrix}_q A_{k,a-1}(q) = \sum_{k\geq 0}\begin{bmatrix} a+b\\ k\end{bmatrix}_q A_{k,b-1}(q) \tag{1.6}$$

for any positive integers a and b.

We shall first give a generating function proof of (1.6) in Section 2 and then a combinatorial proof in Section 3. We conclude the paper with some further extensions and remarks.

2. A generating function proof of (1.6)

It follows from (1.4) that

$$\big(e(tz;q)-t\,e(z;q)\big)\sum_{n\geq 0} A_n(t,q)\frac{z^n}{(q;q)_n} = \sum_{k}\frac{(1-t^k)z^k}{(q;q)_k}. \tag{2.1}$$

Now,

$$\begin{aligned} e(tz;q)\sum_{n\geq 0} A_n(t,q)\frac{z^n}{(q;q)_n} &= \sum_k \frac{(tz)^k}{(q;q)_k}\sum_{n,i} A_{n,i}(q)t^i\frac{z^n}{(q;q)_n} \\ &= \sum_{k,n,i} \begin{bmatrix} n+k \\ k \end{bmatrix}_q A_{n,i}(q)t^{i+k}\frac{z^{n+k}}{(q;q)_{n+k}} \\ &= \sum_{k,n,i} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-k}(q)t^i\frac{z^n}{(q;q)_n}, \end{aligned}$$

and

$$\begin{aligned} t\,e(z;q)\sum_{n\geq 0} A_n(t,q)\frac{z^n}{(q;q)_n} &= t\sum_k \frac{z^k}{(q;q)_k}\sum_{n,i} A_{n,i}(q)t^i\frac{z^n}{(q;q)_n} \\ &= \sum_{k,n,i} \begin{bmatrix} n+k \\ k \end{bmatrix}_q A_{n,i}(q)t^{i+1}\frac{z^{n+k}}{(q;q)_{n+k}} \\ &= \sum_{k,n,i} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-1}(q)t^i\frac{z^n}{(q;q)_n}. \end{aligned}$$

Substituting the last two expressions in (2.1) and identifying the coefficients of $t^iz^n/(q;q)_n$ of both sides, we obtain

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-k}(q) - \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-1}(q) = \begin{cases} 1, & \text{if } i=0\neq n, \\ -1, & \text{if } i=n\neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Setting $i=a$, $n=a+b$ and using the symmetrical property (1.5) we obtain (1.6). □

We can also derive q-analogs of other identities in [2]. For example, let $H_n(t;q) = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q t^i$ be the Rogers-Szegö polynomials, see [1, p. 49]. Then

$$e(tz;q)e(z;q) = \sum_{n\geq 0} \frac{z^n}{(q;q)_n}H_n(t;q).$$

If we multiply (1.4) with $e(tz;q)^2 - t^2e(z;q)^2 = (e(tz;q)+te(z;q))(e(tz;q)-te(z;q))$, then the right-hand side is equal to

$$\begin{aligned} &(e(tz;q)+te(z;q))(e(z;q)-e(tz;q)) \\ &\qquad = \sum_{n\geq 0} \frac{z^n}{(q;q)_n}\left((1-t)H_n(t;q) + (t-t^n)H_n(1;q)\right). \end{aligned}$$

On the other hand, we have

$$(e(tz;q))^2\sum_{n\geq 0}A_n(t,q)\frac{z^n}{(q;q)_n}=\sum_{n,i,k}H_k(1;q)\begin{bmatrix}n\\k\end{bmatrix}_q A_{n-k,i-k}(q)t^i\frac{z^n}{(q;q)_n},$$

and

$$t^2(e(z;q))^2\sum_{n\geq 0}A_n(t,q)\frac{z^n}{(q;q)_n}=\sum_{n,i,k}H_k(1;q)\begin{bmatrix}n\\k\end{bmatrix}_q A_{n-k,i-2}(q)t^i\frac{z^n}{(q;q)_n}.$$

Hence, identifying the coefficients of $t^iz^n/(q;q)_n$ in these expressions yields

$$\begin{aligned}&\sum_k H_k(1;q)\begin{bmatrix}n\\k\end{bmatrix}_q A_{n-k,i-k}(q)-\sum_k H_k(1;q)\begin{bmatrix}n\\k\end{bmatrix}_q A_{n-k,i-2}(q)\\ &=\begin{bmatrix}n\\i\end{bmatrix}_q-\begin{bmatrix}n\\i-1\end{bmatrix}_q+\begin{cases}H_n(1;q), & \text{if } i=1\neq n,\\ -H_n(1;q), & \text{if } i=n\neq 1,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Of course, we can also multiply (1.4) with $e(tz;q)^r-t^re(z;q)^r$ for any integer $r\geq 1$ and derive a q-analog of the more general (and complicated) identity in [2].

3. A combinatorial proof of (1.6)

For each permutation $\pi=\pi_1\pi_2\ldots\pi_n$ of $[n]:=\{1,\ldots,n\}$, define the following four statistics:

$$\begin{aligned}\mathrm{exc}(\pi)&:=|\{i:1\leq i\leq n,\pi_i>i\}|;\\ \mathrm{des}(\pi)&:=|\{i:1\leq i\leq n-1,\pi_i>\pi_{i+1}\}|;\\ \mathrm{maj}(\pi)&:=\sum_{\pi_i>\pi_{i+1}}i;\\ \mathrm{inv}(\pi)&:=|\{(i,j):i<j,\pi_i>\pi_j\}|;\end{aligned}$$

called number of *excedances*, number of *descents*, *major index* and *inversion number*, respectively. It is well-known that the Eulerian number $A_{n,k}$ counts the number of permutation of $[n]$ with k descents or k excedances.

Let $\mathfrak{S}_n$ be the set of permutations of $[n]$. Sherashian and Wachs [6] prove that

$$A_n(t,q)=\sum_{\pi\in\mathfrak{S}_n}q^{(\mathrm{maj}-\mathrm{exc})\pi}t^{\mathrm{exc}\pi}.$$

For our purpose we shall use another interpretation of $A_n(t,q)$ due to Foata and Han [3,4]. This interpretation is based on Gessel's hook factorization of permutations [5], that we recall now. A word $w=x_1x_2\ldots x_m$ is called a *hook* if $x_1>x_2$ and either $m=2$, or $m\geq 3$ and $x_2<x_3<\ldots<x_m$. Clearly, each permutation $\pi=\pi_1\pi_2\ldots\pi_n$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2...\tau_r$, where p is an increasing word and

each factor τ_1, τ_2, $\ldots$, τ_k is a hook. To derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. For each i let $\mathrm{inv}\tau_i$ denote the number of inversions of τ_i and define

$$\mathrm{lec}(\pi) := \sum_{1\le i\le k} \mathrm{inv}(\tau_i). \tag{3.1}$$

For example, the hook factorization of $\pi = 1\,3\,4\,14\,12\,2\,5\,11\,15\,8\,6\,7\,13\,9\,10$ is

$$1\,3\,4\,14\,|12\,2\,5\,11\,15\,|8\,6\,7\,|13\,9\,10.$$

Hence $p = 1\,3\,4\,14$, $\tau_1 = 12\,2\,5\,11\,15$, $\tau_2 = 8\,6\,7$, $\tau_3 = 13\,9\,10$ and

$$\mathrm{lec}(\pi) = \mathrm{inv}(12\,2\,5\,11\,15) + \mathrm{inv}(8\,6\,7) + \mathrm{inv}(13\,9\,10) = 7.$$

Let $p\tau_1\tau_2...\tau_r$ be the hook factorization of a permutation π. Let $\mathcal{A}_0$ (resp. $\mathcal{A}_i$ $(1 \le i \le r)$) denote the set of all letters in the word p (resp. in the hook τ_i). We call $\mathcal{A}_0 = \mathrm{cont}(p)$ (resp. $\mathcal{A}_i = \mathrm{cont}(\tau_i)$ $(1 \le i \le r)$) the *content* of p (resp. of hook τ_i) and *content* of π the sequence $\mathrm{Cont}(\pi) = (\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r)$. The statistic $(\mathrm{inv} - \mathrm{lec})\pi$ is equal to the number of pairs (k, l) such that $k \in \mathcal{A}_i$, $l \in \mathcal{A}_j$, $k > l$ and $i < j$, a number we shall denote by $\mathrm{inv}(\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r)$.

From Foata and Han [3, 4] we derive the following combinatorial interpretations:

$$A_n(t, q) = \sum_{\pi\in\mathfrak{S}_n} q^{(\mathrm{inv}-\mathrm{lec})\pi} t^{\mathrm{lec}\pi}.$$

Therefore

$$A_{n,k}(q) = \sum_{\substack{\pi\in\mathfrak{S}_n\\ \mathrm{lec}\pi=k}} q^{(\mathrm{inv}-\mathrm{lec})\pi}. \tag{3.2}$$

Recall [1] that the q-multinomial coefficient

$$\begin{bmatrix} n \\ a_0, a_1, \ldots, a_k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_{a_0}(q; q)_{a_1}\cdots(q; q)_{a_k}}$$

has the following interpretation

$$\begin{bmatrix} n \\ a_0, a_1, \ldots, a_k \end{bmatrix}_q = \sum_{(\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_k)} q^{\mathrm{inv}(\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_k)}, \tag{3.3}$$

where the sum is over all ordered partitions $(\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k)$ of $[n]$ such that $|\mathcal{A}_i| = a_i$, $0 \le i \le k$.

We will give a combinatorial proof of (1.6) using (3.2) and (3.3). As a warm up we first prove the symmetric property (1.5) by constructing an explicit involution on permutations.

Lemma 2. *There is an involution $\pi \mapsto \sigma$ on $\mathfrak{S}_n$ satisfying*

$$\mathit{lec}(\pi) = n - 1 - \mathit{lec}(\sigma), \quad \text{and} \quad (\mathit{inv} - \mathit{lec})\pi = (\mathit{inv} - \mathit{lec})\sigma.$$

Proof. Let τ be a hook with $\mathrm{inv}(\tau)=k$ and $\mathrm{cont}(\tau)=\{x_1,\ldots,x_m\}$, where $x_1<\ldots<x_m$. Define $d(\tau)=x_{m-k+1}x_1\ldots x_{m-k}x_{m-k+2}\ldots x_m$. Clearly $d(\tau)$ is the unique hook satisfying $\mathrm{cont}(d(\tau))=\mathrm{cont}(\tau)$ and $\mathrm{inv}(d(\tau))=m-k=|\mathrm{cont}(\tau)|-\mathrm{inv}(\tau)$.

Let τ be a word with $\mathrm{inv}(\tau)=k$ and $\mathrm{cont}(\tau)=\{x_1,\ldots,x_m\}$, where $x_1<\ldots<x_m$. Define $d'(\tau)=x_{m-k}x_1\ldots x_{m-k-1}x_{m-k+1}\ldots x_m$. Clearly $d'(\tau)$ is the unique word satisfying $\mathrm{cont}(d'(\tau))=\mathrm{cont}(\tau)$ and $\mathrm{inv}(d'(\tau))=m-k-1=|\mathrm{cont}(\tau)|-\mathrm{inv}(\tau)-1$.

Let $\pi=p\tau_1\tau_2\ldots\tau_r$ be the hook factorization of $\pi\in\mathfrak{S}_n$.

- If $p\neq\emptyset$, let $\sigma=d'(p)d(\tau_1)d(\tau_2),\ldots,d(\tau_r)$.
- If $p=\emptyset$, let $\sigma=d'(\tau_1)d(\tau_2)d(\tau_3)\ldots d(\tau_r)$.

Since d and d' are two involutions, it is routine to check that such a mapping is an involution with the required properties. □

For each fixed positive integer n, a *two-pix-permutations of* $[n]$ is a sequence of words

$$\mathbf{v}=(p_1,\tau_1,\tau_2,\ldots,\tau_{r-1},\tau_r,p_2) \tag{3.4}$$

satisfying the following conditions:

(C1) p_1 and p_2 are two increasing words, possibly empty;
(C2) $\tau_1,\ldots,\tau_r$ are hooks for some positive integer r;
(C3) The concatenation $p_1\tau_1\tau_2\ldots\tau_{r-1}\tau_r p_2$ of all components of $\mathbf{v}$ is a permutation of $[n]$.

We also extend the two statistics to the two-pix-permutations by

$$\begin{aligned}\mathrm{lec}(\mathbf{v}) &= \mathrm{inv}(\tau_1)+\mathrm{inv}(\tau_2)+\cdots+\mathrm{inv}(\tau_r),\\ \mathrm{inv}(\mathbf{v}) &= \mathrm{inv}(p_1\tau_1\tau_2\ldots\tau_{r-1}\tau_r p_2).\end{aligned}$$

It follows that

$$(\mathrm{inv}-\mathrm{lec})\mathbf{v}=\mathrm{inv}(\mathrm{cont}(p_1),\mathrm{cont}(\tau_1),\mathrm{cont}(\tau_2),\ldots,\mathrm{cont}(\tau_r),\mathrm{cont}(p_2)). \tag{3.5}$$

Lemma 3. *The generating function of all two-pix-permutations* $\mathbf{v}$ *of* n *such that* $lec(\mathbf{v})=s$ *by the statistic* $inv-lec$ *is*

$$\sum_{k\geq 0}\begin{bmatrix}n\\k\end{bmatrix}_q A_{k,s}(q). \tag{3.6}$$

Proof. By the hook factorization, the two-pix-permutation $\mathbf{v}$ in (3.4) is in bijection with the pair (σ,p_2), where $\sigma=p_1\tau_1\tau_2\ldots\tau_{r-1}\tau_r$ is a permutation on $[n]\setminus\mathrm{cont}(p_2)$ and p_2 is an increasing word. Thus, by (3.2), (3.3) and (3.5), the generating function of all two-pix-permutations $\mathbf{v}$ of $[n]$ such that $\mathrm{lec}(\mathbf{v})=s$ and $|p_2|=n-k$ with respect to the weight $q^{(\mathrm{inv}-\mathrm{lec})(\mathbf{v})}$ is $\begin{bmatrix}n\\k\end{bmatrix}_q A_{k,s}(q)$. □

Lemma 4. *There is a bijection* $\mathbf{v}\mapsto\mathbf{u}$ *on the set of all two-pix-permutations of* $[n]$ *satisfying*

$$lec(\mathbf{v})=n-2-lec(\mathbf{u}),\quad \text{and}\quad (inv-lec)\mathbf{v}=(inv-lec)\mathbf{u}.$$

Proof. We give an explicit construction of the bijection. Let $\mathbf{v}$ be a two-pix-permutation and write

$$\mathbf{v} = (\tau_0, \tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r, \tau_{r+1}),$$

where $\tau_0 = p_1$ and $\tau_{r+1} = p_2$. If τ_i (resp. τ_j) is the leftmost (resp. rightmost) non-empty word (Clearly $i = 0, 1$ and $j = r, r+1$), we can write $\mathbf{v}$ in the following compact way by removing the empty words at the beginning or at the end

$$\mathbf{v} = (\tau_i, \tau_{i+1}, \ldots, \tau_{j-1}, \tau_j). \tag{3.7}$$

It is easy to see that the above procedure is reversible by adding some necessary empty words at the two ends of the compact form (3.7). Now we work with the compact form. Recall that

$$(\mathrm{inv} - \mathrm{lec})\mathbf{v} = \mathrm{inv}(\mathrm{cont}(\tau_i), \mathrm{cont}(\tau_{i+1}), \ldots, \mathrm{cont}(\tau_{j-1}), \mathrm{cont}(\tau_j)) \tag{3.8}$$

and $\mathrm{lec}(\mathbf{v}) = \sum_{k=i}^{j} \mathrm{lec}(\tau_k)$.

If $i = j$, then only one word τ_i is in the sequence $\mathbf{v}$. We define $\mathbf{u} = (\emptyset, \sigma_i, \emptyset)$, where σ_i is the unique word (hook) with content $[n]$ such that $\mathrm{lec}(\sigma_i) = n - 2 - \mathrm{lec}(\tau_i)$.

If $j > i$, we define the two-pix-permutation $\mathbf{u}$ as follows

$$\mathbf{u} = (d'(\tau_i), d(\tau_{i+1}), d(\tau_{i+2}), \ldots, d(\tau_{j-1}), d'(\tau_j)),$$

where d and d' are two involutions defined in the proof of Lemma 2.

Since $\mathrm{lec}(d'(\tau_i)) = |\mathrm{cont}(\tau_i)| - 1 - \mathrm{lec}(\tau_i)$, $\mathrm{lec}(d'(\tau_j)) = |\mathrm{cont}(\tau_j)| - 1 - \mathrm{lec}(\tau_j)$ and $\mathrm{lec}(d'(\tau_k)) = |\mathrm{cont}(\tau_k)| - \mathrm{lec}(\tau_k =$ for $k \neq i, j$, we have

$$\mathrm{lec}(\mathbf{u}) = \sum |\mathrm{cont}(\tau_k)| - 2 - \mathrm{lec}(\mathbf{v}) = n - 2 - \mathrm{lec}(\mathbf{v}).$$

Finally it follows from (3.8) that $(\mathrm{inv} - \mathrm{lec})\mathbf{u} = (\mathrm{inv} - \mathrm{lec})\mathbf{v}$.

We give an example to illustrate the bijection. Let $\mathbf{v} = (27, 6389, 514, \emptyset)$. Then $\mathbf{v}$ is a two-pix-permutation of $[9]$ and $\mathrm{inv}(\mathbf{v}) = 19, \mathrm{lec}(\mathbf{v}) = 3, (\mathrm{inv} - \mathrm{lec})\mathbf{v} = 16$. The compact form is $(27, 6389, 514)$, so that

$$\mathbf{u} = (d'(27), d(6389), d'(514) = (72, 9368, 145).$$

Since the first word 72 is not increasing, we obtain the standard form by adding the empty word at the beginning. So that $\mathbf{u} = (\emptyset, 72, 9368, 145)$. Hence $\mathrm{inv}(\mathbf{u}) = 20, \mathrm{lec}(\mathbf{u}) = 4, (\mathrm{inv} - \mathrm{lec})\mathbf{u} = 16$. □

Combining Lemmas 2, 3 and 4 we obtain a combinatorial proof of (1.6).

4. Further extensions and remarks

The classical Eulerian polynomials correspond to the generating function of descent numbers of symmetric groups. Let $r \geq 1$ be an integer. As a natural extension of (1.4), we consider the polynomial $A_n^{(r)}(t,q)$ defined by the following generating function

$$\frac{e(z;q^r)-e(t^rz;q^r)}{e(t^rz;q^r)-te(z;q^r)}=\sum_{n\geq 1}A_n^{(r)}(t,q)\frac{z^n}{(q^r;q^r)_n}. \tag{4.1}$$

It is easy to see that $A_n^{(r)}(1,1)=r^n n!$, which is the cardinality of $C_r\wr\mathfrak{S}_n$, that is, the wreath product of a cyclic group C_r of order r with the symmetric group $\mathfrak{S}_n$. Introduce the coefficients $A_{n,i}^{(r)}$ by

$$A_n^{(r)}(t,q)=\sum_i A_{n,i}^{(r)}(q)t^i. \tag{4.2}$$

The generating function proof of (1.6) can be applied to derive immediately the following identity

$$\sum_k \begin{bmatrix} n\\ k\end{bmatrix}_{q^r} A_{k,rn-i-1}^{(r)}(q)=\sum_k \begin{bmatrix} n\\ k\end{bmatrix}_{q^r} A_{k,i-1}^{(r)}(q) \tag{4.3}$$

for positive integers i and n such that $i\neq rn$.

Let r and n be two positive integers. Let $\mathcal{A}=\{a,b,\dots\}$ be any subset of $[n]$. We define the *r-colored* of $\mathcal{A}$ by

$$\mathcal{A}^r:=\{a^1,b^1,\dots,a^2,b^2,\dots,\dots,a^r,b^r,\dots\}.$$

Define the order of $[n]^r$ by

$$1^1<1^2\dots<1^r<2^1<2^2\dots<2^r<\dots<n^1<n^2<\dots<n^r.$$

A *pix-r-colored-word* is a sequence of colored words

$$\mathbf{w}=(p,\tau_1,\tau_2,\dots,\tau_k)$$

satisfying the following conditions:

- (C1) p is an increasing word, with content $\mathcal{A}_0^r$, the r-colored of $\mathcal{A}_0$, possibly empty;
- (C2) τ_i $(1\leq i\leq k)$ are hooks, with content $\mathcal{A}_i^r$, the r-colored of $\mathcal{A}_i$, and the positive integer k is not fixed;
- (C3) $(\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_k)$ is an ordered partition of $[n]$.

Let $\mathscr{W}_{n,r}$ be the set of all pix-r-colored-words of $[n]$. For each $\mathbf{w}\in\mathscr{W}_{n,r}$ we define two statistics by

$$\begin{aligned}\mathrm{inv}_r(\mathbf{w})&=\mathrm{inv}(p\tau_1\tau_2\dots\tau_{r-1}\tau_r),\\ \mathrm{lec}_r(\mathbf{w})&=\mathrm{inv}(\tau_1)+\cdots+\mathrm{inv}(\tau_k).\end{aligned}$$

Proposition 5. *Let $A_n^{(r)}(t,q)$ be defined by (4.1). Then*

$$\sum_{\sigma\in\mathscr{W}_{n,r}} q^{(inv_r-lec_r)(\sigma)}t^{lec_r(\sigma)}=A_n^{(r)}(t,q).$$

Proof. By definition and (3.3) we have

$$\begin{aligned}
\sum_{\sigma\in\mathscr{W}_{n,r}} q^{(\mathrm{inv}_r-\mathrm{lec}_r)(\sigma)} t^{\mathrm{lec}_r(\sigma)} &= \sum_{\substack{a_0+a_1+\ldots+a_k=n\\ a_i\ge 1}} \prod_{1\le i\le k} P_{ra_i}(t) \sum_{\substack{(\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_k)\\ \#\mathcal{A}_i=a_i}} q^{\mathrm{inv}_r(\mathcal{A}_0,\mathcal{A}_1,\ldots,A_k)}\\
&= \sum_{\substack{a_0+a_1+\ldots+a_k=n\\ a_i\ge 1}} \begin{bmatrix} n \\ a_0,a_1,\ldots,a_k\end{bmatrix}_{q^r} \prod_{1\le i\le k} P_{ra_i}(t),
\end{aligned}$$

where $P_m(t) := t+t^2+\ldots+t^{m-1}$. So the generating function is

$$\begin{aligned}
&\sum_{n,i}\sum_{\sigma\in\mathscr{W}_{n,r}} q^{(\mathrm{inv}_r-\mathrm{lec}_r)(\sigma)} t^{\mathrm{lec}_r(\sigma)} \frac{z^n}{(q^r;q^r)_n}\\
&\quad= \sum_{n\ge 0}\sum_{\substack{a_0+a_1+\ldots+a_k=n\\ a_i\ge 1}} \begin{bmatrix} n \\ a_0,a_1,\ldots,a_k\end{bmatrix}_{q^r} \prod_{1\le i\le k} P_{ra_i}(t) \frac{z^n}{(q^r;q^r)_n}\\
&\quad= \left(\sum_{a\ge 0}\frac{z^a}{(q^r;q^r)_a}\right)\left(1-\sum_{b\ge 1} P_{rb}(t)\frac{z^b}{(q^r;q^r)_b}\right)^{-1}\\
&\quad= e(z;q^r)\left(1-\sum_{b\ge 1}\frac{t-t^{rb}}{1-s}\frac{z^b}{(q^r;q^r)_b}\right)^{-1}\\
&\quad= \frac{(1-t)e(z;q^r)}{e(t^r z;q^r)-t\,e(z;q^r)}.
\end{aligned}$$

This completes the proof in view of (4.1). $\square$

Similarly, we can define the *two-pix-r-colored-words* to be sequences of colored words

$$\mathbf{w} = (p_1,\tau_1,\tau_2,\ldots,\tau_k,p_2)$$

with the following conditions:

- (C1) p_1 and p_2 are increasing words, with content $\mathcal{A}_0^r$ and $\mathcal{B}_0^r$, possibly empty;
- (C2) τ_i $(1\le i\le k)$ are hooks, with content $\mathcal{A}_i^r$, the r-colored of $\mathcal{A}_i$, and the positive integer k is not fixed;
- (C3) $(\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_k,\mathcal{B}_0)$ is an ordered partition of $[n]$.

Clearly we can give a combinatorial proof of (4.3) by applying the following generalization of Lemma 4.

Proposition 6. *There is a bijection* $\mathbf{v}\mapsto\mathbf{u}$ *on the set of all two-pix-r-colored-words such that*

$$lec_r\mathbf{v} = rn-2-lec_r\mathbf{u}, \quad \text{and} \quad (inv_r-lec_r)\mathbf{v} = (inv_r-lec_r)\mathbf{u}.$$

When $r = 1$, through Gessel's hook factorization we can translate the statistic $(\mathrm{inv}_r, \mathrm{lec}_r)$ onto $\mathfrak{S}_n$. It would be interesting to see whether there is an analogue hook factorization for general $r \geq 1$ so that we can translate our $(\mathrm{inv}_r, \mathrm{lec}_r)$ defined on $\mathscr{W}_{n,r}$ onto $C_r \wr \mathfrak{S}_n$.

We conclude this paper with another symmetric identity for the Eulerian numbers. Notice that for any positive integers n and k we have

$$\begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_{-1} = 0 \quad \text{and} \quad \begin{bmatrix} 2n \\ 2k \end{bmatrix}_{-1} = \binom{n}{k}.$$

It is known [8, Corollary 6.2] that if $dk = n$ and ω_d is a primitive d^{th} root of unity, then

$$A_n(t, \omega_d) = A_k(t) \left(\frac{1-t^d}{1-t} \right)^k.$$

In particular, if $d = 2$, then $\omega_d = -1$. Hence, assuming that $a + b$ is even, substituting $q = -1$ in (1.6) yields

$$\sum_{k\geq 0} \binom{\frac{a+b}{2}}{k} \sum_{i+j=a-1} \binom{k}{i} A_{k,j} = \sum_{k\geq 0} \binom{\frac{a+b}{2}}{k} \sum_{i+j=b-1} \binom{k}{i} A_{k,j}. \tag{4.4}$$

This identity can be rephrased as follows:

$$\sum_{k\geq 0} \binom{c+d}{k} \sum_{i+j=2c-1} \binom{k}{i} A_{k,j} = \sum_{k\geq 0} \binom{c+d}{k} \sum_{i+j=2d-1} \binom{k}{i} A_{k,j}$$

and

$$\sum_{k\geq 0} \binom{c+d-1}{k} \sum_{i+j=2(c-1)} \binom{k}{i} A_{k,j} = \sum_{k\geq 0} \binom{c+d-1}{k} \sum_{i+j=2(d-1)} \binom{k}{i} A_{k,j}$$

for any positive integers c and d.

The last two symmetrical identities involving binomial coefficients and Eulerian numbers cry out for a combinatorial interpretation.

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